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Exponential image and conjugacy classes in the group $O(3, 2)$

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§ 1. Introduction

Let G be a classical real linear Lie group, $\mathfrak{g}$ its Lie algebra and let $\exp: \mathfrak{g} \to G$ be the exponential map of G. It is now well known the description of conjugacy classes in G and orbits in $\mathfrak{g}$ under the conjugation action of G, as seen in the paper [1] by N. Burgoyne and R. Cushman. In [2], D. Ž. Djoković has studied that which of conjugacy classes lies in the image of the exponential map, and he obtained the many results based on the conjugacy classes. It is of interest to determine which conjugacy classes lie in the interior, boundary or exterior of $\exp \mathfrak{g}$ in G, for the ordinary topology of $\mathfrak{g}$ and G. In this paper, we shall observe this for a special classical group.

In the papers [7] and [8], the author showed the following for $G = GL(n, R)$ or $G = SL(n, R)$: Let x be an element in G. Then (i) x is an interior point of $\exp \mathfrak{g}$ in G if and only if x has no negative eigenvalues, (ii) x is a boundary point of $\exp \mathfrak{g}$ in G if and only if x has negative eigenvalues and the multiplicities of the negative eigenvalues are all even.

Let $O(p, q)$ be the orthogonal group of the signature (p, q), $o(p, q)$ its Lie algebra and let $O_0(p, q)$ be the connected component of the identity element in $O(p, q)$. In the paper [9], for $p \geq q \geq 0$ the author showed that $\exp: o(p, q) \to O_0(p, q)$ is surjective if and only if $q = 0, 1$. Hence $O(2, 2)$ is the simplest one that $\exp: o(p, q) \to O_0(p, q)$ is not surjective.

In this paper, we give the complete table for $G = O(3, 2)$ that shows which of conjugacy classes lies in the interior, boundary or exterior of $\exp \mathfrak{g}$ in G, and we also give similar results on $O(2, 2)$ as a corollary. The main results are Theorem 9 and the corollaries in Section 4. In particular, the boundary of $\exp o(2, 2)$ in $O(2, 2)$ and the boundary of $\exp o(3, 2)$ in $O(3, 2)$ are characterized as follows:

(i) Let $x \in O(2, 2)$. Then x is a boundary point of $\exp o(2, 2)$ in $O(2, 2)$ if and only if eigenvalues of x are all real negative and the multiplicity of each eigenvalue of x is even (2 or 4).

(ii) Let $x \in O(3, 2)$. Then x is a boundary point of $\exp o(3, 2)$ in $O(3, 2)$ if and only if x is conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & x' \end{pmatrix}$ in $O(3, 2)$, where x' is a boundary point of $\exp o(2, 2)$ in $O(2, 2)$.

Recently D. Ž. Djoković ([3], [4]) determined the closure of an arbitrary orbit and the closure of an arbitrary conjugacy class, for a classical group. But it seems, in author's opinion, that his description of the closures of conjugacy classes does not give every information about the boundary of $\exp \mathfrak{g}$ in G. We shall explain this in the following example.

EXAMPLE. Let $G=GL(2, R)$ and take $x=\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$. If there exists an element y in $\exp \mathfrak{g}$ such that the closure $C^{\sim}$ of the conjugacy class C of y contains x, it follows that x is a boundary point of $\exp \mathfrak{g}$ in G. But since the characteristic polynomial of each element in $C^{\sim}$ agrees with that of y, the characteristic polynomials of y and x are equal. Furthermore $y \in \exp \mathfrak{g}$. Hence the Jordan form of y in G is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$ and so $C^{\sim}=\{-I_2\}$. This contradicts the assumption $x \in C^{\sim}$. Therefore there exists no y in $\exp \mathfrak{g}$ such that the closure $C^{\sim}$ of the conjugacy class C of y contains x.

Next, we shall show that x in the above example is a boundary point of $\exp \mathfrak{g}$ in $G=GL(2, R)$. Let $0<\theta<\pi$, and put $S(\theta)=\begin{pmatrix} \pi-\theta & 0 \\ 0 & 1 \end{pmatrix}$, $R(\theta)=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Then we have $\lim_{\theta\to\pi} S(\theta)^{-1}R(\theta)S(\theta)=x$. This procedure plays an essential role in this paper. (See [7] and [8] for the relating discussion.)

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§2. Preliminaries and notation

Let V be a finite-dimensional vector space over the field R of real numbers, equipped with a non-degenerate symmetric bilinear form $\tau: V\times V\to R$. The *orthogonal group*, $O(V, \tau)$, is the group of linear automorphisms of V preserving τ and let $o(V, \tau)$ be its Lie algebra. Then $O(V, \tau)$ is determined up to isomorphism by the signature (p, q) of τ and so we shall write $O(p, q)$ for $O(V, \tau)$. When we make a basis of V be fixed, we identify V with R^{p+q} (the column vector space). Let J be the non-singular symmetric matrix associated with τ. Then we identify $O(V, \tau)$ with $O(J)$, where $O(J)$ is the set of all real matrices A such that ${}^tAJA=J$. We shall denote by $o(J)$ the Lie algebra of $O(J)$. Let $I_{p,q}$ be $\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, where I_p is the identity matrix. Then we note that $O(I_{p,q})$ is isomorphic to $O(J)$ by the map $T\in O(I_{p,q})\to P^{-1}TP\in O(J)$ for some non-singular real matrix P such that ${}^tPI_{p,q}P=J$. The pseudo-orthogonal group $O(p,q)$ (that is, $p>0$ and $q>0$) has four connected components $O(p, q)^{\varepsilon}_{\varepsilon'}$, $(\varepsilon, \varepsilon'=\pm)$. Let $T=\begin{pmatrix} T_1 & T_{12} \\ T_{21} & T_2 \end{pmatrix}\begin{matrix} \}p \\ \}q \end{matrix}$ be an element of $G=O(I_{p,q})$. Then the four connected components $G^{\varepsilon}_{\varepsilon'}$ are given by

$$G^{\varepsilon}_{\varepsilon'} = \{T\in O(I_{p,q});\ \sigma(\det T_1) = \varepsilon,\ \sigma(\det T_2) = \varepsilon'\},$$

where $\sigma(\det T_i)$ is the sign of the determinant of T_i. We note that $G_+^+ = O_0(I_{p,q})$ and also $\det T = (\det T_1)\cdot(\det T_2)^{-1}$, $|\det T_1| = |\det T_2| \geq 1$ (cf. [9]).

For the exponential map $\exp: o(p, q) = \mathfrak{g} \to O(p, q) = G$, we denote the interior, closure, boundary and exterior of $\exp\mathfrak{g}$ in G by $\mathrm{Int}(\exp\mathfrak{g})$, $\mathrm{Cl}(\exp\mathfrak{g})$, $\partial(\exp\mathfrak{g})$ and $(\exp\mathfrak{g})^e$, respectively. Then it is obvious that $\exp\mathfrak{g} \subset G_+^+$, $\partial(\exp\mathfrak{g}) = \mathrm{Cl}(\exp\mathfrak{g})\backslash\mathrm{Int}(\exp\mathfrak{g})$ and $(\exp\mathfrak{g})^e = (G_+^+\backslash\mathrm{Cl}(\exp\mathfrak{g})) \cup G_-^+ \cup G_+^- \cup G_-^-$. We note that $\exp\mathfrak{g}$, $\mathrm{Int}(\exp\mathfrak{g})$, $\mathrm{Cl}(\exp\mathfrak{g})$, $\partial(\exp\mathfrak{g})$, $(\exp\mathfrak{g})^e$ and $G_{\varepsilon'}^{\varepsilon}$ are all normal subsets in $G = O(p, q)$.

Here we shall explain an outline in a form convenient for us, about the meaning and the notation of types introduced by N. Burgoyne and R. Cushman [1].

Let $O(V', \tau')$ be an orthogonal group and let $A \in O(V, \tau)$, $B \in O(V', \tau')$. Then we write $(A, V, \tau) \sim (B, V', \tau')$ if there exists a real linear isomorphism ϕ of V onto V' such that $\phi A = B\phi$ and $\tau(u, v) = \tau'(\phi u, \phi v)$ for all $u, v \in V$. An equivalence class for the above equivalence relation "$\sim$" is called a *type*. If Γ denotes a type and $(A, V, \tau) \in \Gamma$, we put $\dim \Gamma = \dim V$. We denote a Lie group type by Γ and a Lie algebra type by Δ. From [1, Prop. 1], the determination of conjugacy classes is equivalent to the classification of types, that is, for $A, B \in O(V, \tau)$, there exists $x \in O(V, \tau)$ such that $x^{-1}Ax = B$ if and only if $(A, V, \tau) \sim (B, V, \tau)$. Thus, from now on, if $A \in O(V, \tau)$ and $(A, V, \tau) \in \Gamma$, we often use Γ in a sense of the conjugacy class in $O(V, \tau)$ of A.

Let $A \in O(V, \tau)$ and $(A, V, \tau) \in \Gamma$. Suppose that $V = V_1 + V_2$ is a τ-orthogonal disjoint sum of proper A-invariant subspaces. Then the groups $O(V_i, \tau|V_i)$ are well defined and $A|V_i \in O(V_i, \tau|V_i)$. Let Γ_i be the type containing $(A|V_i, V_i, \tau|V_i)$. Then we set $\Gamma = \Gamma_1 + \Gamma_2$.

The type Γ is called *indecomposable* if it cannot be decomposed as the sum of two or more types.

Let $A \in o(V, \tau)$, $(A, V, \tau) \in \Delta$ and let $A = S + N$ be the additive Jordan decomposition of A, that is, $S, N \in o(V, \tau)$, S is semisimple, N is nilpotent and $SN = NS$. If $N^m \neq 0$ and $N^{m+1} = 0$, m is called the *height* of Δ and we write $m = \mathrm{ht}\Delta$.

Now suppose that Δ is an indecomposable type with $\mathrm{ht}\,\Delta = m$. If $(A, V, \tau) \in \Delta$, set $V^- = V/NV$ and for $v \in V$, put $v^- = v + NV$. Define A^- and τ^- by $A^-v^- = (Av)^-$ and $\tau^-(u^-, v^-) = \tau(u, N^m v)$. Of course $A^- = S^-$. Then the proof of [1, Prop. 2] guarantees the following;

(i) V^- can be regarded as an S-invariant subspace of V and then V can be regarded as $V^- + NV^- + \cdots + N^m V^-$ (direct sum).

(ii) $\dim N^i V^- = \dim V^-$ for $0 \leq i \leq m$, and $S = S^- + \cdots + S^-$ ($m+1$ copies, direct sum).

(iii) For $u = \sum_{r=0}^m N^r u_r$ and $v = \sum_{s=0}^m N^s v_s$, where $u_r, v_s \in V^-$, $\tau(u, v) = \sum_{r+s=m} (-1)^r \tau^-(u_r, v_s)$.

Since τ is symmetric, we note that τ^- is symmetric if m is even, and alternating if m is odd. If $\zeta,\ldots$ are eigenvalues of $A^- = S^-$, the indecomposable type Δ with $\mathrm{ht}\,\Delta = m$ is written as the form $\Delta_m(\zeta,\ldots)$. Then $\dim \Delta_m(\zeta,\ldots) = (m+1)\cdot$(number of eigenvalues $\zeta,\ldots$ of A^- with multiplicities counted).

All indecomposable types (of Lie algebra type) for orthogonal groups are given in [1, p. 349, Table II] as follows: (1) $\Delta_m(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$, $\zeta \neq \pm\bar{\zeta}$, (2) $\Delta_m(\zeta, -\zeta)$, $\zeta = \bar{\zeta} \neq 0$, (3) $\Delta_m^{\varepsilon}(\zeta, -\zeta)$, $\zeta = -\bar{\zeta} \neq 0$, (4) $\Delta_m^{\varepsilon}(0)$, m even, and (5) $\Delta_m(0, 0)$, m odd, $(\varepsilon = \pm)$.

Let Δ be one of the above indecomposable types. Then it follows from [1, Appendix 2] that S^- and τ^- associated to a representative $(A, V, \tau) \in \Delta$ can be exactly expressed as follows;

(1) For $\Delta_m(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$, where $\zeta = a + ib$, there exists a basis $\langle e_1, e_2, e_3, e_4\rangle$ of V^- such that the matrices of S^- and τ^- with respect to the basis are given by

$$S^- = \begin{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & 0 \\ 0 & -{}^t\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{pmatrix}, \quad \tau^- = 1/2\begin{pmatrix} 0 & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & 0 \end{pmatrix} \text{ if } m \text{ is even,}$$

$$\tau^- = 1/2\begin{pmatrix} 0 & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & 0 \end{pmatrix} \text{ if } m \text{ is odd.}$$

(2) For $\Delta_m(\zeta, -\zeta)$, there exists a basis $\langle e_1, f_1\rangle$ of V^- such that S^- and τ^- with respect to the basis are given by

$$S^- = \begin{pmatrix} \zeta & 0 \\ 0 & -\zeta \end{pmatrix}, \tau^- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ if } m \text{ is even, } \tau^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ if } m \text{ is odd.}$$

(3) For $\Delta_m^{\varepsilon}(\zeta, -\zeta)$, there exists a basis $\langle e_1, f_1\rangle$ of V^- such that S^- and τ^- with respect to the basis are given by

$$S^- = \begin{pmatrix} 0 & -i\zeta \\ i\zeta & 0 \end{pmatrix}, \tau^- = \varepsilon\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ if } m \text{ is even,}$$

$$\tau^- = \varepsilon(i\zeta)^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ if } m \text{ is odd.}$$

(4) For $\Delta_m^\varepsilon(0)$, m even, there exists a basis $\langle e\rangle$ of V^- such that $S^-=0$ and $\tau^-(e, e)=\varepsilon 1$.

(5) For $\Delta_m(0, 0)$, m odd, there exists a basis $\langle e_1, f_1\rangle$ of V^- such that $S^-=0$ and $\tau^-=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

For the group $O(V, \tau)$, let $A\in O(V, \tau)$ and $(A, V, \tau)\in\Gamma$. Then we can, in a unique way, write $A=S\exp N$, where $S\in O(V, \tau)$, $N\in o(V, \tau)$, S is semisimple, N is nilpotent and $SN=NS$. For this nilpotent N, similar terms as the case $o(V, \tau)$ can be used and similar results of [1] hold. We refer the related results to T. Iwamoto [5]. All indecomposable types for orthogonal groups can be obtained by using the Cayley transformation [1, p. 352] from those of the Lie algebras. An explicit table of all indecomposable types for orthogonal groups is given in [2, p. 83].

Now we shall again give the following theorem [9, Theorem 1] which is a convenient form of a part of the main theorem of [1].

THEOREM 1 (N. Burgoyne and R. Cushman). *In the group $O(p, q)$, or in the Lie algebra $o(p, q)$, the following statements hold.*

(i) *Let Γ be a type of $O(p, q)$. Then the decomposition $\Gamma=\Gamma_1+\cdots+\Gamma_s$ into indecomposable types is unique and we have the relations;*

$$(p+q=)\dim\Gamma = \dim\Gamma_1+\cdots+\dim\Gamma_s$$

and

$$(q=)n_-(\Gamma) = n_-(\Gamma_1)+\cdots+n_-(\Gamma_s),$$

where $(n_+(\Gamma), n_-(\Gamma))$ denotes the signature of Γ. It is noticed that if the signature of a type Γ is $(n_+(\Gamma), n_-(\Gamma))$, Γ can be considered as a type in the group $O(n_+(\Gamma), n_-(\Gamma))$ and we have $\dim\Gamma=n_+(\Gamma)+n_-(\Gamma)$.

(ii) *Conversely if $\Gamma_1,\ldots,\Gamma_s$ are indecomposable types belonging to the same family as $O(p, q)$ satisfying the above restrictions on dimension and n_-, then $\Gamma_1+\cdots+\Gamma_s$ is a well defined type in $O(p, q)$.*

A type Γ of $O(p, q)$ is said to be an *exponential* if $\Gamma=\exp\Delta$ for some type Δ of the Lie algebra $o(p, q)$ (cf. D. Ž. Djoković [2]). We state the following theorem which is a theorem in D. Ž. Djoković [2, p. 84] because it is also essential for our purpose.

THEOREM 2. *A type Γ of $O(p, q)$ is an exponential if and only if the multiplicities of the non-exponential (=exceptional) indecomposable types in Γ are all even.*

REMARK 1. It follows from [2, p. 83] that the non-exponential indecomposable types for $O(p, q)$ are $\Gamma_m(\lambda, \lambda^{-1})$, λ real, $\lambda<0$ and $\lambda \neq -1$, and $\Gamma_m^{\pm}(-1)$, m even.

§3. Indecomposable types and conjugacy classes in $O(3, 2)$

In Table 1, we list up all indecomposable types Δ in $o(p, q)$ arranged in order of $n_-(\Delta)$. We note that an indecomposable type Δ in Table 1 actually occurs if and only if $n_+(\Delta) \leq p$ and $n_-(\Delta) \leq q$ by Theorem 1.

Table 1 (*indecomposable types of* $o(p, q)$)

Δ		$n_+(\Delta)$	$n_-(\Delta)$	$(k \geq 1)$
$\Delta_0^+(0)$		1	0	
$\Delta_0^+(\zeta, -\zeta)$	$\bar{\zeta} = -\zeta \neq 0$	2	0	
$\Delta_{4k-4}^-(0)$		$2k-2$	$2k-1$	
$\Delta_{2k-2}(\zeta, -\zeta)$	$\bar{\zeta} = \zeta \neq 0$	$2k-1$	$2k-1$	
$\Delta_{4k-2}^-(0)$		$2k$	$2k-1$	
$\Delta_{4k-4}^-(\zeta, -\zeta)$	$\bar{\zeta} = -\zeta \neq 0$	$4k-4$	$4k-2$	
$\Delta_{8k-6}^+(0)$		$4k-3$	$4k-2$	
$\Delta_{4k-3}(0, 0)$		$4k-2$	$4k-2$	
$\Delta_{4k-3}(\zeta, -\zeta)$	$\bar{\zeta} = \zeta \neq 0$	$4k-2$	$4k-2$	
$\Delta_{4k-3}^{\varepsilon}(\zeta, -\zeta)$	$\bar{\zeta} = -\zeta \neq 0$	$4k-2$	$4k-2$	
$\Delta_{2k-2}(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\bar{\zeta} \neq \pm\zeta$	$4k-2$	$4k-2$	
$\Delta_{8k-4}^+(0)$		$4k-1$	$4k-2$	
$\Delta_{4k-2}^-(\zeta, -\zeta)$	$\bar{\zeta} = -\zeta \neq 0$	$4k$	$4k-2$	
$\Delta_{4k-2}^+(\zeta, -\zeta)$	$\bar{\zeta} = -\zeta \neq 0$	$4k-2$	$4k$	
$\Delta_{8k-2}^+(0)$		$4k-1$	$4k$	
$\Delta_{4k-1}(0, 0)$		$4k$	$4k$	
$\Delta_{4k-1}(\zeta, -\zeta)$	$\bar{\zeta} = \zeta \neq 0$	$4k$	$4k$	
$\Delta_{4k-1}^{\varepsilon}(\zeta, -\zeta)$	$\bar{\zeta} = -\zeta \neq 0$	$4k$	$4k$	
$\Delta_{2k-1}(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\bar{\zeta} \neq \pm\zeta$	$4k$	$4k$	
$\Delta_{8k}^+(0)$		$4k+1$	$4k$	
$\Delta_{4k}^+(\zeta, -\zeta)$	$\bar{\zeta} = -\zeta \neq 0$	$4k+2$	$4k$	$(\varepsilon = \pm)$

In particular we shall list in Table 2 all indecomposable types which actually occur in $O(3, 2)$.

Table 2 (*indecomposable types of* $O(3, 2)$)

algebra types Δ		group types Γ		n_+	n_-
$\Delta_0^+(0)$		$\Gamma_0^+(1), \Gamma_0^+(-1)$		1	0
$\Delta_0^+(\zeta, -\zeta)$	$\bar{\zeta} = -\zeta \neq 0$	$\Gamma_0^+(\lambda, \lambda^{-1})$	$\lvert\lambda\rvert = 1, \lambda \neq \pm 1$	2	0
$\Delta_0^-(0)$		$\Gamma_0^-(1), \Gamma_0^-(-1)$		0	1
$\Delta_0(\zeta, -\zeta)$	$\bar{\zeta} = \zeta \neq 0$	$\Gamma_0(\lambda, \lambda^{-1})$	$\bar{\lambda} = \lambda \neq \lambda^{-1}$	1	1
$\Delta_2^-(0)$		$\Gamma_2^-(1), \Gamma_2^-(-1)$		2	1
$\Delta_0^-(\zeta, -\zeta)$	$\bar{\zeta} = -\zeta \neq 0$	$\Gamma_0^-(\lambda, \lambda^{-1})$	$\lvert\lambda\rvert = 1, \lambda \neq \pm 1$	0	2
$\Delta_2^+(0)$		$\Gamma_2^+(1), \Gamma_2^+(-1)$		1	2
$\Delta_1(0, 0)$		$\Gamma_1(1, 1), \Gamma_1(-1, -1)$		2	2
$\Delta_1(\zeta, -\zeta)$	$\bar{\zeta} = \zeta \neq 0$	$\Gamma_1(\lambda, \lambda^{-1})$	$\bar{\lambda} = \lambda \neq \lambda^{-1}$	2	2
$\Delta_1^\varepsilon(\zeta, -\zeta)$	$\bar{\zeta} = -\zeta \neq 0$	$\Gamma_1^\varepsilon(\lambda, \lambda^{-1})$	$\lvert\lambda\rvert = 1, \lambda \neq \pm 1$	2	2
$\Delta_0(\zeta, -\zeta, \bar{\zeta}, -\bar{\zeta})$	$\bar{\zeta} \neq \pm\zeta$	$\Gamma_0(\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1})$	$\bar{\lambda} \neq \lambda \neq \bar{\lambda}^{-1}$	2	2
$\Delta_4^+(0)$		$\Gamma_4^+(1), \Gamma_4^+(-1)$		3	2

REMARK 2. It follows from Remark 1 that non-exponential indecomposable types are just $\Gamma_0^+(-1)$, $\Gamma_0^-(-1)$, $\Gamma_2^+(-1)$, $\Gamma_2^-(-1)$, $\Gamma_4^+(-1)$, $\Gamma_0(\lambda, \lambda^{-1})$ and $\Gamma_1(\lambda, \lambda^{-1})$ where λ real, $\lambda < 0$ and $\lambda \neq -1$.

By Theorem 1 we can now describe in Table 3 all conjugacy classes in $O(2, 2)$ and $O(3, 2)$.

Table 3

(I) *All conjugacy classes in* $O(2, 2)$		
(a_1)	$\Gamma_0(\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1})$	$\bar{\lambda} \neq \lambda \neq \bar{\lambda}^{-1}$
(a_2)	$\Gamma_1^\varepsilon(\lambda, \lambda^{-1})$	$\lvert\lambda\rvert = 1, \lambda \neq \pm 1$
(a_3)	$\Gamma_1(\lambda, \lambda^{-1})$	$\bar{\lambda} = \lambda \neq \lambda^{-1}$
(a_4)	$\Gamma_1(1, 1), \Gamma_1(-1, -1)$	
(a_5)	$\Gamma_0^+(\pm 1) + \Gamma_2^+(\pm 1)$	
(a_6)	$\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_0^-(\mu, \mu^{-1})$	$\lvert\lambda\rvert = \lvert\mu\rvert = 1, \lambda \neq \pm 1, \mu \neq \pm 1$
(a_7)	$\Gamma_0^+(\pm 1) + \Gamma_0^+(\pm 1) + \Gamma_0^-(\lambda, \lambda^{-1})$	$\lvert\lambda\rvert = 1, \lambda \neq \pm 1$
(b_1)	$\Gamma_2^-(\pm 1) + \Gamma_0^-(\pm 1)$	
(b_2)	$\Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0(\mu, \mu^{-1})$	$\bar{\lambda} = \lambda \neq \lambda^{-1}, \bar{\mu} = \mu \neq \mu^{-1}$
(b_3)	$\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_0^-(\pm 1) + \Gamma_0^-(\pm 1)$	$\lvert\lambda\rvert = 1, \lambda \neq \pm 1$
(b_4)	$\Gamma_0^+(\pm 1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(\pm 1)$	$\bar{\lambda} = \lambda \neq \lambda^{-1}$
(b_5)	$\Gamma_0^+(\pm 1) + \Gamma_0^+(\pm 1) + \Gamma_0^-(\pm 1) + \Gamma_0^-(\pm 1)$.	
(II) *All conjugacy classes in* $O(3, 2)$		
(a_i')	$\Gamma_0^+(\pm 1) + (a_i)$	$(1 \leq i \leq 7)$
(b_j')	$\Gamma_0^+(\pm 1) + (b_j)$	$(1 \leq j \leq 5)$
(c_1)	$\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_0(\mu, \mu^{-1}) + \Gamma_0^-(\pm 1)$	$\lvert\lambda\rvert = 1, \lambda \neq \pm 1, \bar{\mu} = \mu \neq \mu^{-1}$

(c_2)	$\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_2^+(\pm 1)$	$\lvert\lambda\rvert = 1, \lambda \neq \pm 1$
(c_3)	$\Gamma_0(\lambda, \lambda^{-1}) + \Gamma_2^-(\pm 1)$	$\bar{\lambda} = \lambda \neq \lambda^{-1}$
(c_4)	$\Gamma_4^+(\pm 1)$.	

Next we shall give the following important table (Table 4) which plays a practical role in the last section. Since matrix representations of types in $O(p, q)$ are determined by those of indecomposable types Γ by Theorem 1, in this table we give the matrix representations of the indecomposable types in Table 2. If $(A, R^{n_+ + n_-}, J)$ is a representative of an indecomposable type Γ in Table 2, where J is the symmetric matrix indicating the bilinear form, we simply write $(A, J) \in \Gamma$, and especially in the case $J = I_{n_+, n_-}$ we write (A°, J°) for (A, J). (A, J)'s are mainly described in [6, pp. 486–487]. This notation (A°, J°) is very useful to give explicitly the representative of each conjugacy class for $O(I_{2,2})$ or $O(I_{3,2})$. When two representatives (A°, J°), (A, J) of Γ are given, we shall describe a real matrix P such that ${}^tPJ^\circ P = J$ and $P^{-1}A^\circ P = A$.

Table 4 (*matrix representations of indecomposable types in* $O(3, 2)$)

(1) $\Gamma_0^+(\lambda, \lambda^{-1})$, $\lvert\lambda\rvert = 1, \lambda \neq \pm 1$;

$$A^\circ = R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad J^\circ = I_{2,0}, \quad (\lambda = e^{i\theta}).$$

(1′) $\Gamma_0^-(\lambda, \lambda^{-1})$, $\lvert\lambda\rvert = 1, \lambda \neq \pm 1$; $A^\circ = R(\theta)$, $J^\circ = I_{0,2}$, $(\lambda = e^{i\theta})$.

(2) $\Gamma_0^+(\pm 1)$; $A^\circ = (\pm 1)$, $J^\circ = I_{1,0}$.

(2′) $\Gamma_0^-(\pm 1)$; $A^\circ = (\pm 1)$, $J^\circ = I_{0,1}$.

(3) $\Gamma_0(\lambda, \lambda^{-1})$, $\bar{\lambda} = \lambda \neq \lambda^{-1}$; $A^\circ = 2^{-1}\begin{pmatrix} \lambda + \lambda^{-1} & \lambda - \lambda^{-1} \\ \lambda - \lambda^{-1} & \lambda + \lambda^{-1} \end{pmatrix}$, $J^\circ = I_{1,1}$;

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = \sqrt{(1/2)}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

(4) $\Gamma_2^-(\pm 1)$; $A^\circ = \pm\begin{pmatrix} 0 & -1 & 0 \\ 3 & 0 & 2\sqrt{2} \\ 2\sqrt{2} & 0 & 3 \end{pmatrix}$, $J^\circ = I_{2,1}$;

$$A = \pm\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1/2 & -1 \\ -1/2 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} -\sqrt{2}/8 & 0 & \sqrt{2} \\ \sqrt{2}/8 & -\sqrt{2} & -\sqrt{2} \\ -1/4 & -1 & -2 \end{pmatrix}.$$

(4′) $\Gamma_2^{\ddagger}(\pm 1)$; $A^\circ = \pm \begin{pmatrix} 3 & 0 & 2\sqrt{2} \\ 2\sqrt{2} & 0 & 3 \\ 0 & -1 & 0 \end{pmatrix}$, $J^\circ = I_{1,2}$;

$$A = \pm \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, J = \begin{pmatrix} 0 & 1/2 & 1 \\ 1/2 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, P = \begin{pmatrix} -1/4 & -1 & -2 \\ \sqrt{2}/8 & -\sqrt{2} & -\sqrt{2} \\ -\sqrt{2}/8 & 0 & \sqrt{2} \end{pmatrix}.$$

(5) $\Gamma_1(\varepsilon 1, \varepsilon 1)$; $A^\circ = \varepsilon \begin{pmatrix} 1 & 1/2 & 0 & 1/2 \\ -1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 1/2 & 0 & -1/2 & 1 \end{pmatrix}$, $J^\circ = I_{2,2}$;

$$A = \varepsilon \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & {}^t\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad P = \sqrt{(1/2)} \begin{pmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{pmatrix}.$$

We note that the type $\Gamma_1(-1, -1)$ can be also presented as

$$A^\circ = \begin{pmatrix} -1 & 1/2 & 0 & 1/2 \\ -1/2 & -1 & 1/2 & 0 \\ 0 & 1/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 & -1 \end{pmatrix}, \quad J^\circ = I_{2,2};$$

$$A = \begin{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & {}^t\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

$$P = \sqrt{(1/2)} \begin{pmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{pmatrix}.$$

(6) $\Gamma_1(\lambda, \lambda^{-1})$, $\bar{\lambda} = \lambda \neq \lambda^{-1}$;

$$A^\circ = 2^{-1}\begin{pmatrix} \lambda+\lambda^{-1} & 1 & \lambda-\lambda^{-1} & 1 \\ -\lambda^{-2} & \lambda+\lambda^{-1} & \lambda^{-2} & \lambda-\lambda^{-1} \\ \lambda-\lambda^{-1} & 1 & \lambda+\lambda^{-1} & 1 \\ \lambda^{-2} & \lambda-\lambda^{-1} & -\lambda^{-2} & \lambda+\lambda^{-1} \end{pmatrix}, \quad J^\circ = I_{2,2};$$

$$A = \begin{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} & 0 \\ 0 & {}^t\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^{-1} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad P = \sqrt{(1/2)}\begin{pmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{pmatrix}.$$

(7) $\Gamma_1^\varepsilon(\lambda, \lambda^{-1})$, $|\lambda| = 1$, $\lambda \neq \pm 1$; $A = \begin{pmatrix} R(\theta) & 0 \\ R(\theta) & R(\theta) \end{pmatrix}$,
$(\lambda = e^{i\theta})$

$$J = (-\varepsilon \sin\theta)\,(1-\cos\theta)^{-1}K, \text{ where } K = \begin{pmatrix} 0 & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & 0 \end{pmatrix};$$

$$A^\circ = \begin{cases} PAP^{-1} & \text{if } \; -\varepsilon\sin\theta > 0 \\ P'AP'^{-1} & \text{if } \; -\varepsilon\sin\theta < 0 \end{cases}, \quad J^\circ = I_{2,2},$$

where $P = (-\varepsilon(2-2\cos\theta)^{-1}\sin\theta)^{1/2}\cdot\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$,

and $\quad P' = (\varepsilon(2-2\cos\theta)^{-1}\sin\theta)^{1/2}\cdot\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$.

(8) $\Gamma_0(\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1})$, $\bar{\lambda} \neq \lambda \neq \bar{\lambda}^{-1}$;

$$A^\circ = 2^{-1}\begin{pmatrix} (|\lambda| + |\lambda|^{-1})R(\theta) & (|\lambda| - |\lambda|^{-1})R(\theta) \\ (|\lambda| - |\lambda|^{-1})R(\theta) & (|\lambda| + |\lambda|^{-1})R(\theta) \end{pmatrix}, \quad J^\circ = I_{2,2}; \quad (\lambda = |\lambda|e^{i\theta})$$

$$A = \begin{pmatrix} |\lambda|R(\theta) & 0 \\ 0 & {}^t(|\lambda|R(\theta))^{-1} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad P = \sqrt{(1/2)}\begin{pmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{pmatrix}.$$

(9) $\Gamma_4^\ddagger(\pm 1)$;

$$A^\circ = \pm\begin{pmatrix} 49/48 & -5/12 & \sqrt{2}/4 & 7/12 & 1/48 \\ 5/12 & 3/4 & -\sqrt{2}/2 & -1/4 & 5/12 \\ \sqrt{2}/4 & \sqrt{2}/2 & 1 & \sqrt{2}/2 & \sqrt{2}/4 \\ 7/12 & 1/4 & \sqrt{2}/2 & 5/4 & 7/12 \\ -1/48 & 5/12 & -\sqrt{2}/4 & -7/12 & 47/48 \end{pmatrix}, \quad J^\circ = I_{3,2};$$

$$A = \pm\begin{pmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ 1/2 & 1 & 1 & & \\ 1/6 & 1/2 & 1 & 1 & \\ 1/24 & 1/6 & 1/2 & 1 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & 0 \end{pmatrix},$$

$$P = \sqrt{(1/2)}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

REMARK 3. Since $-I_5 \in O(I_{3,2})^-_+$, we have $\Gamma_4^\ddagger(-1) \subset O(3, 2)^-_+$.

In the following theorem, we shall partition the conjugacy classes for $G = O(2, 2)$ or $G = O(3, 2)$ described in Table 3 into $\exp \mathfrak{g}$, $G^\ddagger_\ddagger \backslash \exp \mathfrak{g}$, $G^\pm_-$, G^-_+ and G^-_-. It can be easily done by Theorem 2.

THEOREM 3. (I) *The case* $G = O(2, 2)$.

Conjugacy classes contained in $\exp \mathfrak{g}$

(α_1) $\Gamma_0(\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1})$ $\bar{\lambda} \neq \lambda \neq \bar{\lambda}^{-1}$

(α_2)	$\Gamma_1^{\varepsilon}(\lambda, \lambda^{-1})$	$\lvert\lambda\rvert = 1,\ \lambda \neq \pm 1$
(α_3)	$\Gamma_1(\lambda, \lambda^{-1})$	$\lambda > 0,\ \lambda \neq 1$
(α_4)	$\Gamma_1(1, 1)$	
(α_5)	$\Gamma_1(-1, -1)$	
(α_6)	$\Gamma_0^+(1) + \Gamma_2^+(1)$	
(α_7)	$\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_0^-(\mu, \mu^{-1})$	$\lvert\lambda\rvert = \lvert\mu\rvert = 1,\ \lambda \neq \pm 1,\ \mu \neq \pm 1$
(α_8)	$2\Gamma_0^+(1) + \Gamma_0^-(\lambda, \lambda^{-1})$	$\lvert\lambda\rvert = 1,\ \lambda \neq \pm 1$
(α_9)	$2\Gamma_0^+(-1) + \Gamma_0^-(\lambda, \lambda^{-1})$	$\lvert\lambda\rvert = 1,\ \lambda \neq \pm 1$
(α_{10})	$\Gamma_2^-(1) + \Gamma_0^-(1)$	
(α_{11})	$\Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0(\mu, \mu^{-1})$	$\lambda > 0,\ \mu > 0,\ \lambda \neq 1,\ \mu \neq 1$
(α_{12})	$2\Gamma_0(\lambda, \lambda^{-1})$	$\lambda < 0,\ \lambda \neq -1$
(α_{13})	$\Gamma_0^+(\lambda, \lambda^{-1}) + 2\Gamma_0^-(1)$	$\lvert\lambda\rvert = 1,\ \lambda \neq \pm 1$
(α_{14})	$\Gamma_0^+(\lambda, \lambda^{-1}) + 2\Gamma_0^-(-1)$	$\lvert\lambda\rvert = 1,\ \lambda \neq \pm 1$
(α_{15})	$\Gamma_0^+(1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(1)$	$\lambda > 0,\ \lambda \neq 1$
(α_{16})	$2\Gamma_0^+(1) + 2\Gamma_0^-(1)$	
(α_{17})	$2\Gamma_0^+(1) + 2\Gamma_0^-(-1)$	
(α_{18})	$2\Gamma_0^+(-1) + 2\Gamma_0^-(1)$	
(α_{19})	$2\Gamma_0^+(-1) + 2\Gamma_0^-(-1).$	

Conjugacy classes contained in $G_+^+ \backslash \exp \mathfrak{g}$

(β_1)	$\Gamma_1(\lambda, \lambda^{-1})$	$\lambda < 0,\ \lambda \neq -1$
(β_2)	$\Gamma_0^+(-1) + \Gamma_2^+(-1)$	
(β_3)	$\Gamma_2^-(-1) + \Gamma_0^-(-1)$	
(β_4)	$\Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0(\mu, \mu^{-1})$	$\lambda < 0,\ \mu < 0,\ \lambda \neq \mu, \mu^{-1}$ $\lambda \neq -1,\ \mu \neq -1$
(β_5)	$\Gamma_0^+(-1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(-1)$	$\lambda < 0,\ \lambda \neq -1.$

Conjugacy classes contained in G_-^+

(γ_1)	$\Gamma_2^-(1) + \Gamma_0^-(-1)$	
(γ_2)	$\Gamma_2^-(-1) + \Gamma_0^-(1)$	
(γ_3)	$\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_0^-(1) + \Gamma_0^-(-1)$	$\lvert\lambda\rvert = 1,\ \lambda \neq \pm 1$
(γ_4)	$\Gamma_0^+(1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(-1)$	$\lambda > 0,\ \lambda \neq 1$
(γ_5)	$\Gamma_0^+(-1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(1)$	$\lambda < 0,\ \lambda \neq -1$
(γ_6)	$2\Gamma_0^+(1) + \Gamma_0^-(1) + \Gamma_0^-(-1)$	
(γ_7)	$2\Gamma_0^+(-1) + \Gamma_0^-(1) + \Gamma_0^-(-1).$	

Conjugacy classes contained in G_+^-

(δ_1)	$\Gamma_0^+(1) + \Gamma_2^+(-1)$	
(δ_2)	$\Gamma_0^+(-1) + \Gamma_2^+(1)$	
(δ_3)	$\Gamma_0^+(1) + \Gamma_0^+(-1) + \Gamma_0^-(\lambda, \lambda^{-1})$	$\lvert\lambda\rvert = 1,\ \lambda \neq \pm 1$
(δ_4)	$\Gamma_0^+(-1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(1)$	$\lambda > 0,\ \lambda \neq 1$
(δ_5)	$\Gamma_0^+(1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(-1)$	$\lambda < 0,\ \lambda \neq -1$

(δ_6) $\Gamma_0^+(1) + \Gamma_0^+(-1) + 2\Gamma_0^-(1)$

(δ_7) $\Gamma_0^+(1) + \Gamma_0^+(-1) + 2\Gamma_0^-(-1)$.

Conjugacy classes contained in G_-^-

(ω_1) $\Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0(\mu, \mu^{-1})$, λ, μ *real*, $\lambda\mu < 0$, $\lambda \neq \pm 1, \mu \neq \pm 1$

(ω_2) $\Gamma_0^+(-1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(-1)$ $\lambda > 0, \lambda \neq 1$

(ω_3) $\Gamma_0^+(1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(1)$ $\lambda < 0, \lambda \neq -1$

(ω_4) $\Gamma_0^+(1) + \Gamma_0^+(-1) + \Gamma_0^-(1) + \Gamma_0^-(-1)$.

(II) *The case* $G = O(3, 2)$.

Conjugacy classes contained in $\exp \mathfrak{g}$

(α_k') $\Gamma_0^+(1) + (\alpha_k)$ $(1 \leq k \leq 19)$

(α_{20}') $2\Gamma_0^+(-1) + \Gamma_2^+(1)$

(α_{21}') $2\Gamma_0^+(-1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(1)$ $\lambda > 0, \lambda \neq 1$

(α_{22}') $\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_0(\mu, \mu^{-1}) + \Gamma_0^-(1)$, $|\lambda| = 1, \lambda \neq \pm 1, \mu > 0, \mu \neq 1$

(α_{23}') $\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_2^+(1)$ $|\lambda| = 1, \lambda \neq \pm 1$

(α_{24}') $\Gamma_0(\lambda, \lambda^{-1}) + \Gamma_2^-(1)$ $\lambda > 0, \lambda \neq 1$

(α_{25}') $\Gamma_4^+(1)$.

Conjugacy classes contained in $G_+^+ \backslash \exp \mathfrak{g}$

(β_l') $\Gamma_0^+(1) + (\beta_l)$ $(1 \leq l \leq 5)$.

Conjugacy classes contained in G_-^+

(γ_m') $\Gamma_0^+(1) + (\gamma_m)$ $(1 \leq m \leq 7)$

(γ_8') $\Gamma_0^+(-1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0(\mu, \mu^{-1})$, $\lambda\mu < 0, \lambda, \mu$ *real*, $\lambda \neq \pm 1, \mu \neq \pm 1$

(γ_9') $2\Gamma_0^+(-1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(-1)$, $\lambda > 0, \lambda \neq 1$

(γ_{10}') $\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_0(\mu, \mu^{-1}) + \Gamma_0^-(-1)$, $|\lambda| = 1, \lambda \neq \pm 1, \mu > 0, \mu \neq 1$

(γ_{11}') $\Gamma_0(\lambda, \lambda^{-1}) + \Gamma_2^-(-1)$ $\lambda > 0, \lambda \neq 1$.

Conjugacy classes contained in G_+^-

(δ_k') $\Gamma_0^+(-1) + (\alpha_k)$ $(1 \leq k \leq 19)$

(δ_{19+l}') $\Gamma_0^+(-1) + (\beta_l)$ $(1 \leq l \leq 5)$

(δ_{25}') $2\Gamma_0^+(1) + \Gamma_2^+(-1)$

(δ_{26}') $2\Gamma_0^+(1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(-1)$ $\lambda < 0, \lambda \neq -1$

(δ_{27}') $\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_0(\mu, \mu^{-1}) + \Gamma_0^-(-1)$, $|\lambda| = 1, \lambda \neq \pm 1, \mu < 0, \mu \neq -1$

(δ_{28}') $\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_2^+(-1)$ $|\lambda| = 1, \lambda \neq \pm 1$

(δ'_{29})	$\Gamma_0(\lambda, \lambda^{-1}) + \Gamma_2^-(-1)$	$\lambda < 0, \lambda \neq -1$
(δ'_{30})	$\Gamma_4^+(-1)$.	

Conjugacy classes contained in G_-^-

(ω'_m)	$\Gamma_0^+(-1) + (\gamma_m)$	$(1 \leq m \leq 7)$
(ω'_8)	$\Gamma_0^+(1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0(\mu, \mu^{-1})$,	$\lambda\mu < 0$, λ, μ *real*, $\lambda \neq \pm 1, \mu \neq \pm 1$
(ω'_9)	$2\Gamma_0^+(1) + \Gamma_0(\lambda, \lambda^{-1}) + \Gamma_0^-(1)$	$\lambda < 0, \lambda \neq -1$
(ω'_{10})	$\Gamma_0^+(\lambda, \lambda^{-1}) + \Gamma_0(\mu, \mu^{-1}) + \Gamma_0^-(1)$,	$\|\lambda\| = 1, \lambda \neq \pm 1, \mu < 0, \mu \neq -1$
(ω'_{11})	$\Gamma_0(\lambda, \lambda^{-1}) + \Gamma_2^-(1)$	$\lambda < 0, \lambda \neq -1$.

In Theorem 3, observing real negative eigenvalues of conjugacy classes contained in $O(2, 2)_+^+$ and $O(3, 2)_+^+$, we obtain the following statements.

COROLLARY. (i) *Eigenvalues of each element of* $O(2, 2)_+^+ \backslash \exp o(2, 2)$ *are all negative.*

(ii) *Any conjugacy class for* $O(2, 2)$ *whose eigenvalues are all real negative, coincides with one of the conjugacy classes* (α_5), (α_{12}), (α_{19}), (β_1), (β_2), (β_3), (β_4) and (β_5).

(iii) *Any conjugacy class for* $O(3, 2)_+^+$ *whose eigenvalues consist of* 1 *and four real negative numbers, coincides with one of the conjugacy classes* (α'_5), (α'_{12}), (α'_{19}), (β'_1), (β'_2), (β'_3), (β'_4) *and* (β'_5).

§4. Int (exp g), ∂(exp g) and (exp g)^e in $G = O$ (3, 2)

In this section we shall observe the structures of the interior, boundary and exterior of the exponential image in G, where G mainly stands for $O(3, 2)$. In what follows, we shall use the same symbols and notations as in Theorem 3 unless otherwise stated.

By Theorem 3, eigenvalues of elements of $G_+^+ \backslash \exp \mathfrak{g}$ are listed up as follows:

(i) $1, \lambda, \lambda, \lambda^{-1}, \lambda^{-1}$; $\lambda < 0, \lambda \neq -1$

(ii) $1, -1, -1, -1, -1$;

(iii) $1, \lambda, \lambda^{-1}, \mu, \mu^{-1}$; $\lambda < 0, \mu < 0, \lambda \neq \mu, \mu^{-1}, \lambda \neq -1, \mu \neq -1$

(iv) $1, -1, -1, \lambda, \lambda^{-1}$; $\lambda < 0, \lambda \neq -1$.

Hence by the continuity of eigenvalues, we have the following propostion.

PROPOSITION 4. *The conjugacy classes contained in* $\exp \mathfrak{g}$ *except for* (α'_5), (α'_{12}) *and* (α'_{19}) *are contained in* Int (exp g).

REMARK 4. For $x \in O(2, 2) = O(J)$, we idenify x with $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in O(J')$, where $J' = \begin{pmatrix} 1 & 0 \\ 0 & J \end{pmatrix}$. Under this identification we regard $O(2, 2)$ as a subgroup of $O(3, 2)$.

Proposition 5. (α'_5): $\Gamma_0^+(1)+\Gamma_1(-1, -1)$, (α'_{12}): $\Gamma_0^+(1)+2\Gamma_0(\lambda, \lambda^{-1})$ *where* $\lambda<0$, $\lambda\neq -1$, *and* (α'_{19}): $\Gamma_0^+(1)+2\Gamma_0^+(-1)+2\Gamma_0^-(-1)$ *are contained in* $\partial(\exp \mathfrak{g}) \cap \exp \mathfrak{g}$.

Proof. We already know that these conjugacy classes are in $\exp \mathfrak{g}$, while (β'_1) is not in $\exp \mathfrak{g}$. We identify (α'_5) and (β'_1) with (α_5) and (β_1), respectively. Then by Table 4 we can choose the representatives of (α_5) and (β_1) as follows:

$$(A, J)\in(\alpha_5);\ A = \begin{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & {}^t\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix},$$

and for the same J,

$$(A_\lambda, J)\in(\beta_1);\ A_\lambda = \begin{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} & 0 \\ 0 & {}^t\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^{-1} \end{pmatrix}.$$

Since $\lim_{\lambda\to -1} A_\lambda = A$, we obtain $(\alpha'_5)\subset\partial(\exp \mathfrak{g}) \cap \exp \mathfrak{g}$.

Similarly, for (α'_{12}), by considering the type (β'_4) we have

$$\lim_{\mu,\nu\to\lambda} (\Gamma_0^+(1) + \Gamma_0(\mu, \mu^{-1}) + \Gamma_0(\nu, \nu^{-1})) = \Gamma_0^+(1) + 2\Gamma_0(\lambda, \lambda^{-1}),$$

where $\mu<0$, $\nu<0$, $\mu\neq -1$, $\nu\neq -1$ and $\mu\neq\nu, \nu^{-1}$.

We identify (α'_{19}) and (β'_4) with (α_{19}) and (β_4), respectively. Then by Table 4, a representative of (β_4) is given by

$$(A_{\lambda,\mu}, I_{2,2})\in(\beta_4);\ A_{\lambda,\mu} = 2^{-1} \begin{pmatrix} \lambda+\lambda^{-1} & 0 & \lambda-\lambda^{-1} & 0 \\ 0 & \mu+\mu^{-1} & 0 & \mu-\mu^{-1} \\ \lambda-\lambda^{-1} & 0 & \lambda+\lambda^{-1} & 0 \\ 0 & \mu-\mu^{-1} & 0 & \mu+\mu^{-1} \end{pmatrix},$$

where $\lambda<0$, $\lambda\neq -1$, $\mu<0$, $\mu\neq -1$ and $\lambda\neq\mu, \mu^{-1}$. Since $\lim_{\lambda,\mu\to -1} A_{\lambda,\mu} = -I_4$, we obtain $(\alpha'_{19})\subset\partial(\exp \mathfrak{g}) \cap \exp \mathfrak{g}$.

Proposition 6. (β'_1): $\Gamma_0^+(1)+\Gamma_1(\lambda, \lambda^{-1})$, $\lambda<0, \lambda\neq -1$ *is contained in*

$\partial(\exp \mathfrak{g}) \cap (G_{+}^{\dagger}\backslash\exp \mathfrak{g})$.

PROOF. We identify (β_1') with (β_1), and we choose (A_λ, J) in the proof of Proposition 5 as a representative of (β_1). Let $0<\theta<\pi$ and put

$$S_\lambda(\theta) = \begin{pmatrix} -\lambda(\pi-\theta) & 0 \\ 0 & 1 \end{pmatrix}, \quad R_\lambda(\theta) = -\lambda\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then we have

$$(S_\lambda(\theta))^{-1}R_\lambda(\theta)S_\lambda(\theta) = \begin{pmatrix} -\lambda\cos\theta & (\pi-\theta)^{-1}\sin\theta \\ -\lambda^2(\pi-\theta)\sin\theta & -\lambda\cos\theta \end{pmatrix}$$

and

$$\lim_{\theta\to\pi}(S_\lambda(\theta))^{-1}R_\lambda(\theta)S_\lambda(\theta) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

We now put

$$Q_\lambda(\theta) = \begin{pmatrix} S_\lambda(\theta) & 0 \\ 0 & {}^t(S_\lambda(\theta))^{-1} \end{pmatrix} \text{ and } T_\lambda(\theta) = \begin{pmatrix} R_\lambda(\theta) & 0 \\ 0 & {}^t(R_\lambda(\theta))^{-1} \end{pmatrix}.$$

Then we get $Q_\lambda(\theta)$, $T_\lambda(\theta) \in O(J)$ and $(T_\lambda(\theta), J) \in \Gamma_0(\mu, \mu^{-1}, \bar{\mu}, \bar{\mu}^{-1})$, where $\mu = |\lambda|e^{i\theta} = -\lambda e^{i\theta}$. Furthermore we obtain that

$$\lim_{\theta\to\pi}(Q_\lambda(\theta))^{-1}T_\lambda(\theta)Q_\lambda(\theta) = \begin{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} & 0 \\ 0 & {}^t\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^{-1} \end{pmatrix} = A_\lambda.$$

Noticing that $\Gamma_0(\mu, \mu^{-1}, \bar{\mu}, \bar{\mu}^{-1})$ is an exponential type, we get $(\beta_1') \subset \partial(\exp \mathfrak{g}) \cap (G_{+}^{\dagger}\backslash\exp \mathfrak{g})$.

PROPOSITION 7. *The conjugacy classes* (β_4'): $\Gamma_0^+(1)+\Gamma_0(\lambda, \lambda^{-1})+\Gamma_0(\mu, \mu^{-1})$, $\lambda<0$, $\mu<0$, $\lambda\neq-1$, $\mu\neq-1$ *and* $\lambda\neq\mu, \mu^{-1}$, *and* (β_5'): $\Gamma_0^+(1)+\Gamma_0^+(-1)+\Gamma_0(\lambda, \lambda^{-1})+\Gamma_0^-(-1)$, $\lambda<0$, $\lambda\neq-1$ *are contained in* $G_{+}^{\dagger}\backslash\mathrm{Cl}(\exp \mathfrak{g})$.

PROOF. By Theorem 3, these conjugacy classes are contained in $G_{+}^{\dagger}\backslash\exp \mathfrak{g}$. Let (A, J) and (A', J') be representatives of (β_4') and (β_5'), respectively. Then both A and A' belong to the exterior of $\exp gl(5, R)$ in $GL(5, R)$ by the facts stated in §1, because A and A' have negative eigenvalues with the multiplicity one. Hence we obtain that (β_4') is contained in $O_0(J)\backslash\mathrm{Cl}(\exp o(J))$ and (β_5') is contained in $O_0(J')\backslash\mathrm{Cl}(\exp o(J'))$.

PROPOSITION 8. *The conjugacy classes* (β_2'): $\Gamma_0^+(1)+\Gamma_0^+(-1)+\Gamma_2^+(-1)$ *and* (β_3'): $\Gamma_0^+(1)+\Gamma_2^-(-1)+\Gamma_0^-(-1)$ *are contained in* $\partial(\exp\mathfrak{g}) \cap (G_+^+\backslash\exp\mathfrak{g})$.

In order to show Proposition 8, we shall consider a representation of the group $SL(2, R)\times SL(2, R)$ in the four dimensional real vector space $M_2(R)$ which is the set of all 2×2 real matrices as usual. We now choose

$$E_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad E_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad E_3=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad E_4=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

as a basis of $M_2(R)$, then an element A in $M_2(R)$ can be written as

$$A = x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4 = \begin{pmatrix} x_1 + x_4 & x_2 + x_3 \\ -x_2 + x_3 & x_1 - x_4\end{pmatrix}.$$

For the fixed basis, we define the symmetric bilinear form τ on $M_2(R)$ by

$$\tau(A, A) = \det A = x_1^2 + x_2^2 - x_3^2 - x_4^2.$$

Then $O(M_2(R), \tau)$ can be identified with $O(I_{2,2})$. For $(g_1, g_2)\in SL(2, R)\times SL(2, R)$, we define $\Phi(g_1, g_2)$: $M_2(R)\to M_2(R)$ by

$$\Phi(g_1, g_2)A = g_1Ag_2^{-1}.$$

Since $\det\Phi(g_1, g_2)A=\det A$, we have $\Phi(g_1, g_2)\in O(M_2(R), \tau)$. It is obvious that Φ: $SL(2, R)\times SL(2, R)\to O(M_2(R), \tau)$ is homomorphism and the image of Φ is contained in $O_0(M_2(R), \tau)$. Let $d\Phi$ be the differential of Φ. Then we have $d\Phi(X, Y)A = XA - AY$ for $A\in M_2(R)$ and $(X, Y)\in sl(2, R)\oplus sl(2, R)$. Since the diagram

$$\begin{array}{ccc} SL(2, R)\times SL(2, R) & \xrightarrow{\Phi} & O_0(M_2(R), \tau) = O_0(I_{2,2}) \\ \uparrow{\scriptstyle \exp\times\exp} & & \uparrow{\scriptstyle \exp} \\ sl(2, R)\oplus sl(2, R) & \xrightarrow{d\Phi} & o(M_2(R), \tau) = o(I_{2,2}) \end{array}$$

is commutative and since $SL(2, R)\times SL(2, R)$ is connected, the mapping Φ: $SL(2, R)\times SL(2, R)\to O_0(M_2(R), \tau)=O_0(I_{2,2})$ is surjective. By explicit calculation one can show the following lemma.

LEMMA. (i) *For* $g_1=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $g_2=\begin{pmatrix}x & y\\ z & w\end{pmatrix}\in SL(2, R)$, the *matrix representation of* $\Phi(g_1, g_2)$: $M_2(R)\to M_2(R)$ *with respect to the basis* $\langle E_1, E_2, E_3, E_4\rangle$ *is given by*

$$\Phi(g_1, g_2)$$

$$= 2^{-1}\begin{pmatrix} aw - bz - cy + dx & -az - bw + cx + dy & -az + bw + cx - dy & aw + bz - cy - dx \\ -ay + bx - cw + dz & ax + by + cz + dw & ax - by + cz - dw & -ay - bx - cw - dz \\ -ay + bx + cw - dz & ax + by - cz - dw & ax - by - cz + dw & -ay - bx + cw + dz \\ aw - bz + cy - dx & -az - bw - cx - dy & -az + bw - cx + dy & aw + bz + cy + dx \end{pmatrix}$$

$$= 4^{-1}\begin{pmatrix} a+d & -b+c & b+c & a-d \\ b-c & a+d & a-d & -b-c \\ b+c & a-d & a+d & -b+c \\ a-d & -b-c & b-c & a+d \end{pmatrix} \cdot \begin{pmatrix} x+w & y-z & -y-z & -x+w \\ -y+z & x+w & x-w & -y-z \\ -y-z & x-w & x+w & -y+z \\ -x+w & -y-z & y-z & x+w \end{pmatrix}.$$

(ii) *The kernel of* Φ *is* $\{(I_2, I_2), (-I_2, -I_2)\}$.

(iii) trace $\Phi(g_1, g_2) = (\text{trace}\, g_1) \cdot (\text{trace}\, g_2)$.

Here we shall give the proof of Proposition 8. Let us identify (β'_2) and (β'_3) with (β_2) and (β_3), respectively and put

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } X(\theta) = \begin{pmatrix} \pi-\theta & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \begin{pmatrix} \pi-\theta & 0 \\ 0 & 1 \end{pmatrix}, \ 0 < \theta < \pi.$$

Then we have that B, $X(\theta) \in sl(2, R)$ and

$$\Phi(\exp B, \exp X(\theta)) = \exp d\Phi(B, X(\theta)) \in \exp o(I_{2,2}).$$

On the other hand, since

$$\exp B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \exp X(\theta) = \begin{pmatrix} x(\theta) & y(\theta) \\ z(\theta) & x(\theta) \end{pmatrix}$$

where $x(\theta)=\cos\theta$, $y(\theta)=(\pi-\theta)^{-1}\sin\theta$ and $z(\theta)=-(\pi-\theta)\sin\theta$, we have

$$\Phi\ (\exp B, \exp X(\theta)) =$$

$$4^{-1}\begin{pmatrix} 2 & -1 & 1 & 0 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 0 & -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2x(\theta) & y(\theta)-z(\theta) & -y(\theta)-z(\theta) & 0 \\ -y(\theta)+z(\theta) & 2x(\theta) & 0 & -y(\theta)-z(\theta) \\ -y(\theta)-z(\theta) & 0 & 2x(\theta) & -y(\theta)+z(\theta) \\ 0 & -y(\theta)-z(\theta) & y(\theta)-z(\theta) & 2x(\theta) \end{pmatrix}.$$

Therefore by noticing $\lim_{\theta\to\pi} x(\theta)=-1$, $\lim_{\theta\to\pi} y(\theta)=1$ and $\lim_{\theta\to\pi} z(\theta)=0$, we obtain

$$\lim_{\theta\to\pi} \Phi(\exp B, \exp X(\theta)) = 4^{-1}\begin{pmatrix} 2 & -1 & 1 & 0 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 0 & -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} -2 & 1 & -1 & 0 \\ -1 & -2 & 0 & -1 \\ -1 & 0 & -2 & -1 \\ 0 & -1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & -1 & 0 \\ -1 & -1/2 & -1/2 & 0 \\ -1 & 1/2 & -3/2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

As a representative of $\Gamma_2^-(-1)+\Gamma_0^-(-1)$, we can choose $(T^\circ, I_{2,2})$ by Table 4 as follows:

$$T^\circ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & -2\sqrt{2} & 0 \\ -2\sqrt{2} & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

If we put

$$Q = \begin{pmatrix} \sqrt{2}/2 & -5\sqrt{2}/8 & -3\sqrt{2}/8 & 0 \\ \sqrt{2}/2 & 5\sqrt{2}/8 & 3\sqrt{2}/8 & 0 \\ 0 & 3/4 & 5/4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then we have $Q\in O(I_{2,2})$ and

$$Q^{-1}T^{\circ}Q=\begin{pmatrix}-1 & 1 & -1 & 0\\ -1 & -1/2 & -1/2 & 0\\ -1 & 1/2 & -3/2 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$$=\lim_{\theta\to\pi}\Phi(\exp B,\exp X(\theta))$$

$$=\lim_{\theta\to\pi}\exp d\Phi(B,X(\theta)).$$

Thus we obtain that the conjugacy class (β_3'): $\Gamma_0^+(1)+\Gamma_2^-(-1)+\Gamma_0^-(-1)\subset\partial(\exp\mathfrak{g})\cap(G_+^+\backslash\exp\mathfrak{g})$.

For (β_2): $\Gamma_0^+(-1)+\Gamma_2^+(-1)$, we can choose its representative $(S^{\circ}, I_{2,2})$ by Table 4 as follows:

$$S^{\circ}=\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & -3 & 0 & -2\sqrt{2}\\ 0 & -2\sqrt{2} & 0 & -3\\ 0 & 0 & 1 & 0\end{pmatrix}.$$

Now if we put $K=\begin{pmatrix}0 & & & 1\\ & & 1 & \\ & 1 & & \\ 1 & & & 0\end{pmatrix}$, then we have $K^2=I_4$, $KT^{\circ}K=S^{\circ}$ and $K\cdot o(I_{2,2})\cdot K=o(I_{2,2})$. Therefore we get

$$S^{\circ}=KT^{\circ}K=KQ(\lim_{\theta\to\pi}\exp d\Phi(B,X(\theta))Q^{-1}K$$

$$=\lim_{\theta\to\pi}\exp(KQ(d\Phi(B,X(\theta)))Q^{-1}K)$$

and $KQ(d\Phi(B,X(\theta)))Q^{-1}K\in o(I_{2,2})$. Thus the conjugacy class (β_2'): $\Gamma_0^+(1)+\Gamma_0^+(-1)+\Gamma_2^+(-1)$ is contained in $\partial(\exp\mathfrak{g})\cap(G_+^+\backslash\exp\mathfrak{g})$. This completes the proof of Proposition 8.

By summarizing from Proposition 4 to Proposition 8, we obtain the main theorem.

THEOREM 9. *Let* $G=O(3,2)$ *and* $\mathfrak{g}=o(3,2)$. *Then conjugacy classes described in Theorem* 3(II) *can be partitioned into* $G=\mathrm{Int}(\exp\mathfrak{g})\cup(\partial(\exp\mathfrak{g})\cap\exp\mathfrak{g})\cup(\partial(\exp\mathfrak{g})\cap(G_+^+\backslash\exp\mathfrak{g}))\cup(G_+^+\backslash\mathrm{Cl}(\exp\mathfrak{g}))\cup G_-^{+}\cup G_+^{-}\cup G_-^{-}$, *where*

$\mathrm{Int}(\exp\mathfrak{g})$: (α_k'), $(1\le k\le 25,\ k\ne 5, 12, 19)$

$\partial(\exp \mathfrak{g}) \cap \exp \mathfrak{g}$	:	(α'_5), (α'_{12}), (α'_{19})
$\partial(\exp \mathfrak{g}) \cap (G^{+}_{+}\backslash\exp \mathfrak{g})$	:	(β'_1), (β'_2), (β'_3)
$G^{+}_{+}\backslash\mathrm{Cl}(\exp \mathfrak{g})$	:	(β'_4), (β'_5)
G^{+}_{-}	:	(γ'_k), $(1 \le k \le 11)$
G^{-}_{+}	:	(δ'_k), $(1 \le k \le 30)$
G^{-}_{-}	:	(ω'_k), $(1 \le k \le 11)$.

Corollary 1. *Let* $G = O(2, 2)$ *and* $\mathfrak{g} = o(2, 2)$. *Then conjugacy classes described in Theorem* 3(I) *are similarly partitioned as in Theorem* 9. *That is,*

$\mathrm{Int}(\exp \mathfrak{g})$	:	(α_k), $(1 \le k \le 18,\ k \ne 5, 12)$
$\partial(\exp \mathfrak{g}) \cap \exp \mathfrak{g}$	:	(α_5), (α_{12}), (α_{19})
$\partial(\exp \mathfrak{g}) \cap (G^{+}_{+}\backslash\exp \mathfrak{g})$	:	(β_1), (β_2), (β_3)
$G^{+}_{+}\backslash\mathrm{Cl}(\exp \mathfrak{g})$	:	(β_4), (β_5)
G^{+}_{-}	:	(γ_k), $(1 \le k \le 7)$
G^{-}_{+}	:	(δ_k), $(1 \le k \le 7)$
G^{-}_{-}	:	(ω_k), $(1 \le k \le 4)$.

In Theorem 9 and Corollary 1, by aiming at real negative eigenvalues and by calculating traces of all conjugacy classes for $O(2, 2)$ and $O(3, 2)$, we get the following cororllary.

Corollary 2. (I) *The case* $G = O(2, 2)$. *For* $x \in O(2, 2)$, *we have the following.*

(i) $x \in O(2, 2)^{+}_{+}\backslash\mathrm{Int}(\exp o(2, 2))$ *if and only if eigenvalues of x are all real negative.*

(ii) $x \in \partial(\exp o(2, 2))$ *if and only if eigenvalues of x are all real negative and the multiplicity of each eigenvalue is even* (2 *or* 4).

(iii) $x \in O(2, 2)^{+}_{+} \cap (\exp o(2, 2))^{e}$ *if and only if eigenvalues of x are all real negative and there exists an eigenvalue with multiplicity one.*

(iv) *Let* $x \in O(2, 2)^{+}_{+}$. *Then* $x \in \mathrm{Int}(\exp o(2, 2))$ *if and only if* $\mathrm{trace}\, x > -4$ *or* $x \in (\alpha_1)$.

(II) *The case* $G = O(3, 2)$. *For* $x \in O(3, 2)$, *we have the following statements, and the assertions* (i)′, (ii)′, (iii)′ *are described by using the same identification as in Remark* 4.

(i)′ $x \in O(3, 2)^{+}_{+}\backslash\mathrm{Int}(\exp o(3, 2))$ *if and only if x is conjugate to* $\begin{pmatrix} 1 & 0 \\ 0 & x' \end{pmatrix}$ *in* $O(3, 2)$, *where* $x' \in O(2, 2)^{+}_{+}\backslash\mathrm{Int}(\exp o(2, 2))$.

(ii)′ *$x \in \partial(\exp o(3, 2))$ if and only if x is conjugate to* $\begin{pmatrix} 1 & 0 \\ 0 & x' \end{pmatrix}$ *in* $O(3, 2)$, *where $x' \in \partial(\exp o(2, 2))$.*

(iii)′ *$x \in O(3, 2)^{\ddagger} \cap (\exp o(3, 2))^e$ if and only if x is conjugate to* $\begin{pmatrix} 1 & 0 \\ 0 & x' \end{pmatrix}$ *in* $O(3, 2)$, *where $x' \in O(2, 2)^{\ddagger} \cap (\exp o(2, 2))^e$.*

(iv)′ *Let $x \in O(3, 2)^{\ddagger}$. Then $x \in \mathrm{Int}(\exp o(3, 2))$ if and only if* $\mathrm{trace}\, x > -3$ *or $x \in (\alpha_1')$.*

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