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On decaying entire solutions of second order sublinear elliptic equations

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1. Introduction

In this paper we consider the semilinear ellitpic equation

$$\Delta u + a(x)u^{\sigma} = 0 \quad \text{in} \quad R^n, \tag{1}$$

where $n \geqq 3$, $x=(x_1,\dots, x_n)$, $\Delta=\sum_{i=1}^{n} \partial^2/\partial x_i^2$, $0<\sigma<1$, and $a(x)$ is a positive locally Hölder continuous function (with exponent $\alpha \in (0, 1)$) in R^n.

We are interested in the existence of positive entire solutions of equation (1). By an entire solution of (1) we mean a function $u \in C_{loc}^{2+\alpha}(R^n)$ which satisfies (1) at every point of R^n. The problem of existence of such solutions has been studied by several authors including [1–5]. Most of them have dealt with bounded positive entire solutions which are bounded away from zero. However, equation (1) may also have positive entire solutions which approach zero as $|x| \to \infty$.

The main objective of this paper is to prove the existence of positive entire solutions of (1) decaying to zero at infinity. Our procedure is to construct solutions of (1) which are squeezed between supersolutions and subsolutions tending to zero as $|x| \to \infty$. The latter are obtained as spherically symmetric solutions of elliptic equations with $a(x)u^{\sigma}$ in (1) replaced by radial majorants and minorants. For this purpose we need a global existence theory of a certain singular boundary value problem for nonlinear ordinary differential equations. We also attempt to extend the main result for (1) to semilinear elliptic systems of the form

$$\begin{cases} \Delta u + a(x)u^{\sigma}v^{\tau} = 0 \\ \Delta v + b(x)u^{\lambda}v^{\mu} = 0, \end{cases} \tag{2}$$

where σ, τ, λ and μ are nonnegative constants and $a(x)$ and $b(x)$ are positive locally Hölder continuous functions in R^n.

2. Main results

We employ the notation:

$$a^*(r) = \max_{|x|=r} a(x), \qquad a_*(r) = \min_{|x|=r} a(x). \tag{3}$$

It is easy to see that $a^*(r)$ and $a_*(r)$ are locally Hölder continuous (with the same exponent α as $a(x)$) on the interval $R_+=[0, \infty)$.

THEOREM 1. *Suppose that*

$$\int^{\infty} t^{1+\varepsilon}a^*(t)dt < \infty \tag{4}$$

for some positive constant ε. *Let* $p=\min\{\varepsilon/(1-\sigma), n-2\}$. *Then equation* (1) *has an entire solution* $u(x)$ *such that*

$$c_1|x|^{2-n} \leqq u(x) \leqq c_2|x|^{-p}, \quad |x| \geqq 1, \tag{5}$$

for some positive constants c_1 *and* c_2.

PROOF. Our proof is based on Theorem 2.10 of Ni [5]: If there exist positive functions v and w in $C^{2+\alpha}_{loc}(R^n)$ which satisfy the inequalities

$$\Delta v + a(x)v^\sigma \leqq 0, \quad \Delta w + a(x)w^\sigma \geqq 0, \quad w \leqq v \tag{6}$$

in R^n, then equation (1) has an entire solution u satisfying

$$w \leqq u \leqq v \quad \text{in} \quad R^n.$$

We construct such v and w as spherically symmetric solutions of the equations

$$\Delta v + a^*(|x|)v^\sigma = 0 \quad \text{and} \quad \Delta w + a_*(|x|)w^\sigma = 0,$$

respectively, by requiring that $v(x)\to 0$ and $w(x)\to 0$ as $|x|\to\infty$. If we let $v(x)=y(|x|)$ and $w(x)=z(|x|)$, then we are led to the following one-dimensional singular boundary value problems for $y(r)$ and $z(r)$:

$$\begin{cases} y'' + \dfrac{n-1}{r}y' + a^*(r)y^\sigma = 0, & r > 0, \\ y'(0) = 0, \quad y(r) \to 0 & \text{as} \quad r \to \infty, \end{cases} \tag{7}$$

$$\begin{cases} z'' + \dfrac{n-1}{r}z' + a_*(r)z^\sigma = 0, & r > 0, \\ z'(0) = 0, \quad z(r) \to 0 & \text{as} \quad r \to \infty. \end{cases} \tag{8}$$

As easily verified, (7) and (8) are equivalent to the integral equations

$$y(r) = \frac{1}{n-2}\int_0^r \left(\frac{t}{r}\right)^{n-2} ta^*(t)y^\sigma(t)dt + \frac{1}{n-2}\int_r^\infty ta^*(t)y^\sigma(t)dt \tag{9}$$

and

$$z(r) = \frac{1}{n-2}\int_0^r \left(\frac{t}{r}\right)^{n-2} ta_*(t)z^\sigma(t)dt + \frac{1}{n-2}\int_r^\infty ta_*(t)z^\sigma(t)dt, \tag{10}$$

respectively. In order to solve (9) and (10), put

$$
(11)\qquad
\begin{aligned}
c_1 &= \left\{\frac{1}{n-2}\int_0^1 t^{n-1}a_*(t)dt + \frac{1}{n-2}\int_1^\infty t^{1-\sigma(n-2)}a_*(t)dt\right\}^{1/(1-\sigma)},\\
c_2 &= \left\{\frac{1}{n-2}\int_0^1 ta^*(t)dt + \frac{1}{n-2}\int_1^\infty t^{1+(1-\sigma)p}a^*(t)dt\right\}^{1/(1-\sigma)},
\end{aligned}
$$

and define

$$
(12)\qquad X = \{(y, z)\in C(R_+)\times C(R_+):\ \zeta(r)\leqq z(r)\leqq y(r)\leqq \eta(r)\},
$$

where

$$
(13)\qquad
\begin{aligned}
\zeta(r) &= \begin{cases} c_1, & 0\leqq r\leqq 1\\ c_1 r^{2-n}, & 1\leqq r<\infty,\end{cases}\\
\eta(r) &= \begin{cases} c_2, & 0\leqq r\leqq 1\\ c_2 r^{-p}, & 1\leqq r<\infty.\end{cases}
\end{aligned}
$$

Clearly, X is a closed convex subset of the Fréchet space $C(R_+)\times C(R_+)$ of continuous vector functions with the topology of uniform convergence on every compact subinterval of R_+. Consider now the operator $F\colon X\to C(R_+)\times C(R_+)$ defined by

$$
(14)\qquad F(y, z) = (\tilde{y}, \tilde{z}),
$$

where

$$
\begin{aligned}
\tilde{y}(r) &= \frac{1}{n-2}\int_0^r \left(\frac{t}{r}\right)^{n-2} ta^*(t)y^\sigma(t)dt + \frac{1}{n-2}\int_r^\infty ta^*(t)y^\sigma(t)dt,\\
\tilde{z}(r) &= \frac{1}{n-2}\int_0^r \left(\frac{t}{r}\right)^{n-2} ta_*(t)z^\sigma(t)dt + \frac{1}{n-2}\int_r^\infty ta_*(t)z^\sigma(t)dt
\end{aligned}
$$

for $r\geqq 0$. We show that F is continuous and maps X into a compact subset of X.

(i) *F maps X into X.* Let $(y, z)\in X$, $F(y, z)=(\tilde{y}, \tilde{z})$ and $F(\eta, \zeta)=(\tilde{\eta}, \tilde{\zeta})$. Then, by (11), (13) and (14), we see that

$$
\begin{aligned}
\tilde{\eta}(r) &\leqq c_2^\sigma\left\{\frac{1}{n-2}\int_0^1 ta^*(t)dt + \frac{1}{n-2}\int_1^\infty t^{1-\sigma p}a^*(t)dt\right\}\\
&\leqq \eta(r),\quad 0\leqq r\leqq 1,\\
\tilde{\eta}(r) &\leqq c_2^\sigma r^{-p}\left\{\frac{1}{n-2}\int_0^1 ta^*(t)dt + \frac{1}{n-2}\int_1^\infty t^{1+(1-\sigma)p}a^*(t)dt\right\}\\
&\leqq \eta(r),\quad 1\leqq r<\infty,
\end{aligned}
$$

$$\tilde{\zeta}(r) \geqq c_1^\sigma \left\{\frac{1}{n-2}\int_0^1 t^{n-1}a_*(t)dt + \frac{1}{n-2}\int_1^\infty t^{1-\sigma(n-2)}a_*(t)dt\right\}$$
$$\geqq \zeta(r), \quad 0 \leqq r \leqq 1,$$
$$\tilde{\zeta}(r) \geqq c_1^\sigma r^{2-n} \left\{\frac{1}{n-2}\int_0^1 t^{n-1}a_*(t)dt + \frac{1}{n-2}\int_1^\infty t^{1-\sigma(n-2)}a_*(t)dt\right\}$$
$$\geqq \zeta(r), \quad 1 \leqq r < \infty,$$

and by (12) and (14), we have

$$\zeta(r) \leqq \tilde{\zeta}(r) \leqq \tilde{z}(r) \leqq \tilde{y}(r) \leqq \tilde{\eta}(r) \leqq \eta(r), \quad r \geqq 0.$$

(ii) *F is continuous.* Let (y, z) and (y_m, z_m), $m=1, 2,\dots$ be vector functions in X such that $(y_m, z_m)\to(y, z)$ in $C(R_+)\times C(R_+)$ as $m\to\infty$. Let $F(y_m, z_m)=(\tilde{y}_m, \tilde{z}_m)$ and $F(y, z)=(\tilde{y}, \tilde{z})$. Then, from (14) we have

$$|\tilde{y}_m(r) - \tilde{y}(r)| \leqq \frac{1}{n-2}\int_0^\infty ta^*(t)|y_m^\sigma(t)-y^\sigma(t)|dt, \quad r \geqq 0.$$

Since the integrand $ta^*(t)|y_m^\sigma(t)-y^\sigma(t)|$ is bounded from above by an integrable function $2c_2^\sigma ta^*(t)$ and converges to zero at every point t of R_+, the Lebesgue dominated convergence theorem implies that $\tilde{y}_m(r)\to\tilde{y}(r)$ uniformly on R_+ as $m\to\infty$. Likewise, $\tilde{z}_m(r)\to\tilde{z}(r)$ uniformly on R_+ as $m\to\infty$. Thus, we have $F(y_m, z_m)\to F(y, z)$ in $C(R_+)\times C(R_+)$ as $m\to\infty$.

(iii) *FX is relatively compact.* This follows from the observation that if $(y, z)\in X$, then $(\tilde{y}, \tilde{z})=F(y, z)$ satisfies

$$0 \geqq \tilde{y}'(r) = -\int_0^r\left(\frac{t}{r}\right)^{n-1} a^*(t)y^\sigma(t)dt \geqq -c_2^\sigma\int_0^\infty a^*(t)dt,$$
$$0 \geqq \tilde{z}'(r) = -\int_0^r\left(\frac{t}{r}\right)^{n-1} a_*(t)z^\sigma(t)dt \geqq -c_2^\sigma\int_0^\infty a_*(t)dt$$

for $r\geqq 0$.

Thus, we are able to apply the Schauder-Tychonoff fixed point theorem, concluding that F has a fixed point (y, z) in X. The components y and z are solutions of (9) and (10) (and hence solutions of (7) and (8)) such that $y(r)\geqq z(r)$ for $r\geqq 0$. It follows that the functions $v(x)=y(|x|)$ and $w(x)=z(|x|)$ satisfy (6) in R^n. Since it is obvious that $v, w\in C_{loc}^{2+\alpha}(R^n)$, using the above-mentioned theorem of Ni [5], we conclude that equation (1) has an entire solution u lying between v and w. The construction of v and w shows that u satisfies (5) with c_1 and c_2 defined by (11). This completes the proof.

THEOREM 2. *Suppose that* (4) *holds for some positive constant* ε. *Let* $p=\min\{\varepsilon/(1-\sigma), n-2\}$. *If* $u(x)>0$ *is an arbitrary solution of* (1) *on* $|x|\geqq r_0$ *such* *that* $u(x)\to 0$ *as* $|x|\to\infty$, *then the spherical mean* $U(r)$ *of* $u(x)$ *on* $|x|=r$ *satisfies*

$$c_1 r^{2-n} \leqq U(r) \leqq c_2 r^{-p} \tag{15}$$

for some positive constants c_1 *and* c_2, *provided* r *is sufficiently large.*

PROOF. The first inequality in (15) follows from the Hadamard three-spheres theorem for superharmonic functions; see Protter and Weinberger [6, p. 131].

To prove the second inequality in (15), we first note that $r^{1-n}(r^{n-1}U'(r))'$ equals the spherical mean of Δu on $|x|=r$, so that $U'(r)$ is eventually negative, say, for $r\geqq r_1\geqq\max\{r_0, 1\}$. Integrating (1) on $|x|=r$ and using Hölder's inequality, we get

$$-r^{1-n}(r^{n-1}U'(r))' \leqq a^*(r)U^{\sigma}(r), \quad r \geqq r_1.$$

Integrating the above inequality twice (first on $[r_1, r]$ and second on $[r, \infty)$) yields

$$\begin{aligned} U(r) \leqq & -\frac{1}{n-2} r_1^{n-1} U'(r_1) r^{2-n} \\ & + \frac{1}{n-2}\left\{\int_{r_1}^{r}\left(\frac{t}{r}\right)^{n-2} ta^*(t)U^{\sigma}(t)dt + \int_r^{\infty} ta^*(t)U^{\sigma}(t)dt\right\} \end{aligned} \tag{16}$$

for $r\geqq r_1$. From (16) it is not hard to see that if

$$U(r) \leqq kr^{-q}, \qquad r \geqq r_1, \tag{17}$$

for some constants $k>0$ and $q\in[0, n-2]$, then

$$\begin{aligned} & U(r) \leqq (A+Bk^{\sigma})r^{2-n}, \quad r \geqq r_1, \qquad \text{if } \varepsilon + q\sigma \geqq n-2, \\ & U(r) \leqq Ar^{2-n} + Bk^{\sigma}r^{-(\varepsilon+q\sigma)}, \quad r \geqq r_1, \qquad \text{if } \varepsilon + q\sigma < n-2, \end{aligned} \tag{18}$$

where

$$A = -\frac{1}{n-2} r_1^{n-1} U'(r_1) \quad \text{and} \quad B = \frac{1}{n-2}\int_{r_1}^{\infty} t^{1+\varepsilon}a^*(t)dt.$$

Now, since $U(r)\leqq k_0=U(r_1)$, $r\geqq r_1$, (17) holds with $k=k_0$ and $q=0$, so that (18) implies that

$$\begin{aligned} & U(r) \leqq (A+Bk_0^{\sigma})r^{2-n} && \text{if } \varepsilon \geqq n-2, \\ & U(r) \leqq Ar^{2-n} + Bk_0^{\sigma}r^{-\varepsilon} && \text{if } \varepsilon < n-2. \end{aligned}$$

Suppose $\varepsilon<n-2$. Then, from the above,

$$U(r) \leqq k_1 r^{-\varepsilon}, \qquad k_1 = A + Bk_0^{\sigma},$$

that is, $U(r)$ satisfies (17) with $k=k_1$ and $q=\varepsilon$. So, (18) implies that for $r \geqq r_1$,

$$U(r) \leqq (A+Bk_1^q)r^{2-n} \qquad \text{if} \quad \varepsilon(1+\sigma) \geqq n-2,$$

$$U(r) \leqq Ar^{2-n} + Bk_1^q r^{-\varepsilon(1+\sigma)} \qquad \text{if} \quad \varepsilon(1+\sigma) < n-2.$$

If $\varepsilon(1+\sigma)<n-2$, then

$$U(r) \leqq k_2 r^{-\varepsilon(1+\sigma)}, \qquad k_2 = A + Bk_1^q,$$

so that from (18) it follows that

$$U(r) \leqq (A+Bk_2^q)r^{2-n} \qquad \text{if} \quad \varepsilon(1+\sigma+\sigma^2) \geqq n-2,$$

$$U(r) \leqq Ar^{2-n} + Bk_2^q r^{-\varepsilon(1+\sigma+\sigma^2)} \qquad \text{if} \quad \varepsilon(1+\sigma+\sigma^2) < n-2.$$

Repeating the same argument, we conclude that either

$$U(r) \leqq k_m r^{2-n}, \qquad r \geqq r_1,$$

for some m, or else

$$U(r) \leqq k_m r^{-\varepsilon(1+\sigma+\cdots+\sigma^{m-1})}, \qquad r \geqq r_1,$$

for all m, where k_m is defined recursively by

$$k_m = A + Bk_{m-1}^\sigma, \qquad m = 1, 2, \ldots .$$

In the latter case we have

$$U(r) \leqq Kr^{-\varepsilon/(1-\sigma)}, \qquad r \geqq r_1,$$

where K is an upper bound for the sequence $\{k_m\}$, the boundedness of which can easily be checked. This establishes the second inequality in (15), and the proof is complete.

Corollary. *Suppose that*

$$\int^\infty t^{1+(1-\sigma)(n-2)} a^*(t)dt < \infty.$$

(i) *There exists a positive solution u of* (1) *such that*

$$c_1|x|^{2-n} \leqq u(x) \leqq c_2|x|^{2-n}, \quad |x| \geqq 1,$$

for some positive constants c_1 and c_2.

(ii) *For any positive solution u of* (1) *which decays to zero at infinity, the spherical mean $U(r)$ of u on $|x|=r$ satisfies*

$$c_1 r^{2-n} \leqq U(r) \leqq c_2 r^{2-n}$$

for some positive constants c_1 *and* c_2, *provided* r *is sufficiently large.*

REMARK. According to Theorem 2.6 of Kawano [1], equation (1) has a positive entire solution which tends to a positive constant as $|x|\to\infty$, if

$$\int^{\infty} ta^*(t)dt < \infty. \tag{19}$$

Since (4) implies (19), under condition (4), there are two types of solutions: solutions tending to positive constants as $|x|\to\infty$, and solutions tending to zero as $|x|\to\infty$. Furthermore, for any given constant $c\geqq 0$, we can construct a positive entire solution which converges to c at infinity. The procedure is similar to the proof of Theorem 1, but we do not develop it here. For closely related results we refer to Naito [4].

Theorems 1 and 2 can be extended without difficulty to sublinear elliptic equations of the form

$$\Delta u + a_1(x)u^{\sigma_1} + \cdots + a_m(x)u^{\sigma_m} = 0 \quad \text{in} \quad R^n, \tag{20}$$

where $n\geqq 3$, $0\leqq\sigma_i<1$, $i=1,\dots, m$, and $a_i(x)$, $i=1,\dots, m$, are positive locally Hölder continuous functions (with exponent $\alpha\in(0, 1)$) in R^n.

THEOREM 3. *Suppose that there exist positive constants* ε_i, $i=1,\dots, m$, *such that*

$$\int^{\infty} t^{1+\varepsilon_i}a_i^*(t)dt < \infty, \qquad i = 1,\dots, m,$$

where

$$a_i^*(r) = \max_{|x|=r} a_i(x), \qquad i = 1,\dots, m.$$

Let

$$p = \min\{\varepsilon_1/(1-\sigma_1),\dots, \varepsilon_m/(1-\sigma_m), n-2\}.$$

Then:

(i) *There exists a positive entire solution of* (20) *such that*

$$c_1|x|^{2-n} \leqq u(x) \leqq c_2|x|^{-p}, \quad |x| \geqq 1,$$

for some positive constants c_1 *and* c_2.

(ii) *For any positive solution* u *of* (20) *which decays to zero at infinity, the spherical mean* $U(r)$ *of* u *on* $|x|=r$ *satisfies*

$$c_1r^{2-n} \leqq U(r) \leqq c_2r^{-p}$$

for some positive constants c_1 *and* c_2, *provided* r *is sufficiently large.*

3. Elliptic systems

Let us now consider the semilinear elliptic system (2). We assume throughout that $\sigma+\tau<1$ and $\lambda+\mu<1$, and $a(x)$ and $b(x)$ are positive locally Hölder continuous functions (with exponent $\alpha\in(0, 1)$) in R^n, $n\geqq 3$.

Kawano [1] has recently given sufficient conditions for (2) to possess entire solutions (u, v) such that both u and v are positive and tend to positive constants as $|x|\to\infty$. See also Kawano and Kusano [2]. It is the purpose of this paper to obtain conditions under which (2) has positive entire solutions (u, v), both components of which decay to zero as $|x|\to\infty$.

Below we use the functions $a^*(r)$, $a_*(r)$ defined by (3), and

$$b^*(r) = \max_{|x|=r} b(x), \qquad b_*(r) = \min_{|x|=r} b(x).$$

THEOREM 4. *Suppose that*

$$\int^\infty t^{1+\varepsilon}a^*(t)dt < \infty \quad \text{and} \quad \int^\infty t^{1+\delta}b^*(t)dt < \infty \tag{21}$$

for some positive constants ε and δ. Let S denote the set of solutions $(p, q)\in [0, n-2]\times[0, n-2]$ of the system of inequalities

$$\begin{cases} (1-\sigma)p - \tau q \leqq \varepsilon \\ -\lambda p + (1-\mu)q \leqq \delta \end{cases} \tag{22}$$

and let

$$p^* = \max\{p\colon (p, q)\in S\} \quad \text{and} \quad q^* = \max\{q\colon (p, q)\in S\}. \tag{23}$$

Then:

(i) *There exists a positive entire solution (u, v) of (2) such that*

$$\begin{aligned} &c_1|x|^{2-n} \leqq u(x) \leqq c_2|x|^{-p^*}, \\ &c_3|x|^{2-n} \leqq v(x) \leqq c_4|x|^{-q^*}; \quad |x| \geqq 1, \end{aligned} \tag{24}$$

for some positive constants c_1, c_2, c_3 and c_4.

(ii) *For any positive solution (u, v) of (2) which decays to zero at infinity, the spherical means $U(r)$ and $V(r)$ of u and v on $|x|=r$ satisfy*

$$\begin{aligned} &c_1 r^{2-n} \leqq U(r) \leqq c_2 r^{-p^*}, \\ &c_3 r^{2-n} \leqq V(r) \leqq c_4 r^{-q^*} \end{aligned} \tag{25}$$

for some positive constants c_1, c_2, c_3 and c_4, provided r is sufficiently large.

OUTLINE OF THE PROOF. (i) We make use of a theorem of Kawano [1, Theorem 5.1] which asserts that if there exist positive functions (u_1, v_1) and (u_2, v_2) in $C^{2+\alpha}_{loc}(R^n)\times C^{2+\alpha}_{loc}(R^n)$ satisfying the inequalities

$$\begin{cases} \Delta u_1 + a(x)u_1^\sigma v_1^\tau \geqq 0 \\ \Delta v_1 + b(x)u_1^\lambda v_1^\mu \geqq 0, \end{cases} \tag{26}$$

$$\begin{cases} \Delta u_2 + a(x)u_2^\sigma v_2^\tau \leqq 0 \\ \Delta v_2 + b(x)u_2^\lambda v_2^\mu \leqq 0, \end{cases} \tag{27}$$

$$u_1 \leqq u_2, \quad v_1 \leqq v_2 \tag{28}$$

in R^n, then equation (2) has an entire solution (u, v) such that

$$u_1 \leqq u \leqq u_2 \text{ and } v_1 \leqq v \leqq v_2 \text{ in } R^n. \tag{29}$$

We wish to construct the desired (u_1, v_1) and (u_2, v_2) as spherically symmetric vector functions. For this purpose we put

$$A_1 = \frac{1}{n-2}\int_0^1 t^{n-1}a_*(t)dt + \frac{1}{n-2}\int_1^\infty t^{1-(\sigma+\tau)(n-2)}a_*(t)dt,$$

$$B_1 = \frac{1}{n-2}\int_0^1 t^{n-1}b_*(t)dt + \frac{1}{n-2}\int_1^\infty t^{1-(\lambda+\mu)(n-2)}b_*(t)dt,$$

$$A_2 = \frac{1}{n-2}\int_0^1 ta^*(t)dt + \frac{1}{n-2}\int_1^\infty t^{1+(1-\sigma)p-\tau q}a^*(t)dt,$$

$$B_2 = \frac{1}{n-2}\int_0^1 tb^*(t)dt + \frac{1}{n-2}\int_1^\infty t^{1-\lambda p+(1-\mu)q}b^*(t)dt,$$

and define

$$\nu = (1-\sigma)(1-\mu) - \lambda\tau,$$

$$c_1 = A_1^{(1-\mu)/\nu}B_1^{\tau/\nu}, \qquad c_2 = A_2^{(1-\mu)/\nu}B_2^{\tau/\nu},$$

$$c_3 = A_1^{\lambda/\nu}B_1^{(1-\sigma)/\nu}, \qquad c_4 = A_2^{\lambda/\nu}B_2^{(1-\sigma)/\nu}.$$

It is easy to see that $c_1 \leqq c_2$ and $c_3 \leqq c_4$. We consider the functions $\eta_i(r)$, $\zeta_i(r)$, $i=1, 2$, defined as follows:

$$\eta_1(r) = c_1, \qquad \eta_2(r) = c_2,$$

$$\zeta_1(r) = c_3, \qquad \zeta_2(r) = c_4,$$

for $0 \leqq r \leqq 1$, and

$$\eta_1(r) = c_1 r^{2-n}, \qquad \eta_2(r) = c_2 r^{-p^*},$$

$$\zeta_1(r) = c_3 r^{2-n}, \qquad \zeta_2(r) = c_4 r^{-q^*},$$

for $1 \leqq r < \infty$, and define X to be the subset of $Y = C(R_+) \times C(R_+) \times C(R_+) \times C(R_+)$ consisting of vector functions (y_1, z_1, y_2, z_2) such that

$$\eta_1(r) \leqq y_1(r) \leqq y_2(r) \leqq \eta_2(r),$$
$$\zeta_1(r) \leqq z_1(r) \leqq z_2(r) \leqq \zeta_2(r), \quad r \geqq 0.$$

Clearly X is a closed convex subset of the Fréchet space Y. Finally, let $F: X \to Y$ denote the operator defined by

$$F(y_1, z_1, y_2, z_2) = (\tilde{y}_1, \tilde{z}_1, \tilde{y}_2, \tilde{z}_2),$$

where

$$\tilde{y}_1(r) = \frac{1}{n-2}\int_0^r \left(\frac{t}{r}\right)^{n-2} t a_*(t) y_1^\sigma(t) z_1^\tau(t) dt + \frac{1}{n-2}\int_r^\infty t a_*(t) y_1^\sigma(t) z_1^\tau(t) dt,$$

$$\tilde{z}_1(r) = \frac{1}{n-2}\int_0^r \left(\frac{t}{r}\right)^{n-2} t b_*(t) y_1^\lambda(t) z_1^\mu(t) dt + \frac{1}{n-2}\int_r^\infty t b_*(t) y_1^\lambda(t) z_1^\mu(t) dt,$$

$$\tilde{y}_2(r) = \frac{1}{n-2}\int_0^r \left(\frac{t}{r}\right)^{n-2} t a^*(t) y_2^\sigma(t) z_2^\tau(t) dt + \frac{1}{n-2}\int_r^\infty t a^*(t) y_2^\sigma(t) z_2^\tau(t) dt,$$

$$\tilde{z}_2(r) = \frac{1}{n-2}\int_0^r \left(\frac{t}{r}\right)^{n-2} t b^*(t) y_2^\lambda(t) z_2^\mu(t) dt + \frac{1}{n-2}\int_r^\infty t b^*(t) y_2^\lambda(t) z_2^\mu(t) dt,$$

for $r \geqq 0$. Then it can be shown that F is continuous and maps X into a compact subset of X, so that from the Schauder-Tychonoff fixed point theorem it follows that F has a fixed point (y_1, z_1, y_2, z_2) in X. Differentiation of the integral equations satisfied by (y_1, z_1, y_2, z_2) shows that $(u_1(x), v_1(x)) = (y_1(|x|), z_1(|x|))$ and $(u_2(x), v_2(x)) = (y_2(|x|), z_2(|x|))$ are positive entire solutions of the systems

$$\begin{cases} \Delta u_1 + a_*(|x|) u_1^\sigma v_1^\tau = 0 \\ \Delta v_1 + b_*(|x|) u_1^\lambda v_1^\mu = 0, \end{cases}$$

$$\begin{cases} \Delta u_2 + a^*(|x|) u_2^\sigma v_2^\tau = 0 \\ \Delta v_2 + b^*(|x|) u_2^\lambda v_2^\mu = 0, \end{cases}$$

respectively, and so they satisfy the inequalities (26) and (27). Since (27) holds, the above-mentioned theorem of Kawano implies that system (2) has an entire solution (u, v) satisfying (29). From the construction of $y_i(r)$ and $z_i(r)$, $i = 1, 2$, it is obvious that (u, v) has the asymptotic behavior (24).

(ii) The proof proceeds exactly as in the proof of Theorem 2. We first note that $U'(r)$ and $V'(r)$ are eventually negative, say, for $r \geqq r_1$. Integrating (2) and using Hölder's inequality, we obtain

$$-r^{1-n}(r^{n-1}U'(r))' \leqq a^*(r) U^\sigma(r) V^\tau(r),$$

$$- r^{1-n}(r^{n-1}V'(r))' \leqq b^*(r)U^{\lambda}(r)V^{\mu}(r), \quad r \geqq r_1.$$

We then integrate the above inequalities twice and obtain two integral inequalities similar to (16). Finally, with the help of the last inequalities we observe that if

$$U(r) \leqq kr^{-s} \quad \text{and} \quad V(r) \leqq kr^{-t}, \quad r \geqq r_1,$$

for some constants $k>0$ and $s, t \in [0, n-2]$, then the following inequalities hold for $r \geqq r_1$:

$$\begin{aligned}
&U(r) \leqq (A+Bk^{\sigma+\tau})r^{2-n} && \text{if} \quad \varepsilon + \sigma s + \tau t \geqq n-2,\\
&V(r) \leqq (A+Bk^{\lambda+\mu})r^{2-n} && \text{if} \quad \delta + \lambda s + \mu t \geqq n-2,\\
&U(r) \leqq Ar^{2-n} + Bk^{\sigma+\tau}r^{-(\varepsilon+\sigma s+\tau t)} && \text{if} \quad \varepsilon + \sigma s + \tau t < n-2,\\
&V(r) \leqq Ar^{2-n} + Bk^{\lambda+\mu}r^{-(\delta+\lambda s+\mu t)} && \text{if} \quad \delta + \lambda s + \mu t < n-2,
\end{aligned}$$

where A and B are positive constants. This leads us to the upper bounds for $U(r)$ and $V(r)$ in (25).

COROLLARY. *Suppose that*

$$\int^{\infty} t^{1+(1-\sigma-\tau)(n-2)}a^*(t)dt < \infty,$$

$$\int^{\infty} t^{1+(1-\lambda-\mu)(n-2)}b^*(t)dt < \infty.$$

(i) *There exists a positive entire solution* (u, v) *of* (2) *such that*

$$c_1|x|^{2-n} \leqq u(x) \leqq c_2|x|^{2-n},$$
$$c_3|x|^{2-n} \leqq v(x) \leqq c_4|x|^{2-n}, \quad |x| \geqq 1,$$

for some positive constants c_1, c_2, c_3 *and* c_4.

(ii) *For any positive solution* (u, v) *of* (2) *which decays to zero at infinity, the spherical means* $U(r)$ *and* $V(r)$ *of* u *and* v *on* $|x|=r$ *satisfy*

$$c_1r^{2-n} \leqq U(r) \leqq c_2r^{2-n},$$
$$c_3r^{2-n} \leqq V(r) \leqq c_4r^{2-n}$$

for some positive constants c_1, c_2, c_3 *and* c_4, *provided* r *is sufficiently large.*

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