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Coexistence problem for two competing species models with density-dependent diffusion

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Abstract. We study the pattern formation of the Gause-Lotka-Volterra system of competition and nonlinear diffusion. This problem is related to segregation patterns between two competing species. It is shown that coexistence is possible by the effect of cross-population pressure in the situation where the inter-specific competition is stronger than the intra-specific one.

1. Introduction

In recent years, reaction-diffusion equation models have been proposed for the study of population dynamics. Shigesada et al. [17] proposed a one dimensinonal model of two competing species with self- and cross-population pressures

$$
(1.1)\qquad \begin{aligned}
u_t &= [(d_{11}+d_{12}v)u]_{xx} + (r_1 - a_{11}u - a_{12}v)u,\\
v_t &= [(d_{22}+d_{21}u)v]_{xx} + (r_2 - a_{21}u - a_{22}v)v,
\end{aligned}
$$

where u, v are the population densities of the two competing species, d_{ii} and d_{ij} $(i\neq j)$ are the self- and cross-diffusion rates, r_i are the intrinsic growth rates, a_{ii} and a_{ij} $(i\neq j)$ are the intra- and interspecific coefficients of competition. If $d_{ij}=0$ $(i\neq j)$, (1.1) is reduced to a normal competition-diffusion equation that has been extensively investigated. Kishimoto [9] proved that any nonnegative non-constant steady state solutions are unstable under zero flux boundary conditions. This result ecologically interprets that there occurs no spatial segregation between two competing species. On the other hand, Mimura and Kawasaki [12] showed that for suitable $d_{12}>0$ and/or $d_{21}>0$ there exist new non-constant steady state solutions bifurcating from a trivial solution

$$
(\bar{u}, \bar{v}) = \left(\frac{r_1a_{22}-r_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}}, \frac{r_2a_{11}-r_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}\right)
$$

when r_i, a_{ij} $(i, j=1, 2)$ are chosen to satisfy $a_{12}/a_{22}<r_1/r_2<a_{11}/a_{21}$. This occurs on the basis of the cross-diffusion induced instability.

From an ecological point of view, it is quite interesting to study coexistence problem under $a_{11}/a_{21}<a_{12}/a_{22}$. The reason is that when $d_{ij}=0$ $(i\neq j)$, (1.1) never exhibit coexistence of two species and only one species can survive in com-

petition. This indicates the competitive exclusion principle (Gause [6]). However, still for $d_{ij}=0\,(i\neq j)$ it is known that coexistence of two species is possible in related models to (1.1) under $a_{11}/a_{21}<r_1/r_2<a_{12}/a_{22}$. Levin [10] examined spatially discretized models of (1.1), Matano and Mimura [11] considered 2-dimensional space models of (1.1) in suitable non-convex domains. Although these results are established, the study of (1.1) $(d_{ij}>0)$ was still left open.

In this paper, we consider (1.1) for r_i, a_{ij} except $a_{12}/a_{22}<r_1/r_2<a_{11}/a_{21}$. To do so, we deal with a simple case when $d_{21}=0$, and taking

$$\frac{a_{11}r_2}{a_{21}r_1}=\lambda,\quad \frac{a_{12}r_2}{a_{22}r_1}=\mu,\quad \frac{r_2d_{12}}{a_{22}d_{11}}=\alpha,$$

$$\frac{r_1}{d_{11}}=\beta,\quad \frac{d_{22}}{d_{11}}=d,\quad \frac{r_2}{d_{22}}=\gamma,$$

rewrite (1.1) as

$$\begin{aligned} u_t &= [(1+\alpha v)u]_{xx} + \beta(1-\lambda u-\mu v)u, \\ v_t &= dv_{xx} \qquad\qquad\quad + \gamma(1-u-v)v, \end{aligned} \qquad t>0,\ x\in(0,1)\equiv I. \tag{1.2}$$

The boundary conditions are assumed to be

$$u_x=v_x=0,\ t>0,\ x\in\partial I. \tag{1.3}$$

The asymptotic behavior of solutions of (1.2), (1.3) with $\alpha=0$ can be qualitatively classified into four cases (Figure 1);

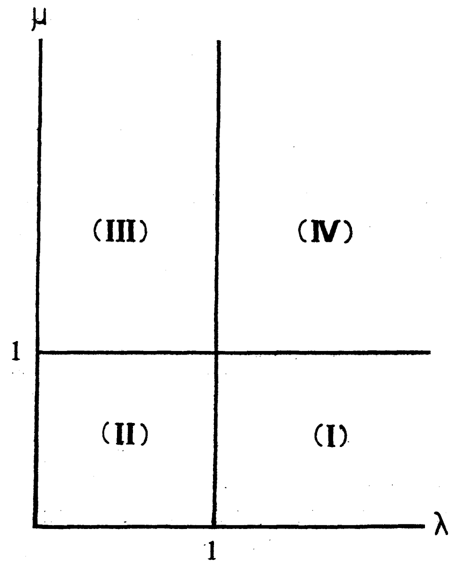

Figure 1. Classification of constant steady states of (1.1) with $\alpha=0$ in (λ,μ)-space.

(I) ($\mu<1<\lambda$) There is no non-constant steady state solution and

$$\lim_{t\to\infty}(u(t, x), v(t, x)) = (\bar{u}, \bar{v}) \text{ uniformly in } x\in I.$$

(II) (μ, $\lambda<1$) There is no non-constant steady state solution and

$$\lim_{t\to\infty}(u(t, x), v(t, x)) = (1/\lambda, 0) \text{ uniformly in } x\in I.$$

(III) ($\lambda<1<\mu$) There may be non-constant steady state solutions but only $(1/\lambda, 0)$ and $(0,1)$ are stable steady states. Which species can survive in competition depends on initial data.

(IV) ($1<\mu$, λ) There is no non-constant steady state solution and

$$\lim_{t\to\infty}(u(t, x), v(t, x)) = (0, 1) \text{ uniformly in } x\in I.$$

The proofs are shown in, for instance, Kishimoto [9], Hsu [8].

There are a few difficulties in showing the existence of (stable, if possible) non-constant steady state solutions of (1.2), (1.3) with $\alpha>0$. First (1.2) does not possess the property of order preserving, though the system with $\alpha=0$ has this property. Secondly, for the case (II), (IV), there occurs no bifurcation from the trivial solutions. On the other hand, for (III), there is a bifurcation from the trivial coexisting steady state $(\bar{u}, \bar{v})$. However, the resulting new non-constant solutions are unstable. Therefore we must trace the secondary bifurcation which seems to be a tough problem from an analytical viewpoint.

To be free from these difficulties, we restrict $\beta\ll 1$ for mathematical simplicity, so that we are able to study the stationary problem of (1.2), (1.3) in the limit $\beta\to 0$, following the approach by Nishiura [14]. In Sections 2 and 3, using the singular perturbation methods in the case when d is sufficiently small and the (finite dimensional) degree theory, we show the existence of non-constant steady state solutions of (1.2), (1.3) with $\beta\to 0$ for some λ, μ in the cases (II) and (III). In Section 4, we deal with the case β is not zero but sufficiently small and for sufficiently small d, construct non-constant, nonnegative steady state solutions exhibiting spatial segregation. In Section 5, we give the proofs of the results. Unfortunately, the stability problem is not yet able to be discussed here. Therefore, in Section 6, we will show some numerical simulations of (1.1), which confirm that there exist stable non-constant steady state solutions for some λ, μ in the cases (II) and (III). We would like to emphasize that coexistence of two competing is possible due to the migration of cross-population pressure (see Figure 7).

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2. The shadow system ($\beta \downarrow 0$).

Through the transformation

$$(1+\alpha v)u = w, \tag{2.1}$$

we rewrite the stationary problem of (1.2), (1.3) as

$$\begin{cases} 0 = w_{xx} + \beta\left(1 - \dfrac{\lambda w}{1+\alpha v} - \mu v\right)\dfrac{w}{1+\alpha v}, \\ 0 = Dv_{xx} + \left(1 - \dfrac{w}{1+\alpha v} - v\right)v, \end{cases} \quad x \in I, \tag{2.2}$$

$$w_x = v_x = 0, \quad x \in \partial I, \tag{2.3}$$

where $D = d/\gamma$. If u, v are both nonnegative, the stationary problem of (1.2), (1.3) is equivalent to (2.2) and (2.3) through (2.1). Therefore, henceforth we consider the problem (2.2) and (2.3). Put, for simplicity,

$$f(w, v) = \left(1 - \frac{\lambda w}{1+\alpha v} - \mu v\right)\frac{w}{1+\alpha v}$$

and

$$g(w, v) = \left(1 - \frac{w}{1+\alpha v} - v\right)v.$$

The nullclines of f and g look like Figure 2.

In this section, we consider the limiting case of (2.2) as $\beta \to 0$. The resulting system is

$$\int_I f(w, v)dx = 0, \tag{2.4}$$

$$0 = Dv_{xx} + g(w, v), \quad x \in I, \tag{2.5}$$

$$v_x = 0, \quad x \in \partial I, \tag{2.6}$$

where $w = c$ is a constant function because of the boundary condition (2.3).

DEFINITION 1. We call a solution of (2.4)–(2.6) by a triplet (D, c, v) satisfying

i) v is non-negative, non-constant and is of class $C^1(\bar{I}) \cap C^2(I)$,

ii) D, c are positive constants,

iii) (D, c, v) satisfies (2.4)–(2.6).

When $v(x)$ is strictly monotone increasing, $u(x)$ is strictly monotone decreasing from (2.1). Such inhomogeneity of (u, v) exhibits spatial segregation of two competing species. Thus, we will be concerned with solutions of (2.4)–(2.6), where one of the components $v(x)$ is strictly monotone increasing in $\bar{I}$.

LEMMA 1. *Suppose that* α, c *satisfy* i) $0<\alpha\leqq 1$ *or* ii) $0<c\leqq 1$ *or* iii) *for fixed* $\alpha(>1)$, $(\alpha+1)^2/4\alpha\leqq c$. *Then there is no solution of* (2.4)–(2.6).

Henceforth, we fix $\alpha(>1)$ arbitrarily. Then c must satisfy $1<c<(\alpha+1)^2/4\alpha$.

Following [14], we investigate the problem (2.4)–(2.6). We first fix c and then consider the problem (2.5), (2.6). This problem can be fully analyzed, since it is simplified to be the scalar equation with respect to v. Let $v_-(c)$, $v_0(c)$, $v_+(c)$ $(v_-<v_0<v_+)$ be three solutions of $g(c, v)\equiv(1-c/(1+\alpha v)-v)v=0$ for fixed c, that is, $v_-(c)\equiv 0$, $v_0(c)=[(\alpha-1)-\{(\alpha+1)^2-4\alpha c\}^{1/2}]/2\alpha$ and $v_+(c)=[(\alpha-1)+\{(\alpha+1)^2-4\alpha c\}^{1/2}]/2\alpha$. Define $G(v; c)$ by

$$G(v;\, c) = \int_{v_0(c)}^{v} g(c,\, s)ds,$$

and then define $E_\pm(c)$ and $E^*(c)$ by $E_\pm(c)=G(v_\pm(c); c)$ and $E^*(c)=\min\,(E_-(c), E_+(c))$, respectively. Put $A_\delta=(1+\delta, (\alpha+1)^2/4\alpha-\delta)$ for any $\delta\geqq 0$ and write A_0 as A simply. Define T, $\bar{T}$, $\bar{T}_\delta$ $(\delta>0)$ by

$$T = \cup_{c\in A}(0,\, E^*(c)) \times \{c\},$$

$$\bar{T} = \cup_{c\in A}[0,\, E^*(c)) \times \{c\},$$

and

$$\bar{T}_\delta = \cup_{c\in A_\delta}[0,\, E^*(c)) \times \{c\}.$$

From Lemma 3.1 in Nishiura [14], we have

LEMMA 2. *Consider* (2.5), (2.6) *for fixed* c $(1<c<(\alpha+1)^2/4\alpha)$. *Then there exists an E-parameter family of solutions* $(D(E, c), v(x; E, c))$ *for* $0<E<E^*(c)$ *where* v *is nonnegative and strictly monotone increasing with respect to* $x\in\bar{I}$. *Moreover*, (D, v) *satisfies*

(i) $v(x; E, c)\in C^0(\bar{I}\times\bar{T}_\delta)\cap C^\infty(\bar{I}\times T)$,

$$\frac{\partial v}{\partial c}(x;\, E,\, c)\in C^0(\bar{I}\times\bar{T}_\delta)$$

for sufficiently small $\delta>0$,

(ii) $\lim_{E\downarrow 0} v(x; E, c) = v_0(c)$,

$$\lim_{E\uparrow E^*(c)} v(x;E,c) = \begin{cases} v_-(c) \text{ compact uniformly in } [0,1) \text{ if } E_-(c)<E_+(c); \\ v_+(c) \text{ compact uniformly in } (0,1] \text{ if } E_-(c)>E_+(c), \\ v_m(c) = \begin{cases} v_-(c)\ (0\leqq x<m) \\ \qquad \text{compact uniformly in } \bar{I}\backslash\{x=m\}, \\ v_+(c)\ (m<x\leqq 1) \end{cases} \end{cases}$$

where m *is defined by* $m=m(c_0)=m_+(c_0)^{1/2}/(m_+(c_0)^{1/2}+m_-(c_0)^{1/2})$ *in which* $m_\pm(c_0)=|(\partial/\partial v)g(c_0, v_\pm(c_0))|$ *with* c_0 *uniquely determined by* $E_-(c_0)=E_+(c_0)$,

(iii) $D(E, c)^{-1/2}=\int_{\xi_-(E,c)}^{\xi_+(E,c)} \{2(E-G(c, v))\}^{-1/2}dv,$

where $\xi_\pm(E, c)$ $(\xi_-<v_0<\xi_+)$ *are two consecutive zeros of* $E-G(c, v)=0$,

(iv) $D(E, c)\in C^0(\bar{T}_\delta)\cap C^\infty(T)$ *for any* $\delta>0$.

REMARK 1. From (iii) of Lemma 2, we find $\lim_{E\uparrow E^*(c)} D(E, c)=0$. Then we find by (ii) of Lemma 2 that if E is close to $E_-(c)$ $(1<c<c_0)$, there occurs a boundary layer at the right hand side $x=1$, if E is close to $E^*(c_0)$, an internal layer in the neighborhood of $x=m$, and if E is close to $E_+(c)$ $(c_0<c<(\alpha+1)^2/4\alpha)$, a boundary layer at the left hand side $x=0$.

Next we substitute the solution $v(x; E, c)$ of (2.5), (2.6) obtained in Lemma 2 into the equation (2.4)

$$0=\int_I f(c, v(x; E, c))dx = F(E, c)$$

and then look for a pair (E, c) satisfying $F(E, c)=0$ for $(E, c)\in T$.

THEOREM 1. *Let* λ *be* $1/c_0<\lambda<1$. *Then, generically, there exists a one-dimensional submanifold* S *(in* T*) of pairs* (E, c) *satisfying* $F(E, c)=0$ *such that* $\bar{S}$ *connects the two points of* $\bar{S}\cap\partial T$. *Moreover,*

(A) *when* $0<\mu<\lambda+\dfrac{1-\lambda}{v_+(c_0)}$,

$\bar{S}\cap\partial T=\{(E_-(1/\lambda), 1/\lambda), (E^*(c_0), c_0)\}$,

(B) *when* $\lambda+\dfrac{1-\lambda}{v_+(c_0)}<\mu<\lambda+\dfrac{2(1-\alpha)}{1-1/\alpha}$,

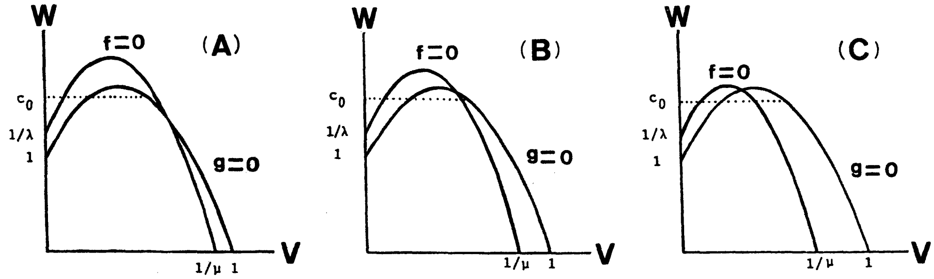
$\bar{S}\cap\partial T=\{(E_-(1/\lambda), 1/\lambda), (E_+(\bar{c}), \bar{c})\}$,

(C) *when* $\lambda+\dfrac{2(1-\lambda)}{1-1/\alpha}<\mu$,

$\bar{S}\cap\partial T=\{(E_-(1/\lambda), 1/\lambda), (0, \bar{c})\}$,

where $\bar{c}=(1+\alpha\bar{v})\bar{u}$ with $(\bar{u}, \bar{v})=((\mu-1)/(\mu-\lambda), (1-\lambda)/(\mu-\lambda))$ (*Figures* 2 (A)–(C) *show the curves* $f=g=0$ *in the cases* (A)–(C) *respectively*).

By the relation $(1+\alpha v)u=c$ and (iii) in Lemma 2, Theorem 1 implies the existence of the solution branch for (D, u, v) in $X=\mathbf{R}\times C^2(I)\times C^2(I)$. In the cases (A) and (B), it is expected from Remark 1 that there exist at least two solutions (u, v) for sufficiently small D. In the former case, one solution has a



Figures 2. The curves $f=0$, $g=0$ for different values of λ, μ, α.

boundary layer at the right hand side corresponding to $(E_-(1/\lambda), 1/\lambda)$ and the other has an internal layer in the neighborhood of $x=m^*$ (which will be defined in Theorem 3) corresponding to $(E^*(c_0), c_0)$. In the latter case, one has a boundary layer at the right hand side corresponding to $(E_-(1/\lambda), 1/\lambda)$ and the other has a boundary layer at the left hand side corresponding to $(E_+(\bar{c}), \bar{c})$. The common feature to the cases (A), (B) is that, taking D as a bifurcation parameter, primary bifurcations from the trivial solutions do not occur but a "spontaneous" bifurcation does, whereby the solution branch in X connects the two different types of singularly perturbed solutions when D is small. On the other hand, for the case (C), there occurs a primary bifurcation from the trivial solution $(\bar{u}, \bar{v}) = ((\mu-1)/(\mu-\lambda), (1-\lambda)/(\mu-\lambda))$ at $D=\{\alpha(\bar{u}-\bar{v})-1\}\bar{v}/(1+\alpha\bar{v})\pi^2$, and hence the solution branch in X connects the singularly perturbed solution and the bifurcating solution from (u, v). Thus, we find that the cases (A), (B) and the case (C) exhibit different types of bifurcations. (An analogous phenomenon is observed for prey-predator models, see Nishiura [15].)

THEOREM 2. *Let λ be $0<\lambda<1/c_0$. Then, generically, for $\mu>\lambda+\dfrac{1-\lambda}{v_+(c_0)}$ there exists a one-dimensional submanifold $\bar{S}$ (in T) of pairs (E, c) satisfying $F(E, c)=0$ such that S connects the two points of $\bar{S}\cap\partial T$. Moreover,*

(D) *when* $\lambda + \dfrac{1-\lambda}{v_+(c_0)} < \mu < \lambda + \dfrac{2(1-\lambda)}{1-1/\alpha}$,

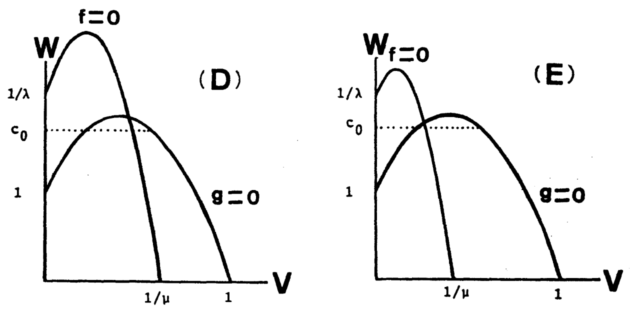
$$\bar{S}\cap\partial T = \{(E^*(c_0), c_0), (E_+(\bar{c}), \bar{c})\},$$

(E) *when* $\lambda + \dfrac{2(1-\lambda)}{1-1/\alpha} < \mu$,

$$\bar{S}\cap\partial T = \{(E^*(c_0), c_0), (0, \bar{c})\}$$

(Figures 2 (D)–(E) show the curves $f=g=0$ in the cases (D)–(E) respectively).

We find that, by taking D as a bifurcation parameter, the case (D) exhibits a



Figures 2. The curves $f=0$, $g=0$ for different values of λ, μ, α.

spontaneous bifurcation and (E) does the usual bifurcation from the trivial solution $(\bar{u}, \bar{v})$.

PROPOSITION 1. *If*

$$\text{(F)}\quad \mu \leqq \frac{\alpha-\lambda}{\alpha-1} \quad \textit{and} \quad 0<\lambda \leqq \frac{4\alpha}{(\alpha+1)^2}$$

or

$$\text{(G)}\quad \lambda \geqq \frac{1}{2}+\frac{1}{4}\left(\frac{\mu}{\alpha}+\frac{\alpha}{\mu}\right),$$

then there is no solution of (2.4)–(2.6).

Figure 3 shows the regions of (A)–(G) in the (λ, μ)-space. It is interesting

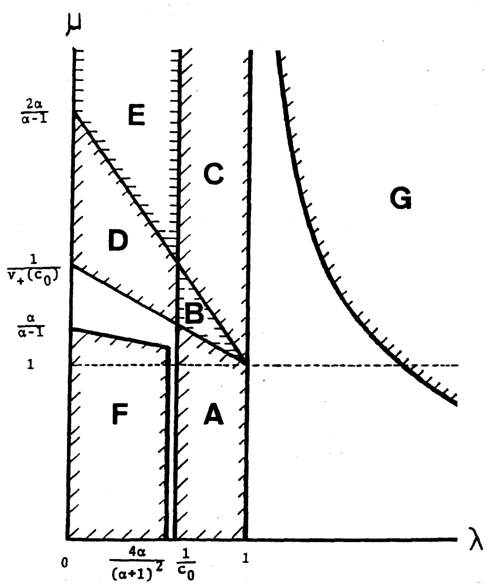


Figure 3. Existence and non-existence regions of non-constant steady states of (1.1) in (λ, μ)-space.

to compare Figure 1 ($\alpha=0$) with Figure 3 (any fixed $\alpha>1$). For any (λ, μ) satisfying (A) and (II) there exist at least two non-constant solutions, though the case $\alpha=0$ does not, and for (λ, μ) satisfying (A) and (III), there exist at least two non-constant solutions in addition to the constant solutions $(1/\lambda, 0)$, $(0, 1)$.

3. Spatial pattern of solutions in the limit $\beta \downarrow 0$.

In the preceding section, we showed the existence of non-constant solutions of (2.4)–(2.6) in implicit forms. In this section, we study the precise spatial forms of these solutions. Unfortunately, we are not able to study this problem except the case when D is sufficiently small. From Theorem 1, we already know that in the case (A) there are at least two nonnegative, non-constant solutions for sufficiently small D, one is the boundary layer solution, and the other is the internal layer one. We will construct these solutions by using singular perturbation techniques. We only consider the case (A), since (B)–(E) can be treated in a similar way to (A).

THEOREM 3. *Put $D=\varepsilon^2$, and fix arbitrarily $(\lambda, \mu)\in(\mathrm{A})$. Then there exists $\varepsilon_0>0$ such that an ε-parameter family of solutions $(\varepsilon^2, c(\varepsilon), v(x; \varepsilon))$ of (2.4)–(2.6) exists for $0<\varepsilon<\varepsilon_0$, and satisfies*

(3.1) $\qquad \lim_{\varepsilon\downarrow 0} c(\varepsilon) = c_0,$

(3.2) $\qquad \lim_{\varepsilon\downarrow 0} v(x; \varepsilon) = v_{m^*}(c_0)$ *compact uniformly in* $x\in \bar{I}\backslash\{x=m^*\}$,

where

$$v_{m^*}(c_0) = \begin{cases} v_-(c_0) & (0\leqq x<m^*) \\ v_+(c_0) & (m^*<x\leqq 1) \end{cases}$$

in which m^* is uniquely determined by the relation $\int_I f(c_0, v_{m^*}(c_0))dx=0$.

THEOREM 4. *Let $(\lambda, \mu)\in(\mathrm{A})$. Then there exists $\ell_0>0$ such that an ℓ-parameter family of solutions $(D(\ell), c(\ell), v(x; \ell))$ of (2.4)–(2.6) exists for $0<\ell<\ell_0$ and satisfies*

(3.3) $\qquad \lim_{\ell\downarrow 0} D(\ell) = 0,$

(3.4) $\qquad \lim_{\ell\downarrow 0} c(\ell) = \dfrac{1}{\lambda},$

(3.5) $\qquad \lim_{\ell\downarrow 0} v(x; \ell) = 0$ *compact uniformly in* $x\in[0, 1)$.

4. Spatial pattern of solutions for $0<\beta\ll 1$.

We showed the existence of solutions of (2.4)–(2.6) for suitable λ, μ and α

and explicitly constructed them for sufficiently small D. Our original problem is to solve (2.2), (2.3) with $\beta>0$. Intuitively, we may approach it by the perturbation procedure with a power series expansion for $w(x;\beta)$, $v(x;\beta)$ of the form

$$w(x;\ \beta) = \sum_{n=0}^{\infty} w_n(x)\beta^n, \quad v(x;\ \beta) = \sum_{n=0}^{\infty} v_n(x)\beta^n,$$

where (w_0, v_0) is apparently one of the solutions of (2.4)–(2.6). Theorems 1 and 2 show that (w_0, v_0) is isolated. We could, therefore, invoke the implicit function theorem to obtain a solution $(w(x;\beta), v(x;\beta))$ of (2.2), (2.3), and we might get the solution branch for $\beta>0$ by extending the shadow branch S. In this section we will only treat the case when D is sufficiently small by using singular perturbation methods. We only consider the case (A). Other cases (B)–(E) will be treated similarly.

THEOREM 5. *Let $(\lambda, \mu)\in$(A). Then there exist ε_0 and β_0 such that for $0<\beta<\beta_0$ a family of nonnegative solutions $(\varepsilon^2, w(x;\varepsilon,\beta), v(x;\varepsilon,\beta))$ of (2.2), (2.3) exists for $0<\varepsilon<\varepsilon_0$, and satisfies*

(4.1) $\lim_{\beta\downarrow 0}\lim_{\varepsilon\downarrow 0} w(x;\ \varepsilon,\ \beta) = c_0$,

(4.2) $\lim_{\beta\downarrow 0}\lim_{\varepsilon\downarrow 0} v(x;\ \varepsilon,\ \beta) = v_{m^*}(c_0)$ *compact uniformly in* $x\in\bar{I}\backslash\{x=m^*\}$.

THEOREM 6. *Let $(\lambda, \mu)\in$(A). Then there exist ℓ_0 and β_0 such that for $0<\beta<\beta_0$ a family of nonnegative solutions $(D(\ell,\beta), w(x;\ell,\beta), v(x;\ell,\beta))$ of (2.2), (2.3) exists for $0<\ell<\ell_0$ and satisfies*

(4.3) $\lim_{\beta\downarrow 0}\lim_{\ell\downarrow 0} D(\ell,\ \beta) = 0$,

(4.4) $\lim_{\beta\downarrow 0}\lim_{\ell\downarrow 0} w(x;\ \ell,\ \beta) = \dfrac{1}{\lambda}$,

(4.5) $\lim_{\beta\downarrow 0}\lim_{\ell\downarrow 0} v(x;\ \ell,\ \beta) = 0$ *compact uniformly in* $x\in[0,\ 1)$.

The shapes of u $(=w/(1+\alpha v))$, v obtained in Theorems 5 and 6 are illustrated in Figures 4 and 5, respectively.

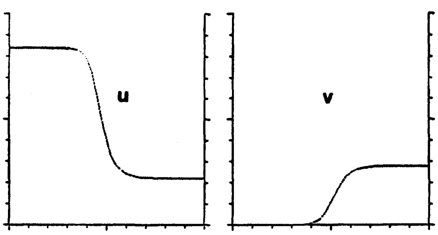


Figure 4. Internal transition layer solutions when D is small.

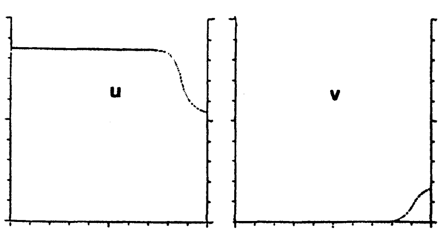


Figure 5. Boundary layer solutions when D is small.

5. Proofs.

PROOF of LEMMA 1.

Suppose $1 \geqq \alpha > 0$, then we fine that if $1 \leqq c$, then $g(c, v) < 0$ for $v > 0$ and if $1 > c > 0$, then there is ξ such that $g(c, v) > 0$ $(0 < v < \xi)$, $g(c, v) < 0$ $(\xi < v)$. For the former case, it is obvious to see that there is no solution of (2.4)–(2.6). Consider the latter case. Suppose a monotone increasing solution $v(x)$ of (2.4)–(2.6). Then there is x_0 such that $v(x_0) = \xi$. From (2.5)

$$\frac{d}{dx} v(x_0) = -\frac{1}{D}\int_0^{x_0} g(c, v(x))dx < 0.$$

This is a contradiction to the monotone increasing nature of v. Cases (ii), (iii) can be easily treated so we omit the proofs. ∎

PROOF of THEOREM 1.

From Lemma 3.2 in Nishiura [14], we have

LEMMA 3.

(i) $F(E, c) \in C^0(\bar{T}) \cap C^\infty(T)$,

(ii) $\dfrac{\partial F}{\partial c}(E, c) \in C^0(T_\delta)$ *for any* $\delta > 0$,

(iii) $F(0, c) = f(c, v_0(c))$,

(iv) $\lim_{(E,c)\to(E^*(\zeta),\zeta)} F(E, c) = \begin{cases} f(\zeta, v_-(\zeta)) & (1 < \zeta < c_0), \\ f(\zeta, v_+(\zeta)) & (c_0 < \zeta < (\alpha+1)^2/4\alpha), \end{cases}$

(v) *when* (E, c) $(\in T)$ *belongs to a sufficiently small neighborhood of* $(E^*(c_0), c_0)$,

$$\frac{\partial F}{\partial c}(E, c)\cdot[f(c, \xi_+(E, c)) - f(c, \xi_-(E, c))] > 0.$$

We first show the following lemma:

LEMMA 4. *Suppose that* (λ, μ) *satisfies* (A). *Then there exists* E_0 *such that there exists a continuous function* $c(E)$ *for* $E_0 < E < E^*(c_0)$ *and*

(i) $(D(E, c(E)), c(E), v(x; E, c(E)))$ *satisfies* (2.4)–(2.6),

(ii) $\lim_{E\uparrow E^*(c_0)} v(x; E, c(E)) = v_{m^*}(c_0)$ *compact uniformly in* $\bar{I}\backslash\{x = m^*\}$,

(iii) $\lim_{E\uparrow E^*(c_0)} c(E) = c_0$,

(iv) $\lim_{E\uparrow E^*(c_0)} D(E, c(E)) = 0$.

PROOF. Fix arbitrarily (λ, μ) satisfying (A). Then we find that $f(c_0, v_-(c_0))<0$ and $f(c_0, v_+(c_0))>0$, and therefore for sufficiently small $k>0$,

$$f(c, v_-(c)) < 0 \qquad \text{for} \quad c\in[c_0-k, c_0]$$

and

$$f(c, v_+(c)) > 0 \qquad \text{for} \quad c\in[c_0, c_0+k].$$

Put $E'=\max(E_-(c_0-k), E_+(c_0+k))$ and for any fixed E $(E'\leqq E<E^*(c_0))$, we define $c_\pm(E)$ by the inverses of $E_\pm(c)=E$, by using the strict monotonicity of $E_\pm(c)$ with respect to c. Then it turns out that

$$\left.\begin{array}{l} f(c_-(E), v_-(c_-(E))) < 0 \\ \text{and} \\ f(c_+(E), v_+(c_+(E))) > 0 \end{array}\right\} \quad \text{for} \quad E' \leqq E < E^*(c_0).$$

From (v) of Lemma 3, there exists E'' such that

$$\frac{\partial F}{\partial c}(E, c) \neq 0 \qquad \text{for} \quad c\in \mathrm{Sec}_E\, T$$

for any E satisfying $E''\leqq E<E^*(c_0)$, where $\mathrm{Sec}_E\, T=\{c|(E, c)\in T\}$. That is, there uniquely exists $c\in \mathrm{Sec}_E\, T$ satisfying $F(E, c)=0$ for any E $(\max(E', E'')=E_0\leqq E<E^*(c_0))$. The continuity of $c(E)$ is also proved by using the usual implicit function theorem. (ii) is proved in Theorem 3.3 in Nishiura [14]. (iii) is derived from the relations $c_-(E)<c(E)<c_+(E)$ and $\lim_{E\uparrow E^*(c_0)} c_\pm(E)=c_0$. (iv) is obvious from (iii) of Lemma 2. ∎

We will prove (A) of Theorem 1. Recalling the functional form of F,

$$\text{(5.1)} \qquad F(E, c; \lambda, \mu) = \int_I \left[1-\frac{\lambda c}{1+\alpha v(x; E, c)}-\mu v(x; E, c)\right]\frac{c}{1+\alpha v(x; E, c)}dx,$$

we first note that there is only one point $(E_-(1/\lambda), 1/\lambda)$ in $\partial T\backslash\{(E^*(c_0), c_0)\}$ such that $F(E, c)=0$. From the continuity of F((i) of Lemma 3), we can take two points $P_1=(E_1, c_1)$, $P_2=(E_2, c_2)$ in T satisfying

$$\text{(5.2)} \qquad 1 < c_1 < 1/\lambda < c_2 < c_0,\ 0 < E_1 < E_-(1/\lambda) < E_2 < E^*(c_0),$$

and

$$\text{(5.3)} \qquad \begin{cases} F(E, c_1) > 0 & \text{for} \quad E_1 \leqq E \leqq E_-(c_1), \\ F(E, c_2) < 0 & \text{for} \quad E_2 \leqq E \leqq E_-(c_2). \end{cases}$$

On the other hand, it follows from (i) of Lemma 3 that there are c_3 and c_4 satisfying

$$c_-\left(\frac{E_0+E^*(c_0)}{2}\right) < c_3 < c\left(\frac{E_0+E^*(c_0)}{2}\right) < c_4 < c_+\left(\frac{E_0+E^*(c_0)}{2}\right).$$

Define two points P_3, P_4 by $P_3=((E_0+E^*(c_0))/2, c_3)$, $P_4=((E_0+E^*(c_0))/2, c_4)$. We take smooth simple curves A_1, A_2, A_3 and A_4 in T with the properties that i) P_1 and P_4 are connected by A_2 and P_2 and P_3 by A_4, on which $F \neq 0$, ii) P_3 and P_4 are connected by A_1 and P_1 and P_2 by A_3, iii) the curve made of A_1-A_4 is a simple closed one in T. Here we use a finite dimensional degree theory. From the facts that $\deg(F, A_1, 0) \neq 0$ (Lemma 4) and $F \neq 0$ on A_2 and A_4, it follows that $\deg\cdot(F, A_3, 0) \neq 0$. Thus we find that there is at least one pair $(E, c) \in A_3$ satisfying $F(E, c)=0$, and that there is a connected component S of solutions of (2.4)–(2.6) in T. Also, $\bar{S} \cap \partial T \ni \{(E^*(c_0), c_0)\}$ follows from (iii) of Lemma 4. Finally we show $\bar{S} \cap \partial T \ni \{(E_-(1/\lambda), 1/\lambda)\}$. Consider two sequences (E_{1n}, c_{1n}), (E_{2n}, c_{2n}) which satisfy $E_{10}=E_1$, $E_{20}=E_2$, $c_{10}=c_1$, $c_{20}=c_2$, and $\lim_{n\to\infty} E_{1n}=\lim_{n\to\infty} E_{2n}=E_-(1/\lambda)$, $\lim_{n\to\infty} c_{1n}=\lim_{n\to\infty} c_{2n}=1/\lambda$. We construct A_{2n}, A_{3n} A_{4n} in the same way as A_2, A_3 and A_4 in the above, and then obtain the connected components S_n in T. Thus, we find that, taking a subsequence $\{S_{n(k)}\}$ of $\{S_n\}$, $\lim_{k\to\infty} S_{n(k)}=S$, whose closure contains $(E_-(1/\lambda), 1/\lambda)$.

We next show (B) of Theorem 1. For this case, we find that there are only two points $(E_-(1/\lambda), 1/\lambda)$, $(E_+(\bar{c}), \bar{c})$ (in ∂T) satisfying $F(E, c)=0$. Fix μ_1, μ_2 arbitrarily such that

$$\mu_1 < \lambda + \frac{1-\lambda}{v_+(c_0)} < \mu_2 < \lambda + \frac{2(1-\lambda)}{1-1/\alpha}.$$

By noting that for fixed $\lambda (1/c_0<\lambda<1)$,

$$\lim_{(E,c)\to(E_-(1/\lambda),1/\lambda)} F(E, c) = 0 \quad \text{uniformly in } \mu,$$

we can take (E_1, c_1), (E_2, c_2) in T, satisfying (5.2), (5.3) suitably in a similar way to the case (A). Hence, by the homotopy invariance on the parameter $\mu(\mu_1 \leqq \mu \leqq \mu_2)$, we have

$$0 \neq \deg(F(\cdot\,; \mu_1), A_3, 0) = \deg(F(\cdot\,; \mu_2), A_3, 0). \tag{5.4}$$

On the other hand, for $\mu=\mu_2$, we can take two points $P_5=(E_5, c_5)$, $P_6=(E_6, c_6)$ in T satisfying

$$c_0 < c_5 < \bar{c} < c_6 < (\alpha+1)^2/4\alpha, \quad 0 < E_6 < E_+(\bar{c}) < E_5 < E^*(c_0), \tag{5.5}$$

and

$$\begin{cases} F(E, c_5) < 0 & \text{for} \quad E_5 \leqq E \leqq E_+(c_5), \\ F(E, c_6) > 0 & \text{for} \quad E_6 \leqq E \leqq E_+(c_6). \end{cases} \tag{5.6}$$

From $f(c_0, v_\pm(c_0))<0$ for $\mu=\mu_2$, it follows that $F(E, c; \lambda, \mu_2) \neq 0$ in a small neighborhood of $(E^*(c_0), c_0)$. Thus, there are four smooth simple curves A_3, A_5, A_6 and A_1 in T with the properties that i) P_2 and P_5 are connected by A_5 and P_6 and P_1 by A_7, on which $F \neq 0$, ii) P_1 and P_2 are connected by A_3 and P_5 and P_6 by A_6, iii) the curve made of A_3, A_5, A_6 and A_7 is a simple closed one in T. Then, by (5.4), $0 \neq \deg(F, A_3, 0) = \deg(F, A_6, 0)$. Thus, there exists the connected component S of solutions of (2.4)–(2.6) in T. It is also seen that $\bar{S} \cap \partial T = \{(E_-(1/\lambda), 1/\lambda), (E_+(\bar{c}), \bar{c})\}$. Case (C) is similarly proved. Finally, we show that the connected component S in each case is generically a one-dimensional submanifold in T, i.e., it consists of smooth curves which do not interesect one another in T. From (5.1), we can see that $(\partial/\partial\lambda)F(E, c; \lambda, \mu)|_{\lambda=0}$ is equal to $-\int_I c^2(1+\alpha v(x; E, c))^{-2}dx$ which is not equal to zero in T. From Theorem 3.9 in [14], therefore, it follows that S is generically a one-dimensional submanifold in T. ∎

Theorem 2 can be proved in a similar way to Theorem 1, so we omit the proof.

When (λ, μ) belongs to (A), the nonlinearities of f and g are qualitatively analogous to those in Mimura, Tabata and Hosono [13], who treated prey-predator systems. Therefore, Theorem 5 and incidentally Theorem 3 can be proved by using Theorem 2 in [14]. We first prove Theorem 6.

PROOF of THEOREM 6.

It is expected that $w(x)$ is almost constant but $v(x)$ has a boundary layer at $x=1$ for sufficiently small D, β. Following the approach by Hosono [7], we construct boundary layer solutions. For some $\ell>0$, let us divide I into two subintervals $I_1=(0, 1-\ell)$, $I_2=(1-\ell, 1)$ and consider two boundary layer problem:

$$(5.7)\qquad \begin{cases} 0 = w_{xx} + \beta f(w, v), & \\ 0 = Dv_{xx} + g(w, v), & x \in I_1, \\ w_x(0) = v_x(0) = 0, & \\ w(1-\ell) = \xi,\ v(1-\ell) = v_0(\xi), & \end{cases}$$

and

$$(5.8)\qquad \begin{cases} 0 = w_{xx} + \beta f(w, v), & \\ 0 = Dv_{xx} + g(w, v), & x \in I_2, \\ w(1-\ell) = \xi,\ v(1-\ell) = v_0(\xi), & \\ w_x(1) = v_x(1) = 0,\ v(1) = \eta, & \end{cases}$$

where ξ, η are positive constants. The relation of ξ, η, D to ℓ will become clear later.

By the transformation $y=x/(1-\ell)$, (5.7) is written as

$$(5.9)\qquad \begin{cases} 0 = w_{yy} + \beta\tilde{f}(w, v), & \\ & y \in I, \\ 0 = Dv_{yy} + \tilde{g}(w, v), & \\ w_y(0) = v_y(0) = 0,\ w(1) = \xi,\ v(1) = v_0(\xi), \end{cases}$$

where $\tilde{f}(w, v)=(1-\ell)^2 f(w, v)$ and $\tilde{g}(w, v)=(1-\ell)^2 g(w, v)$. In the limit $D \downarrow 0$, the second equation of (5. 9) is formally reduced to $0=\tilde{g}(w, v)$, from which we take $v \equiv 0$. Substituting it into the first equation, we have

$$(5.10)\qquad \begin{aligned} &0 = w_{yy} + \beta\tilde{f}(w, 0), \qquad y \in I, \\ &w_y(0) = 0, \quad w(1) = \xi. \end{aligned}$$

LEMMA 5. *For any fixed* ξ_0, ξ_1 $(1/2\lambda<\xi_0<1/\lambda<\xi_1)$ *and* $\beta>0$, (5.10) *has a unique monotone solution* $w_0(y;\ \ell, \xi, \beta)$ *satisfying*

$$(5.11)\qquad \frac{\partial}{\partial \xi}\left[\frac{d}{dy} w_0(1;\ \ell, \xi, \beta)\right] > 0 \qquad \text{for} \quad \xi_0 < \xi < \xi_1.$$

PROOF. Let $w(y;\ a, \beta)$ be a solution of the first equation in (5.10) subject to $w(0)=a$, $w_y(0)=0$. Fix ξ_0, ξ_1 satisfying $1/2\lambda<\xi_0<1/\lambda<\xi_1$ and $\beta>0$ and take a_0, a_1 $(1/2\lambda<a_0<1/\lambda<a_1)$ such that $w(y;\ a_0, \beta)$ (resp. $w(y;\ a_1, \beta)$) is strictly monotone decreasing (resp. increasing) and $w(1;\ a_0, \beta)=\xi_0$ (resp. $w(1;\ a_1, \beta)=\xi_1$). Then we know that if a^-, a_- are chosen to satisfy $a_1>a^->a_->a_0$, $w(y;\ a^-, \beta)>w(y;\ a_-, \beta)>1/2\lambda$ holds for any $y \in I$, which shows the existence of a unique solution $w_0(y;\ \ell, \xi, \beta)$ of (5.10). We will show (5.11). Let $a(\xi)$ be $a(\xi)=w_0(1;\ \ell, \xi, \beta)$ and write $w_0(y;\ \ell, \xi, \beta)$ as $w_0(y;\ a(\xi))$ simply. Note that $\Theta(y;\ \xi)=(\partial/\partial a)w_0(y;\ a(\xi))$ is a strictly monotone increasing function, because it satisfies

$$\begin{cases} \Theta_{yy} = -\tilde{f}_w(w_0(y;\ a(\xi)), 0)\Theta \qquad y \in I, \\ \Theta_y(0) = 0,\ \Theta(1) = 1. \end{cases}$$

$$\begin{aligned} \frac{\partial}{\partial \xi}\left(\frac{d}{dy} w_0(y;\ a(\xi))\Big|_{y=1}\right) &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial \xi} w_0(1;\ a(\xi))\right)\Big|_{y=1} \\ &= \frac{\partial}{\partial y}\left(\frac{da}{d\xi}\left(\frac{\partial}{\partial a} w_0(y;\ a)\right)\right)\Big|_{y=1} = \frac{\partial}{\partial y}\Theta(y;\ \xi)\Big|_{y=1}\frac{da}{d\xi} \\ &= \frac{\partial}{\partial y}\Theta(y;\ \xi)\Big|_{y=1}\left(\frac{\partial}{\partial a} w_0(1;\ a)\right)^{-1} = \frac{\partial\Theta}{\partial y}\frac{1}{\Theta}\Big|_{y=1} > 0, \end{aligned}$$

which shows (5.11). ∎

Thus, we obtain an outer solution $(w, v)=(w_0(y;\ \ell, \xi, \beta), 0)$ of (5.9) in the limit $D \downarrow 0$. When D is not zero, one can expect from the boundary condition $v(1)=v_0(\xi)$ that there occurs a boundary layer at $y=1$. In order to obtain a lowest approximation to the solution of (5.9) in the whole interval, we consider the boundary layer equation derived from the second equation of (5.9) by using the stretched variable $\eta=(1-y)/\varepsilon$ (we put $D=\varepsilon^2$ for simplicity),

$$(5.12) \qquad \begin{cases} 0 = \hat{z}_{\eta\eta} + \tilde{g}(\xi, \hat{z}), & (-\infty<\eta<0), \\ \hat{z}(0) = v_0(\xi), \quad \hat{z}(-\infty) = 0. \end{cases}$$

From Lemma 2.1 in Fife [2], we know

LEMMA 6. *Fix arbitrarily* ℓ_0, ξ_0, ξ_1 *satisfying* $0<\ell_0<1$ *and* $1<\xi_0<\xi_1<c_0$. *Then* (5.12) *has a unique monotone solution* $\hat{z}(\eta; \ell, \xi)$ *for* $0<\ell<\ell_0$ and $\xi_0<\xi<\xi_1$. *Furthermore,* $\hat{z}$, $\hat{z}_\eta$ *exponentially decay uniformly in* $0<\ell<\ell_0$ and $\xi_0<\xi<\xi_1$ *as* $\eta \to -\infty$.

Using a C^∞ cutoff function $\zeta(t)$ satisfying $\zeta\equiv 1\left(0<t<\frac{1}{4}\right)$, $\zeta\equiv 0\ \left(t>\frac{1}{2}\right)$, we define $z(y; \varepsilon, \ell, \xi)$ by $z(y; \varepsilon, \ell, \xi)=\hat{z}((1-y)/\varepsilon; \ell, \xi)\zeta(1-y)$. Thus, we obtain the lowest approximation $(w^0, v^0)=(w_0(y; \ell, \xi, \beta), z(y; \varepsilon, \ell, \xi))$ to the solution of (5.9). We then seek a pair (r, s) such that $w_{(1)}=w^0+r$, $v_{(1)}=v^0+s$ become an exact solution of (5.9). Since this was done in [13, Lemma 4.3], we only state the result.

LEMMA 7. *There exist* ε_0, β_0 *such that* (5.9) *has a solution* $(w_{(1)}, v_{(1)})$ *for* $0<\varepsilon<\varepsilon_0$ *and* $0<\beta<\beta_0$ *which satisfies*

(i) $\|w_{(1)} - w^0\|_{H^2(I)} \longrightarrow 0, \qquad (\varepsilon \to 0)$,

(ii) $\|v_{(1)} - v^0\|_{C^2_{\varepsilon 0}(I)} \longrightarrow 0, \qquad (\varepsilon \to 0)$,

uniformly in ℓ $(0<\ell<\ell_0)$ *and* ξ $(\xi_0<\xi<\xi_1)$, *where* $H^2(I)$ *is the Sobolev space with the norm*

$$\|u\|_{H^2(I)} = \left(\sum_{k=0}^2 \int_I \left|\left(\frac{d}{dx}\right)^k u(x)\right|^2 dx\right)^{1/2}$$

and $C^2_{\varepsilon 0}(I)$ *is the space of 2-times continuously differentiable functions on* I *with* $u_x(0)=0=u(1)$ *and with the norm*

$$\|u\|_{C^2_{\varepsilon 0}(I)} = \sum_{k=0}^2 \max_{x\in I}\left|\left(\varepsilon\frac{d}{dx}\right)^k u(x)\right|.$$

REMARK 2. From (ii) of Lemma 7, we find that

$$\lim_{\varepsilon\downarrow 0}\left(\varepsilon\frac{dv_{(1)}}{dy}\bigg|_{y=1}\right)^2 = -2\int_0^{v_0(\xi)} \tilde{g}(\xi, s)ds.$$

We next consider (5.8). By the transformation $y=(x-1)/\ell+1$, (5.8) is reduced to

$$(5.13)\qquad \begin{cases} 0 = w_{yy} + \ell^2\beta f(w, v), & \\ 0 = \delta v_{yy} + g(w, v), & y \in I, \\ w(0) = \xi,\ v(0) = v_0(\xi), & \\ w_y(1) = v_y(1) = 0,\ v(1) = \eta, & \end{cases}$$

where $\delta=D/\ell^2$. Assume D and ℓ satisfy $\delta=O(1)$ as $\ell \downarrow 0$. It turns out that, when $\ell \downarrow 0$ (or $D \downarrow 0$) in (5.13), $w(y)\equiv\xi$ and then v satisfies

$$(5.14)\qquad \begin{cases} 0 = \delta v_{yy} + g(\xi, v), & y \in I, \\ v(0) = v_0(\xi),\ v(1) = \eta, & v_y(1) = 0. \end{cases}$$

Here we look for a pair (δ, v) satisfying (5.14) for a given η.

LEMMA 8. *Fix arbitrarily ξ satisfying $1<\xi<c_0$. For any η ($v_0(\xi)<\eta<v_+(\xi)$), there uniquely exists a monotone increasing solution $v_2(y; \eta, \xi)$ of* (5.14) *with $\delta=\delta_0(\eta, \xi)$*, where

$$\delta_0(\eta, \xi) = \left[\int_{v_0(\xi)}^{\eta} \frac{dv}{\left(2\int_v^{\eta} g(\eta, s)ds\right)^{1/2}}\right]^{-2}.$$

PROOF. It follows from (5.14) that

$$y = \int_v^{\eta} \frac{d\tau}{\left(2\delta^{-1}\int_\tau^{\eta} g(\xi, s)ds\right)^{1/2}}.$$

The lemma is proved by using this representation. ∎

Thus, we find the lowest approximation to the solution of (5.13),

$$(w, v) = (\xi, v_2(y; \eta, \xi)) \qquad \text{for} \quad \delta = \delta_0(\xi, \eta).$$

Let us seek a solution $(\delta, w_{(2)}, v_{(2)})$ of the problem (5.13) which takes the form

$$\begin{aligned} &\delta = \delta_0(\eta, \xi) + \tau, \\ &w_{(2)}(y; \tau, \ell, \eta, \xi, \beta) = \xi + r(y; \tau, \ell, \eta, \xi, \beta), \\ &v_{(2)}(y; \tau, \ell, \eta, \xi, \beta) = v_2(y; \eta, \xi) + s(y; \tau, \ell, \eta, \xi, \beta), \end{aligned}$$

where (r, s) is a remainder to be determined. From (5.13), we know that (r, s) satisfies

$$(5.15)\qquad \begin{cases} R_2(r, s; \tau, \ell, \eta, \xi, \beta) = r_{yy} + \ell^2\beta f(\xi+r, v_2+s) = 0, \\ S_2(r, s; \tau, \ell, \eta, \xi, \beta) = (\delta_0+\tau)(v_2+s)_{yy} + g(\xi+r, v_2+s) = 0, \end{cases}$$

with the boundary conditions

$$(5.16)\qquad \begin{cases} r(0) = s(0) = 0, \\ r_y(1) = s_y(1) = 0, \end{cases}$$

$$(5.17)\qquad s(1) = 0.$$

Set

$$T_2(t; \tau, \ell, \eta, \xi, \beta) = (R_2(t; \tau, \ell, \eta, \xi, \beta), S_2(t; \tau, \ell, \eta, \xi, \beta))$$

with $t=(r, s)$. We first consider the problem (5.15), (5.16). That is, we look for a solution $t \in (C_0^2(I))^2$ satisfying $T_2=0$, where $C_0^2(I)$ is the space of 2-times continuously differentiable functions on I with $u(0)=u_x(1)=0$.

LEMMA 9. *Let $\beta>0$, $1<\xi<c_0$ and $v_0(\xi)<\eta<v_+(\xi)$ be arbitrarily fixed.*
(i) *$T_2=T_2(t; \tau, \ell)$ is a continuously diffentiable mapping from $(C_0^2(I))^2 \times \mathbf{R}^2$ into $(C^0(\bar{I}))^2$,*
(ii) *$T_2(0; 0, 0) = 0$,*
(iii) *$T_{2_t}(0; 0, 0)$ is an isomorphism of $(C_0^2(I))^2$ onto $(C^0(\bar{I}))^2$,*
where T_{2_t} is the Fréchet derivative of T_2.

PROOF. (i), (ii) are obvious. We only show (iii). $T_{2_t}(0; 0, 0)$ is described by

$$T_{2_t}(0; 0, 0) = \begin{pmatrix} \left(\frac{d}{dy}\right)^2 & 0 \\ g_w(\xi, v_2) & \delta_0\left(\frac{d}{dy}\right)^2 + g_v(\xi, v_2) \end{pmatrix}.$$

The operator $(d/dy)^2$ is an isomorphism of $C_0^2(I)$ onto $C^0(\bar{I})$. On the other hand, Lemma 8 shows that $\phi(y) \equiv (d/dy)v_2(y; \eta, \xi)$ is positive and satisfies

$$L_{\delta_0}\phi \equiv \delta_0\left(\frac{d}{dy}\right)^2\phi + g_v(\xi, v_2)\phi = 0, \quad \phi(0) = 0.$$

Thus we find the other solution $\psi(y)$ independent of $\phi(y)$

$$\psi(y) = \phi(y)\int_1^y \frac{dx}{\phi(x)^2} \qquad y \in I,$$

which satisfies $L_{\delta_0}\psi=0$, $\psi(1)=0$ and $\lim_{y\downarrow 0}\psi(y) = -1/\phi'(0)$. Using two functions ϕ, ψ, we can construct the bounded Green's kernel of the operator L_{δ_0}, which shows that L_{δ_0} is an isomorphism. ∎

Thus, from Lemma 9 we can apply the implicit function theorem to $T_2=0$.

LEMMA 10. *Let $B_q=\{(\tau, \ell)\in \mathbf{R}^2 \mid |\tau|<q, 0<\ell<q\}$. There exist a positive number q_0 and a unique continuous mapping $t(\tau, \ell)$ from B_{q_0} into $\mathbf{R}^2$ such that*
(i) $t(0, 0) = 0$,
(ii) $T_2(t(\tau, \ell), \tau, \ell) = 0$,
(iii) $t(\tau, \ell) = O(|\tau|+|\ell^2|)$.

Thus, we obtain the solution $(w_{(2)}, v_{(2)})$ of (5.8) in the absence of $v_{(2)}(1)=\eta$. We next give the relation between τ and ℓ to satisfy $v_{(2)}(1)=\eta$ i.e. $s(1; \tau, \ell, \eta, \xi, \beta)=0$. Rewrite the second equation of (5.15) as

$$(5.18)\qquad 0 = L_{\delta_0}s + g_u(\xi, v_2)r + \tau(v_2+s)_{yy} + N_g(r, s),$$

where $L_{\delta_0}\equiv\delta_0(d/dy)^2+g_v(\xi, v_2)$ and N_g is the higher order term. Since L_{δ_0} is invertible from $C_0^2(I)$ into $C^0(I)$, (5.18) is rewritten as

$$(5.19)\qquad s = L_{\delta_0}^{-1}[-g_u(\xi, v_2)r - \tau(v_2+s)_{yy} - N_g(r, s)].$$

On the other hand, the first of (5.15) is written as

$$(5.20)\qquad r = - L^{-1}[\ell^2\beta f(\xi+r, v_2+s)],$$

in which we used the fact that $L\equiv(d/dy)^2$ is invertible from $C_0^2(I)$ into $C^0(I)$. By using (5.20), (5.19) becomes

$$(5.21)\qquad \begin{aligned} s = L_{\delta_0}^{-1}[&g_u(\xi, v_2)(L^{-1}[\ell^2\beta f(\xi, v_2) + \ell^2\beta N_f(r, s)]) \\ &- \tau(v_2+s)_{yy} - N_g(r, s)], \end{aligned}$$

where N_f is the higher order term. Define $\Xi(\tau, \ell)$ by $\Xi(\tau, \ell)=s(1; \tau, \ell, \eta, \xi, \beta)$ for fixed η, ξ and β.

LEMMA 11. *$\Xi(\tau, \ell)$ is a mapping from B_{q_0} into $\mathbf{R}$ which satisfies*
(i) $\Xi(0, 0) = 0$,
(ii) $\dfrac{\partial}{\partial\tau}\Xi(0, 0) = - L_{\delta_0}^{-1}\left[\left(\dfrac{d}{dy}\right)^2 v_2\right] \neq 0.$

PROOF. Since $s(y; 0, 0)=0$ from (i) of Lemma 10, (i) is obvious. Noting that $t=O(|\tau|+|\ell^2|)$ ((iii) of Lemma 10), we find

$$\begin{aligned}\frac{\partial}{\partial\tau}\Xi(0, 0) &= - L_{\delta_0}^{-1}\left[\left(\frac{d}{dy}\right)^2 v_2(1; 0, 0, \xi, \eta, \beta)\right] \\ &= - L_{\delta_0}^{-1}\left[-\frac{1}{\delta_0}g(\xi, \eta)\right] \neq 0,\end{aligned}$$

which gives (ii). ∎

LEMMA 12. *Consider the equation* $\Xi(\tau, \ell)=0$. *Then there exist* ℓ_0 *such that* $\Xi(\tau, \ell)=0$ *has a unique solution* $\tau(\ell)$ *with* $\tau(0)=0$ *for* $0\leqq \ell<\ell_0$.

PROOF. The proof is achieved by using the standard implicit function theorem. ∎

Thus, we have

LEMMA 13. *Let* $\beta>0$, $1<\xi<c_0$ *and* $v_0(\xi)<\eta<v_+(\xi)$ *be arbitrarily fixed. Then there exists* $\ell_0>0$ *such that, for* $\delta(\ell; \eta, \xi)=\delta_0(\eta, \xi)+\tau(\ell; \eta, \xi)$, $(w_{(2)}(y; \tau(\ell; \eta, \xi), \ell, \eta, \xi, \beta), v_{(2)}(y; \tau(\ell; \eta, \xi), \ell, \eta, \xi, \beta))$ *is a unique solution of* (5.8) *which satisfies*

$$\begin{cases} \lim_{\ell\downarrow 0} \|w_{(2)}-\xi\|_{C_0^2(I)}=0, \\ \lim_{\ell\downarrow 0} \|v_{(2)}-v_2\|_{C_0^2(I)}=0, \end{cases} \qquad \textit{uniformly in } \eta \textit{ and } \xi.$$

REMARK 3. From Lemma 13, it follows that

(i) $\lim_{\ell\downarrow 0}\dfrac{1}{\ell}\dfrac{d}{dy}w_{(2)}|_{y=0}=0$,

(ii) $\lim_{\ell\downarrow 0}\dfrac{1}{2}\left(\dfrac{dv_{(2)}}{dy}\Big|_{y=0}\right)^2=\delta_0(\eta, \xi)^{-1}\displaystyle\int_{v_0(\xi)}^{\eta} g(\xi, s)ds.$

Lemmas 7 and 13 construct the solutions $(w_{(1)}, v_{(1)})$ and $(w_{(2)}, v_{(2)})$ of the problems (5.7) and (5.8) respectively. Therefore, in order to complete the proof of Theorem 6 we determine two parameters η and ξ so that these two functions are matched in the C^1-sense. Noting $D=\varepsilon^2=\delta\ell^2$, we write $w_{(1)}, v_{(1)}$ as $w_{(1)}(x; \ell, \xi, \beta)$, $v_{(1)}(x; \ell, \xi, \beta)$, respectively. From Lemmas 7 and 13, we know

$$w_{(1)}(1-\ell; \ell, \xi, \beta)=w_{(2)}(1-\ell; \ell, \eta, \xi, \beta)=\xi,$$
$$v_{(1)}(1-\ell; \ell, \xi, \beta)=v_{(2)}(1-\ell; \ell, \eta, \xi, \beta)=v_0(\xi).$$

Define Φ, Ψ by

$$\Phi(\ell, \eta, \xi, \beta)=\frac{d}{dx}w_{(1)}(1-\ell; \ell, \xi, \beta)-\frac{d}{dx}w_{(2)}(1-\ell; \ell, \xi, \beta),$$
$$\Psi(\ell, \eta, \xi, \beta)=\frac{1}{2}\left[\left(\ell\frac{d}{dx}v_{(1)}(1-\ell; \ell, \xi, \beta)\right)^2-\left(\ell\frac{d}{dx}v_{(2)}(1-\ell; \ell, \xi, \beta)\right)^2\right],$$

respectively. When $\ell\downarrow 0$ and $\xi\to 1/\lambda$, it follows that by Lemma 5,

$$\Phi(0, \eta, 1/\lambda, \beta)=\frac{d}{dx}w_0(1; 0, 1/\lambda, \beta)=0 \tag{5.22}$$

and by Remarks 2 and 3

$$\Psi(0, \eta, 1/\lambda, \beta) = -\delta_0(\eta, 1/\lambda)^{-1}\int_0^\eta g(1/\lambda, s)ds. \tag{5.23}$$

Here we determine $\eta = \eta^*$ by

$$\int_0^{\eta^*} g(1/\lambda, s)ds = 0.$$

Furthermore, it follows that by Lemma 5,

$$\frac{\partial}{\partial \xi}\Phi(0, \eta, 1/\lambda, \beta) = \frac{\partial}{\partial \xi}\left(\frac{d}{dx}w_0(1; 0, \xi, \beta)\right)\Big|_{\xi=1/\lambda} > 0 \tag{5.24}$$

and by (5.14),

$$\frac{\partial}{\partial \eta}\Psi(0, \eta^*, 1/\lambda, \beta) = -\left[\left(\frac{\partial}{\partial \eta}(\delta_0(\eta, 1/\lambda)^{-1})\right)\int_0^\eta g(1/\lambda, s)ds + (\delta_0(\eta, 1/\lambda)^{-1})g(1/\lambda, \eta)\right]\Big|_{\eta=\eta^*} > 0, \tag{5.25}$$

because $g(1/\lambda, \eta^*) > 0$. Thus, we can apply the implicit function theorem presented by Fife (Theorem 4.3 in [3]) to the problem $\Phi = \Psi = 0$, and then we conclude that for sufficiently small ℓ (>0) there exist $\eta(\ell)$, $\xi(\ell)$ such that

$$\Phi(\ell, \eta(\ell), \xi(\ell), \beta) = \Psi(\ell, \eta(\ell), \xi(\ell), \beta) = 0 \tag{5.26}$$

and

$$\lim_{\ell \downarrow 0}\eta(\ell) = \eta^*, \lim_{\ell \downarrow 0}\xi(\ell) = 1/\lambda.$$

Thus, we can prove the existence of an ℓ-family of $(w(x; \ell, \beta), v(x; \ell, \beta))$ satisfying (2.2), (2.3) with $D = D(\ell)$.

Finally we show that there exist ℓ_1, β_1 such that w, v obtained in the above are nonnegative functions on I for $0 < \ell < \ell_1$, $0 < \beta < \beta_1$.

It suffices to show the nonnegativity of v. For sufficiently small ℓ, we find that $v(x) > 0$ in a neighborhood of $x = 1$. Presuppose that $v(x) < 0$ for some x (<1). Then, there is $x_0 \in I$ such that $v(x_0) = 0$, $v_x(x_0) \geqq 0$. $v_x(x_0) = 0$ is excluded because of the uniquness of the solution. Therefore $v_x(x_0) > 0$, which, together with the boundary condition, implies the existence of $x_1 \geqq 0$ such that $v_x(x_1) \leqq 0$, $v(x) < 0$ for $x_1 < x < x_0$. Thus, integrating the second equation of (2.2) with respect to x, we have

$$D\frac{dv}{dx}(x_0) = D\frac{dv}{dx}(x_1) - \int_{x_1}^{x_0} g(w, v)dx \leqq -\int_{x_1}^{x_0} g(w, v)dx.$$

Noting $g(w, v) > 0$ for sufficiently small ℓ and β, we know $v_x(x_0) < 0$, which is a contradiction. ∎

PROOF of THEOREM 4.

In order to construct a boundary layer solution, we divide I into two sub-intervals $I_1=(0, 1-\ell)$, $I_2=(1-\ell, 1)$ and for fixed $w=c$, consider the following two boundary layer problems for v:

$$(5.27)\qquad \begin{cases} 0 = Dv_{xx} + g(c, v) \\ v_x(0) = 0,\ v(1-\ell) = v_0(c), \end{cases} \qquad x \in I_1,$$

and

$$(5.28)\qquad \begin{cases} 0 = Dv_{xx} + g(c, v) \\ v(1-\ell) = v_0(c),\ v_x(1) = 0,\ v(1) = \eta, \end{cases} \qquad x \in I_2,$$

where η is a positive constant. (5.27), (5.28) can be solved in a similar way to the proof of Theorem 6. Then it turns out that the solutions v_1 of (5.27) and v_2 of (5.28) are functions of (η, c, ℓ) on I_1 and I_2, respectively. Next we determine two parameters η and c as functions of ℓ so that v_1 and v_2 are matched in C^1-sense and v_1, v_2 and c satisfy (2.4). To do so, consider the equations

$$\Phi(\eta, c, \ell) = \int_0^{1-\ell} f(c, v_1)dx + \int_{1-\ell}^1 f(c, v_2)dx = 0,$$

$$\Psi(\eta, c, \ell) = \frac{1}{2}\left[\left(\ell\frac{d}{dx}v_1(1-\ell; \eta, c, \ell)\right)^2 - \left(\ell\frac{d}{dx}v_2(1-\ell; \eta, c, \ell)\right)^2\right] = 0.$$

These equations can be solved similarly to (5.26), so we omit the details. ∎

6. Conclusion

We have shown that there exist non-constant solutions of (2.2), (2.3) when D, β are sufficiently small. In the case when (λ, μ) belongs to (A) $\cap$ (III) (for instance, see Theorem 1) we constructed two different types of singularly perturbed solutions when D is small. One is of internal transition layer type and the other is of boundary layer type (Figures 4 and 5). As expected from the case $\beta=0$ (Theorem 3), we address ourselves to an interesting question "Is there any inter-relation between these two solutions?". By numerical computations, we observe that there is the global branch of the solution (D, u, v) in the space $X=\mathbf{R}_+ \times C^2(I) \times C^2(I)$, connecting the two singular perturbed solutions, being separated from the constant solution branches $(D, 1/\lambda, 0)$, $(D, 0, 1)$ (Figure 6). Furthermore it is numerically observed that the solution of internal transition layer type is thought to be *stable* and the one of boundary layer type is thought to be *unstable* (Figure 7). Though the problem is different from the competitive type, we should refer to the recent works on prey-predator type by Fujii et al. [4], Nishiura [15] [16], Fujii and Nishiura [5] from a global bifurcation point of view, and Aronson,

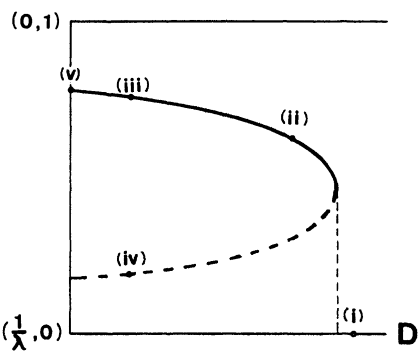


Figure 6a. The spontaneous bifurcation with respect to D. A solid line represents an internal transition layer solution and a dashed line a boundary layer one.

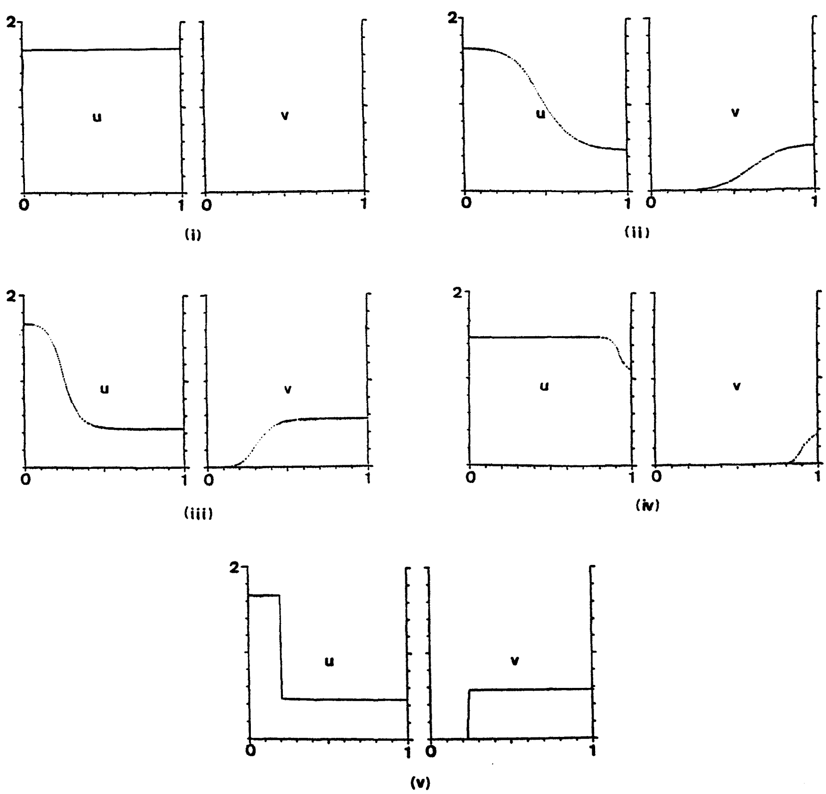


Figure 6b. Spatial patterns of u and v when D varies. $\alpha=5$, $\beta=1$, $\lambda=0.68$, $\mu=1.2$ (this case corresponds to case (III)) and (i) $D=0.005$, (ii) $D=0.004$, (iii), (iv) $D=0.001$, (v) $D=0.0$.

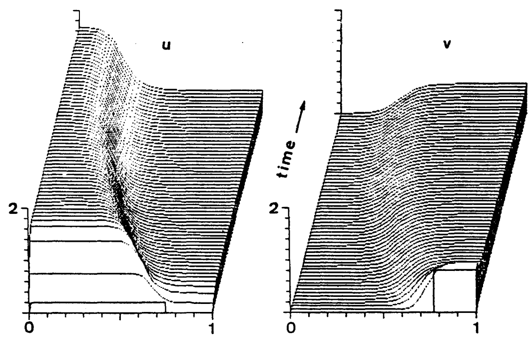


Figure 7. Spatio-temporal profiles of solution of (1.1). The values of the coefficients correspond to the case (iii) in Figure 6b.

Tesei and Weinberger [1] for the stability on non-constant solutions of (2.2), (2.3) with $D=0$. Finally, we would like to conclude that even in the situation where the inter-specific competition is stronger than the intra-specific one, the system treated in this paper exhibits coexistence of two competing species on the basis of suitable cross-population pressure.

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