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Allowable delays for positive diffusion processes

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1. Introduction Many diffusion processes are well modelled by parabolic equations of the form

$$u_t = a(t)u_{xx} - p(x, t)u + q(x, t)u \tag{1.1}$$

where a, p, and q are nonnegative coefficients representing phenomena which underlie the diffusion process. For example, in population dynamics the term au_{xx} corresponds to diffusion due to local concentration, while $-pu$ and qu correspond to death and birth rates, respectively.

Since such phenomena may not lead to instantaneous changes in population size, it is natural to include delays in the models under consideration. Thus in problems of population dynamics, chemical reactions, etc., it is important to be able to generalize (1.1) to delay parabolic equations of the form

$$u_t = a(t)u_{xx} - p(x, t)u(x, t-\sigma) + q(x, t)u(x, t-\rho) \tag{1.2}$$

where the delays σ and ρ are nonnegative constants.

The values to be assigned to such delays will depend largely on an understanding of the mechanics of the diffusion process itself. However, in many situations an additional constraint arises from the fact that the solution of the diffusion process is inherently positive. The purpose of the present paper is to establish upper bounds on delays which result from the requirement that $u(x, t)>0$ for $t\geq 0$.

In case the coefficients of (1.2) are constants such conditions can sometimes be obtained by assuming solutions of the form $u(x, t) = e^{\lambda t}e^{\mu x}$, leading to the characteristic equation

$$\lambda = a\mu^2 - pe^{-\lambda\sigma} + qe^{-\lambda\rho}. \tag{1.3}$$

If (1.2) is accompanied by boundary conditions such as

$$u(0, t) - \alpha u_x(0, t) = 0$$
$$u(L, t) + \beta u_x(L, t) = 0$$

with α, $\beta\geq 0$, then it follows that $\mu^2\leq 0$ and the absence of real solutions to

$$\lambda + pe^{-\lambda\sigma} \leq qe^{-\lambda\rho} \tag{1.4}$$

is inconsistent with the inherent positivity of $u(x, t)$. For example, in case $q=0$ it is readily shown that (1.4) has no real solution for λ whenever $\sigma > 1/pe$, and we see that $\sigma \leq 1/pe$ is a necessary condition for a delay of σ to be allowable.

We shall be interested in establishing some general upper bounds on the allowable delays in (1.2) in the case of nonconstant coefficients.

To determine a unique solution of the delay partial differential equation (1.2) it is necessary to impose spatial boundary conditions such as

$$u(0, t) = u(L, t) = 0$$

or

$$u_x(0, t) = u_x(L, t) = 0 \tag{1.5}$$

or

$$\lim_{x\to\infty} u(x, t) = \lim_{x\to-\infty} u(x, t) = 0$$

and initial data of the form

$$u(x, t) = \varphi(x, t) \qquad \text{for } -M \leq t \leq 0 \text{ and } x \in I \tag{1.6}$$

where $M=\max\{\sigma, \rho\}$ and $I=[0, L]$ or $(-\infty, \infty)$, as determined by (1.5). Existence and uniqueness for (1.2), (1.5) and (1.6) then follow from the fact that there is a unique "heat kernel" $g(x, t; \xi, \tau)$ associated with the differential operator $L[u]=u_t-a(t)u_{xx}$ and the boundary conditions (1.5). This heat kernel satisfies

$$L[g] = \delta(x-\xi)\delta(t-\tau)$$
$$g(x, t; \xi, \tau) \equiv 0 \qquad \text{for} \quad \tau > t$$

and

$$\lim_{\tau\downarrow t} g(x, t; \xi, \tau) = \delta(x-\xi).$$

The usual conclusion drawn from the existence of $g(x, t; \xi, \tau)$ is that the problem

$$L[u] = F(x, t),$$

accompanied by (1.5) and initial data $u(x, 0)=f(x)$, is equivalent to the integral equation

$$u(x, t) = \int_I g(x, t; \xi, 0)f(\xi)d\xi + \int_0^t \int_I g(x, t; \xi, \tau)F(\xi, \tau)d\xi\, d\tau$$

and therefore well posed. In the case of delay equations we can similarly conclude that (1.2), (1.5) and (1.6) are equivalent to the integral equation

(1.7) $$u(x, t) = \int_I g(x, t; \xi, 0)\varphi(\xi, 0)d\xi$$
$$- \int_0^t \int_I g(x, t; \xi, \tau)[p(\xi, \tau)u(\xi, \tau-\sigma) - q(\xi, \tau)u(\xi, \tau-\rho)]d\xi\, d\tau.$$

Since (1.7) can be solved by the "method of steps", using steps of size $m = \min\{\sigma, \rho\}$, it follows that (1.2), (1.5) and (1.6) does indeed have a unique solution.

2. Sufficient conditions for oscillations. For $(x, t) \in [0, L] \times \boldsymbol{R}^+$, consider the delay parabolic equation

(2.1) $$u_t = a(t)u_{xx} - p(x, t)u(x, t-\sigma) + q(x, t)u(x, t-\rho)$$

subject to boundary conditions such as

(2.2) $$u_x(0, t) - u_x(L, t) = 0.$$

Here the delays σ and ρ are positive constants and the coefficients a, p, and q nonnegative continuous functions. We say that a solution $u(x, t)$ of (2.1) *oscillates* in the strip $[0, L] \times \boldsymbol{R}^+$ if for every $t_0 \geq 0$ there exists an $(x_1, t_1) \in (0, L) \times [t_0, \infty)$ such that $u(x_1, t_1) = 0$.

In this section we begin with some properties of ordinary delay inequalities which underlie the oscillation criteria for (2.1). The following lemma is an extension of Theorem 3 in Ladas and Sficas [5] and, in the case of equations with constant coefficients, was proved by Arino, Ladas and Sficas [1].

2.1 LEMMA. *Consider the delay differential inequalities*

(2.3) $$U'(t) + P(t)U(t-\sigma) - Q(t)U(t-\rho) \leq 0, \quad t \geq t_0$$

and

(2.4) $$U'(t) + P(t)U(t-\sigma) - Q(t)U(t-\rho) \geq 0, \quad t \geq t_0$$

where the delays σ and ρ are positive constants and the functions P and Q are continuous and nonnegative for $t \geq t_0$. Assume that

(2.5) $$p \equiv \lim_{t\to\infty} P(t) \quad \textit{and} \quad q \equiv \lim_{t\to\infty} Q(t) \textit{ exist,}$$

(2.6) $$q < p,$$

(2.7) $$\rho \leq \sigma,$$

(2.8) $$(p-q)\sigma e > 1,$$

and for t sufficiently large

(2.9) $$\int_{t-\sigma}^{t-\rho} Q(s)ds \leq 1.$$

Then inequality (2.3) *cannot have an eventually positive solution and inequality* (2.4) *cannot have an eventually negative solution.*

PROOF. Since the negative of a solution of (2.3) is a solution of (2.4), it suffices to prove the result for inequality (2.3). Thus assume, for the sake of contradiction, that (2.3) has an eventually positive solution $U(t)$. Set

$$z(t) = U(t) - \int_{t-\sigma}^{t-\rho} Q(s+\rho)U(s)ds. \tag{2.10}$$

Then

$$z'(t) + [P(t)-Q(t-\sigma+\rho)]U(t-\sigma) \leq 0 \tag{2.11}$$

and in view of (2.5) and (2.6), $z(t)$ is eventually strictly decreasing. We also claim that $z(t)$ is bounded below. Otherwise, $\lim_{t\to\infty} z(t) = -\infty$ and from (2.10) and (2.5) if follows that $U(t)$ is not bounded. Hence there exists a t_1 sufficiently large such that (2.9) is satisfied for $t=t_1$ and also

$$z(t_1) < 0 \quad \text{and} \quad U(t_1) = \max_{s\leq t_1} U(s).$$

Then, from (2.10),

$$\begin{aligned} 0 > z(t_1) &= U(t_1) - \int_{t_1-\sigma}^{t_1-\rho} Q(s+\rho)U(s)ds \geq U(t_1) - U(t_1)\int_{t_1-\sigma}^{t_1-\rho} Q(s+\sigma)ds \\ &= U(t_1)\left[1 - \int_{t_1-\sigma}^{t_1-\rho} Q(s+\sigma)ds\right] \geq 0 \end{aligned}$$

and this contradiction proves our claim that $z(t)$ is bounded below. Since $z(t)$ is decreasing, it follows that $z(t)$ is bounded; that is, there exists a $B>0$ such that

$$|z(t)| \leq B, \quad t \geq t_0.$$

Integrating (2.11) from t_2 to t_3 with $t_2 < t_3$ and t_2 sufficiently large we find

$$0 \leq \int_{t_2}^{t_3} [P(t)-Q(t-\sigma+\rho)]U(t-\sigma)dt \leq z(t_2) - z(t_3) \leq 2B$$

which in view of (2.5) and (2.6) implies that $U \in L_1(t_0, \infty)$. Then, from (2.3) and (2.5), it follows that $U' \in L_1(t_0, \infty)$ which implies that

$$\lim_{t\to\infty} U(t) \equiv L$$

exists. Furthermore L must be zero because $U \in L_1(t_0, \infty)$. Hence, from (2.10) and (2.5), $\lim_{t\to\infty} z(t) = 0$ and since $z(t)$ is also decreasing, it follows that, eventually

$$z(t) > 0. \tag{2.12}$$

Clearly $z(t) \leq U(t)$ and therefore, from (2.11) and for t sufficiently large, we have

$$z'(t) + [P(t) - Q(t-\sigma+\rho)]z(\rho-\sigma) \leq 0. \tag{2.13}$$

But (2.5) and (2.8) imply, (see Ladas and Stavroulakis [7]), that (2.13) cannot have an eventually positive solution. This contradicts (2.12) and completes the proof of the theorem.

In order to establish analogous results for the parabolic equation (2.1) we shall employ an "averaging technique" which was first used for hyperbolic equations by Yoshida [8] and later by Kreith, Kusano and Yoshida [4]. It has also been used to study *delay* hyperbolic equations by Georgiou and Kreith [2].

2.2 THEOREM. *Set*

$$P(t) = \min_{0 \leq x \leq L} p(x, t) \quad \text{and} \quad Q(t) = \max_{0 \leq x \leq L} q(x, t) \tag{2.14}$$

and assume that P and Q satisfy the conditions (2.5)–(2.9) *of Lemma* 2.1. *Then every solution of* (2.1) *and* (2.2) *oscillates in the strip* $[0, L] \times \boldsymbol{R}^+$.

PROOF. Since the negative of a solution of (2.1) and (2.2) is also a solution, it suffices to show that there is no $t_0 \geq 0$ such that a solution of (2.1) and (2.2) is positive in $(0, L) \times [t_0, \infty)$. Assume, for the sake of contradiction, that $u(x, t)$ is a solution of (2.1) and (2.2) and that

$$u(x, t) > 0 \qquad \text{for} \quad (x, t) \in (0, L) \times [t_0, \infty)$$

for some $t_0 \geq 0$. Set

$$U(t) = \int_0^L u(x, t)dx, \quad t \geq t_0.$$

Then $U(t) > 0$ for $t \geq t_0$, and integrating (2.1), (2.2) with respect to x from 0 to L we find,

$$\begin{aligned} U'(t) &= -\int_0^L p(x, t)u(x, t-\sigma)dx + \int_0^L q(x, t)u(x, t-\rho)dx \\ &\leq -P(t)U(t-\sigma) + Q(t)U(t-\rho). \end{aligned}$$

That is,

$$U'(t) + P(t)U(t-\sigma) - Q(t)U(t-\rho) \leq 0. \tag{2.15}$$

But on the basis of Lemma 2.1, (2.15) cannot have a positive solution. This contradicts the fact that $U(t) > 0$ and completes the proof of the theorem.

3. Additional results and remarks. Using the idea of Theorem 2.2 and known oscillation results for ordinary delay differential equations one can obtain further oscillation theorems for partial delay differential equations of the form

$$u_t = a(t)u_{xx} - \sum_{i=1}^{n} p_i(x, t)u(x, t-\sigma_i) \tag{3.1}$$

under the condition (2.2) or boundary conditions of the form

$$\begin{aligned} &u(0, t) - \alpha u_x(0, t) = 0 \quad \text{and} \\ &u(L, t) + \beta u_x(L, t) = 0 \end{aligned} \tag{3.2}$$

with α, $\beta \geq 0$. In (3.1) the coefficients p_i are assumed to be nonnegative and the oscillation of its solutions can be established by utilizing the known sufficient conditions of Ladas and Stavroulakis [6] or Hunt and Yorke [3] for delay differential equations of the form

$$U'(t) + \sum_{i=1}^{n} q_i(t)U(t-\sigma_i) = 0$$

where

$$q_i(t) = \min_{0 \leq x \leq L} p_i(x, t), \qquad i = 1, 2, \ldots, n.$$

In (3.1) it is also possible to have variable delays $\sigma_i = \sigma_i(t)$.

Additional oscillation results can be obtained for the delay parabolic equation

$$u_t = a(t)u_{xx} + P(x, t)u(x, t-\sigma) \tag{3.3}$$

where the coefficient P is not of fixed sign. Decomposing P into its positive and negative part (as a function of t),

$$P(x, t) = p^+(x, t) - p^-(x, t),$$

and using the results in Arino, Ladas and Sficas [1] we obtain sufficient conditions for all solutions of (3.3) which satisfy the boundary conditions (2.2) or (3.2) to oscillate in the strip $[0, L] \times \boldsymbol{R}^+$.

From a physical point of view it is also tempting to consider parabolic equations with delays in the diffusion term u_{xx}. Unfortunately, there seems to be no satisfactory existence theory for equations of the form

$$u_t = a(t)u_{xx}(x, t-\nu) - p(x, t)u(x, t-\sigma)$$

when $\nu > 0$. While one can use techniques such as those used above to establish the *non*existence of positive solutions, such results run the risk of being vacuous in the present state of knowledge concerning existence.

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