HIROSHIMA MATH. J.
14 (1984), 401–406

Boundary limit of discrete Dirichlet potentials

Takashi KAYANO and Maretsugu YAMASAKI

(Received January 11, 1984)

§ 1. Introduction

Let X be a countable set of nodes, Y be a countable set of arcs, K be the node-arc incidence function and r be a strictly positive real function on Y. The quartet $N=\{X, Y, K, r\}$ is called an infinite network if the graph $\{X, Y, K\}$ is connected, locally finite and has no self-loop. For notation and terminology, we mainly follow [4], [6] and [7].

Let $L(X)$ be the set of all real functions on X and $L_0(X)$ be the set of all $u \in L(X)$ with finite support. For $u \in L(X)$, its Dirichlet integral $D_p(u)$ of order p $(1<p\leq\infty)$ is defined by

$$D_p(u) = \sum_{y\in Y} r(y)^{1-p}|\sum_{x\in X} K(x, y)u(x)|^p \quad (1 < p < \infty),$$

$$D_\infty(u) = \sup_{y\in Y} r(y)^{-1}|\sum_{x\in X} K(x, y)u(x)|.$$

Denote by $\boldsymbol{D}^{(p)}(N)$ the set of all $u \in L(X)$ with finite Dirichlet integral of order p. It is easily seen that $\boldsymbol{D}^{(p)}(N)$ is a Banach space with the norm $\|u\|_p=[D_p(u)+|u(b)|^p]^{1/p}$ $(1<p<\infty)$ and $\|u\|_\infty=D_\infty(u)+|u(b)|(b\in X)$.

Denote by $\boldsymbol{D}_0^{(p)}(N)$ the closure of $L_0(X)$ in $\boldsymbol{D}^{(p)}(N)$ with respect to the norm $\|u\|_p$. This $\boldsymbol{D}_0^{(p)}(N)$ is determined independently of the choice of b. As in the continuous potential theory, we may call an element of $\boldsymbol{D}_0^{(p)}(N)$ a (discrete) Dirichlet potential of order p.

A typical Dirichlet potential of order 2 is the Green function g_a of N with pole at a (cf. [1]). This is defined by the conditions: $g_a \in \boldsymbol{D}_0^{(2)}(N)$ and $\Delta g_a(x)=-\varepsilon_a(x)$ on X, where Δ is the discrete Laplace operator and ε_a is the characteristic function of the set $\{a\}$. It was shown in [7] that the Green function g_a exists if and only if N is of hyperbolic type of order 2, or equivalently $\boldsymbol{D}^{(2)}(N)\neq\boldsymbol{D}_0^{(2)}(N)$.

In the case where $\{X, Y, K\}$ is the lattice domain in the 3-dimensional Euclidean space and $r=1$, Duffin [2] showed by means of Fourier analysis that g_a vanishes at the ideal boundary ∞ of N. In the general case, $g_a(x)$ does not always have limit 0 as x tends to the ideal boundary ∞ of N along a path from a to ∞.

In this paper, we are concerned with the boundary behavior of Dirichlet potentials of order p. Namely, we aim to show that for every $u \in \boldsymbol{D}_0^{(p)}(N)$ the set of all paths along which $u(x)$ does not have limit 0 as x tends to the ideal boundary ∞ of N is a small set in some sense. As in the continuous case (cf. [5]), we use the notion of extremal length of order p. We say that a property holds for p-almost every path of a family Γ of paths if it does for the member of Γ except for those belonging to a subfamily with infinite extremal length of order p. Some elementary properties of the extremal length of a family of paths will be discussed in §2. We shall prove in §3 that every Dirichlet potential $u(x)$ of order p has limit 0 as x tends to the ideal boundary ∞ of N along p-almost every infinite path.

§2. Extremal length of a family of paths

Let $L(Y)$ be the set of all real functions on Y and $L^+(Y)$ be the subset of $L(Y)$ that consists of non-negative functions. For $w \in L(Y)$, its energy $H_p(w)$ of order p $(1<p\leq\infty)$ is defined by

$$H_p(w) = \sum_{y\in Y} r(y)|w(y)|^p \quad (1 < p < \infty),$$
$$H_\infty(w) = \sup_{y\in Y} |w(y)|.$$

Let us recall the notion of paths. A path P from a to the ideal boundary ∞ of N is the triple $\{C_X(P), C_Y(P), p\}$ of an infinite ordered set $C_X(P)=\{x_n; n\geq 0\}$ of nodes, an infinite ordered set $C_Y(P)=\{y_n; n\geq 1\}$ of arcs and a function p on Y called the path index of P which satisfy the conditions: $x_0=a$, $x_i\neq x_j$ if $i\neq j$, $\{x\in X; K(x, y_i)\neq 0\}=\{x_{i-1}, x_i\}$, $p(y_i)=-K(x_{i-1}, y_i)$ if $y_i\in C_Y(P)$ and $p(y)=0$ if $y\notin C_Y(P)$. Denote by $P_{a,\infty}$ the set of all paths from a to the ideal boundary ∞ of N and by P_∞ the union of $P_{x,\infty}$ for all $x\in X$. We call an element of P_∞ an infinite path.

The extremal length $\lambda_p(\Gamma)$ of order p of a set Γ of paths in N is defined by

$$\lambda_p(\Gamma)^{-1} = \inf\{H_p(W); W\in E(\Gamma)\},$$

where $E(\Gamma)$ is the set of all $W\in L^+(Y)$ such that $\sum_P r(y)W(y)=\sum_{y\in C_Y(P)} r(y)W(y) \geq 1$ for all $P\in\Gamma$. In case Γ is empty, we set $\lambda_p(\Gamma)=\infty$.

The following fundamental properties of the extremal length can be proved analogously to the continuous case (cf. [5]):

LEMMA 2.1. *Let Γ_1 and Γ_2 be sets of paths. If $\Gamma_1\subset\Gamma_2$, then $\lambda_p(\Gamma_1)\geq\lambda_p(\Gamma_2)$.*

LEMMA 2.2. *Let $\{\Gamma_n; n=1, 2,\ldots\}$ be a family of sets of paths in N. Then $\sum_{n=1}^\infty \lambda_p(\Gamma_n)^{-1}\geq\lambda_p(\cup_{n=1}^\infty \Gamma_n)^{-1}$.*

PROOF. We may assume that $\lambda_p(\Gamma_n)>0$ for all n. For any $\varepsilon>0$, there exist $W_n\in E(\Gamma_n)\,(n=1, 2,\ldots)$ such that $H_p(W_n)<\lambda_p(\Gamma_n)^{-1}+2^{-n}\varepsilon$. Let $W(y)=\sup\{W_n(y); n=1, 2,\ldots\}$. Then $\sum_P r(y)W(y)\geq 1$ for all $P\in\cup_{n=1}^\infty \Gamma_n$ and

$$\begin{aligned}\lambda_p(\cup_{n=1}^\infty \Gamma_n)^{-1} \leq H_p(W) &\leq \sum_{n=1}^\infty H_p(W_n)\\ &\leq \sum_{n=1}^\infty \lambda_p(\Gamma_n)^{-1} + \varepsilon.\end{aligned}$$

Since ε is arbitrary, we obtain our inequality.

LEMMA 2.3. *Let Γ be a set of paths. Then $\lambda_p(\Gamma)=\infty$ if and only if there exists $W\in L^+(Y)$ such that $H_p(W)<\infty$ and $\sum_P r(y)W(y)=\infty$ for all $P\in\Gamma$.*

PROOF. Assume that there exists $W\in L^+(Y)$ such that $H_p(W)<\infty$ and $\sum_P r(y)W(y)=\infty$ for all $P\in\Gamma$. For any $\varepsilon>0$, we have $\varepsilon W\in E(\Gamma)$, so that $\lambda_p(\Gamma)^{-1}\leq H_p(\varepsilon W)=\varepsilon^p H_p(W)$ $(1<p<\infty)$ and $\lambda_\infty(\Gamma)^{-1}\leq H_\infty(\varepsilon W)=\varepsilon H_\infty(W)$. Letting $\varepsilon\to 0$, we obtain $\lambda_p(\Gamma)=\infty$. Assume that $\lambda_p(\Gamma)=\infty$. Then there exists a sequence $\{W_n\}$ in $E(\Gamma)$ such that $H_p(W_n)<4^{-np}$ in case $1<p<\infty$ and $H_p(W_n)<2^{-n}$ in case $p=\infty$. Put $W(y)=[\sum_{n=1}^{\infty}(2^n W_n(y))^p]^{1/p}$ in case $1<p<\infty$ and $W(y)=\sup\{2^n W_n(y);\ n=1, 2,\dots\}$ in case $p=\infty$. Then it is easily seen that W satisfies our requirement.

LEMMA 2.4. *Let Γ be a set of paths. Then $\lambda_\infty(\Gamma)<\infty$ if and only if there exists a path $P\in\Gamma$ such that $\sum_P r(y)<\infty$.*

PROOF. The "only if" part is clear by Lemma 2.3. Assume that there exists a path $P\in\Gamma$ such that $\sum_P r(y)<\infty$. For any $W\in E(\Gamma)$, we have

$$1\leq \sum_P r(y)W(y)\leq H_\infty(W)\sum_P r(y),$$

so that $\lambda_\infty(\Gamma)\leq\sum_P r(y)$.

§3. Main results

For each $u\in L(X)$, let us define $du\in L(Y)$ by

$$(du)(y) = -r(y)^{-1}\sum_{x\in X}K(x, y)u(x).$$

Then we have $H_p(du)=D_p(u)$.

LEMMA 3.1. *Let $u\in \boldsymbol{D}_0^{(\infty)}(N)$ and $\{N_n\}$ $(N_n=\langle X_n, Y_n\rangle)$ be an exhaustion of N. Then $\sup_{y\in Y-Y_n}|(du)(y)|\to 0$ as $n\to\infty$.*

PROOF. For any $\varepsilon>0$, there exists $f\in L_0(X)$ such that $D_\infty(u-f)<\varepsilon$. We can find n_0 such that $f(x)=0$ on $X-X_n$ for all $n\geq n_0$. Thus we have $\sup_{y\in Y-Y_n}|(du)(y)|\leq D_\infty(u-f)<\varepsilon$ for all $n>n_0$.

LEMMA 3.2. *Let $P\in P_{a,\infty}$ and p be the path index of P. If $f\in L_0(X)$, then $\sum_P r(y)p(y)[(df)(y)]=f(a)$.*

PROOF. Let us put $C_X(P)=\{x_n;\ n\geq 0\}$ and $C_Y(P)=\{y_n;\ n\geq 1\}$. There exists m such that $f(x_n)=0$ for all $n\geq m$, since $f\in L_0(X)$. We have $(df)(y_n)=0$ for all $n>m$ and

$$\sum_P r(y)p(y)[(df)(y)] = \sum_{n=1}^m r(y_n)p(y_n)[(df)(y_n)]$$
$$= -\sum_{n=1}^m p(y_n)[K(x_{n-1}, y_n)f(x_{n-1}) + K(x_n, y_n)f(x_n)]$$
$$= \sum_{n=1}^m [f(x_{n-1})-f(x_n)] = f(a) - f(x_m) = f(a).$$

Similarly we can prove

LEMMA 3.3. *Let $P \in P_{a,\infty}$ and $C_X(P)=\{x_n;\ n\geq 0\}$. If $u \in L(X)$ and $\sum_P \cdot r(y)|(du)(y)| < \infty$, then $\sum_P r(y)p(y)[(du)(y)] = \lim_{n\to\infty}[u(a)-u(x_n)]$.*

We have

THEOREM 3.1. *Let $u \in \boldsymbol{D}^{(p)}(N)$. Then $u(x)$ has a limit as x tends to the ideal boundary ∞ of N along p-almost every $P \in P_{a,\infty}$.*

PROOF. Let us put $\Gamma=\{P\in P_{a,\infty};\ \sum_P r(y)|(du)(y)|=\infty\}$. Then $u(x)$ has a limit as x tends to the ideal boundary ∞ of N along any path $P \in P_{a,\infty}-\Gamma$ by Lemma 3.3. We see by Lemma 2.3 that $\lambda_p(\Gamma)=\infty$, since $W(y)=|(du)(y)|$ satisfies $H_p(W)=D_p(u)<\infty$ and $\sum_P r(y)W(y)=\infty$ for all $P\in\Gamma$.

This is a discrete analogue to Theorem 2.28 in [5].

For Dirichlet potentials of order p, we have

THEOREM 3.2. *Let $u \in \boldsymbol{D}_0^{(p)}(N)$. Then, for p-almost every $P \in P_{a,\infty}$, $\sum_P \cdot r(y)p(y)[(du)(y)]$ exists and is equal to $u(a)$.*

PROOF. There exists a sequence $\{f_k\}$ in $L_0(X)$ such that $\|u-f_k\|_p \to 0$ as $k\to\infty$. Let us put $w(y)=(du)(y)$ and $w_k(y)=(df_k)(y)$. Then we see that $w_k(y)\to w(y)$ as $k\to\infty$ for each $y\in Y$ and $H_p(w-w_k)\to 0$ as $k\to\infty$. Let Γ_0 be the set of all $P\in P_{a,\infty}$ such that $\sum_P r(y)|w(y)|<\infty$. Then $\lambda_p(P_{a,\infty}-\Gamma_0)=\infty$ (cf. the proof of Theorem 3.1) and $w(P)=\sum_P r(y)p(y)w(y)$ exists for any $P\in\Gamma_0$. Let $\{N_n\}(N_n=\langle X_n, Y_n\rangle)$ be an exhaustion of N with $a\in X_1$. For any $\varepsilon>0$, let us put

$$\Gamma(\varepsilon) = \{P\in\Gamma_0;\ |u(a)-w(P)| \geq \varepsilon\},$$

$$\Gamma_n(\varepsilon) = \{P\in\Gamma(\varepsilon);\ |w(P)-\textstyle\sum_{y\in C_Y(P)\cap Y_m} r(y)p(y)w(y)| < \varepsilon/4 \quad \text{for all} \quad m\geq n\}.$$

Let $n<m$ and $P\in\Gamma_n(\varepsilon)$. Since Y_m is a finite set, there exists k_1 such that

$$\textstyle\sum_{y\in Y_m} r(y)|w(y)-w_k(y)| < \varepsilon/4$$

for all $k\geq k_1$. Since $f_k(a)\to u(a)$ as $k\to\infty$, there exists k_2 such that $|u(a)-f_k(a)|<\varepsilon/4$ for all $k\geq k_2$. Note that k_1 and k_2 are independent of P. For each $k\geq \max\{k_1, k_2\}$, we have

$$\textstyle\sum_{y\in C_Y(P)\cap(Y-Y_m)} r(y)|w_k(y)| \geq |\sum_{y\in C_Y(P)\cap(Y-Y_m)} r(y)p(y)w_k(y)|$$
$$= |\textstyle\sum_P r(y)p(y)w_k(y) - \sum_{y\in C_Y(P)\cap Y_m} r(y)p(y)w_k(y)|$$
$$= |f_k(a) - \textstyle\sum_{y\in C_Y(P)\cap Y_m} r(y)p(y)w_k(y)| \quad \text{(by Lemma 3.2)}$$
$$> |u(a) - \textstyle\sum_{y\in C_Y(P)\cap Y_m} r(y)p(y)w(y)| - \varepsilon/2$$
$$> |u(a) - w(P)| - 3\varepsilon/4 \geq \varepsilon/4,$$

since $P\in\Gamma_n(\varepsilon)$. Define $W_k\in L^+(Y)$ by $W_k(y)=4|w_k(y)|/\varepsilon$ if $y\in Y-Y_m$ and $W_k(y)=0$ if $y\in Y_m$. Then $\sum_P r(y)W_k(y)\geq 1$ for all $P\in\Gamma_n(\varepsilon)$ by the above observation, so that

$$\lambda_p(\Gamma_n(\varepsilon))^{-1}\leq H_p(W_k)=(4/\varepsilon)^p\sum_{y\in Y-Y_m}r(y)|w_k(y)|^p\quad(1<p<\infty)$$

$$\lambda_\infty(\Gamma_n(\varepsilon))^{-1}\leq H_\infty(W_k)=(4/\varepsilon)\sup_{y\in Y-Y_m}|w_k(y)|\,.$$

Letting $k\to\infty$, we have

$$\lambda_p(\Gamma_n(\varepsilon))^{-1}\leq(4/\varepsilon)^p\sum_{y\in Y-Y_m}r(y)|w(y)|^p\quad(1<p<\infty),$$

$$\lambda_\infty(\Gamma_n(\varepsilon))^{-1}\leq(4/\varepsilon)\sup_{y\in Y-Y_m}|w(y)|\,,$$

since $H_p(w-w_k)\to 0$ as $k\to\infty$. By letting $m\to\infty$, we obtain $\lambda_p(\Gamma_n(\varepsilon))=\infty$ $(1<p<\infty)$ and $\lambda_\infty(\Gamma_n(\varepsilon))=\infty$ by Lemma 3.1. From the relation $\Gamma(\varepsilon)=\bigcup_{n=1}^{\infty}\Gamma_n(\varepsilon)$ and Lemma 2.2, it follows that $\lambda_p(\Gamma(\varepsilon))=\infty$. Let $\Gamma^*=\{P\in\Gamma_0;\ u(a)\neq w(P)\}$. Since $\Gamma^*=\bigcup_{n=1}^{\infty}\Gamma(1/n)$, we have $\lambda_p(\Gamma^*)=\infty$ and hence $\lambda_p((P_{a,\infty}-\Gamma_0)\cup\Gamma^*)=\infty$ by Lemma 2.2. Thus $w(P)=u(a)$ for p-almost every $P\in P_{a,\infty}$.

REMARK 3.1. An essential idea of the proof of Theorem 3.2 can be found in the proof of Theorem 2.10 in [5].

By Lemma 3.3 and Theorem 3.2, we have

THEOREM 3.3. *Let $u\in \boldsymbol{D}_0^{(p)}(N)$. Then $u(x)$ has limit 0 as x tends to the ideal boundary ∞ of N along p-almost every path from a to ∞.*

COROLLARY 1. *Let $u\in \boldsymbol{D}_0^{(p)}(N)$. Then $u(x)$ has limit 0 as x tends to the ideal boundary ∞ along p-almost every infinite path.*

PROOF. For each $x\in X$, let Γ_x be the set of all $P\in P_{x,\infty}$ such that $u(x)$ does not have limit 0 as x tends to the ideal boundary ∞ of N along P. Then $\lambda_p(\Gamma_x)=\infty$ by Theorem 3.3. By Lemma 2.2, we have $\lambda_p(\cup\{\Gamma_x;\ x\in X\})=\infty$. Thus $u(x)$ has limit 0 as x tends to the ideal boundary ∞ of N along p-almost every infinite path.

COROLLARY 2. *Let N be of hyperbolic type of order 2. Then the Green function $g_a(x)$ of N with pole at a has limit 0 as x tends to the ideal boundary ∞ of N along 2-almost every infinite path.*

PROOF. Since N is of hyperbolic type of order 2, the Green function g_a of N with pole at a exists and $g_a\in \boldsymbol{D}_0^{(2)}(N)$ (cf. [7]). Thus our assertion follows from Corollary 1 of Theorem 3.3.

REMARK 3.2. In case N is of parabolic type of order p, i.e., $\boldsymbol{D}_0^{(p)}(N)=\boldsymbol{D}^{(p)}(N)$, $1\in \boldsymbol{D}^{(p)}(N)$ does not have limit 0 as x tends to the ideal boundary ∞

of N along any infinite path. However we have $\lambda_p(P_{a,\infty})=\infty$ for any $a\in X$ in this case (cf. [6]), so that $\lambda_p(P_\infty)=\infty$.

We show by an example that $g_a(x)$ does not always have limit 0 as x tends to the ideal boundary ∞ of N along an infinite path.

EXAMPLE 3.1. Let Z be the set of all integers and let $X=\{x_n;\ n\in Z\}$ and $Y=\{y_n;\ n\in Z\}$. Define K by

$$K(x_{n-1}, y_n) = -1 \quad \text{and} \quad K(x_n, y_n) = 1 \qquad \text{for all} \quad n\in Z,$$

$$K(x, y) = 0 \qquad \text{for any other pair.}$$

Then $\{X, Y, K\}$ may be considered as the lattice domain of the real line. Let us define r by $r(y_n)=1$ if $n\leq 0$ and $r(y_n)=2^{-n}$ if $n>0$. Then $N=\{X, Y, K, r\}$ is an infinite network. Let $a=x_0$. Since $\lambda_2(P_{a,\infty})=\sum_{n=1}^{\infty} r(y_n)=1$, N is of hyperbolic type of order 2. We see that $g_a(x_n)=1$ if $n\leq 0$ and $g_a(x_n)=\sum_{k=n+1}^{\infty} r(y_k)=2^{-n}$ if $n>0$. Let P be the path defined by $C_X(P)=\{x_n;\ n\leq 0\}$, $C_Y(P)=\{y_n;\ n\leq 0\}$, $p(y_n)=-1$ if $n\leq 0$ and $p(y_n)=0$ if $n>0$. Then $P\in P_{a,\infty}$ and $g_a(x)$ has limit 1 as x tends to the ideal boundary ∞ of N along P. Note that $\lambda_2(\{P\})=\infty$.

Finally we show that the converse of Theorem 3.3 does not hold in general for $p=\infty$.

EXAMPLE 3.2. Let N be the same as in Example 3.1. Consider $u\in L(X)$ defined by $u(x_n)=n$ for $n\leq 0$ and $u(x_n)=2^{-n}$ for $n>0$. Then $(du)(y_n)=-r(y_n)^{-1}[u(x_n)-u(x_{n-1})]=-1$ for $n\leq 1$ and $(du)(y_n)=1$ for $n\geq 2$, so that $u\in \boldsymbol{D}^{(\infty)}(N)$. We see by Lemma 2.4 that $u(x)$ has limit 0 as x tends to the ideal boundary ∞ along ∞-almost every path from $a=x_0$ to ∞. On the other hand, it follows from Lemma 3.1 that $u\notin \boldsymbol{D}_0^{(\infty)}(N)$.

References

[1] C. R. Deeter and J. M. Gray, The discrete Green's function and the discrete kernel function, Discrete Math. **10** (1974), 29–42.

[2] R. J. Duffin, Discrete potential theory, Duke Math. J. **20** (1953), 233–251.

[3] R. J. Duffin, The extremal length of a network, J. Math. Anal. Appl. **5** (1962), 200–215.

[4] T. Nakamura and M. Yamasaki, Generalized extremal length of an infinite network, Hiroshima Math. J. **6** (1976), 95–111.

[5] M. Ohtsuka, Dirichlet problem, extremal length and prime ends, Van Nostrand, 1970.

[6] M. Yamasaki, Parabolic and hyperbolic infinite networks, Hiroshima Math. J. **7** (1977), 135–146.

[7] M. Yamasaki, Discrete potentials on an infinite network, Mem. Fac. Sci. Shimane Univ. **13** (1979), 31–44.

Department of Mathematics,
Faculty of Science,
Shimane University