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On a product formula for a class of nonlinear evolution equations

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§ 0. Introduction

This work is concerned with the initial value problem of the form

$$(\mathrm{IVP};\ u_0) \qquad (d/dt)u(t) + \partial\phi u(t) \ni Bu(t), \quad t > 0,$$
$$u(0) = u_0,$$

where φ is a proper lower semicontinuous (l.s.c.) convex functional on an abstract real Hilbert space $\mathbf{H}$, $\partial\varphi$ is its subdifferential and B is a single-valued operator in $\mathbf{H}$ with domain $D(B)$ containing the effective domain $D(\varphi)$ of φ. Initial value problems of this type have been studied by many authors (e.g. [**5**, **11**, **12**, **13**]).

Let $\{S(\tau); 0 \leq \tau < \infty\}$ be the nonlinear contraction semigroup on $\overline{D(\varphi)}$ generated by $-\partial\varphi$, and $\{V(\tau); 0 \leq \tau < \infty\}$ a one-parameter family of single-valued operators $V(\tau)$ in $\mathbf{H}$ with $D(V(\tau)) \supset D(B)$ such that $\tau^{-1}(V(\tau)-1) \to B$ as $\tau \downarrow 0$ in a certain sense. (However the family $\{V(\tau)\}$ is not assumed to be a contraction semigroup on $\overline{D(B)}$.) In this paper it is our main interest to establish an existence theorem for (IVP; u_0) by showing that

$$(0.1) \qquad u_n(t) = [V(\tau(n))S(\tau(n))P]^{[t/\tau(n)]}u_0 \longrightarrow u(t) \quad \text{in} \quad \mathbf{H} \quad \text{as} \quad n \to \infty$$

and

$$\varphi(S(\tau(n))Pu_n(t)) \longrightarrow \varphi(u(t)) \quad \text{in} \quad \mathbf{R} \quad \text{as} \quad n \to \infty$$

for a suitable subsequence $\{\tau(n)\}$ with $\tau(n) \downarrow 0$ as $n \to \infty$, where P is the projection from $\mathbf{H}$ onto $\overline{D(\varphi)}$ and $[s]$ denotes the greatest integer in $s \in \mathbf{R}$.

In case $-B$ is the subdifferential of a proper l.s.c. convex functional φ on $\mathbf{H}$, i.e. $B = -\partial\varphi$, it was shown by Kato and Masuda [7] that

$$[S'(\tau)P'S(\tau)P]^{[t/\tau]}u_0 \longrightarrow u(t) \quad \text{in} \quad \mathbf{H} \quad \text{as} \quad \tau \downarrow 0 \qquad \text{for} \quad t \in [0, \infty)$$

and the convergence is uniform on $[0, T]$ for each $0 < T < \infty$, where $\{S'(\tau); 0 \leq \tau < \infty\}$ is the nonlinear contraction semigroup on $\overline{D(\psi)}$ generated by $-\partial\psi$ and P' the projection from $\mathbf{H}$ onto $\overline{D(\psi)}$. This result is a nonlinear analogue of Trotter's product formula for linear nonnegative self-adjoint operators (cf. [2, 6]) and the family $\{T(\tau); 0 \leq \tau < \infty\}$ defined by $T(t)u_0 = u(t)$, $u_0 \in \overline{D(\varphi) \cap D(\psi)}$, gives the contraction semigroup on $\overline{D(\varphi)\cap D(\psi)}$ generated by $-\partial(\varphi+\psi)$.

However, our problem (IVP; u_0) involves the case in which the solution may blow up in a finite time. In fact, such cases may occur when $-B$ is not a monotone operator (cf. [5, 13]). Therefore the methods employed in [2] and [6] cannot be applied directly to prove the convergence (0.1). In this paper, with the aid of the compactness argument and some techniques evolved in the theory of ordinary differential equations, we show that (0.1) and (0.2) hold uniformly on some interval $[0, T]$, and that the limit u gives a solution of (IVP; u_0) on $[0, T]$. If (IVP; u_0) has a unique solution u on $[0, T^*)$, we are able to show without taking a subsequence $\{\tau(n)\}$ that

$$(0.1)' \qquad \lim_{\tau\downarrow 0}\,[V(\tau)S(\tau)P]^{[t/\tau]}u_0 = u(t) \quad \text{in} \quad \mathbf{H}$$

and

$$(0.2)' \qquad \lim_{\tau\downarrow 0}\,\varphi(S(\tau)P[V(\tau)S(\tau)P]^{[t/\tau]}u_0) = \varphi(u(t)) \quad \text{in} \quad \mathbf{R},$$

exist uniformly with respect to t in every compact subinterval of $[0, T^*)$. The formula (0.1)$'$ is a nonlinear version in our case of the Trotter's product formula; and for the linear case, this type of formula was earlier given by Ichinose and Koyama [4, 8].

§1. Main results

Let $\mathbf{H}$ be a Hilbert space, φ a proper l.s.c. convex functional on $\mathbf{H}$, and let B be a single-valued operator from $D(B)\subset\mathbf{H}$ into $\mathbf{H}$. Given u_0 in $\mathbf{H}$, we consider the initial value problem

$$(\text{IVP}; u_0) \qquad u'(t) + \partial\varphi u(t) \ni Bu(t), \quad 0 < t < T, \quad u(0) = u_0,$$

where $u'(t)$ $(=(d/dt)u(t))$ denotes the strong derivative of $u(t)$ in $\mathbf{H}$.
An $\mathbf{H}$-valued function u on $[0, T]$ $(0<T<\infty)$ is called *a solution of* (IVP; u_0) *on*

(i) $u\in W^{1,2}(0, T; \mathbf{H})$ and $u(0) = u_0$,

(ii) $Bu(\cdot)\in L^2(0, T; \mathbf{H})$, and

(iii) $Bu(t) - u'(t)\in\partial\varphi u(t)$ for a.e. $t\in[0, T]$.

An $\mathbf{H}$-valued function u on $[0, T)$ $(0<T\leq\infty)$ is called *a solution of* (IVP; u_0) *on* $[0, T)$ if it is a solution of (IVP; u_0) on $[0, T']$ for every $0<T'<T$.

REMARK 1.1. Let u be a solution of (IVP; u_0) on $[0, T]$. Then it is known that the function $t\to\varphi(u(t))$ is absolutely continuous on $[0, T]$. See Brézis [1; Lemma 3.3].

In what follows, $|\cdot|_{\mathbf{H}}$ and $(\cdot, \cdot)_{\mathbf{H}}$ denote the norm and the inner product in $\mathbf{H}$, respectively.

Our objective here is to discuss the construction of solutions of the problem (IVP; u_0) in terms of product formula of Trotter's type under the following assumptions.

(A) *For each* $r \geq 0$, *the set* $L_\phi(r) = \{z \in \mathbf{H};\ |z|_\mathbf{H} \leq r,\ \varphi(z) \leq r\}$ *is compact in* $\mathbf{H}$.

(B) *The operator* B *satisfies the following conditions*:

(b.1) $D(\varphi) \subset D(B)$;

(b.2) $|Bz|_\mathbf{H}^2 \leq l(\varphi(z))$ *for any* $z \in D(\varphi)$, *where* $l(\cdot)$ *is a non-decreasing continuous function from* $\mathbf{R}$ *into* $[0, \infty)$;

(b.3) B *is demicontinuous on each level set of* φ, *namely*: *if* $z_n \to z$ *strongly in* $\mathbf{H}$ *and* $\{\varphi(z_n)\}$ *is bounded, then* $Bz_n \to Bz$ *weakly in* $\mathbf{H}$.

(C) *There exists a family* $V = \{V(\tau);\ 0 < \tau < \infty\}$ *of single-valued operators* $V(\tau)$ *from* $D(V(\tau)) \subset \mathbf{H}$ *into* $\mathbf{H}$ *satisfying the following conditions*:

(c.1) $D(B) \subset D(V(\tau))$ *for* $\tau > 0$;

(c.2) $\tau^{-1}|(V(\tau)-1)z|_\mathbf{H} \leq L|Bz|_\mathbf{H}$ *for* $\tau > 0$ *and* $z \in D(B)$, *where* L *is a non-negative constant*;

(c.3) $\tau^{-1}(V(\tau)-1)z \to Bz$ *weakly in* $\mathbf{H}$ *as* $\tau \downarrow 0$ *and, for each* $r \geq 0$, *the convergence holds uniformly with respect to* z *in* $L_\phi(r)$.

As is well-known (cf. Brezis [1; Theorems 3.4, 3.6]), $-\partial\varphi$ generates a non-linear semigroup $S = \{S(\tau);\ 0 < \tau < \infty\}$ on $\overline{D(\varphi)}$. We denote by P the projection from $\mathbf{H}$ onto the closed convex set $\overline{D(\varphi)}$.

To state our main theorems we introduce the scalar initial value problem

$$\rho'(t) = (L^2/2)l(\rho(t)), \quad \rho(0) = K, \tag{1.1}$$

where K is a given real number, $l(\cdot)$ is as in (b.2) of (B) and L is as in (c.2) of (C). It is well known (cf. Coddington-Levinson [3; Ch. 2, Theorem 1.2]) that (1.1) admits a unique maximal solution. We denote it by $f(\cdot, K)$ and write $[0, t^*(K))$ $(0 < t^*(K) \leq \infty)$ for the associated maximal interval of existence.

The main results are now stated as follows.

THEOREM 1.2. *Suppose that* (A), (B) *and* (C) *hold. Let* u_0 *be an element of* $D(\varphi)$. *Then there exist a solution* u *of* (IVP; u_0) *on* $[0, t^*(\varphi(u_0)))$ *and a sequence* $\{\tau(n)\}$ *in* $(0, \infty)$ *with* $\tau(n) \downarrow 0$ *as* $n \to \infty$ *such that*

$$\lim_{n\to\infty} [V(\tau(n))S(\tau(n))P]^{[t/\tau(n)]}u_0 = u(t) \quad \text{in} \quad \mathbf{H} \tag{1.2}$$

and

$$\lim_{n\to\infty} \varphi(S(\tau(n))P[V(\tau(n))S(\tau(n))P]^{[t/\tau(n)]}u_0) = \varphi(u(t)) \tag{1.3}$$

for each $t \in [0, t^*(\varphi(u_0)))$ *and the limits exist uniformly with respect to* t *in any finite interval* $[0, T]$ *with* $0 < T < t^*(\varphi(u_0))$.

REMARK 1.3. Suppose that the operator B satisfies condition (B) and that the family $\{V(\tau); 0<\tau<\infty\}$ is defined by $V(\tau)=1+\tau B$ for $\tau>0$. Then $\{V(\tau)\}$ satisfies automatically condition (C). Therefore, in order to obtain a local solution of (IVP; u_0) in this particular case by Theorem 1.2, it is sufficient to assume that only (A) and (B) hold. However, Theorem 1.2 not only asserts the existence of local solutions, but also guarantees that the solutions of (IVP; u_0) may be approximated in different ways according to the choice of the family $\{V(\tau)\}$ satisfying (C).

THEOREM 1.4. *Suppose that* (A), (B) *and* (C) *hold. Let* u_0 *be an element of* $D(\varphi)$. *Further suppose that* (IVP; u_0) *has a unique solution* u *on* $[0, T^*)$. *Then*

$$\lim_{\tau\downarrow 0}\,[V(\tau)S(\tau)P]^{[t/\tau]}u_0 = u(t) \quad \textit{strongly in} \quad \mathbf{H} \tag{1.4}$$

and

$$\lim_{\tau\downarrow 0}\,\varphi(S(\tau)P[V(\tau)S(\tau)P]^{[t/\tau]}u_0) = \varphi(u(t)) \tag{1.5}$$

for $t\in[0, T^*)$ *and the limits exist uniformly in* $t\in[0, T]$ *for every* $0<T<T^*$.

The proofs of the above results are given in §3 and §4.

Since φ is proper, l.s.c., and convex on $\mathbf{H}$, there exist $a^*\in\mathbf{H}$ and constants $\alpha, \alpha^*, \beta, \beta^*>0$ such that

$$\phi(z) \geq (a^*, z)_{\mathbf{H}} - \beta \geq -\alpha|z|_{\mathbf{H}} - \beta \geq -\alpha^*|z|_{\mathbf{H}}^2 - \beta^* \qquad \text{for any} \quad z\in\mathbf{H}. \tag{1.6}$$

Also, on account of Kato-Masuda [7; Example 2.5], we have

$$\varphi(z) \geq \varphi(S(\tau)Py) + \tau^{-1}(y-S(\tau)Py, z-y)_{\mathbf{H}} + (2\tau)^{-1}|y-S(\tau)Py|_{\mathbf{H}}^2, \tag{1.7}$$

for $z, y\in\mathbf{H}$. These inequalities are used in later arguments.

LEMMA 1.5. *Let* $\{\tau(n)\}$ *be a sequence in* $(0, \infty)$ *with* $\tau(n)\downarrow 0$ (*as* $n\to\infty$) *and* $\{v_n\}$ *a sequence in* $L^2(0, T; \mathbf{H})$, $0<T<\infty$, *such that*

$$v_n \longrightarrow v \qquad \textit{strongly in} \quad L^2(0, T; \mathbf{H})$$

and

$$\tau(n)^{-1}(v_n(\cdot) - S(\tau)Pv_n(\cdot)) \longrightarrow v^* \qquad \textit{weakly in} \quad L^2(0, T; \mathbf{H}).$$

Then $v^*(t)\in\partial\varphi v(t)$ *for a.e.* $t\in[0, T]$.

PROOF. Since $\{\tau(n)^{-1}(v_n(\cdot)-S(\tau(n))Pv_n(\cdot))\}$ is bounded in $L^2(0, T; \mathbf{H})$, we have $v_n(\cdot)-S(\tau(n))Pv_n(\cdot)\to 0$ strongly in $L^2(0, T; \mathbf{H})$. Therefore $S(\tau(n))Pv_n\to v$ strongly in $L^2(0, T; \mathbf{H})$. Next, it follows from (1.7) that

$$(1.8)\quad \int_0^T \varphi(w(t))dt \geq \int_0^T \varphi(S(\tau(n))Pv_n(t))dt + \tau(n)^{-1}\int_0^T (v_n(t) - S(\tau(n))Pv_n(t),\ w(t) - v_n(t))_{\mathbf{H}}dt$$

for every $w \in L^2(0, T; \mathbf{H})$. Letting $n \to \infty$ in (1.8) yields

$$\int_0^T \varphi(w(t))dt \geq \int_0^T \varphi(v(t))dt + \int_0^T (v^*(t),\ w(t) - v(t))_{\mathbf{H}}dt$$

for every $w \in L^2(0, T; \mathbf{H})$, which means that $v^*(t) \in \partial\varphi v(t)$ for a.e. $t \in [0, T]$. Q.E.D.

LEMMA 1.6. *Suppose that* (A), (B) *and* (C) *hold. Let* $\{\tau(n)\}$ *be a sequence in* $(0, \infty)$ *with* $\tau(n) \downarrow 0$ *(as* $n \to \infty$*),* $0 < \tau < \infty$, *and let* $\{v_n\}$ *be a bounded sequence in* $L^\infty(0, T; \mathbf{H})$ *such that*

$$v_n \longrightarrow v \qquad \textit{strongly in} \quad L^2(0, T; \mathbf{H})$$

and

$$\{\varphi(v_n(t))\} \quad \textit{is uniformly bounded on} \quad [0, T].$$

Then $v(t) \in D(B)$ *for a.e.* $t \in [0, T]$ *and*

$$(1.9)\quad \tau(n)^{-1}(V(\tau(n))v_n(\cdot) - v_n(\cdot)) \longrightarrow Bv(\cdot) \qquad \textit{weakly in} \quad L^2(0, T; \mathbf{H}).$$

PROOF. By assumption there is a number $r > 0$ such that $v_n(t) \in L_\phi(r)$ for a.e. $t \in [0, T]$ and $n = 1, 2, \ldots$. Hence, by (b.3), $v(t) \in D(B)$ for a.e. $t \in [0, T]$ and

$$\int_0^T (\tau(n)^{-1}(V(\tau(n))v_n(t) - v_n(t)) - Bv(t),\ w(t))_{\mathbf{H}}dt$$
$$= \int_0^T (\tau(n)^{-1}(V(\tau(n))v_n(t) - v_n(t)) - Bv_n(t),\ w(t))_{\mathbf{H}}dt + \int_0^T (Bv_n(t) - Bv(t),\ w(t))_{\mathbf{H}}dt$$

for any $w \in L^2(0, T; \mathbf{H})$. Now conditions (b.2) and (c.2) imply that both $\{\tau(n)^{-1}(V(\tau(n))v_n(\cdot) - v_n(\cdot)) - Bv_n(\cdot)\}$ and $\{Bv_n(\cdot) - Bv(\cdot)\}$ are bounded in $L^\infty(0, T; \mathbf{H})$. By virtue of (b.3), (c.3) and Lebesgue's dominated convergence theorem, the right hand side of the above equality tends to zero as $n \to \infty$. Q.E.D.

§2. Approximate solutions and their estimates

Throughout this section, we suppose that (A), (B) and (C) hold, and that $u_0 \in D(\varphi)$. We here construct a sequence of approximate solutions of (IVP; u_0) and give energy estimates for them.

For each $\tau > 0$, we define a function $F(\tau, \cdot)$; $[0, \infty) \to \mathbf{H}$ by

$$F(\tau, k\tau) = [S(\tau)PV(\tau)]^k S(\tau)u_0 \qquad \text{for} \quad k = 0, 1,\dots,$$

and

$$F(\tau, t) = F(\tau, k\tau) + (t-k\tau)\tau^{-1}\{F(\tau, (k+1)\tau)-F(\tau, k\tau)\}$$
$$\text{for} \quad k\tau < t < (k+1)\tau \quad \text{and} \quad k = 0, 1,\dots.$$

In what follows we regard the functions $F(\tau, \cdot)$, $\tau>0$ small, as approximate solutions to the problem (IVP; u_0). Note that the function $F(\tau, \cdot)$ is also written as follows:

$$F(\tau, t) = ([t/\tau]+1-(t/\tau))F(\tau, [t/\tau]\tau)$$
$$+ ((t/\tau)-[t/\tau])F(\tau, ([t/\tau]+1)\tau) \qquad \text{for} \quad t \geq 0.$$

Clearly, $F(\tau, t)$ is absolutely continuous in $t\geq 0$ and differentiable in t except for the points $t=k\tau$ $(k=0, 1,\dots)$. By definition we have

$$F(\tau, k\tau) \in D(\partial\varphi) \qquad \text{for} \quad k = 0, 1,\dots,$$
$$F(\tau, t) \in D(\varphi) \qquad \text{for} \quad t \geq 0$$

and

(2.1) $$F_t(\tau, t)(=(\partial/\partial t)F(\tau, t)) = \tau^{-1}\{F(\tau, ([t/\tau]+1)\tau)-F(\tau, [t/\tau]\tau)\}$$
$$\text{for} \quad t \neq k\tau, k = 0, 1,\dots.$$

In order to give the energy estimate for $F(\tau, \cdot)$, it is convenient to employ the maximal solution $f(\cdot\,; K)$ of (1.1) and the corresponding Cauchy's polygon $f(\tau, \cdot\,; K)$ which is given by

$$f(\tau, 0; K) = K$$

and

$$f(\tau, t; K) = f(\tau, k\tau; K) + (L^2/2)(t-k\tau)l(f(\tau, k\tau; K))$$
$$\text{for} \quad k\tau < t \leq (k+1)\tau \quad \text{and} \quad k = 0, 1,\dots.$$

Note that $f(\tau, t; K)$ and $f(t, K)$ are nondecreasing in t and K,

(2.2) $$f(\tau, t; K) = K + (L^2/2)\int_0^t l(f(\tau, [s/\tau]\tau; K))ds,$$

and

$$f(\tau, [t/\tau]\tau; K) \leq f(\tau, t; K) \leq f(t; K) \qquad \text{for} \quad 0 \leq t < t^*(K).$$

LEMMA 2.1. (i) *For any $\tau>0$ and any pair of non-negative integers m, n with $m<n$, we have*

(2.3) $$\varphi(F(\tau, n\tau)) + (2\tau)^{-1} \sum_{k=m}^{n-1} |F(\tau, (k+1)\tau) - F(\tau, k\tau)|_{\mathbf{H}}^2 \le \varphi(F(\tau, m\tau)) + (L^2/2)\tau \sum_{k=m}^{n-1} l(\varphi(F(\tau, k\tau))).$$

(ii) *For any $\tau > 0$ and any $T, T' \ge 0$ with $0 \le T < T' < \infty$ we have*

(2.4) $$2^{-1}\int_T^{T'} |F_t(\tau, t)|_{\mathbf{H}}^2 dt + \varphi(F(\tau, T')) \le f(\tau, T' - [T/\tau]\tau; \varphi(F(\tau, [T/\tau]\tau))).$$

PROOF. The inequality (1.7) implies that

$$\varphi(z) - \varphi(S(\tau)Py) \ge (2\tau)^{-1}|z - S(\tau)Py|_{\mathbf{H}}^2 - (2\tau)^{-1}|z - y|_{\mathbf{H}}^2 \quad \text{for} \quad y, z \in \mathbf{H}.$$

In this inequality, put $z = F(\tau, k\tau)$ and $y = V(\tau)z$. Then, since

$$F(\tau, k\tau) = S(\tau)P[V(\tau)S(\tau)P]^k u_0$$

and

$$y = [V(\tau)S(\tau)P]^{k+1}u_0, \quad S(\tau)Py = F(\tau, (k+1)\tau),$$

it follows from conditions (b.2) and (c.2) that

(2.5) $$\begin{aligned} \varphi(F(\tau, (k+1)\tau)) &+ (2\tau)^{-1}|F(\tau, (k+1)\tau) - F(\tau, k\tau)|_{\mathbf{H}}^2 \\ &\le \varphi(F(\tau, k\tau)) + (2\tau)^{-1}|(V(\tau) - 1)F(\tau, k\tau)|_{\mathbf{H}}^2 \\ &\le \varphi(F(\tau, k\tau)) + (L^2/2)\tau|BF(\tau, k\tau)|_{\mathbf{H}}^2 \\ &\le \varphi(F(\tau, k\tau)) + (L^2/2)\tau l(\varphi(F(\tau, k\tau))). \end{aligned}$$

Summation of (2.5) over $k = m, m+1, \dots, n-1$ yields (2.3). Comparing $\varphi(F(\tau, k\tau))$ with $f(\tau, k\tau; \varphi(F(\tau, m\tau)))$, we infer from (2.3) that

(2.6) $$\varphi(F(\tau, n\tau)) \le f(\tau, (n-m)\tau; \varphi(F(\tau, m\tau))) \quad \text{for} \quad n > m.$$

Suppose that $m\tau \le T < (m+1)\tau$ and $n\tau \le T' < (n+1)\tau$. It follows from the convexity of φ and (2.5) that

$$\begin{aligned} \varphi(F(\tau, T')) &\le \varphi(F(\tau, n\tau)) + (T' - n\tau)\tau^{-1}\{\varphi(F(\tau, (n+1)\tau)) - \varphi(F(\tau, n\tau))\} \\ &\le \varphi(F(\tau, n\tau)) + (T' - n\tau)\tau^{-1}\{(L^2/2)\tau l(\varphi(F(\tau, n\tau))) \\ &\qquad - (2\tau)^{-1}|F(\tau, (n+1)\tau) - F(\tau, n\tau)|_{\mathbf{H}}^2\} \end{aligned}$$

and

$$\begin{aligned} &2^{-1}\int_T^{T'} |F_t(\tau, t)|_{\mathbf{H}}^2 dt \\ &\le (2\tau)^{-1} \sum_{k=m}^{n-1} |F(\tau, (k+1)\tau) - F(\tau, k\tau)|_{\mathbf{H}}^2 \\ &\quad + 2^{-1}(T' - n\tau)\tau^{-2}|F(\tau, (n+1)\tau) - F(\tau, n\tau)|_{\mathbf{H}}^2. \end{aligned}$$

Therefore, noting (2.3) and (2.6), we get

$$\begin{aligned}
&\varphi(F(\tau, T')) + 2^{-1}\int_T^{T'} |F_t(\tau, t)|_{\mathbf{H}}^2 dt \\
&\le \varphi(F(\tau, m\tau)) + (L^2/2)\int_{m\tau}^{T'} l(\varphi(F(\tau, [t/\tau]\tau)))dt \\
&\le f(\tau, T'-m\tau; \varphi(F(\tau, m\tau))).
\end{aligned}$$

Thus (2.4) is obtained. Q.E.D.

LEMMA 2.2. *Let* $0<T<t^*(\varphi(u_0))$ *and* $0<\tau_0<\infty$. *Then there is a constant* $M_1 \ge 0$ *such that*

(2.7) $\qquad |F(\tau, t)|_{\mathbf{H}} \le M_1 \quad$ *for* $\tau \in (0, \tau_0]$ *and* $t \in [0, T]$,

(2.8) $\qquad |\varphi(F(\tau, t))| \le M_1 \quad$ *for* $\tau \in (0, \tau_0]$ *and* $t \in [0, T]$

and

(2.9) $\qquad |F_t(\tau, \cdot)|_{L^2(0,T;\mathbf{H})} \le M_1 \quad$ *for* $\tau \in (0, \tau_0]$.

PROOF. Let $0<\tau\le\tau_0$. Put $T=0$ and $T'=t$ in (2.4). Then

$$\begin{aligned}
(2.10) \qquad 2^{-1}\int_0^t |F_s(\tau, s)|_{\mathbf{H}}^2 ds + \varphi(F(\tau, t)) &\le f(\tau, t; \varphi(S(\tau)u_0)) \\
&\le f(\tau, T; \varphi(u_0)) \le f(T; \varphi(u_0))
\end{aligned}$$

for all $t \in [0, T]$. Using (1.6) and (2.10), we have

$$\begin{aligned}
|F(\tau, t)|_{\mathbf{H}} &\le |F(\tau, 0)|_{\mathbf{H}} + \int_0^t |F_s(\tau, s)|_{\mathbf{H}} ds \\
&\le |F(\tau, 0)|_{\mathbf{H}} + t^{1/2}\left(\int_0^T |F_s(\tau, s)|_{\mathbf{H}}^2 ds\right)^{1/2} \\
&\le |F(\tau, 0)|_{\mathbf{H}} + \alpha T + (4\alpha)^{-1}\int_0^t |F_s(\tau, s)|_{\mathbf{H}}^2 ds \\
&\le |S(\tau)u_0|_{\mathbf{H}} + \alpha T + 2^{-1}|F(\tau, t)|_{\mathbf{H}} + \beta/(2\alpha) + (2\alpha)^{-1} f(T; \varphi(u_0))
\end{aligned}$$

for all $t \in [0, T]$, so that

$$\begin{aligned}
2^{-1}|F(\tau, t)|_{\mathbf{H}} \le \sup\{|S(\tau)u_0|_{\mathbf{H}};\ 0<\tau<\tau_0\} + \alpha T + \beta/(2\alpha) \\
+ (2\alpha)^{-1} f(T; \varphi(u_0)) \equiv R_1 \qquad \text{for all} \quad t \in [0, T].
\end{aligned}$$

Also, on account of (1.6) and (2.4) we have

$$|\varphi(F(\tau, t))| \le \varphi(F(\tau, t)) + 2\alpha|F(\tau, t)|_{\mathbf{H}} + 2\beta \le f(T; \varphi(u_0)) + 4\alpha R_1 + 2\beta \equiv R_1'$$

for all $t \in [0, T]$, and

$$\int_0^t |F_s(\tau, s)|_{\mathbf{H}}^2 ds \leq 2f(T;\, \varphi(u_0)) + 2\alpha|F(\tau, t)|_{\mathbf{H}} + 2\beta$$
$$\leq 2f(T;\, \varphi(u_0)) + 4\alpha R_1 + 2\beta \equiv R_1''$$

for all $t \in [0, T]$. Hence the desired estimates (2.7), (2.8) and (2.9) are obtained with $M_1 \equiv \max\{2R_1, R_1', R_1''\}$. Q.E.D.

LEMMA 2.3. (i) *For* $\tau > 0$ *and* $t \neq k\tau$, $k = 0, 1, \ldots$, *we have*

(2.11) $F_t(\tau, t) + \tau^{-1}(1 - S(\tau)P)V(\tau)F(\tau, [t/\tau]\tau) = \tau^{-1}(V(\tau) - 1)F(\tau, [t/\tau]\tau)$.

(ii) *Let* T *and* τ_0 *be as in Lemma* 2.2. *Then there is a constant* $M_2 \geq 0$ *such that*

(2.12) $|\tau^{-1}(V(\tau) - 1)F(\tau, [t/\tau]\tau)|_{\mathbf{H}} \leq M_2$ for $\tau \in (0, \tau_0]$ and $t \in [0, T]$

and

(2.13) $|\tau^{-1}(1 - S(\tau)P)V(\tau)F(\tau, [\cdot/\tau]\tau)|_{L^2(0,T;\mathbf{H})} \leq M_2$ for $\tau \in (0, \tau_0]$.

PROOF. The equality (2.11) follows immediately from (2.1) and the definition of $F(\tau, k\tau)$. By (b.2), (c.2) and Lemma 2.2 we have

$$|\tau^{-1}(V(\tau) - 1)F(\tau, [t/\tau]\tau)|_{\mathbf{H}} \leq L|BF(\tau, [t/\tau]\tau)|_{\mathbf{H}}$$
$$\leq L\{l(\varphi(F(\tau, [t/\tau]\tau)))\}^{1/2} \leq L\{l(M_1)\}^{1/2} \equiv R_2$$

for $\tau \in (0, \tau_0]$ and $t \in [0, T]$. Besides, it follows from (2.11) that

$$|\tau^{-1}(1 - S(\tau)P)V(\tau)F(\tau, [t/\tau]\tau)|_{\mathbf{H}}$$
$$\leq |F_t(\tau, t)|_{\mathbf{H}} + |\tau^{-1}(V(\tau) - 1)F(\tau, [t/\tau]\tau)|_{\mathbf{H}}$$

for $t \neq k\tau$, $k = 0, 1, \ldots$. Therefore,

$$|\tau^{-1}(1 - S(\tau)P)V(\tau)F(\tau, [\cdot/\tau]\tau)|_{L^2(0,T;\mathbf{H})} \leq M_1 + R_2 T,$$

where M_1 is as in Lemma 2.2. Thus we obtain (2.12) and (2.13) with $M_2 = M_1 + R_2T + R_2$. Q.E.D.

§ 3. Proof of Theorem 1.2

In order to prove the existence of a solution of (IVP; u_0), we show that $\{F(\tau(n), \cdot)\}$ converges as $n \to \infty$ for some suitable sequence $\{\tau(n)\}$ with $\tau(n) \downarrow 0$. Suppose that (A), (B) and (C) hold, $u_0 \in D(\varphi)$, and let $0 < T < t^*(\varphi(u_0))$. Then, by condition (A) and Lemma 2.2, $\{F(\tau, t);\, 0 < \tau < \tau_0,\, 0 \leq t \leq T\}$ is compact in $\mathbf{H}$ and $\{F(\tau, \cdot);\, 0 < \tau < \tau_0\}$ is bounded in $W^{1,2}(0, T; \mathbf{H})$. Firstly, by the Ascoli-Arzerà theorem, there exists a sequence $\{\tau(n)\}$ with $\tau(n) \downarrow 0$ such that

$$F(\tau(n), \cdot) \longrightarrow u \qquad \text{strongly in} \quad C([0, T]; \mathbf{H})$$

and

$$F_t(\tau(n), \cdot) \longrightarrow u' \qquad \text{weakly in} \quad L^2(0, T; \mathbf{H})$$

for some $u \in W^{1,2}(0, T; \mathbf{H})$. Secondly,

$$F(\tau(n), [t/\tau(n)]\tau(n)) \longrightarrow u(t) \quad \text{strongly in } \mathbf{H} \text{ and uniformly in} \quad t \in [0, T]$$

and

$$V(\tau(n))F(\tau(n), [t/\tau(n)]\tau(n)) \longrightarrow u(t) \quad \text{strongly in } \mathbf{H} \text{ and uniformly in } t \in [0, T]$$

as $n \to \infty$, and the functions

$$t \longrightarrow \varphi(F(\tau(n), [t/\tau(n)]\tau(n))), \quad n = 1, 2, \ldots$$

are uniformly bounded on $[0, T]$. Thirdly, on account of (ii) of Lemma 2.3, we may assume (by taking a subsequence if necessary) that

$$\tau(n)^{-1}(V(\tau(n)) - 1)F(\tau(n), [\cdot/\tau(n)]\tau(n)) \longrightarrow v^* \quad \text{weakly in} \quad L^2(0, T; \mathbf{H})$$

and

$$\tau(n)^{-1}(1 - S(\tau(n))P)V(\tau(n))F(\tau(n), [\cdot/\tau(n)]\tau(n)) \longrightarrow w^* \quad \text{weakly in} \quad L^2(0, T; \mathbf{H}) \tag{3.1}$$

for some $v^*, w^* \in L^2(0, T; \mathbf{H})$. Therefore, it follows from (i) of Lemma 2.3 that

$$u'(t) + w^*(t) = v^*(t) \qquad \text{for a.e.} \quad t \in [0, T].$$

On the other hand, Lemmas 1.5 and 1.6 together imply that

$$w^*(t) \in \partial\varphi u(t) \quad \text{and} \quad v^*(t) = Bu(t) \qquad \text{for a.e.} \quad t \in [0, T].$$

Since $F(\tau(n), 0) = S(\tau(n))u_0 \to u_0$ strongly in $\mathbf{H}$, we have $u(0) = u_0$, and consequently, u is a solution of (IVP; u_0) on $[0, T]$.

Next, from the fact that

$$[V(\tau)S(\tau)P]^{[t/\tau]}u_0 = \begin{cases} u_0 & \text{for} \quad 0 \leq t < \tau, \\ V(\tau)F(\tau, ([t/\tau] - 1)\tau) & \text{for} \quad \tau \leq t \leq T, \end{cases}$$

it follows that (1.2) holds for the sequence $\{\tau(n)\}$ and for the function u obtained above.

For the proof of (1.3), it is sufficient to show that

$$\liminf_{n \to \infty} \varphi(F(\tau(n), [t/\tau(n)]\tau(n))) \geq \varphi(u(t)) \quad \text{uniformly in} \quad t \in [0, T] \tag{3.2}$$

and

(3.3) $\limsup_{n\to\infty} \varphi(F(\tau(n), [t/\tau(n)]\tau(n))) \leq \varphi(u(t))$ uniformly in $t \in [0, T]$.

The inequality (3.2) follows immediately from the lower semicontinuity of φ and the compactness of $\{F(\tau, t); 0<\tau<\tau_0, 0\leq t\leq T\}$ in $\mathbf{H}$. Hence we prove (3.3). To this end we need the following lemma.

LEMMA 3.1. *For each measurable set E in* $[0, T]$, *we have*

$$\lim_{n\to\infty} \int_E \varphi(F(\tau(n), [t/\tau(n)]\tau(n)))dt = \int_E \varphi(u(t))dt. \tag{3.4}$$

PROOF. Let $z=u(t)$ and $y=[V(\tau(n))S(\tau(n))P]^{[t/\tau(n)]}u_0$ in (1.7) for $t \in [0, T]$ and integrate the resultant inequality over E. Then we have

$$\begin{aligned}\int_E \varphi(u(t))dt \geq \int_E \varphi(F(\tau(n), [t/\tau(n)]\tau(n))dt \\ + \int_E \tau(n)^{-1}((1-S(\tau(n))P)[V(\tau(n))S(\tau(n))P]^{[t/\tau(n)]}u_0, \\ u(t) - [V(\tau(n))S(\tau(n))P]^{[t/\tau(n)]}u_0)_{\mathbf{H}}dt.\end{aligned} \tag{3.5}$$

The second term of the right hand side of (3.5) tends to 0 as $n\to\infty$ by (2.3) and (3.1). This, together with (3.2), yields (3.4). Q.E.D.

Suppose that (3.3) does not hold. Then there are a number $\varepsilon>0$, a subsequence $\{\tau(n_k)\}$ of $\{\tau(n)\}$ and a sequence $\{t_k\}\subset[0, T]$ such that t_k converges to some t_0 as $k\to\infty$ and

$$\varphi(F(\tau(n_k), [t_k/\tau(n_k)]\tau(n_k))) \geq \varphi(u(t_k)) + 3\varepsilon \quad \text{for} \quad k = 1, 2,\dots. \tag{3.6}$$

First we have $t_0>0$. In fact, by (i) of Lemma 2.1 we have

$$\begin{aligned}&\varphi(F(\tau(n_k), [t_k/\tau(n_k)]\tau(n_k))) \\ &\quad\leq \varphi(F(\tau(n_k), 0)) + (L^2/2)t_k l(M_1), \quad k = 1, 2,\dots,\end{aligned} \tag{3.7}$$

where M_1 is the same constant as in Lemma 2.2. Also, since $\varphi(u(t))$ is continuous, we infer from (3.6) and (3.7) that

$$\begin{aligned}\varphi(u(t_0)) + 3\varepsilon &\leq \limsup_{k\to\infty} \varphi(F(\tau(n_k), [t_k/\tau(n_k)]\tau(n_k))) \\ &\leq \lim_{k\to\infty} \varphi(S(\tau(n_k))u_0) + (L^2/2)t_0 l(M_1) \\ &\leq \varphi(u_0) + (L^2/2)t_0 l(M_1).\end{aligned}$$

Clearly this is impossible for $t_0=0$, and so t_0 must be positive.

Next, choose a number $\delta>0$ and an integer k_0 so that

$$2\delta < t_0,$$

(3.8) $|\varphi(u(t)) - \varphi(u(s))| \le \varepsilon$ for any $s, t \in [t_0 - 2\delta, t_0 + 2\delta] \cap [0, T]$,

(3.9) $|t_k - t_0| \le \delta$ for any $k \ge k_0$,

(3.10) $(3/2)L^2 l(M_1)\delta < \varepsilon$.

Then, from (3.6), (3.8), (3.9), (3.10) and from (i) of Lemma 2.1, it follows that

$$\begin{aligned}\varphi(u(t)) + 2\varepsilon \le \varphi(u(t_k)) + 3\varepsilon &\le \varphi(F(\tau(n_k), [t_k/\tau(n_k)]\tau(n_k))) \\ &\le \varphi(F(\tau(n_k), [t/\tau(n_k)]\tau(n_k))) + (3/2)L^2\delta l(M_1) \\ &\le \varphi(F(\tau(n_k), [t/\tau(n_k)]\tau(n_k))) + \varepsilon,\end{aligned}$$

for $k = 1, 2, \dots$, and $t \in E = [t_0 - 2\delta, t_0 - \delta]$. Therefore, on account of Lemma 3.1.

$$\begin{aligned}\int_E \varphi(u(t))dt + \varepsilon\delta &\le \lim_{k\to\infty} \int_E \varphi(F(\tau(n_k), [t/\tau(n_k)]\tau(n_k)))dt \\ &= \int_E \varphi(u(t))dt.\end{aligned}$$

This is a contradiction. Thus (3.3) must hold, and the proof of Theorem 1.2 is thereby complete.

§4. Proof of Theorem 1.4

Suppose that all the assumptions of Theorem 1.4 are satisfied. Let T_0^* be the supremum of all $T \ge 0$ such that both of the limit relations (1.4) and (1.5) hold uniformly on $[0, T]$.

As is easily seen from Theorem 1.2 and the uniqueness assumption for the solution of (IVP; u_0), we have

$$T_0^* \ge t^*(\varphi(u_0)).$$

It is sufficient to prove that $T_0^* \ge T^*$, and this can be shown in the following way.

Suppose $T_0^* < T^*$. Fix any $T_0 \in (T_0^*, T^*)$ and put

$$K = \max_{0 \le t \le T_0} \varphi(u(t)).$$

Choose positive numbers δ_0 and ε_0 such that

$$3\delta_0 < t^*(K + \varepsilon_0)$$

and

$$0 < s_0 \equiv T_0^* - \delta_0 < T_0^* + \delta_0 < T_0.$$

Further, choose a positive number τ_0 ($\le \delta_0$) such that

$$\varphi(F(\tau, t)) \leq K + \varepsilon_0 \quad \text{for} \quad 0 < \tau < \tau_0 \quad \text{and} \quad 0 \leq t \leq s_0;$$

such τ_0 does exist since $\varphi(F(\tau, t)) \to \varphi(u(t))$ as $\tau \downarrow 0$ uniformly in $t \in [0, s_0]$. Then, on account of Lemma 2.1, we have

$$\begin{aligned} 2^{-1}\int_{s_0}^{t} |F_s(\tau, s)|_{\mathbf{H}}^2 ds &+ \varphi(F(\tau, t)) \\ &\leq f(\tau, t-[s_0/\tau]\tau; \varphi(F(\tau, [s_0/\tau]\tau))) \\ &\leq f(\tau, t-s_0+\tau; K+\varepsilon_0) \\ &\leq f(3\delta_0; K+\varepsilon_0) < \infty \end{aligned}$$

for all $0<\tau<\tau_0$ and $t \in [s_0, s_0+2\delta_0] = [T_0^*-\delta_0, T_0^*+\delta_0]$. In the same way as in Lemma 2.2, we infer from the above inequalities that there exists a constant M_1 such that

$$|F(\tau, t)|_{\mathbf{H}} \leq M_1 \quad \text{for any} \quad T_0^* - \delta_0 \leq t \leq T_0^* + \delta_0 \quad \text{and} \quad 0 < \tau < \tau_0,$$

$$|\varphi(F(\tau, t))| \leq M_1 \quad \text{for any} \quad T_0^* - \delta_0 \leq t \leq T_0^* + \delta_0 \quad \text{and} \quad 0 < \tau < \tau_0$$

and

$$|F_t(\tau, \cdot)|_{L^2(T_0^*-\delta_0, T_0^*+\delta_0: \mathbf{H})} \leq M_1 \quad \textit{for any} \quad 0 < \tau < \tau_0.$$

Therefore, in a way similar to the proof of Theorem 1.2, we can conclude that

$$F(\tau, t) \longrightarrow u(t) \quad \text{as } \tau \downarrow 0 \text{ strongly in } \mathbf{H} \text{ and uniformly in } \quad t \in [0, T_0^*+\delta_0]$$

and that

$$\varphi(F(\tau, t)) \longrightarrow \varphi(u(t)) \quad \text{as } \tau \downarrow 0 \text{ uniformly in } \quad t \in [0, T_0^*+\delta_0].$$

In fact, by the uniqueness of the solution, the limit relations (1.2) and (1.3) hold uniformly on $[0, T_0^*+\delta_0]$ for all sequences $\{\tau(n)\}$ with $\tau(n) \downarrow 0$. This contradicts the definition of T_0^*. Thus $T_0^* \geq T^*$.

§ 5. Application

Throughout this section, let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary Γ. We denote by $|\cdot|_q$ (resp. $|\cdot|_{1,q}$), $1 \leq q \leq \infty$, the norm of the space $L^q(\Omega)$ (resp. the Sobolev space $W^{1,q}(\Omega)$) and by $W_0^{1,q}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\Omega)$. For simplicity we often write u_{x_i} for the partial derivative $(\partial/\partial x_i)u$ and u_t for $(\partial/\partial t)u$.

We use the symbol Δ_p, $2 \leq p < \infty$, to denote the following nonlinear differential operator:

$$\Delta_p u = \textstyle\sum_{i=1}^N (|u_{x_i}|^{p-2} u_{x_i})_{x_i}.$$

We here consider the following obstacle problem:

$$
\begin{aligned}
(5.1)\qquad & u_t \geq \Delta_p u + \operatorname{div} \beta(u) + f(u) \quad \text{on} \quad (0, T) \times \Omega, \\
& u \geq h \quad \text{on} \quad (0, T) \times \Omega, \\
& u_t = \Delta_p u + \operatorname{div} \beta(u) + f(u) \quad \text{on} \quad \{u > h\}, \\
& u(0, \cdot) = u_0 \quad \text{on} \quad \Omega, \quad \text{and} \\
& u = 0 \quad \text{on} \quad (0, T) \times \Omega,
\end{aligned}
$$

where $2 \leq p < \infty$; h is a given obstacle function in $W^{1,p}(\Omega)$ satisfying $h \leq 0$ a.e. on Γ; and the initial function u_0 belongs to $W_0^{1,p}(\Omega)$ and satisfies $u_0 \geq h$ a.e. on Ω; $\beta(\cdot) = (\beta_1(\cdot), \beta_2(\cdot), \dots, \beta_N(\cdot))$ is a function from $\mathbf{R}$ into $\mathbf{R}^N$; and f is a function from $\mathbf{R}$ into itself.

We impose the following conditions (β) and (f) on β and f, respectively.

(β) (i) *If $p > N$, then each β_i is locally Lipschitz continuous on R;*

(ii) *If $2 < p = N$, then each β_i is locally Lipschitz continuous on R and there exist a number $2 \leq q < \infty$ and constants $C_1, C_2 \geq 0$ such that*

$$(5.2)\qquad |\beta_i'(\lambda)| \leq C_1 |\lambda|^q + C_2 \qquad \text{for} \quad \lambda \in \mathbf{R},\ i = 1, 2, \dots, N;$$

and if $2 = p = N$, then each β_i is locally Lipschitz continuous and (5.2) holds with $q = 0$;

(iii) *If $2 \leq p < N$, then each β_i is locally Lipschitz continuous and (5.2) holds for $q = N(p-2)/(2(N-p))$.*

(f) *There exist monotone nondecreasing functions f_1 and f_2 satisfying $f = f_1 - f_2$ and the following conditions:*

(i) *If $p > N$, then f_1 and f_2 are continuous on* $\mathbf{R}$;

(ii) *If $p = N$, then f_1 and f_2 are continuous on* $\mathbf{R}$ *and there exist a number $2 \leq r < \infty$ and constants $C_3, C_4 \geq 0$ such that*

$$(5.3)\qquad |f_i(\lambda)| \leq C_3 |\lambda|^r + C_4 \qquad \text{for} \quad \lambda \in \mathbf{R},\ i = 1, 2;$$

(iii) *If $2 \leq p < N$, then f_1 and f_2 are continuous on* $\mathbf{R}$ *and* (5.3) *holds for $r = Np/(2(N-p))$.*

We then define a closed convex set K in $W_0^{1,p}(\Omega)$ by

$$K = \{z \in W_0^{1,p}(\Omega);\ z \geq h \text{ a.e. on } \Omega\},$$

and a proper l.s.c. convex functional φ on $\mathbf{H} = L^2(\Omega)$ by

$$
(5.4)\qquad \varphi(z) = \begin{cases} (1/p) \sum_{i=1}^N \int_\Omega |z_{x_i}|^p dx & \text{if} \quad z \in K, \\ \infty & \text{otherwise.} \end{cases}
$$

Moreover, we employ the operator B defined by

$$(5.5) \qquad Bz = \operatorname{div} \beta(z) + f(z), \quad D(B) = W_0^{1,p}(\Omega).$$

LEMMA 5.1. *B maps each bounded subset of $W_0^{1,p}(\Omega)$ into a bounded subset of* $\mathbf{H}$, *namely: B: $W_0^{1,p}(\Omega) \to \mathbf{H}$ is bounded, Further, B satisfies condition* (b.3) *of* (B).

PROOF. Since β can be approximated by smooth functions satisfying (β), we may assume without loss of generality that β is smooth. First we show the lemma in the case of $2 \leq p < N$. If $2 \neq p$, then for any $z \in W_0^{1,p}(\Omega)$ we apply (5.2) to get

$$(5.6) \qquad \begin{aligned} |\beta_i(z)_{x_i}|_{\mathbf{H}}^2 &= |\beta_i'(z) z_{x_i}|_{\mathbf{H}}^2 \leq |z_{x_i}|_p^2 |\beta_i'(z)|_{2p/(p-2)}^2 \\ &\leq |z_{x_i}|_p^2 \{m_1 |z|_{Np/(N-p)}^{N(p-2)/(2(N-p))} + m_2\}^2 \\ &\leq |z_{x_i}|_p^2 \{m_3 (\textstyle\sum_{i=1}^N |z_{x_i}|_p)^{N(p-2)/(2(N-p))} + m_4\}^2, \end{aligned}$$

where $m_1 \sim m_4$ are positive constants independent of z and, in the last inequality, the Gagliardo-Nirenberg inequality [10] is used. In case $2 = p < N$, a similar estimate holds since $\beta(\cdot)$ is bounded. Also, in the case of $2 \leq p < N$, the application of (5.3) yields

$$(5.7) \qquad |f_i(z)|_{\mathbf{H}}^2 \leq m_5 |z|_{Np/(N-p)}^{Np/(N-p)} + m_6 \leq m_7 |z|_{1,p}^{Np/(N-p)} + m_8, \qquad i = 1, 2,$$

where constants $m_5 \sim m_8$ are independent of z. Combining (5.6) and (5.7), the first assertion of the lemma is obtained.

In the case of $p = N$ (resp. $p > N$), the inclusion $W_0^{1,p}(\Omega) \to L^l(\Omega)$ (resp. $W_0^{1,p}(\Omega) \to B(\Omega)$) is bounded for any $1 \leq l < \infty$, where $B(\Omega)$ is the set of all bounded continuous functions on Ω. Therefore the boundedness of B: $W_0^{1,p}(\Omega) \to \mathbf{H}$ can be proved just as in the case of $p < N$.

Suppose that $z_n \to z$ strongly in $\mathbf{H}$ and the set $\{\varphi(z_n)\}$ is bounded. Then we have $|z|_{1,p} < \infty$ and thus $z \in D(B)$. It is easy to show that $f_i(z_n) \to f_i(z)$ weakly in $\mathbf{H}$ by virtue of the maximal monotonicity of $f_i(\cdot)$. Further, by a simple calculation, we have $\beta_i(z_n) \to \beta_i(z)$ strongly in $\mathbf{H}$ and thus $\beta_i(z_n)_{x_i} \to \beta_i(z)_{x_i}$ weakly in $\mathbf{H}$. Therefore it is concluded that B satisfies condition (b.3). Q. E. D.

By the definition of φ, condition (A) is clearly satisfied. Besides, by virtue of Lemma 5.1 and Poincaré's Lemma, condition (B) is satisfied. Now the initial value problem

$$(5.8) \qquad \begin{aligned} &(d/dt)u(t) + \partial\varphi u(t) \ni Bu(t), \qquad 0 \leq t \leq T, \\ &u(0) = u_0 \end{aligned}$$

is a variational formulation for the system (5.1), namely: the function $u(t) = u(t, \cdot)$, u being a solution of (5.8), can be regarded as a generalized solution of the system (5.1).

Applying Theorem 1.2 with $V(\tau)=1+\tau B$, we obtain the following result: There exist at least one solution u of (5.8) on $[0, T]$ for some $0<T<\infty$ and a sequence $\{\tau(n)\}$ with $\tau(n)\downarrow 0$ as $n\to\infty$ such that

$$[(1+\tau(n)B)S(\tau(n))P]^{[t/\tau(n)]}u_0 \longrightarrow u(t) \quad \text{strongly in} \quad \mathbf{H} \tag{5.9}$$

and

$$\varphi(S(\tau(n))P[(1+\tau(n)B)S(\tau(n))P]^{[t/\tau(n)]}u_0) \longrightarrow \varphi(u(t)) \tag{5.10}$$

uniformly for $t\in[0, T]$. Here $\{S(\tau);\ 0<\tau<\infty\}$ is the contraction semigroup on $\overline{D(\varphi)}$ generated by $-\partial\varphi$, and P the projection from $\mathbf{H}$ onto $\overline{D(\varphi)}$.

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