HIROSHIMA MATH. J.
15 (1985), 109–116

Explicit two-step methods with one off-step node

Hisayoshi SHINTANI

(Received August 16, 1984)

1. Introduction

Consider the initial value problem

(1.1) $$y' = f(x, y), \quad y(x_0) = y_0,$$

where $f(x, y)$ is assumed to be sufficiently smooth. Let $y(x)$ be the solution of (1.1) and

(1.2) $$x_n = x_0 + nh \qquad (n=1, 2,\dots; h>0),$$

where h is a stepsize. Let y_1 be an approximation of $y(x_1)$ obtained by some appropriate method. We are concerned with the case where the approximations y_j $(j=2, 3,\dots)$ of $y(x_j)$ are obtained by two-step methods. Conventional two-step methods such as linear two-step methods [1], pseudo-Runge-Kutta methods [1, 3] and so on [4] require starting values y_0 and y_1 to generate y_j $(j=2, 3,\dots)$.

In our previous paper [4] we introduced a set of subsidiary nodes

(1.3) $$x_{n+v} = x_0 + (n+v)h \quad (n=0, 1,\dots; 0<v<1)$$

and at the cost of providing an additional starting value y_v we proposed two-step methods for computing y_{n+1} $(n=1, 2,\dots)$ together with subsidiary approximations y_{n+v} of $y(x_{n+v})$, which are of the form

(1.4) $$y_{n+v} = y_n + b_{r-1}(y_n - y_{n-1}) + d_{r-1}(y_n - y_{n-1+v}) + h\textstyle\sum_{j=0}^{r-1} c_{r-1\,j}k_{jn},$$

(1.5) $$y_{n+1} = y_n + b_r(y_n - y_{n-1}) + h\textstyle\sum_{j=0}^{r} c_{rj}k_{jn},$$

where

(1.6) $$k_{0n} = k_{2n-1}, \quad k_{1n} = f(x_{n-1+v}, y_{n-1+v}), \quad k_{2n} = f(x_n, y_n),$$

(1.7) $$k_{in} = f(x_n + a_ih,\ y_n + b_i(y_n - y_{n-1}) + d_i(y_n - y_{n-1+v}) + h\textstyle\sum_{j=0}^{i-1} c_{ij}k_{jn}),$$

(1.8) $$a_i = b_i + (1-v)d_i + \textstyle\sum_{j=0}^{i-1} c_{ij}, \quad 0 < a_i \leqq 1 \quad (3 \leqq i \leqq r),$$

and a_i, b_i, d_i, and c_{ij} $(j=0, 1,\dots, i-1;\ i=3, 4,\dots, r)$ are real constants. It has been shown that for $r=4, 5$ there exist a method (1.5) of order $r+1$ and a method (1.4) of order r with $r-2$ function evaluations per step.

In this paper, at the expense of supplying one more starting value y_{1+v}, we propose two-step methods of the form

$$(1.9)\qquad y_{n+1} = y_n + b_r(y_n - y_{n-1}) + h\sum_{j=0}^{r-1} c_{rj}k_{jn},$$

$$(1.10)\qquad y_{n+1+v} = y_n + b_{r+1}(y_n - y_{n-1}) + h\sum_{j=0}^{r} c_{r+1\,j}k_{jn},$$

where

$$(1.11)\quad k_{0n} = k_{2n-1},\quad k_{1n} = k_{3n-1},\quad k_{2n} = f(x_n, y_n),\quad k_{3n} = f(x_{n+v}, y_{n+v}),$$

$$(1.12)\quad k_{in} = f(x_n + a_i h,\ y_n + b_i(y_n - y_{n-1}) + d_i(y_n - y_{n-1+v}) + h\sum_{j=0}^{i-1} c_{ij}k_{jn}),$$

$$(1.13)\quad a_i = b_i + (1-v)d_i + \sum_{j=0}^{i-1} c_{ij},\quad 0 < a_i \leqq 1\quad (4 \leqq i \leqq r+1),$$

a_i, b_i, d_i and c_{ij} ($j=0, 1,\dots, i-1$; $i=4, 5,\dots, r+1$) are real constants, $d_i=0$ ($i=5, 6$) and $d_4=0$ unless $r=5$. Convergence of these methods is studied in [5]. A stepsize control is implemented by comparing the method (1.9) with the method

$$(1.14)\qquad z_{n+1} = y_n + z(y_n - y_{n-1}) + h\sum_{j=0}^{r} w_j k_{jn},$$

where z and w_j ($j=0, 1,\dots, r$) are real constants. It is shown that for $r=4, 5$ there exist methods (1.9) and (1.10) of order $r+2$ and a method (1.14) of order $r+1$ with $r-2$ function evaluations per step. Finally methods (1.9) and (1.5) are illustrated by numerical examples.

2. Preliminaries

Let

$$(2.1)\quad a_0 = -1,\quad a_1 = v - 1,\quad a_2 = 0,\quad a_3 = v,\quad a_r = 1,\quad a_{r+1} = 1 + v,$$

$$(2.2)\quad t_{n+1} = u(y_n - y_{n-1}) + h\sum_{j=0}^{r+1} v_j k_{jn},$$

$$(2.3)\quad z_{n+1} = y_{n+1} + t_{n+1},$$

$$(2.4)\quad y(x) + b_i(y(x) - y(x-h)) + d_i(y(x) - y(x+(v-1)h)) + h\sum_{j=0}^{i-1} c_{ij}y'(x+a_j h) - y(x+a_i h) = \sum_{k=1}^{8} e_{ik}(h^k/k!)y^{(k)}(x) + O(h^9)\quad (i=4, 5,\dots, r+1),$$

$$(2.5)\quad u(y(x) - y(x-h)) + h\sum_{j=0}^{r} v_j y'(x+a_j h) = \sum_{k=1}^{8} U_k(h^k/k!)y^{(k)}(x) + O(h^9).$$

Then we have

$$(2.6)\quad (-1)^{k-1}b_i - (v-1)^k d_i + k\sum_{j=0}^{i-1} a_j^{k-1}c_{ij} - a_i^k = e_{ik}\quad (i=4, 5,\dots, r+1),$$

$$(2.7)\quad (-1)^{k-1}u + k\sum_{j=0}^{r} a_j^{k-1}v_j = U_k\quad (k=1, 2,\dots, 8).$$

Let

(2.8) $k_{in}^{*} = y'(x_n + a_i h) \quad (i=0, 1, 2, 3),$

(2.9) $k_{in}^{*} = f(x_n + a_i h,\ y(x_n) + b_i(y(x_n) - y(x_{n-1})) + d_i(y(x_n) - y(x_{n-1+v}))$
$+ h\sum_{j=0}^{i-1} c_{ij} k_{jn}^{*}) \quad (i=4, 5, \ldots, r+1),$

(2.10) $T(x_n) = y(x_n) + b_r(y(x_n) - y(x_{n-1})) + h\sum_{j=0}^{r-1} c_{rj} k_{jn}^{*} - y(x_{n+1}),$

(2.11) $T_v(x_n) = y(x_n) + b_{r+1}(y(x_n) - y(x_{n-1})) + h \sum_{j=0}^{r} c_{r+1\,j} k_{jn}^{*} - y(x_{n+1+v}),$

(2.12) $R(x_n) = u(y(x_n) - y(x_{n-1})) + h\sum_{j=0}^{r} v_j k_{jn}^{*},$

(2.13) $F_8 = \sum_{j=4}^{r-1} c_{rj} e_{j7}, \quad L_8 = \sum_{j=4}^{r} c_{r+1\,j} e_{j7}, \quad g(x) = f_y(x, y(x)),$

(2.14) $A_j = a_j(a_j+1), \quad B_j = A_j(a_j - a_1), \quad C_j = B_j(a_j - a_3), \quad D_j = C_j(a_j - a_4)$
$(j=1, 2, \ldots, 6),$

(2.15) $g_1 = v^2 - v - 1, \quad g_2 = v^2 - 3v + 1, \quad g_3 = 2v - 1, \quad g_4 = v^2 + 3v + 1,$
$g_5 = 56v^2 + 14v - 11, \quad g_6 = 5v^2 + v - 1, \quad g_7 = 15v^2 + v - 3,$
$g_8 = 5v^2 - 1, \quad g_9 = 21v^2 + 7v - 4, \quad g_{10} = 4v^2 + 13v + 4,$

(2.16) $r_1 = g_3 + a_4, \quad r_2 = a_4 g_3 + A_1 - 1, \quad r_3 = a_4 A_1 - r_1,$
$r_4 = a_4 g_3 + A_1, \quad r_5 = a_4 A_1,$

(2.17) $p_1 = 15v^4 - 36v^3 + 14v^2 + 9v - 4, \quad p_2 = 21v^4 - 70v^3 + 55v^2 + 2v - 8,$
$p_3 = 42v^4 - 98v^3 + 25v^2 + 37v - 12, \ p_4 = 4v^3 - 5v^2 - v + 1,$
$p_5 = 5v^4 - 20v^3 + 8v^2 + 5v - 2, \ p_6 = 15v^4 - 24v^3 - v^2 + 6v - 1,$
$p_7 = 28v^2 + 12v - 5, \ p_8 = 189v^2 - 497v + 284,$
$p_9 = 308v^2 - 828v + 485,$

(2.18) $m_1 = 2a_1 + a_4, \ m_2 = 2a_4 g_8 + g_6, \ m_3 = a_4 g_2 - A_1, \ m_4 = 2a_1 a_4 + g_2,$
$m_5 = 7a_4 g_7 + g_5, \ m_6 = 2a_4 g_9 + p_7, \ m_7 = 2a_4 g_5 + 70v^2 + 26v - 13.$

Choosing $e_{ij} = 0$ ($j=1, 2, \ldots, 6$; $i=4, 5, 6$), we have

(2.19) $T(x) = \sum_{j=1}^{8} e_{rj}(h^j/j!) y^{(j)}(x) + F_8(h^8/7!) g(x) y^{(7)}(x) + O(h^9),$

(2.20) $T_v(x) = \sum_{j=1}^{8} e_{r+1\,j}(h^j/j!) y^{(j)}(x) + L_8(h^8/7!) g(x) y^{(7)}(x) + O(h^9),$

(2.21) $R(x) = \sum_{j=1}^{7} U_j(h^j/j!) y^{(j)}(x) + O(h^8),$

(2.22) $\sum_{j=0}^{i-1} c_{ij} = a_i - b_i + a_1 d_i, \quad 2\sum_{j=0}^{i-1} a_j c_{ij} = b_i + a_1 d_i + a_i^2,$
$6\sum_{j=1}^{i-1} A_j c_{ij} = b_i + a_1^2(2a_3+1) d_i + (2a_i+3) a_i^2,$

$$12\sum_{j=3}^{i-1} B_j c_{ij} = -g_3 b_i - a_1^3(a_3+1)d_i + 3a_i^4 - 4(a_1-1)a_i^3 - 6a_1 a_i^2,$$
$$30\sum_{j=4}^{i-1} C_j c_{ij} = -g_8 b_i + g_4 a_1^3 d_i + 6a_i^5 - 15a_1 a_i^4 + 10g_2 a_i^3 + 15A_1 a_i^2,$$

(2.23) $$60\sum_{j=4}^{i-1} a_j C_j c_{ij} = -g_6 b_i + g_4 a_1^4 d_i + 10a_i^6 - 24a_1 a_i^5 + 15g_2 a_i^4 + 20A_1 a_i^3,$$

(2.24) $$420\sum_{j=4}^{i-1} a_j^2 C_j c_{ij} - g_9 b_i - g_{10} a_1^5 d_i - 60a_i^7 + 140a_1 a_i^6 - 84g_2 a_i^5 - 105A_1 a_i^4 = 60e_{i7},$$

(2.25) $$840\sum_{j=5}^{i-1} a_j^2 D_j c_{ij} + m_6 b_i - 105a_i^8 + 120m_1 a_i^7 - 140m_4 a_i^6 + 168m_3 a_i^5 + 210A_1 a_4 a_i^4 = 105e_{i8} - 120m_1 e_{i7} \quad (r=5,\ i\geqq 5).$$

Setting $U_i=0$ $(i=1, 2,\dots, 5)$, we have

(2.26) $$\sum_{j=0}^{r} v_j = -u, \quad 2\sum_{j=0}^{r} a_j v_j = u, \quad 6\sum_{j=1}^{r} A_j v_j = u,$$
$$12\sum_{j=3}^{r} B_j v_j = -g_3 u, \quad 30\sum_{j=4}^{r} C_j v_j = g_8 u,$$

(2.27) $$60\sum_{j=5}^{r} D_j v_j + m_2 u = 10U_6,$$

(2.28) $$-m_5 u = 60U_7 - 70(a_4 g_6 + g_3)U_6 \quad (r=5).$$

From (1.9) and (1.10) we have

(2.29) $$y_{n+2+\sigma} - (1+b_r)u_{n+1+\sigma} + b_r y_{n+\sigma} = h\Phi_\sigma(x_n, y_{n-2},\dots, y_{n+2}, y_{n-2+\nu},\dots, y_{n+1+\nu}, h) \quad (\sigma=0, \nu).$$

Thus the method (1.9)–(1.10) is stable if and only if $-1\leqq b_r<1$ [5].

3. Construction of the methods

We shall show the following

THEOREM. *For* $r=4, 5$ *there exist methods* (1.9) *and* (1.10) *of order* $r+2$ *and a method* (1.14) *of order* $r+1$.

3.1. Case $r=4$

Choosing $e_{i4}=e_{i5}=0$ $(i=1, 2,\dots, 6)$ and $U_j=0$ $(j=1, 2,\dots, 5)$, we have (2.22), (2.26) and

(3.1) $$p_1 = 0,$$

(3.2) $$g_7 b_5 = 10a_5^6 - 12g_3 a_5^5 + 15g_1 a_5^4 + 20g_3 a_5^3 - 30A_1 a_5^2,$$

(3.3) $$-g_9 b_4 - p_8 = 60e_{47},$$

(3.4) $$-g_5 b_4 - 60a_5^7 + 70g_3 a_5^6 - 84g_1 a_5^5 - 105g_3 a_5^4 + 140A_1 a_5^3 = 60e_{57},$$

(3.5) $g_7 u = 10U_6$.

The choice $a_3 = 0.7809341293$ and $u = 0.5$ yields

(3.6) $b_4 = 0.2974663081, \quad c_{40} = -0.05882026395, \quad c_{41} = -0.7654544608,$
$c_{42} = 0.9861380498, \quad c_{43} = 0.5406703668, \quad e_{47} = -0.256,$

(3.7) $b_5 = 14.62235196, \quad c_{50} = -3.101021799, \quad c_{51} = -27.71076926,$
$c_{52} = 20.43975629, \quad c_{53} = -10.01943672, \quad c_{54} = 7.550053666,$
$e_{57} = -27.0$

(3.8) $v_0 = -0.1141782932, \quad v_1 = -0.7877301552, \quad v_2 = 0.4668968977,$
$v_3 = -0.1289354826, \quad v_4 = 0.0639470333, \quad U_6 = 0.346.$

3.2. Case $r=5$

Setting $e_{4i} = U_i = 0$ $(i = 1, 2, \ldots, 6)$ and $e_{5j} = e_{6j} = 0$ $(j = 1, 2, \ldots, 7)$, we have (2.22), (2.26), (2.27), (2.28) and

(3.9) $7p_1 a_4^2 - p_2 a_4 - p_3 = 0,$

(3.10) $A_1^3 g_4 d_4 = -2g_8 a_4^6 + 6p_4 a_4^5 - 3p_5 a_4^4 - 2p_6 a_4^3 - 3A_1 g_6 a_4^2,$

(3.11) $m_2 b_5 = 35v^2 - 89v + 49 - 2(25v^2 - 60v + 31)a_4,$

(3.12) $m_5 b_6 = -60a_4^7 + 70r_1 a_6^6 - 84r_2 a_6^5 + 105r_3 a_6^4 + 140r_4 a_6^3 - 210r_5 a_6^2,$

(3.13) $60D_5 c_{65} + m_2 b_6 = 10a_6^6 - 12m_1 a_6^5 + 15m_4 a_6^4 - 20m_3 a_6^3 - 30A_1 a_4 a_6^2,$

(3.14) $-g_9 b_4 - g_{10} a_1^5 d_4 - 60a_4^7 + 140a_1 a_4^6 - 84g_2 a_4^5 - 150A_1 a_4^4 = 60e_{47},$

(3.15) $m_6 b_5 + 2p_8 a_4 - p_9 = 105e_{58},$

(3.16) $m_7 b_6 - 105a_6^8 + 120r_1 a_6^7 - 140r_2 a_6^6 + 168r_3 a_6^5 + 210r_4 a_6^4$
$- 280r_5 a_6^3 = 105e_{68}.$

The choice $a_3 = 0.40672$ and $u = 10$ yields

(3.17) $a_4 = 0.8657843991, \quad b_4 = 30.98333961, \quad d_4 = 1.016093933,$
$c_{40} = -3.838607752, \quad c_{41} = -18.57698904, \quad c_{42} = -9.134128108,$
$c_{43} = 2.034997901, \quad e_{47} = -1.88,$

(3.18) $b_5 = -0.1204316125, \quad c_{50} = 0.01514095607, \quad c_{51} = 0.07018877773,$
$c_{52} = 0.1881115637, \quad c_{53} = 0.5157103308, \quad c_{54} = 0.3312799843,$

$e_{58} = -0.0412,$

(3.19) $b_6 = -21.90884112, \quad c_{60} = 2.667191773, \quad c_{61} = 13.47599688,$

$c_{62} = 6.458007992, \quad c_{63} = 0.5064857425, \quad c_{64} = -2.111264358,$

$c_{65} = 2.319143086, \quad e_{68} = -5.15,$

(3.20) $v_0 = -1.233009566, \quad v_1 = -6.079604056, \quad v_2 = -3.163209656,$

$v_3 = 0.5612643282, \quad v_4 = -0.1284989354, \quad v_5 = 0.04305788532,$

$U_7 = -0.547.$

4. Methods with two starting values

We shall show examples of methods (1.4), (1.5) and (2.1) for $r=4$, 5.

4.1. Case $r=4$

Choosing $e_{3i}=U_i=0\,(i=1,2,3,4)$, $e_{4j}=0\,(j=1,2,\ldots,5)$, $d_3=0$, $v=(6-\sqrt{5})/5$ and $u=-4$, we have

(4.1) $b_3 = 7.777670546, \quad c_{30} = -1.884916221, \quad c_{31} = -9.252172127,$

$c_{32} = 4.112204206, \quad e_{35} = -2.06$

(4.2) $b_4 = 0, \quad c_{40} = 0.008771516898, \quad c_{41} = -0.2363723741,$

$c_{42} = 0.6293739900, \quad c_{43} = 0.5982268672, \quad e_{46} = -0.184,$

(4.3) $v_0 = 0.9866804749, \quad v_1 = 4.487884326, \quad v_2 = -1.602285618,$

$v_3 = 0.1277208178, \quad U_5 = 1.22.$

4.2. Case $r=5$

Setting $e_{3i}=0\,(i=,1,2,3,4)$, $e_{4j}=U_j=0\,(j=1,2,\ldots,5)$, $e_{5k}=0\,(k=1,2,\ldots,6)$, $c_{53}=d_3=d_4=0$, $a_3=1$ and $u=0.5$, we have

(4.4) $b_3 = 16.35731965, \quad c_{30} = -4.120588739, \quad c_{31} = -20.80685171,$

$c_{32} = 9.570120798, \quad e_{35} = -5.49,$

(4.5) $a_4 = 0.7809341293, \quad b_4 = 1.872205855, \quad c_{40} = -0.4177985443,$

$c_{41} = -3.260557343, \quad c_{42} = 2.478127554, \quad c_{43} = 0.1089566073,$

$e_{46} = 1.07$

(4.6) $b_5 = 0.2974663081$, $c_{40} = -0.05882026395$, $c_{41} = -0.7654544608$,

$c_{42} = 0.9861380498$, $c_{43} = 0.5406703668$, $e_{47} = -0.256$,

(4.7) $v_0 = -0.1141782932$, $v_1 = -0.7877301552$, $v_2 = 0.4668968977$,
$v_3 = 0.06394703330$, $v_4 = -0.1289354826$, $U_6 = 0.346$.

5. Numerical examples

The following six problems are tested:

Problem 1. $y' = y$, $y(0) = 1$. $y(x) = \exp(x)$.

Problem 2. $y' = 2xy$, $y(0) = 1$. $y(x) = \exp(x^2)$.

Problem 3. $y' = -5y$, $y(0) = 1$. $y(x) = \exp(-5x)$.

Problem 4. $y' = -y^2$, $y(0) = 1$. $y(x) = 1/(1+x)$.

Problem 5. $y' = y - 2x/y$, $y(0) = 1$. $y(x) = (1+2x)^{1/2}$.

Problem 6. $y' = 1 - y^2$, $y(0) = 0$. $y(x) = \tanh(x)$.

Computation by methods (3.6) and (3.17) is carried out by the following program.

(i) Compute y_i, $y_{i+\nu}$, f_i anf $f_{i+\nu}$ $(i = 0, 1)$.

(ii) *Compute* y_2, f_2 and t_2.

(iii) If $|t_2| > \varepsilon \max(1, |y_2|)$, then halve the stepsize and go to (i).

(iv) If $|t_2| < \varepsilon_1 \max(1, |t_2|)$, then replace y_0, f_o and h by y_2, f_2 and $2h$ respectively and go to (i).

(v) Compute $y_{2+\nu}$ and $f_{2+\nu}$, replace y_i, $y_{i+\nu}$, f_i and $f_{i+\nu}$ $(i = 0, 1)$ by y_{i+1}, $y_{i+1+\nu}$, f_{i+1} and $f_{i+1+\nu}$ respectively and go to (ii).

Here $\varepsilon = 10^{-r-3}/2$ and $\varepsilon_1 = 2^{-r-4}\varepsilon$ $(r = 4, 5)$. Nyström formula, Butcher (6, 8) formula [1] and Shanks (7, 9) formula [2] are used for computing starting values for methods of order 5, 6 and 7 respectively.

Computation by methods (4.1) and (4.4) is carried out in an analogous manner. The errors at $x = 3$ are listed in Table 1.

Table 1.

Meth \ Prob	1	2	3	4	5	6
(3.6)	$-4.31E-8$	$-1.58E-5$	$1.07E-9$	$-8.04E-9$	$2.43E-7$	$1.16E-8$
(3.17)	$2.14E-9$	$4.07E-5$	$3.62E-10$	$-8.03E-11$	$3.46E-8$	$-1.47E-11$
(4.1)	$-8.05E-6$	$-6.58E-3$	$-1.57E-8$	$-1.61E-8$	$-8.27E-7$	$4.55E-8$
(4.4)	$5.68E-8$	$-2.59E-5$	$-8.38E-10$	$-3.42E-10$	$-2.75E-8$	$1.27E-8$

References

[1] L. Lapidus and J. H. Seinfeld, *Numerical solution of ordinary differential equations*, Academic Press, New York and London, 1971.

[2] E. B. Shanks, *Solution of differential equations by evaluations of functions*, Math. Comp., **20** (1966), 21–38.

[3] H. Shintani, *On pseudo-Runge-Kutta methods of the third kind*, Hiroshima Math. J., **11** (1981), 247–254.

[4] H. Shintani, *Two-step methods with one off-step node*, Hiroshima Math. J., **14** (1984), 479–487.

[5] H. Shintani, *Convergence, consistency and stability of step-by-step methods for ordinary differential equations*, Hiroshima Math. J. (To appear.)

Department of Mathematics,
Faculty of School Education,
Hiroshima University