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Lie algebras in which every soluble subalgebra is either abelian or almost-abelian

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Introduction

In this paper we shall investigate the structure of locally finite Lie algebras in which every soluble subalgebra is either abelian or almost-abelian.

Varea [8] has introduced the concept of $\mathfrak{C}^*$-algebras termed C-algebras, namely, Lie algebras in which every subalgebra of a nilpotent subalgebra H of L is an ideal in the idealizer of H in L, and he has shown for finite-dimensional Lie algebras that $\mathfrak{C}^*$-algebras are precisely Lie algebras in which every soluble subalgebra is either abelian or almost-abelian. Also Varea has introduced the concept of $\mathfrak{C}$-algebras termed c-algebras, namely, Lie algebras in which every 1-dimensional subideal is an ideal. A Lie algebra L is called a $\mathfrak{T}$-algebra if every subideal of L is an ideal of L. The relation among $\mathfrak{C}^*$-algebras, $\mathfrak{C}$-algebras and $\mathfrak{T}$-algebras, and their structure are investigated in [8]. Infinite-dimensional $\mathfrak{C}^*$-algebras are considered in [2]. A Lie algebra L is called an (A)-algebra in [5] if any pair of elements x and y of L such that $[x, y, y] = 0$ satisfies $[x, y] = 0$. Finite-dimensional (A)-algebras are investigated in [6]. Let Δ be one of the relations asc, wsi, wasc and $\leq^{\omega}$. Following [2] and [3], we call a Lie algebra L a $\mathfrak{T}(\Delta)$-algebra (resp. $\mathfrak{C}(\Delta)$-algebra) when we replace the relation "subideal" by Δ in the above definition of $\mathfrak{T}$-algebra (resp. $\mathfrak{C}$-algebra). We call a Lie algebra L a $\mathfrak{T}_0(\Delta)$-algebra (resp. $\mathfrak{C}_0(\Delta)$-algebra) if every Δ-subalgebra H of L satisfies $[L, H] = H^2$. For a class $\mathfrak{X}$ we call a Lie algebra L an $\mathfrak{X}^s$-algebra if every subalgebra of L is an $\mathfrak{X}$-algebra.

In this paper we shall introduce the classes $\mathfrak{C}_0^*$ and $\mathfrak{C}^{(*)}$: A Lie algebra L is a $\mathfrak{C}_0^*$-algebra if every soluble subalgebra of L is abelian, and L is a $\mathfrak{C}^{(*)}$-algebra if any pair of elements x and y of L such that $[x, y, y] \in \langle y \rangle$ satisfies $[x, y] \in \langle y \rangle$.

In Section 2, we shall show characterizations of $\mathfrak{C}_0^*$, $\mathfrak{C}^{(*)}$ and (A)-algebras: $\mathfrak{T}_0(\mathrm{asc})^s = \mathfrak{T}_0(\mathrm{si})^s$ (Lemma 2.1). $\mathfrak{C}_0^* = \mathfrak{C}_0(\mathrm{asc})^s = \mathfrak{C}_0(\mathrm{si})^s$ (Proposition 2.2). $\mathfrak{T}(\mathrm{wasc})^s = \mathfrak{T}(\leq^{\omega})^s$ (Lemma 2.3). $\mathfrak{C}^{(*)} = \mathfrak{C}(\mathrm{wasc})^s = \mathfrak{C}(\leq^{\omega})^s$ (Proposition 2.4). $\mathfrak{T}_0(\mathrm{wasc})^s = \mathfrak{T}_0(\leq^{\omega})^s$ (Lemma 2.5). (A) $= \mathfrak{C}_0(\mathrm{wasc})^s = \mathfrak{C}_0(\leq^{\omega})^s$ (Proposition 2.6).

In Section 3, we shall show that $\mathrm{L}\mathfrak{F} \cap \mathfrak{C}_0^* = \mathrm{L}\mathfrak{F} \cap \mathfrak{T}_0(\mathrm{si})^s$ (Theorem 3.3),

$\mathrm{L}\mathfrak{F}\cap\mathfrak{C}^{(*)}=\mathrm{L}\mathfrak{F}\cap\mathfrak{T}(\mathrm{wsi})^s$ (Theorem 3.4), and $\mathrm{L}\mathfrak{F}\cap(\mathrm{A})=\mathrm{L}\mathfrak{F}\cap\mathfrak{T}_0(\mathrm{wsi})^s$ (Theorem 3.6).

In Section 4, we shall determine the structure of locally finite $\mathfrak{C}^*$-algebras. If L is a locally finite $\mathfrak{C}^*$-algebra over a field of characteristic zero, then L is a locally finite (A)-algebra, an almost-abelian Lie algebra, or a three-dimensional split simple Lie algebra (Theorem 4.2). If L is a $\mathrm{L}(\mathrm{wser})\mathfrak{F}\cap\mathfrak{C}^*$-algebra over a field of characteristic zero, then L is a reductive (A)-algebra, a finite-dimensional almost-abelian Lie algebra, or a three-dimensional split simple Lie algebra (Theorem 4.5).

In Section 5, we shall investigate other properties of the classes $\mathfrak{C}^*$, $\mathfrak{C}_0^*$, $\mathfrak{C}^{(*)}$ and (A): $\mathfrak{C}^*\cap\{\mathrm{L},\acute{\mathrm{E}}\}\mathfrak{A}=\mathfrak{A}_0$ and $\mathfrak{C}_0^*\cap\{\mathrm{L},\acute{\mathrm{E}}\}\mathfrak{A}=\mathfrak{A}$ (Corollary 5.2). Over an algebraically closed field, $\mathrm{L}\mathfrak{F}\cap\mathfrak{C}^{(*)}=\mathfrak{A}_0$ and $\mathrm{L}\mathfrak{F}\cap\mathfrak{C}_0^*=\mathfrak{A}$ (Proposition 5.3).

In Section 6, we shall give exmples and show the following: $(\mathrm{A})\not\leq\mathfrak{T}$ (Example 6.1). $(\mathrm{A})\cup\mathfrak{A}_0<\mathfrak{C}^{(*)}$ (Examples 6.1 and 6.2). $\mathfrak{C}^{(*)}<\mathfrak{C}(\mathrm{wasc})$ (Example 6.3).

1. Notations

Throughout the paper Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified. We mostly follow [1] for the use of notations and terminology.

Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. For an ordinal σ, H is a σ-step ascendant (resp. weakly ascendant) subalgebra of L, denoted by $H\lhd^\sigma L$ (resp. $H\leq^\sigma L$), if there exists an ascending series (resp. chain) $(H_\alpha)_{\alpha\leq\sigma}$ of subalgebras (resp. subspaces) of L such that

(1) $H_0=H$ and $H_\sigma=L$,

(2) $H_\alpha\lhd H_{\alpha+1}$ (resp. $[H_{\alpha+1},H]\subseteq H_\alpha$) for any ordinal $\alpha<\sigma$,

(3) $H_\lambda=\bigcup_{\alpha<\lambda}H_\alpha$ for any limit ordinal $\lambda\leq\sigma$.

H is an ascendant (resp. a weakly ascendant) subalgebra of L, denoted by H asc L (resp. H wasc L), if $H\lhd^\sigma L$ (resp. $H\leq^\sigma L$) for some ordinal σ. When σ is finite, H is a subideal (resp. weak subideal) of L and denoted by H si L (resp. H wsi L). For a totally ordered set Σ, a series (resp. weak series) from H to L of type Σ is a collection $\{\Lambda_\sigma, V_\sigma\colon \sigma\in\Sigma\}$ of subalgebras (resp. subspaces) of L such that

(1) $H\subseteq V_\sigma\subseteq\Lambda_\sigma$ for all $\sigma\in\Sigma$,

(2) $L\backslash H=\bigcup_{\sigma\in\Sigma}(\Lambda_\sigma\backslash V_\sigma)$,

(3) $\Lambda_\tau\subseteq V_\sigma$ if $\tau<\sigma$,

(4) $V_\sigma\lhd\Lambda_\sigma$ (resp. $[\Lambda_\sigma,H]\subseteq V_\sigma$) for all $\sigma\in\Sigma$.

H is a serial (resp. weakly serial) subalgebra of L, denoted by H ser L (resp. H wser L), if there exists a series (resp. weak series) from H to L of type Σ for some Σ.

Let Δ be any of the relations si, asc, ser, $\lhd^{\sigma}$, wsi, wasc, wser and $\leq^{\sigma}$. $\mathfrak{T}(\Delta)$ is the class of Lie algebras L in which every Δ-subalgebra of L is an ideal of L. $\mathfrak{C}(\Delta)$ is the class of Lie algebras L in which every 1-dimensional Δ-subalgebra of L is an ideal of L. In particular we write $\mathfrak{T}$ and $\mathfrak{C}$ for $\mathfrak{T}$(si) and $\mathfrak{C}$(si) respectively. $\mathfrak{F}$, $\mathfrak{A}$ and $\mathfrak{N}$ are the classes of Lie algebras which are finite-dimensional, abelian and nilpotent respectively.

Let $\mathfrak{X}$ be a class of Lie algebras and let Δ be any of the relations $\leq$, $\lhd$, si, ser, wsi and wser. A Lie algebra L is said to lie in $\mathrm{L}(\Delta)\mathfrak{X}$ if for any finite subset X of L there exists an $\mathfrak{X}$-subalgebra K of L such that $X \subseteq K \Delta L$. We write $\mathrm{L}\mathfrak{X}$ for $\mathrm{L}(\leq)\mathfrak{X}$. When $L \in \mathrm{L}\mathfrak{F}$, L is called locally finite. For an ordinal σ, $\acute{\mathrm{E}}_{\sigma}(\Delta)\mathfrak{X}$ is the class of Lie algebras L having an ascending series $(L_{\alpha})_{\alpha \leq \sigma}$ of Δ-subalgebras such that

(1) $L_0 = 0$ and $L_{\sigma} = L$,

(2) $L_{\alpha} \lhd L_{\alpha+1}$ and $L_{\alpha+1}/L_{\alpha} \in \mathfrak{X}$ for any ordinal $\alpha < \sigma$,

(3) $L_{\lambda} = \bigcup_{\alpha < \lambda} L_{\alpha}$ for any limit ordinal $\lambda \leq \sigma$.

We write $\acute{\mathrm{E}}(\Delta)\mathfrak{X} = \bigcup_{\sigma > 0} \acute{\mathrm{E}}_{\sigma}(\Delta)\mathfrak{X}$ and $\mathrm{E}(\Delta)\mathfrak{X} = \bigcup_{n < \omega} \acute{\mathrm{E}}_{n}(\Delta)\mathfrak{X}$. In particular we write $\acute{\mathrm{E}}\mathfrak{X}$ and $\mathrm{E}\mathfrak{X}$ for $\acute{\mathrm{E}}(\leq)\mathfrak{X}$ and $\mathrm{E}(\leq)\mathfrak{X}$ respectively. Thus $\mathrm{E}\mathfrak{A}$ is the class of soluble Lie algebras. $\mathrm{Q}\mathfrak{X}$ is the class of Lie algebras consisting of all homomorphic images of $\mathfrak{X}$-algebras. $\mathrm{S}\mathfrak{X}$ is the class of Lie algebras consisting of all subalgebras of $\mathfrak{X}$-algebras. We say that $\mathfrak{X}$ is A-closed if $\mathfrak{X} = \mathrm{A}\mathfrak{X}$, where A is L, E, É, Q or S. We denote by $\mathfrak{X}^{s}$ the largest S-closed subclass of $\mathfrak{X}$, that is, L belongs to $\mathfrak{X}^{s}$ if and only if every subalgebra of L belongs to $\mathfrak{X}$.

Let H be a subalgebra of L. We denote by $C_L(H)$ (resp. $I_L(H)$) the centralizer (resp. idealizer) of H in L. For $x \in L$ we put $H^{x} = \sum_{n \geq 0} [H, {}_{n}x]$, where $[H, {}_{n}x] = [H, \underbrace{x, x, \cdots, x}_{n}]$. The Hirsch-Plotkin radical $\rho(L)$ of L is the unique maximal locally nilpotent ideal of L [1].

2. Characterizations

The class $\mathfrak{C}^{*}$ is introduced in [8] as the class of Lie algebras L satisfying (4) of the following equivalent conditions ([2, Proposition 3.2 and Theorem 3.5]):

(1) If $\langle x \rangle$ asc $H \leq L$, then $\langle x \rangle \lhd H$.

(2) If $\langle x \rangle$ si $H \leq L$, then $\langle x \rangle \lhd H$.

(3) For $x, y \in L$, if $[x, {}_{n}y, x] \in \langle x \rangle$ for any $n \geq 1$, then $\langle x \rangle \lhd \langle x, y \rangle$.

(4) If H is a nilpotent subalgebra of L and K is a subalgebra of H, then $K \lhd I_L(H)$.

(5) Every soluble subalgebra of L is either abelian or almost-abelian.

The equivalence has been shown in [8] for finite-dimensional Lie algebras and generalized in [2] for infinite-dimensional Lie algebras.

We shall introduce the classes $\mathfrak{C}_0^*$ and $\mathfrak{C}^{(*)}$: A Lie algebra L belongs to $\mathfrak{C}_0^*$ if every soluble subalgebra of L is abelian; A Lie algebra L belongs to $\mathfrak{C}^{(*)}$ if any pair of elements x and y of L such that $[x, y, y] \in \langle y \rangle$ satisfies $[x, y] \in \langle y \rangle$. A Lie algebra L belongs to (A) if any pair of elements x and y of L such that $[x, y, y] = 0$ satisfies $[x, y] = 0$ [5]. It is easy to see that the classes $\mathfrak{C}^*$, $\mathfrak{C}_0^*$, $\mathfrak{C}^{(*)}$ and (A) are s-closed and L-closed. We shall give characterizations of the classes $\mathfrak{C}_0^*$, $\mathfrak{C}^{(*)}$ and (A) which are similar to [2, Proposition 3.2, Lemma 3.3 and Theorem 3.5] and will be used in later sections. We define the following classes of Lie algebras. Let Δ be any of the relations si, asc, ser, $\lhd^{\sigma}$, wsi, wasc, wser and $\leq^{\sigma}$. Let $\mathfrak{T}_0(\Delta)$ denote the class of Lie algebras L in which every Δ-subalgebra H satisfies $[L, H] = H^2$. Let $\mathfrak{C}_0(\Delta)$ denote the class of Lie algebras L in which every 1-dimensional Δ-subalgebra H satisfies $[L, H] = 0$.

First, we shall investigate the class $\mathfrak{C}_0^*$.

LEMMA 2.1. *Let L be a Lie algebra and let K be a subalgebra of L. Then the following are equivalent:*

(1) *If K asc $H \leq L$, then $[H, K] = K^2$.*
(2) *If K si $H \leq L$, then $[H, K] = K^2$.*
(3) *If $K \lhd^2 H \leq L$, then $[H, K] = K^2$.*
(4) *For $x \in L$, if $[K, {}_n x, K] \subseteq K$ for any $n \geq 1$, then $[x, K] \subseteq K^2$.*

PROOF. $(1) \Rightarrow (2) \Rightarrow (3)$ is clear.

$(3) \Rightarrow (4)$: Let $x \in L$ such that $[K, {}_n x, K] \subseteq K$ for all $n \geq 1$. Since $K \lhd K^x \lhd \langle K, x \rangle$, we obtain $[x, K] \subseteq K^2$.

$(4) \Rightarrow (1)$: Let K asc $H \leq L$ and let $(A_\alpha)_{\alpha \leq \sigma}$ be an ascending series from K to H. We show by transfinite induction on α that $[A_\alpha, K] \subseteq K^2$. Let $\alpha > 0$ and assume that $[A_\beta, K] \subseteq K^2$ for all $\beta < \alpha$. If α is a limit ordinal, then $[A_\alpha, K] = [\bigcup_{\beta < \alpha} A_\beta, K] \subseteq K^2$. Otherwise by induction hypothesis $K \lhd A_{\alpha-1} \lhd A_\alpha$. Let $x \in A_\alpha$. Since $[K, {}_n x, K] \subseteq K$ for any $n \geq 1$, it follows that $[x, K] \subseteq K^2$. Hence we have $[A_\alpha, K] \subseteq K^2$. □

By using the concept of subideals and ascendant subalgebras we can characterize $\mathfrak{C}_0^*$-algebras.

PROPOSITION 2.2. *Let L be a Lie algebra. Then the following are equivalent:*

(1) $L \in \mathfrak{C}_0(\mathrm{asc})^s$.
(2) $L \in \mathfrak{C}_0(\mathrm{si})^s$.
(3) $L \in \mathfrak{C}_0(\lhd^2)^s$.
(4) *For $x, y \in L$, if $[x, {}_n y, x] \in \langle x \rangle$ for any $n \geq 1$ then $[y, x] = 0$.*
(5) *If H is a nilpotent subalgebra of L, then $I_L(H) = C_L(H)$.*

(6) $L \in \mathfrak{C}_0^*$.

PROOF. The equivalence of (1)–(4) can be proved by Lemma 2.1.

$(2) \Rightarrow (5)$: Let H be a nilpotent subalgebra of L and let x be any element of H. Since $\langle x \rangle$ is a subideal of $I_L(H)$, we have $[I_L(H), x] = 0$. Hence $I_L(H) = C_L(H)$.

$(5) \Rightarrow (6)$: Let H be a soluble subalgebra of L and denote N by the Hirsch-Plotkin radical $\rho(H)$ of H. For any $x, y \in N$, $\langle x, y \rangle$ is nilpotent since N is locally nilpotent. Since $I_L(\langle x, y \rangle) = C_L(\langle x, y \rangle)$, $\langle x, y \rangle$ is abelian. It follows that N is abelian and $H = I_H(N) = C_H(N)$. Furthermore by [1, Lemma 9.1.2(c)] we have $C_H(N) \leq N$. Therefore H is abelian.

$(6) \Rightarrow (4)$: Suppose that $[x, {}_n y, x] \in \langle x \rangle$ for any $n \geq 1$. We put $M_n = \sum_{i=0}^{n} \langle [x, {}_i y] \rangle$ for any $n \geq 0$ and $M = \bigcup_{n=0}^{\infty} M_n$. Then $M_n \lhd M \leq L$ for all $n \geq 0$. Since $M_n = M_{n-1} + \langle [x, {}_n y] \rangle$, we obtain $M_n^{(1)} \leq M_{n-1}$. Therefore $M_n^{(n+1)} = 0$. We conclude that M_n is abelian for all $n \geq 0$ and so M is abelian. Now we set $K = M + \langle y \rangle$. Then K is soluble and therefore abelian. Hence we have $[y, x] = 0$. □

Second, we shall investigate $\mathfrak{C}^{(*)}$-algebras.

LEMMA 2.3. *Let L be a Lie algebra and let K be a subalgebra of L. Then the following are equivalent*:

(1) *If* K wasc $H \leq L$, *then* $K \lhd H$.

(2) *If* $K \leq^{\omega} H \leq L$, *then* $K \lhd H$.

(3) *For* $x \in L$, *if* $[x, K, K] \subseteq K$, *then* $[x, K] \subseteq K$.

PROOF. $(1) \Rightarrow (2)$ is clear.

$(2) \Rightarrow (3)$: Let x be an element of L such that $[x, K, K] \subseteq K$. Put $H = \{y \in L : [y, {}_n K] \subseteq K$ for some integer $n \geq 1\}$. By [7, Lemma 1] we have $K \leq^{\omega} H \leq L$. Hence $[x, K] \subseteq K$ since $x \in H$.

$(3) \Rightarrow (1)$: Let K wasc $H \leq L$ and let $(A_\alpha)_{\alpha \leq \sigma}$ be a weakly ascending series from K to H. We show by transfinite induction on α that $[A_\alpha, K] \subseteq K$. Let $\alpha > 0$ and assume that $[A_\beta, K] \subseteq K$ for all $\beta < \alpha$. If α is a limit ordinal, then $[A_\alpha, K] = [\bigcup_{\beta < \alpha} A_\beta, K] \subseteq K$. Otherwise by induction hypothesis $[A_{\alpha-1}, K] \subseteq K$. Let $x \in A_\alpha$. Since $[x, K, K] \subseteq K$, it follows that $[x, K] \subseteq K$. Hence we have $[A_\alpha, K] \subseteq K$. □

The following result can be proved by using Lemma 2.3.

PROPOSITION 2.4. *Let L be a Lie algebra. Then the following are equivalent*:

(1) $L \in \mathfrak{C}(\text{wasc})^s$.

(2) $L \in \mathfrak{C}(\leq^{\omega})^s$.

(3) $L \in \mathfrak{C}^{(*)}$.

Third, we consider (A)-algebras. The following results can be proved as in Lemma 2.3 and Proposition 2.4.

LEMMA 2.5. *Let L be a Lie algebra and let K be a subalgebra of L. Then the following are equivalent*:

(1) *If* K wasc $H \leq L$, *then* $[H, K] = K^2$.
(2) *If* $K \leq^{\omega} H \leq L$, *then* $[H, K] = K^2$.
(3) *For* $x \in L$, *if* $[x, K, K] \subseteq K$, *then* $[x, K] \subseteq K^2$.

PROPOSITION 2.6. *Let L be a Lie algebra. Then the following are equivalent*:

(1) $L \in \mathfrak{C}_0(\text{wasc})^s$.
(2) $L \in \mathfrak{C}_0(\leq^{\omega})^s$.
(3) $L \in (A)$.

A Lie algebra L is said to be almost-abelian if L is the split extension of an abelian algebra by the 1-dimensional algebra of scalar multiplications. We denote by $\mathfrak{A}_0$ the class of abelian or almost-abelian Lie algebras. It follows from Propositions 2.2, 2.4, 2.6 and [3, Lemma 2.1] that

$$\begin{array}{ccccc} \mathfrak{A}_0 & \leq & \mathfrak{C}^{(*)} & \leq & \mathfrak{C}^* \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} \\ \mathfrak{A} & \leq & (\mathrm{A}) & \leq & \mathfrak{C}_0^* . \end{array}$$

It is easy to see that $\mathfrak{C}_0^* \cap \mathfrak{C}^{(*)} = (\mathrm{A})$.

Almost-abelian Lie algebras belong to $\mathfrak{C}^{(*)} \backslash \mathfrak{C}_0^*$. A 3-dimensional simple Lie algebra L over a field $\mathfrak{k}$ is called split if L contains an element h such that ad h has a non-zero characteristic root in $\mathfrak{k}$ ([4, p. 14]). If char $\mathfrak{k} \neq 2$, then a 3-dimensional simple Lie algebra L is split if and only if L has a basis $\{e, f, h\}$ such that $[h, e] = e$, $[h, f] = -f$, $[e, f] = h$. Split 3-dimensional simple Lie algebras belong to $\mathfrak{C}^* \backslash (\mathfrak{C}_0^* \cup \mathfrak{C}^{(*)})$. Hence we have

$$(\mathrm{A}) < \mathfrak{C}^{(*)}, \quad \mathfrak{C}_0^* \cup \mathfrak{C}^{(*)} < \mathfrak{C}^*.$$

By [3, Lemma 4.1] a 3-dimensional simple Lie algebra is either a split 3-dimensional simple Lie algebra or an (A)-algebra.

3. Locally finite Lie algebras

We consider locally finite Lie algebras. By [2, Theorem 3.9], $\mathrm{L}\mathfrak{F} \cap \mathfrak{C}^* = \mathrm{L}\mathfrak{F} \cap \mathfrak{T}^s$. We shall show some results which correspond to this. It is necessary to show some obvious equalities.

PROPOSITION 3.1. (1) $\mathrm{L}\mathfrak{F}\cap\mathfrak{T}_0(\mathrm{ser})^s=\mathrm{L}\mathfrak{F}\cap\mathfrak{T}_0(\mathrm{si})^s$.
(2) $\mathrm{L}\mathfrak{F}\cap\mathfrak{C}_0(\mathrm{ser})^s=\mathrm{L}\mathfrak{F}\cap\mathfrak{C}_0(\mathrm{si})^s$.
(3) $\mathrm{L}\mathfrak{F}\cap\mathfrak{T}(\mathrm{wser})^s=\mathrm{L}\mathfrak{F}\cap\mathfrak{T}(\mathrm{wsi})^s$.
(4) $\mathrm{L}\mathfrak{F}\cap\mathfrak{C}(\mathrm{wser})^s=\mathrm{L}\mathfrak{F}\cap\mathfrak{C}(\mathrm{wsi})^s$.
(5) $\mathrm{L}\mathfrak{F}\cap\mathfrak{T}_0(\mathrm{wser})^s=\mathrm{L}\mathfrak{F}\cap\mathfrak{T}_0(\mathrm{wsi})^s$.
(6) $\mathrm{L}\mathfrak{F}\cap\mathfrak{C}_0(\mathrm{wser})^s=\mathrm{L}\mathfrak{F}\cap\mathfrak{C}_0(\mathrm{wsi})^s$.

PROOF. We only show (5) because the others can be proved similarly. Let $L\in\mathrm{L}\mathfrak{F}\cap\mathfrak{T}_0(\mathrm{wsi})^s$ and let K wser $H\leq L$. We can find a finite-dimensional subalgebra F of L which contains x and y for any $x\in K$ and $y\in H$. Then $F\cap K$ wsi $F\cap H$. Hence $[H,K]=K^2$. This shows that $L\in\mathfrak{T}_0(\mathrm{wser})^s$. The converse is clear. □

REMARK. Almost-abelian Lie algebras belong to the classes of (3), (4) but none of (1), (2), (5), (6) of Proposition 3.1.

We consider locally finite $\mathfrak{C}_0^*$-algebras.

LEMMA 3.2. *Let L be a locally finite $\mathfrak{C}_0^*$-algebra and let N be an ideal of L. Then*:
(1) $[L,N]=N^\omega$.
(2) *L/N is a $\mathfrak{C}_0^*$-algebra.*

PROOF. (1) We first assume that L is finite-dimensional and $L=N+\langle x\rangle$. Let h be an element of N. Let H be a maximal soluble subalgebra of L containing h. Then H is abelian. We can consider the Fitting decomposition of L relative to ad H, say $L=L_0\dotplus L_1$. It turns out that $L=H+N$, since H is a Cartan subalgebra of L and $L_1\subseteq L^2\subseteq N$. Hence $[L,h]\subseteq N^2$ and $[L,N]\subseteq N^2$ since h can be taken as an arbitrary element of N. It follows that

$$L^2\subseteq N^2+[x,N]\subseteq N^2.$$

By induction we have

$$L^{n+1}\subseteq[L,N^n]\subseteq N^{n+1}$$

for any $n\geq 1$. Consequently $L=H+N^\omega$ and $[L,h]\subseteq N^\omega$. Hence we have $[L,N]=N^\omega$.

Now we go back to the general case. Let $y\in L$ and $z\in N$. Since $[(N\cap\langle y,z\rangle)+\langle y\rangle, N\cap\langle y,z\rangle]=(N\cap\langle y,z\rangle)^\omega$, we have $[y,z]\in N^\omega$. Hence we have $[L,N]=N^\omega$.

(2) Let H/N be a nilpotent subalgebra of L/N and let $x\in I_L(H)$. Then we have $[x,H]\subseteq H^\omega\subseteq N$ by (1). Therefore $I_{L/N}(H/N)\subseteq C_{L/N}(H/N)$ and L/N satisfies the condition (5) of Proposition 2.2. Hence L/N is a $\mathfrak{C}_0^*$-algebra. □

The following result corresponds to [2, Theorem 3.9].

THEOREM 3.3. $L\mathfrak{F}\cap\mathfrak{C}_0^* = L\mathfrak{F}\cap\mathfrak{T}_0(\mathrm{si})^s$.

PROOF. By Proposition 2.2, it suffices to prove $L\mathfrak{F}\cap\mathfrak{C}_0^* \leq L\mathfrak{F}\cap\mathfrak{T}_0(\mathrm{si})^s$. Suppose that $L\in L\mathfrak{F}\cap\mathfrak{C}_0^*$ and K si $H \leq L$. Then $K^\omega \lhd H$ by [1, Lemma 1.3.2] and K/K^ω si H/K^ω. By using Lemma 3.2 we see that H/K^ω is a $\mathfrak{C}_0^*$-algebra. For any elements x and y of K, $(\langle x, y\rangle + K^\omega)/K^\omega$ is nilpotent and so abelian. Therefore K/K^ω is abelian and by Proposition 2.2

$$[H/K^\omega, (\langle x\rangle + K^\omega)/K^\omega] = 0$$

for any $x\in K$. Hence $[H, K] = K^\omega$ and $L\in\mathfrak{T}_0(\mathrm{si})^s$. □

For $\mathfrak{C}^{(*)}$-algebras we have the following

THEOREM 3.4. $L\mathfrak{F}\cap\mathfrak{C}^{(*)} = L\mathfrak{F}\cap\mathfrak{T}(\mathrm{wsi})^s$.

PROOF. It suffices to prove that $L\mathfrak{F}\cap\mathfrak{C}^{(*)} \leq L\mathfrak{F}\cap\mathfrak{T}(\mathrm{wasc})^s$. Let $L\in L\mathfrak{F}\cap\mathfrak{C}^{(*)}$, and let K be a subalgebra of L. Suppose that x is an element of L such that $[x, K, K]\subseteq K$. Let y be an element of K. Then $[x, y, y]\in K$. Since $\langle x, y\rangle$ is finite-dimensional, there exist $\alpha_1, \alpha_2,\dots,\alpha_n\in\mathfrak{k}$ such that

$$\alpha_1[x, {}_2y] + \alpha_2[x, {}_3y] + \cdots + \alpha_n[x, {}_{n+1}y] = 0$$

and at least one $\alpha_i \neq 0$. Since $L\in\mathfrak{C}^{(*)}$,

$$\alpha_1[x, y] + \alpha_2[x, {}_2y] + \cdots + \alpha_n[x, {}_ny]\in\langle y\rangle$$

and we may assume that α_1 is not zero. Hence $[x, y]\in K$ and $L\in\mathfrak{T}(\mathrm{wasc})^s$ by Lemma 2.3. □

COROLLARY 3.5. *Let L be a locally finite $\mathfrak{C}^{(*)}$-algebra and let N be an ideal of L. Then L/N is a $\mathfrak{C}^{(*)}$-algebra.*

PROOF. Let K/N wsi $H/N \leq L/N$. Then K wsi $H \leq L$. It follows from Theorem 3.4 that $K\lhd H$. Hence $K/N\lhd H/N$ and L/N is a $\mathfrak{C}^{(*)}$-algebra by Theorem 3.4. □

We also consider (A)-algebras. The following results can be proved as in Theorem 3.4 and Corollary 3.5.

THEOREM 3.6. $L\mathfrak{F}\cap(\mathrm{A}) = L\mathfrak{F}\cap\mathfrak{T}_0(\mathrm{wsi})^s$.

COROLLARY 3.7. *Let L be a locally finite* (A)*-algebra and let N be an ideal of L. Then L/N is an* (A)*-algebra.*

REMARK. In Theorems 3.3, 3.4 and 3.6 the "local finiteness" is necessary and the classes $\mathfrak{C}_0^*$, $\mathfrak{C}^{(*)}$ and (A) are not Q-closed in general (Example 6.1).

4. Structure theorems

In this section we shall investigate locally finite $\mathfrak{C}^*$-algebras over a field of characteristic zero. The structure of $\mathfrak{C}$-algebras and $\mathfrak{C}$(wsi)-algebras are shown in [2] and [3]. By using the properties of $\mathfrak{C}$-algebras shown in [2] and the concept of (A)-algebras, we determine the structure of locally finite $\mathfrak{C}^*$-algebras and L(wser) $\mathfrak{F} \cap \mathfrak{C}^*$-algebras, which are main results of this paper.

We first show properties of $\mathfrak{C}_0^*$-algebras and (A)-algebras.

LEMMA 4.1. *Let L be a Lie algebra.*

(1) *If $L = \prod_{\lambda \in \Lambda} L_\lambda$ and each L_λ is a $\mathfrak{C}_0^*$-algebra, then L is a $\mathfrak{C}_0^*$-algebra.*

(2) *If $L = H \oplus K$ is a $\mathfrak{C}^*$-algebra, $H \neq 0$ and $K \neq 0$, then L is a $\mathfrak{C}_0^*$-algebra.*

(3) *If $L = \prod_{\lambda \in \Lambda} L_\lambda$ and each L_λ is an* (A)*-algebra, then L is an* (A)*-algebra.*

(4) *If $L = H \oplus K$ is a $\mathfrak{C}^{(*)}$-algebra, $H \neq 0$ and $K \neq 0$, then L is an* (A)*-algebra.*

PROOF. (1) Let x and y be elements of L such that $[x, {}_ny, x] \in \langle x \rangle$ for any integer $n \geq 1$. Put $x = (x_\lambda)_{\lambda \in \Lambda}$ and $y = (y_\lambda)_{\lambda \in \Lambda}$. Then for any $\lambda \in \Lambda$, $[x_\lambda, {}_ny_\lambda, x_\lambda] \in \langle x_\lambda \rangle$ for any integer $n \geq 1$. Since L_λ is a $\mathfrak{C}_0^*$-algebra, $[x_\lambda, y_\lambda] = 0$ and $[x, y] = 0$. Therefore $L = \mathfrak{C}_0^*$ by Proposition 2.2.

(2) Let M be a non-zero soluble subalgebra of H. Let x be a non-zero element of K. Since L is a $\mathfrak{C}^*$-algebra, $N = M + \langle x \rangle$ is either abelian or almost-abelian. If N is almost-abelian, then $\dim N/N^2 = 1$, which is a contradiction since $N^2 = M^2 < M$. Therefore M must be abelian and so H is a $\mathfrak{C}_0^*$-algebra. We can show similarly that K is a $\mathfrak{C}_0^*$-algebra. Hence L is a $\mathfrak{C}_0^*$-algebra by (1).

(3) Clear by definition of an (A)-algebra.

(4) Let x and y be elements of H such that $[x, y, y] = 0$. We have $[x, y] \in \langle y \rangle$ since L is a $\mathfrak{C}^{(*)}$-algebra. Let z be a non-zero element of K. Then $[x, y + z, y + z] = 0$ and $[x, y + z] \in \langle y + z \rangle$ since L is a $\mathfrak{C}^{(*)}$-algebra. Therefore $[x, y] \in \langle y \rangle \cap \langle y + z \rangle = 0$. Hence H is an (A)-algebra and L is an (A)-algebra. □

We shall show a characterization of the class $\mathfrak{C}^*$ for locally finite Lie algebras.

THEOREM 4.2. *Let L be a Lie algebra over a field of characteristic zero. Then L is a locally finite $\mathfrak{C}^*$-algebra if and only if one of the following holds:*

(1) *L is a locally finite* (A)*-algebra.*

(2) *L is almost-abelian.*

(3) *L is a* 3*-dimensional split simple Lie algebra.*

PROOF. If (1), (2) or (3) holds, then clearly L is a locally finite $\mathfrak{C}^*$-algebra. To show the converse, first we shall show the finite-dimensional case by induction on dim L, and then we shall show the locally finite case.

(a) Let L be a finite-dimensional $\mathfrak{C}^*$-algebra and assume that every proper subalgebra of L satisfies one of (1)–(3). By [2, Theorem 2.3] we have $L = R \oplus S$, where R is an abelian or almost-abelian ideal of L and S is a semisimple ideal of L. If $S = 0$, then L satisfies (1) or (2). Assume that $S \neq 0$. If $R \neq 0$ or S is not simple, then R and S belong to $\mathfrak{C}_0^*$ by Lemma 4.1 (2). Therefore R is abelian and S is an (A)-algebra. Hence L satisfies (1) by Lemma 4.1 (3). Assume that L is simple and that L does not belong to (A). Then there are x and y in L such that $[x, y, y] = 0$ and $[x, y] \neq 0$. Put $z = [x, y]$. Then ad z is nilpotent by [4, Lemma 4 in Chapter 2], and there are non-zero elements h and e in L such that $[h, e] = e$ by [4, Theorem 17 in Chapter 3]. Assume that $I_L(\langle h \rangle) \neq \langle h \rangle$ and take $c \in I_L(\langle h \rangle) \setminus \langle h \rangle$. For $\alpha \in \mathfrak{k}$, put $L_\alpha = \{v \in L: v(\text{ad } h - \alpha \cdot 1)^n = 0$ for some $n\}$ and set $H = L_1 + L_2 + L_3 + \cdots$. Since $\langle c \rangle + H$ is soluble, it is either abelian or almost-abelian. If $\langle c \rangle + H$ is abelian, then $[c, e] = 0$. If $\langle c \rangle + H$ is almost-abelian, then $[c, e] \in \langle e \rangle$ since $(\langle c \rangle + H)^2 = H$. Therefore $\langle c \rangle + \langle h \rangle + \langle e \rangle$ is soluble and so almost-abelian. Consequently $(\langle c \rangle + \langle h \rangle + \langle e \rangle)^2 = \langle h \rangle + \langle e \rangle$ is abelian, which is a contradiction. Hence $\langle h \rangle$ is a Cartan subalgebra of L and L is 3-dimensional. It follows that L satisfies (3).

(b) Let L be a locally finite $\mathfrak{C}^*$-algebra. First assume that L includes a subalgebra S of type (3). By (a), $\langle S, x \rangle$ is of type (3) for any $x \in L$ and therefore $L = S$ is of type (3). Assume that L includes no subalgebras of type (3), and assume that L includes a subalgebra of type (2). Then there are non-zero elements $u, v \in L$ such that $[u, v] = v$. For any elements x and y of L, $\langle u, v, x, y \rangle$ is of type (2). Hence $\langle x, y \rangle \in \mathfrak{A}_0$. By [3, Lemma 2.1 (1)], $L \in \mathfrak{A}_0$ and L is of type (2). Finally assume that L does not include subalgebras of type (2) or (3). Then for any elements x and y of L, $\langle x, y \rangle$ is of type (1). Therefore L is of type (1). □

REMARK. In Theorem 4.2 we cannot remove the condition that L is locally finite (Example 6.1). Also we cannot remove the condition that the field is of characteristic zero (Example 6.2).

By Theorem 4.2 we have characterizations of $\text{L}\mathfrak{F} \cap \mathfrak{C}_0^*$ and $\text{L}\mathfrak{F} \cap \mathfrak{C}^{(*)}$.

COROLLARY 4.3. *Let L be a Lie algebra over a field of characteristic zero. Then L is a locally finite $\mathfrak{C}^{(*)}$-algebra if and only if either L is a locally finite* (A)*-algebra or L is almost-abelian.*

PROOF. Lie algebras satisfying (3) of Theorem 4.2 are not $\mathfrak{C}^{(*)}$-

algebras. Thus a locally finite $\mathfrak{C}^{(*)}$-algebra must satisfy (1) or (2). □

COROLLARY 4.4. *Let L be a locally finite Lie algebra over a field of characteristic zero. Then L is a $\mathfrak{C}_0^*$-algebra if and only if L is an* (A)-*algebra.*

PROOF. Lie algebras satisfying (2) or (3) of Theorem 4.2 are not $\mathfrak{C}_0^*$-algebras. Thus a locally finite $\mathfrak{C}_0^*$-algebra must be an (A)-algebra. □

By Theorem 4.2 and [3, Corollary 3.4] we have a structure theorem of $\mathfrak{C}^*$-algebras. We call a Lie algebra L reductive if $L = R \oplus (\bigoplus_{\lambda\in\Lambda} S_\lambda)$, where R is an abelian ideal of L and each S_λ is a finite-dimensional simple ideal of L.

THEOREM 4.5. *Let L be a Lie algebra over a field of characteristic zero. Then L belongs to $\mathrm{L}(\mathrm{wser})\mathfrak{F} \cap \mathfrak{C}^*$ if and only if one of the following holds*:

(1) *L is a reductive* (A)-*algebra.*

(2) *L is a finite-dimensional almost-abelian Lie algebra.*

(3) *L is a 3-dimensional split simple Lie algebra.*

PROOF. By Theorem 4.2, L satisfies one of (1)–(3) of Theorem 4.2. If L satisfies (3) of Theorem 4.2, then L satisfies (3). We shall show that a Lie algebra L which satisfies (1) and (2) of Theorem 4.2 satisfies (1) and (2) respectively.

(1) By Proposition 3.1 and Theorem 3.6 we have $L \in \mathrm{L}(\lhd)\mathfrak{F}$. It follows from [3, Corollary 3.4] that $L = R \oplus S$, where R is an abelian ideal of L and S is a semisimple ideal of L. By [1, Theorem 13.4.2] we have $S = \bigoplus_{\lambda\in\Lambda} S_\lambda$, where each S_λ is a finite-dimensional simple ideal of S. Hence (1) holds.

(2) By [3, Lemma 2.1 (2)] we have $L \in \mathrm{L}(\lhd)\mathfrak{F}$. Let $x \in L \backslash L^2$. Then there is a finite-dimensional ideal H of L containing x. We have $L = H$ and therefore L is finite-dimensional.

COROLLARY 4.6. *Let L be a Lie algebra over a field of characteristic zero. Then L belongs to $\mathrm{L}(\mathrm{wser})\mathfrak{F} \cap \mathfrak{C}^{(*)}$ if and only if either L is a reductive* (A)-*algebra or L is a finite-dimensional almost-abelian Lie algebra.*

PROOF. Lie algebras satisfying (3) of Theorem 4.5 are not $\mathfrak{C}^{(*)}$-algebras. Hence the assertion holds. □

Let L be a Lie algebra over a field $\mathfrak{k}$. An element x of L is ad-semisimple if there is a basis $\{e_\lambda\}_{\lambda\in\Lambda}$ for $L \otimes_{\mathfrak{k}} \bar{\mathfrak{k}}$ and if there are elements $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ of $\bar{\mathfrak{k}}$ such that $[e_\lambda, x] = \alpha_\lambda e_\lambda$ for any $\lambda \in \Lambda$. We call L ad-semisimple if x is ad-semisimple for any $x \in L$.

COROLLARY 4.7. *Let L be a $\mathrm{L}(\mathrm{wser})\mathfrak{F}$-algebra over a field of characteristic zero. Then the following are equivalent*:

(1) $L \in \mathfrak{C}_0^*$.

(2) $L \in (\mathrm{A})$.
(3) *L is a reductive* (A)*-algebra.*
(4) *Every subalgebra of L is reductive.*
(5) *L is ad-semisimple.*

PROOF. Implications (5) ⇒ (2) ⇒ (1), (3) ⇒ (2) and (4) ⇒ (1) are trivial. Assume (1). Then L satisfies (1) of Theorem 4.5 by Theorem 4.5. Therefore L satisfies (3) and (4). By [6, Theorem 1] each direct summand S_λ is ad-semisimple. Hence L satisfies (5). □

We can generalize [6, Theorems 1 and 2] in the following

COROLLARY 4.8. *Let L be a Lie algebra over a field of characteristic zero. If* $L \in \mathrm{L}(\mathrm{wsi})\mathfrak{F} \cap \mathrm{L}(\mathrm{ser})\mathfrak{F}$ (*resp.* $L \in \mathrm{L}(\mathrm{ser})\mathfrak{F}$), *then the conditions* (1)–(5) *of Corollary* 4.7 *and the condition* $L \in \mathfrak{C}_0(\mathrm{wsi})$ (*resp.* $L \in \mathfrak{C}_0(\mathrm{wasc})$) *are equivalent.*

PROOF. The assertion follows from [3, Corollary 3.9] (resp. [3, Corollary 3.4]). □

REMARK. Since 3-dimensional split simple Lie algebras belong to $\mathfrak{C}_0(\mathrm{ser}) \setminus \mathfrak{C}_0^*$, the above conditions are not equivalent to "$L \in \mathfrak{C}_0(\mathrm{si})$" or "$L \in \mathfrak{C}_0(\mathrm{asc})$" even if L is finite-dimensional.

5. Conditions to be abelian or almost-abelian

The structure of generalized soluble $\mathfrak{C}^*$-algebras over any field and locally finite $\mathfrak{C}^*$-algebras over an algebraically closed field are investigated in [2]. In this section we shall generalize them and apply to the classes $\mathfrak{C}_0^*$, $\mathfrak{C}^{(*)}$ and (A).

First we shall generalize [2, Proposition 3.11]. Let $\mathfrak{X}$ be a class of Lie algebras. We define the class $\{\mathrm{L}, \acute{\mathrm{E}}\}\mathfrak{X}$ to be the smallest L-closed and É-closed class containing $\mathfrak{X}$. For any ordinal α, we inductively define the class $(\mathrm{L}\acute{\mathrm{E}})^\alpha\mathfrak{X}$ as follows: $(\mathrm{L}\acute{\mathrm{E}})^0\mathfrak{X} = \mathfrak{X}$, $(\mathrm{L}\acute{\mathrm{E}})^{\alpha+1}\mathfrak{X} = \mathrm{L}\acute{\mathrm{E}}((\mathrm{L}\acute{\mathrm{E}})^\alpha\mathfrak{X})$ for an ordinal α, $(\mathrm{L}\acute{\mathrm{E}})^\lambda\mathfrak{X} = \bigcup_{\alpha<\lambda}(\mathrm{L}\acute{\mathrm{E}})^\alpha\mathfrak{X}$ for each limit ordinal λ. We denote by $(\mathrm{L}\acute{\mathrm{E}})^*\mathfrak{X}$ the class of Lie algebras L such that $L \in (\mathrm{L}\acute{\mathrm{E}})^\alpha\mathfrak{X}$ for some ordinal α. It is easy to verify that $(\mathrm{L}\acute{\mathrm{E}})^*\mathfrak{X}$ is L-closed and É-closed. Hence a Lie algebra L belongs to $\{\mathrm{L}, \acute{\mathrm{E}}\}\mathfrak{X}$ if and only if L belongs to $(\mathrm{L}\acute{\mathrm{E}})^\alpha\mathfrak{X}$ for some ordinal α.

PROPOSITION 5.1. *Let* $\mathfrak{X}$ *be a class of Lie algebras. If* $\mathfrak{C}^* \cap \mathfrak{X} = \mathfrak{A}_0$, *then* $\mathfrak{C}^* \cap \{\mathrm{L}, \acute{\mathrm{E}}\}\mathfrak{X} = \mathfrak{A}_0$.

PROOF. Assume that $\mathfrak{C}^* \cap \mathfrak{X} = \mathfrak{A}_0$. By the above remark it suffices to show that $\mathfrak{C}^* \cap \mathrm{L}\acute{\mathrm{E}}\mathfrak{X} = \mathfrak{A}_0$. Let $L \in \mathfrak{C}^* \cap \mathrm{L}\acute{\mathrm{E}}\mathfrak{X}$, and let x be any element of $L^{(2)}$. Then there is an $\acute{\mathrm{E}}\mathfrak{X}$-subalgebra H of L such that $x \in H^{(2)}$. Let $(H_\alpha)_{\alpha \leq \sigma}$ be an ascending $\mathfrak{X}$-series of H. We shall show by transfinite induction on α

that $H_\alpha \in \mathfrak{A}_0$ for any ordinal $\alpha \leq \sigma$. Assume that $H_\beta \in \mathfrak{A}_0$ for all $\beta < \alpha$. If α is a limit ordinal, then $H_\alpha = \bigcup_{\beta<\alpha} H_\beta$ is soluble. Therefore H_α is either abelian or almost-abelian. Otherwise, if $M/H_{\alpha-1}$ is a soluble subalgebra of $H_\alpha/H_{\alpha-1}$, then M is soluble. Hence we see that $H_\alpha/H_{\alpha-1}$ is a $\mathfrak{C}^*$-algebra. Therefore $H_\alpha/H_{\alpha-1} \in \mathfrak{C}^* \cap \mathfrak{X} = \mathfrak{A}_0$. It follows that H_α is soluble and therefore H_α is either abelian or almost-abelian. Hence we have $x \in H^{(2)} = 0$. Therefore $L^{(2)} = 0$. We can conclude that L is either abelian or almost-abelian by a characterization of $\mathfrak{C}^*$. □

Let $\mathfrak{X} = \mathrm{E}\mathfrak{A}$ in Proposition 5.1. Then we obtain the following result.

COROLLARY 5.2. (1) $\mathfrak{C}^* \cap \{\mathrm{L}, \acute{\mathrm{E}}\}\mathfrak{A} = \mathfrak{C}^{(*)} \cap \{\mathrm{L}, \acute{\mathrm{E}}\}\mathfrak{A} = \mathfrak{A}_0$.
(2) $\mathfrak{C}_0^* \cap \{\mathrm{L}, \acute{\mathrm{E}}\}\mathfrak{A} = (\mathrm{A}) \cap \{\mathrm{L}, \acute{\mathrm{E}}\}\mathfrak{A} = \mathfrak{A}$.

Next we shall show the structure of $\mathfrak{C}_0^*$, $\mathfrak{C}^{(*)}$ and (A)-algebras in a locally finite case over an algebraically closed field.

PROPOSITION 5.3. *Over an algebraically closed field*
(1) $\mathrm{L}\mathfrak{F} \cap \mathfrak{C}_0^* = \mathfrak{A}$.
(2) $\mathrm{L}\mathfrak{F} \cap \mathfrak{C}^{(*)} = \mathfrak{A}_0$.
(3) $\mathrm{L}\mathfrak{F} \cap (\mathrm{A}) = \mathfrak{A}$.

PROOF. By [2, Proposition 3.10] over an algebraically closed field locally finite $\mathfrak{C}^*$-algebras are abelian, almost-abelian or 3-dimensional split simple, but a 3-dimensional split simple Lie algebra does not belong to $\mathfrak{C}_0^* \cup \mathfrak{C}^{(*)}$. Therefore the assertion is clear. □

REMARK. An ad-semisimple Lie algebra over an algebraically closed field is always abelian. If char $\mathfrak{k} = 0$, then there is a non-abelian (A)-algebra over $\mathfrak{k}$ (Example 6.1).

6. Examples

In this section we shall give examples.

EXAMPLE 6.1. Let W_0 be a Witt algebra, that is, a Lie algebra over a field of characteristic zero with basis $\{w_0, w_1, w_2, \dots\}$ and multiplication $[w_i, w_j] = (i-j)w_{i+j}$. Then $W_0 \notin \mathfrak{C}_0^* \cup \mathfrak{A}_0$. Let W be the subalgebra of W_0 generated by $w_1, w_2, \dots$. For a non-zero element $x = \sum_{i=0}^{\infty} \alpha_i w_i$ of W_0, put $\max(x) = \max\{n: \alpha_n \neq 0\}$. Let $x, y \in W_0$ such that $[x, y, y] \in \langle y \rangle$ and $[x, y] \notin \langle y \rangle$. Put $m = \max([x, y])$ and $n = \max(y)$. Since $W_0^2 = W$, we have $m \neq 0$. Therefore we have $m = n$. Let $[x, y] = \sum_{i=0}^{\infty} \alpha_i w_i$ and $y = \sum_{i=0}^{\infty} \beta_i w_i$. Put $z = \beta_m [x, y] - \alpha_m y$. Then we have $[z, y] \in \langle y \rangle$. We have

$\max(z) = 0$ since $\max(z) < m$. Consequently $z \in \langle w_0 \rangle$ and $y \in \langle w_m \rangle$. We have $[x, y] \in (\langle w_0 \rangle + \langle w_m \rangle) \cap W = \langle y \rangle$. Hence W_0 is a $\mathfrak{C}^{(*)}$-algebra. By [3, Example 4] $W \in (\mathrm{A})$. We easily see that $W/\langle w_4, w_5, \dots \rangle \notin \mathfrak{C}$ and therefore $W \notin \mathfrak{T}$. Hence over a field of characteristic zero the classes $\mathfrak{C}_0^*$, $\mathfrak{C}^{(*)}$ and (A) are not Q-closed and

$$(\mathrm{A}) \not\leq \mathfrak{T}, \ (\mathrm{A}) \cup \mathfrak{A}_0 < \mathfrak{C}^{(*)}.$$

EXAMPLE 6.2. Let $\mathfrak{k}$ be a field of characteristic 2 and let $\mathfrak{k}_1$ be the field of rational functions $\mathfrak{k}(\lambda, \mu)$. Let L be a Lie algebra over $\mathfrak{k}_1$ with basis $\{w, x, y, z\}$ and multiplication $[x, y] = \lambda z$, $[y, z] = \mu x$, $[z, x] = y$, $[w, x] = 0$, $[w, y] = y$, $[w, z] = z$. Clearly L does not belong to $\mathfrak{C}_0^* \cup \mathfrak{A}_0$. Let $H = \langle x, y, z \rangle$ and let $u, v \in L$ such that $[u, v, v] \in \langle v \rangle$. Then $[u, v] \in H$. We shall show that $[u, v] \in \langle v \rangle$. Put $v = \alpha w + \beta x + \gamma y + \delta z$ and $L_0 = \{t \in L\colon [t, {}_n v] = 0$ for some integer $n\}$. Then the characteristic polynomial of ad v is

$$X^4 + (\alpha^2 + \beta^2\lambda + \gamma^2\lambda\mu + \delta^2\mu)X^2 + \alpha(\gamma^2\lambda + \delta^2)\mu X.$$

If $\alpha(\gamma^2\lambda + \delta^2)\mu \neq 0$, then $\dim L_0 = 1$. Hence $u \in \langle v \rangle$. We consider the case $\alpha(\gamma^2\lambda + \delta^2)\mu = 0$. If $\alpha = 0$, then $v \in H$ and the characteristic polynomial of ad $v|_H$ is

$$X^3 + (\beta^2\lambda + \gamma^2\lambda\mu + \delta^2\mu)X.$$

If $\beta^2\lambda + \gamma^2\lambda\mu + \delta^2\mu \neq 0$, then $\dim(L_0 \cap H) = 1$ and $[u, v] \in \langle v \rangle$. Otherwise, put $\beta = \beta_1/\beta_2$, $\gamma = \gamma_1/\gamma_2$ and $\delta = \delta_1/\delta_2$, where $\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$ are polynomials of λ and μ in $\mathfrak{k}$. Then we have

$$\beta_1^2\gamma_2^2\delta_2^2\lambda + \beta_2^2\gamma_1^2\delta_2^2\lambda\mu + \beta_2^2\gamma_2^2\delta_1^2\mu = 0.$$

Since $\beta_1^2\gamma_2^2\delta_2^2$, $\beta_2^2\gamma_1^2\delta_2^2$ and $\beta_2^2\gamma_2^2\delta_1^2$ are polymonials of λ^2 and μ^2, we have $\beta = \gamma = \delta = 0$. Hence $v = 0$. Finally we consider the case $\gamma^2\lambda + \delta^2 = 0$. Then $\gamma = \delta = 0$. Therefore $v \in \langle w, x \rangle$. The characteristic polynomial of ad v is

$$X^4 + (\alpha^2 + \beta^2\lambda)X^2.$$

If $\alpha^2 + \beta^2\lambda \neq 0$, then $\dim L_0 = 2$. Hence $u \in \langle w, x \rangle$ and $[u, v] = 0$. If $\alpha^2 + \beta^2\lambda = 0$, then $v = 0$. Hence L is a $\mathfrak{C}^{(*)}$-algebra. Over $\mathfrak{k}_1$ we have

$$(\mathrm{A}) \cup \mathfrak{A}_0 < \mathfrak{C}^{(*)}.$$

EXAMPLE 6.3. Let $\mathfrak{k}$ be a subfield of the field of real numbers or a field like $\mathfrak{k}_1$ in Example 6.2. Then there is a 3-dimensional non-split simple Lie algebra over $\mathfrak{k}$. Let us construct $L = R \oplus S$, where R is an almost-abelian ideal of L and S is a 3-dimensional non-split simple ideal of L. By [3, Lemma 3.1], L belongs to $\mathfrak{C}$(wasc), and by Lemma 4.1 (4), L does not belong to $\mathfrak{C}^{(*)}$. Hence over $\mathfrak{k}$ we have

$$\mathfrak{C}^{(*)} < \mathfrak{C}(\text{wasc}).$$

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