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A constructive existence proof for first level formal solutions of meromorphic differential equations

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0. Introduction

In [1], [2], [3] the author has introduced and studied a generalized type of formal solutions (*formal fundamental solutions of first level*; see Section 3 for the definition) for meromorphic differential equations. Compared to the usual kind of formal solutions, first level formal solutions have the advantage of always being "summable" in terms of Laplace integrals or, equivalently, generalized factorial series and in this way generating a family of (proper) solutions of the differential equation with natural asymptotic properties in certain sectors [3]. On the other hand, the existence proof for first level formal solutions, given in [1], [2], made use of the Asymptotic Existence Theorem as well as a theorem on the existence of differential equations with a prescribed Stokes' phenomenon. In the present paper we give a completely different proof for the existence of first level formal fundamental solutions which is much more elementary than the original one, since it only uses some results from the *formal* theory of meromorphic differential equations and Banach's Fixed Point Theorem. At the same time the proof is completely constructive and may therefore be made a basis for actually calculating such formal solutions, although it is very likely that there may be more effective ways for calculating them than following all the steps of the proof.

The main idea in the proof is to obtain an improved version of a well-known result upon formal block-diagonalization of a meromorphic differential equation: Let the leading term of the coefficient matrix (in the Laurent expansion about a pole) be a direct sum of blocks such that each two of them have no common eigenvalue. Then a formal meromorphic transformation may be constructed which block-diagonalizes the given equation, however the resulting equation is, in general, a *formal* equation in the sense that the resulting Laurent series for its coefficient matrix does not converge. It is shown in this paper that a modification of the usual construction of the formal transformation leads to a converging (in fact terminating) Laurent series for the resulting differential equation, and at the same time an estimate upon the coefficients of the formal transformation is achieved.

1. Formal reduction to equations with terminating expansions

For arbitrarily fixed positive real numbers δ and d, let $M_{\delta,d}$ denote the set of all sequences $F=(F_k)$, $k=1, 2,\dots$, where the F_k are $n\times n$ constant matrices (with fixed $n\geq 1$), such that

$$\|F\|_{\delta,d} = \sum_{k=1}^{\infty} \|F_k\| \frac{\delta^k}{\Gamma(k/d)} < \infty. \tag{1.1}$$

(Here and throughout, we consider an arbitrarily fixed submultiplicative matrix norm $\|\cdot\|$). Clearly, $M_{\delta,d}$ is a Banach space under the norm $\|\cdot\|_{\delta,d}$.

LEMMA 1. *For $F, G\in M_{\delta,d}$, let $H=(H_k)$ with $H_1=0$ satisfy*

$$\|H_k\| \leq \sum_{j=1}^{k-1} \|F_j\|\,\|G_{k-j}\|,\ k\geq 2.$$

Then $H\in M_{\delta,d}$, and for $c_d=(\Gamma(1/d))^2/\Gamma(2/d)$

$$\|H\|_{\delta,d} \leq c_d \|F\|_{\delta,d}\|G\|_{\delta,d}.$$

PROOF. Since for $j=1,\dots, k-1$

$$\frac{1}{\Gamma(k/d)} = \int_0^1 \frac{(1-x)^{(k-j)/d-1}}{\Gamma((k-j)/d)} \frac{x^{j/d-1}}{\Gamma(j/d)} dx$$

$$\leq \int_0^1 \frac{(1-x)^{1/d-1}}{\Gamma((k-j)/d)} \frac{x^{1/d-1}}{\Gamma(j/d)} dx = c_d \frac{1}{\Gamma((k-j)/d)} \frac{1}{\Gamma(j/d)},$$

we have

$$\|H\|_{\delta,d} \leq c_d \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} \|F_j\| \frac{\delta^j}{\Gamma(j/d)} \|G_{k-j}\| \frac{\delta^{k-j}}{\Gamma((k-j)/d)}$$

$$= c_d \|F\|_{\delta,d}\|G\|_{\delta,d}.$$

By a *meromorphic differential equation* we mean a homogeneous linear system

$$zx' = A(z)x, \quad A(z) = z^r \sum_0^{\infty} A_j z^{-j}, \tag{1.2}$$

where $r\geq 0$ is an integer (the Poincaré rank), the coefficients A_j are constant $n\times n$ matrices, $A_0\neq 0$ if $r\geq 1$, and the series converges for $|z|>a$ with suitable $a\geq 0$. If $A(z)$ is just a *formal series* of the above type (which may or may not converge), then (1.2) is called a *formal meromorphic differential equation*, and if, on the other hand, all but finitely many A_j are zero, we speak of a *terminating meromorphic differential equation.* In all three cases we mostly write $[A(z)]$ for the differential equation (1.2).

A *meromorphic transformation* is an $n \times n$ matrix function $F(z)$ which is analytic and invertible for $|z|>b$ (with suitably large $b \geq 0$) and has at most a pole at $z=\infty$, hence has a Laurent expansion

$$F(z) = \sum F_k z^{-k},\ F_{-k} = 0 \quad \text{for sufficiently large } k. \tag{1.3}$$

As for differential equations, we use the phrases *terminating* resp. *formal* meromorphic transformation, if $F_k=0$ for sufficiently large k, resp. if $F(z)$ is just a formal series (1.3) having a (formal) inverse of the same type. We say that *$F(z)$ transforms $[A(z)]$ into $[B(z)]$* iff (formally)

$$zF'(z) = A(z)F(z) - F(z)B(z). \tag{1.4}$$

Observe that if $F(z)$ and $A(z)$ (resp. $B(z)$) converge (for sufficiently large z), then $B(z)$ (resp. $A(z)$) also converges, whereas $A(z)$ and $B(z)$ may both converge and $F(z)$ may not.

PROPOSITION 1. *Let a formal equation* (1.2) *be given, and for some $\delta>0$, $d>r$, suppose $A=(A_1, A_2,\dots)\in M_{\delta,d}$. Then for sufficiently large natural N and (in case $r\geq 1$) suitable constant matrices $B_{N+1},\dots, B_{N+r}$, there exists $F=(F_k)\in M_{\delta,d}$ with $F_k=0$, $1\leq k\leq N$, such that the formal meromorphic transformation*

$$F(z) = I + \textstyle\sum_{N+1}^{\infty} F_k z^{-k}$$

transforms $[A(z)]$ into $[B(z)]$, $B(z)=z^r \sum_0^\infty B_k z^{-k}$, with

$$B_k = \begin{cases} A_k & (0\leq k\leq N) \quad, \\ 0 & (k\geq N+r+1), \end{cases}$$

and $B_{N+1},\dots, B_{N+r}$ as above (if $r\geq 1$).

REMARK 1.1. Proposition 1 and its proof are completely analogous to a result (resp. its proof) of Y. Sibuya [7], p. 154, which in a sense may be considered as the case $d=\infty$ of Proposition 1 (defining $M_{\delta,\infty}$ in a suitable way).

PROOF of PROPOSITION 1. We only consider the case $r\geq 1$, the proof for $r=0$ essentially follows the same lines.

If we take $N\geq r-1$, then it is easily seen that (1.4), for $F(z)$ and $B(z)$ as above, is equivalent to

$$B_k = A_k + \textstyle\sum_{j=N+1}^{k} (A_{k-j}F_j - F_jA_{k-j}),\ N+1\leq k\leq N+r,$$

and

$$-(k-r)F_{k-r} = A_k + \textstyle\sum_{j=N+1}^{k} (A_{k-j}F_j - F_jB_{k-j}),\ k\geq N+r+1.$$

For given $F=(F_k)\in M_{\delta,d}$, we define $B_k^{(F)} (k\geq 0)$ and $G_k^{(F)}$ $(k\geq 1)$ by

$$B_k^{(F)} = \begin{cases} A_k & (0 \le k \le N), \\ A_k + \sum_{j=N+1}^{k} (A_{k-j}F_j - F_jA_{k-j}) & (N+1 \le k \le N+r), \\ 0 & (k \ge N+r+1), \end{cases}$$

$G_1^{(F)} = \cdots = G_N^{(F)} = 0$, and

$$-(k-r)G_{k-r}^{(F)} = H_k^{(F)} = A_k + \sum_{j=N+1}^{k} (A_{k-j}F_j - F_jB_{k-j}^{(F)}),\ k \ge N+r+1.$$

If $H^{(F)} = (H_k^{(F)})$, with $H_k^{(F)}$ as above for $k \ge N+r+1$ and $H_1^{(F)} = \cdots = H_{N+r}^{(F)} = 0$, then it follows from straightforward estimates, using Lemma 1, that there exists a constant $K > 0$, independent of N, such that for every $F \in M_{\delta,d}$ with $\|F\|_{\delta,d} \le 1$ we have

$$\|H^{(F)}\|_{\delta,d} \le K.$$

Therefore we find for F as above

$$\begin{aligned} \|G^{(F)}\| &= \sum_{k=N+r+1}^{\infty} \frac{\delta^{k-r}}{\Gamma((k-r)/d)} \frac{\|H_k^{(F)}\|}{k-r} \\ &\le C_N \|H^{(F)}\|_{\delta,d} \le C_N K, \end{aligned}$$

where

$$C_N = \delta^{-r} \sup_{k \ge N+r+1} \frac{\Gamma(k/d)}{(k-r)\Gamma((k-r)/d)}.$$

Using Sterling's formula and the fact that (by assumption) $r/d < 1$, we find $C_N \to 0$ (as $N \to \infty$). Hence we may take N so large that the mapping $F \mapsto G^{(F)}$ maps the closed unit sphere of $M_{\delta,d}$ into itself. In a very similar manner, one proves the mapping to be contractive on the closed unit sphere (if N is sufficiently large). Hence by Banach's Fixed Point Theorem, there exists an $F \in M_{\delta,d}$, $\|F\|_{\delta,d} \le 1$, for which $G^{(F)} = F$, and this completes the proof.

2. Formal block-diagonalization

Let a fixed formal equation (1.2) with $r \ge 1$ be given and assume

$$A_0 = \begin{bmatrix} A_1^{(0)} & 0 \\ 0 & A_2^{(0)} \end{bmatrix}, \tag{2.1}$$

where $A_1^{(0)}$ and $A_2^{(0)}$ *do not have an eigenvalue in common.* Then (see W. Wasow [8]) there exists a *unique* formal transformation

$$F(z) = I + \sum_1^{\infty} F_k z^{-k}, \tag{2.2}$$

where the diagonal blocks of F_k vanish (when blocked like A_0), for every $k \geq 1$, such that the transformed equation $[B(z)]$, $B(z) = z^r \sum_0^\infty B_k z^{-k}$, is diagonally blocked (in the block structure of A_0).

LEMMA 2. *Let $A(z)$, $F(z)$, $B(z)$ be as described above. If for sufficiently large $c_1 > 0$*

$$\|A_k\| \leq c_1^k \Gamma(k/r), \quad k \geq 1, \tag{2.3}$$

then for sufficiently large $c_2 > 0$

$$\|F_k\| \leq c_2^k \Gamma(k/r), \quad k \geq 1, \tag{2.4}$$

and

$$\|B_k\| \leq c_2^k \Gamma(k/r), \quad k \geq 1. \tag{2.5}$$

PROOF. Since $F(z)$ has I as its first term, we find $B_0 = A_0$. If we formally define

$$C(u) = \sum_1^\infty A_k \frac{u^{k/r-1}}{\Gamma(k/r)}, \quad D(u) = \sum_1^\infty B_k \frac{u^{k/r-1}}{\Gamma(k/r)},$$

then it can be seen as in [4], proof of Proposition 1, that $F(z)$ satisfies (1.4) iff the series

$$\Psi(u) = \sum_1^\infty F_k \frac{u^{k/r-1}}{\Gamma(k/r)}$$

formally satisfies

$$\begin{aligned} \Psi(u)(A_0 - ruI) - A_0 \Psi(u) \\ = C(u) - D(u) + \int_0^u \{C(u-w)\Psi(w) - \Psi(w)D(u-w)\} dw \end{aligned} \tag{2.6}$$

(if we integrate straight and select $\arg(u-w) = \arg w = \arg u$). Obviously, (2.3), (2.4), and (2.5) are equivalent to the convergence of $C(u)$, resp. $\Psi(u)$, resp. $D(u)$ for $|u|$ sufficiently small, and if we observe the uniqueness of $F(z)$ (and $B(z)$), then we see that Lemma 2 is equivalent to showing the existence of a diagonally blocked $D(u)$ and a matrix $\Psi(u)$ whose diagonal blocks vanish, such that $uD(u)$ and $u\Psi(u)$ both are analytic in the variable $u^{1/r}$ (for $u^{1/r}$ in a sufficiently small neighborhood of zero), for which (2.6) holds.

If we write (in the block structure of A_0)

$$C(u) = \begin{bmatrix} C_{11}(u) & C_{12}(u) \\ C_{21}(u) & C_{22}(u) \end{bmatrix},$$

$$D(u) = \begin{bmatrix} D_1(u) & 0 \\ 0 & D_2(u) \end{bmatrix},$$

$$\Psi(u) = \begin{bmatrix} 0 & \Psi_2(u) \\ \Psi_1(u) & 0 \end{bmatrix},$$

then (2.6) implies

$$D_1(u) = C_{11}(u) + \int_0^u C_{12}(u-w)\Psi_1(w)dw, \tag{2.7}$$

$$\Psi_1(u)(A_1^{(0)}-ruI) - A_2^{(0)}\Psi_1(u) = C_{21}(u) + \int_0^u \{C_{22}(u-w)\Psi_1(w) - \Psi_1(w)D_1(u-w)\}dw, \tag{2.8}$$

plus two equations for $\Psi_2(u)$, $D_2(u)$ which may be treated analogously. Starting with $\Psi_1^{(0)}(u)\equiv 0$, define inductively $D_1^{(n)}(u)$, $\Psi_1^{(n)}(u)$ by

$$D_1^{(n)}(u) = C_{11}(u) + \int_0^u C_{12}(u-w)\Psi_1^{(n)}(w)dw, \; n \geq 0, \tag{2.9}$$

$$\Psi_1^{(n)}(u)(A_1^{(0)}-ruI) - A_2^{(0)}\Psi_1^{(n)}(u) = C_{21}(u) + \int_0^u \{C_{22}(u-w)\Psi_1^{(n-1)}(w) - \Psi_1^{(n-1)}(w)D_1^{(n-1)}(u-w)\}dw, \; n \geq 1, \tag{2.10}$$

(note that (2.10) is a system of linear equations for the components of $\Psi_1^{(n)}(u)$ that has a unique solution iff $A_1^{(0)}-ruI$ and $A_2^{(0)}$ do not have a common eigenvalue; this is the case, according to our assumptions, for $|u|$ sufficiently small). By induction, one shows that $u\Psi_1^{(n)}(u)$ and $uD_1^{(n)}(u)$ are analytic functions of the variable $u^{1/r}$ (for $|u|<\rho_0$ and $n\geq 0$), if ρ_0 is so small that $C(u)$ converges and $A_1^{(0)}-ruI$ and $A_2^{(0)}$ do not have a common eigenvalue for $|u|<\rho_0$. Hence it suffices to show the uniform convergence of $u^{1-1/r}\Psi_1^{(n)}(u)$ for $|u|\leq\rho_1$ (with suitably small ρ_1, $0<\rho_1<\rho_0$), which then implies the uniform convergence of $u^{1-1/r}D_1^{(n)}(u)$, according to (2.9).

For $K>0$ sufficiently large, we have

$$\|C_{\nu\mu}(u)\| \leq K\frac{|u|^{1/r-1}}{\Gamma(1/r)}, \quad 0 < |u| \leq \rho_0/2, \; 1 \leq \nu, \mu \leq 2,$$

and we take $c>0$ to be the maximal value (for $|u|\leq\rho_0/2$) of the norm of the inverse of the coefficient matrix in (2.10), when regarded as a system of equations in the elements of $\Psi_1^{(n)}(u)$ as unknowns. Since

$$\|\Psi_1^{(1)}(u) - \Psi_1^{(0)}(u)\| = \|\Psi_1^{(1)}(u)\| \leq c\|C_{21}(u)\|, \; 0 < |u| \leq \rho_0/2,$$

we may, as an induction assumption, assume

$$\|\Psi_1^{(m)}(u) - \Psi_1^{(m-1)}(u)\| \leq 4^{m-1}(cK)^m \frac{|u|^{m/r-1}}{\Gamma(m/r)}, \quad 0 < |u| \leq \rho_1$$

(where $\rho_1 \leq \rho_0/2$ will be selected later) for $1 \leq m \leq n$, with some fixed $n \geq 1$. Then for every such m

$$\|D_1^{(m)}(u) - D_1^{(m-1)}(u)\| \leq 4^{m-1}c^m K^{m+1} \int_0^{|u|} \frac{(|u|-x)^{1/r-1}}{\Gamma(1/r)} \frac{x^{m/r-1}}{\Gamma(m/r)} dx$$
$$= 4^{m-1}c^m K^{m+1} \frac{|u|^{(m+1)/r-1}}{(\Gamma(m+1)/r)}, \quad 0 < |u| \leq \rho_1.$$

Moreover, for fixed u as above,

$$\|\Psi_1^{(n-1)}(u)\| \leq \textstyle\sum_{m=1}^{n-1} \|\Psi_1^{(m)}(u) - \Psi_1^{(m-1)}(u)\|$$
$$\leq \textstyle\sum_{m=1}^{n-1} 4^{m-1}(cK)^m \frac{|u|^{m/r-1}}{\Gamma(m/r)}.$$

For ρ_1 small enough (and every $n \geq 1$)

$$\textstyle\sum_{m=2}^{n-1} 4^{m-1}(cK)^m \frac{|u|^{(m-1)/r}}{\Gamma(m/r)} \leq \sum_{m=2}^{\infty} 4^{m-1}(cK)^m \frac{\rho_1^{(m-1)/r}}{\Gamma(m/r)}$$
$$\leq \frac{cK}{\Gamma(1/r)},$$

hence

$$\|\Psi_1^{(n-1)}(u)\| \leq 2cK \frac{|u|^{1/r-1}}{\Gamma(1/r)}, \quad 0 < |u| \leq \rho_1.$$

Similarly,

$$\|D_1^{(n)}(u)\| \leq \|D_1^{(0)}(u)\| + \textstyle\sum_{m=1}^{n} \|D_1^{(m)}(u) - D_1^{(m-1)}(u)\|$$
$$\leq 2K \frac{|u|^{1/r-1}}{\Gamma(1/r)}, \quad 0 < |u| \leq \rho_1,$$

if we make ρ_1 so small that

$$\textstyle\sum_{m=1}^{\infty} 4^{m-1}c^m K^{m+1} \frac{|u|^{m/r}}{\Gamma((m+1)/r)} \leq \frac{K}{\Gamma(1/r)}, \quad 0 < |u| \leq \rho_1.$$

For $0 < |u| \leq \rho_1$

$$\|\Psi_1^{(n+1)}(u) - \Psi_1^{(n)}(u)\| \leq c\left\{\left\|\int_0^u C_{22}(u-w)(\Psi_1^{(n)}(w) - \Psi_1^{(n-1)}(w))\,dw\right\|\right.$$
$$+ \left\|\int_0^u (\Psi_1^{(n)}(w) - \Psi_1^{(n-1)}(w))D_1^{(n)}(u-w)\,dw\right\|$$
$$\left. + \left\|\int_0^u \Psi_1^{(n-1)}(w)(D_1^{(n)}(u-w) - D_1^{(n-1)}(u-w))\,dw\right\|\right\}$$

$$\leq c\left\{3\cdot 4^{n-1}c^n K^{n+1}\frac{|u|^{(n+1)/r-1}}{\Gamma((n+1)/r)} + 2\cdot 4^{n-1}c^{n+1}K^{n+2}\frac{|u|^{(n+2)/r-1}}{\Gamma((n+2)/r)}\right\}$$

$$\leq 4^n c^{n+1}K^{n+1}\frac{|u|^{(n+1)/r-1}}{\Gamma((n+1)/r)},$$

if ρ_1 is so small that

$$2\cdot cK\rho_1^{1/r}\frac{\Gamma((n+1)/r)}{\Gamma((n+2)/r)} \leq 1 \quad (\text{for every } n \geq 1).$$

Since none of the conditions upon ρ_1 depended upon n, we therefore obtain for every $n \geq 1$

$$\|\Psi_1^{(n)}(u) - \Psi_1^{(n-1)}(u)\| \leq \tilde{K}^n \frac{|u|^{n/r-1}}{\Gamma(n/r)} \qquad (0 < |u| \leq \rho_1)$$

with sufficiently large $\tilde{K}$ (independent of n), hence the sequence $u^{1-1/r}\Psi_1^{(n)}(u)$ converges uniformly for u as above. This completes the proof.

The above result may be iterated to block-diagonalize a given system into several diagonal blocks:

COROLLARY 1. *For a given formal equation* (1.2) *with* $r \geq 1$ *suppose that*

$$A_0 = \operatorname{diag}[A_1^{(0)}, \ldots, A_l^{(0)}] \qquad (l \geq 2),$$

where two blocks $A_j^{(0)}$ *and* $A_k^{(0)}$ $(1 \leq j < k \leq l)$ *do not have an eigenvalue in common. Then there is a unique formal transformation*

$$F(z) = I + \textstyle\sum_1^\infty F_k z^{-k},$$

where all coefficients F_k $(k \geq 1)$ *have zero diagonal blocks (in the block structure of* A_0*), such that the transformed equation* $[B(z)]$ *is diagonally blocked. If* $A(z)$ *satisfies* (2.3), *then* $F(z)$ *resp.* $B(z)$ *satisfy* (2.4) *resp.* (2.5).

PROOF. Existence and uniqueness of $F(z)$ as described are already known [8]. In order to find (2.4), (2.5) (in case (2.3) holds), apply Lemma 2 several times, first to split (1.2) into two diagonal blocks, then splitting one of the blocks into two subblocks, etc. Since the product of all these (finitely many) formal transformations can be seen to be of the form required for $F(z)$, we conclude from the uniqueness of $F(z)$ that $F(z)$ equals the product of these transformations, and since every factor, according to Lemma 2, satisfies (2.4) (if (2.3) holds), we find that $F(z)$ also satisfies (2.4) (compare [4], Lemma 1). The estimate (2.5) also holds, since every application of Lemma 2 leads to an equation satisfying (2.5).

For later use, we prove

COROLLARY 2. *Let* $[A(z)]$, $F(z)$, *and* $[B(z)]$ *be as in Corollary* 1, *and, in* *addition, let every block of* A_0 *have only one eigenvalue. If for a suitable natural* p *and a permutation matrix* R *(with* $\varepsilon = e^{2\pi i/p}$*)*

$$A(z\varepsilon) = R^{-1}A(z)R, \tag{2.11}$$

then

$$F(z\varepsilon) = R^{-1}F(z)R, \tag{2.12}$$

and

$$B(z\varepsilon) = R^{-1}B(z)R. \tag{2.13}$$

PROOF. From (2.11) we conclude

$$RA_0\varepsilon^r = A_0R,$$

hence if we block $R = [R_{jk}]$ in the block structure of A_0, then

$$\varepsilon^r R_{jk}A_k^{(0)} - A_j^{(0)}R_{jk} = 0, \quad 1 \leq j,\, k \leq l.$$

Therefore we find that to every k, $1 \leq k \leq l$, there corresponds a unique $j = \pi(k)$ such that $A_j^{(0)}$ and $\varepsilon^r A_k^{(0)}$ have the same eigenvalue, hence

$$R_{ik} = 0, \quad i \neq \pi(k),$$

and $R_{\pi(k),k}$ must be square and invertible (since otherwise R could not be invertible; compare [6], p. 21). This shows

$$R = R_1R_2 \tag{2.14}$$

where R_1 is a diagonally blocked permutation matrix and R_2 is a block-permutation matrix, i.e. a permutation matrix which, in the block structure of A_0, has blocks which are either zero or identity matrices.

$$\text{Let } F_1(z) = F(z\varepsilon) \quad \text{and} \quad F_2(z) = R^{-1}F(z)R, \text{ then}$$

$F_j(z) = I + \sum_1^\infty F_k^{(j)}z^{-k}$, $j = 1, 2$, and due to (2.14) we find that $F_k^{(1)}$, as well as $F_k^{(2)}$, for $k \geq 1$, have vanishing diagonal blocks. Obviously, $F_1(z)$ transforms $[A(z\varepsilon)]$ into $[B(z\varepsilon)]$ and $F_2(z)$ takes $[R^{-1}A(z)R]$ into $[R^{-1}B(z)R]$. Since both equations $[B(z\varepsilon)]$ and $[R^{-1}B(z)R]$ are diagonally blocked, we conclude from (2.11) and the uniqueness of transformations which block-diagonalize $[A(z\varepsilon)]$ and have trivial diagonal blocks, that (2.12) holds, which then implies (2.13).

3. First level formal solutions for normalized equations

Throughout this section, we consider a *normalized* meromorphic equation;

by this we here mean an equation (1.2) with $r \geq 1$, where the series converges for $|z|>a$ with suitably large $a \geq 0$, and

$$A_0 = \operatorname{diag}[\lambda_1 I_{s_1}, \ldots, \lambda_l I_{s_l}], \tag{3.1}$$

where $l \geq 2$ and $s_1, \ldots, s_l$ are natural numbers (by I_s we always denote the s-dimensional unit matrix), and $\lambda_1, \ldots, \lambda_l$ are *distinct* complex numbers. We will see in the next section that every meromorphic equation can be transformed into a normalized or a regular singular one by means of "elementary" transformations.

For a normalized equation, a formal meromorphic transformation is called *of first level*, if its coefficients satisfy and estimate (2.4) with sufficiently large constant $c_2 > 0$.

PROPOSITION 2. *Every normalized meromorphic equation may be transformed by means of a first level formal transformation*

$$F(z) = I + \sum_1^\infty F_k z^{-k}$$

into a diagonally blocked equation $[B(z)]$ *(in the block structure of* A_0*), such that each diagonal block of the transformed equation terminates. Moreover, if for a suitable natural number p and a permutation matrix R we have* (2.11), *then* $F(z)$ *may be chosen so that, in addition, we have* (2.12) *and* (2.13).

PROOF. Due to Corollary 1, there exists a (unique) formal transformation of first level

$$F_1(z) = I + \sum_1^\infty F_k^{(1)} z^{-k}$$

(where $F_k^{(1)}$ has vanishing diagonal blocks, $k \geq 1$) such that for the transformed formal equation $[\hat{A}(z)]$ we have

$$\hat{A}(z) = \operatorname{diag}[\hat{A}_1(z), \ldots, \hat{A}_l(z)],$$

and the coefficients of $\hat{A}(z)$ satisfy an estimate analogous to (2.5). If p and R are such that (2.11) holds, then Corollary 2 implies

$$F_1(z\varepsilon) = R^{-1} F_1(z) R, \quad \hat{A}(z\varepsilon) = R^{-1} \hat{A}(z) R.$$

Defining R_1, R_2 as in (2.14) and observing

$$R_1 = \operatorname{diag}[R_{1,1}, \ldots, R_{1,l}],$$

we find that to every j $(1 \leq j \leq l)$ there corresponds a unique $k = \pi(j)$ $(1 \leq k \leq l)$ such that

$$\hat{A}_j(z\varepsilon) = R_{1,k}^{-1} \hat{A}_k(z) R_{1,k}$$

(in fact, k is determined as the unique index for which $\lambda_k = \lambda_j \varepsilon^r$).

Since $F_1(z)$ begins with I, we find for every fixed j, $1 \leq j \leq l$

$$\hat{A}_j(z) = z^r \lambda_j I_{s_j} + \tilde{A}_j(z),$$
$$\tilde{A}_j(z) = z^{r_j} \sum_{k=0}^{\infty} \tilde{A}_k^{(j)} z^{-k}$$

(where $0 \leq r_j \leq r-1$ may be taken such that either $r_j = 0$ or $\tilde{A}_0^{(j)} \neq 0$). From the estimate of the coefficients of $\hat{A}(z)$ we easily conclude

$$(\tilde{A}_1^{(j)}, \tilde{A}_2^{(j)}, \ldots) \in M_{\delta,r}$$

(for sufficiently small $\delta > 0$), hence according to Proposition 1 there exists a formal transformation

$$F_j^{(2)}(z) = I + \sum_1^{\infty} F_k^{(2,j)} z^{-k}$$

transforming $[\tilde{A}_j(z)]$ into a terminating equation $[\tilde{B}_j(z)]$, and $(F_1^{(2,j)}. F_2^{(2,j)}, \ldots) \in M_{\delta,r}$. For $k = \pi(j)$, the transformation

$$F_k^{(2)}(z) = R_{1,k} F_j^{(2)}(z\varepsilon) R_{1,k}^{-1}$$

is then seen to transform $[\tilde{A}_k(z)] = [R_{1,k} \tilde{A}_j(z\varepsilon) R_{1,k}^{-1}]$ into the terminating equation $[\tilde{B}_k(z)] = [R_{1,k} \tilde{B}_j(z\varepsilon) R_{1,k}^{-1}]$. In this way, once $F_j^{(2)}(z)$ is given, we can define transformations $F_k^{(2)}(z)$, for every k being of the form $\pi^{(\mu)}(j)$ with suitable integer μ, which tranform $[\tilde{A}_k(z)]$ into a terminating equation. Hence applying the above discussion to every cycle (in a representation of the permutation π as a product of disjoint cycles), we finally obtain the existence of a formal transformation

$$F_2(z) = \operatorname{diag}[F_1^{(2)}(z), \ldots, F_l^{(2)}(z)]$$
$$= I + \sum_1^{\infty} F_k^{(2)} z^{-k},$$

such that $(F_1^{(2)}, F_2^{(2)}, \ldots) \in M_{\delta,r}$, which transforms $[\tilde{A}(z) = \operatorname{diag}[\tilde{A}_1(z), \ldots, \tilde{A}_l(z)]]$ into $[\tilde{B}(z) = \operatorname{diag}[\tilde{B}_1(z), \ldots, \tilde{B}_l(z)]]$, and by construction

$$F_2(z\varepsilon) = R^{-1} F_2(z) R,$$
$$\tilde{B}(z\varepsilon) = R^{-1} \tilde{B}(z) R.$$

Since A_0 and $F_2(z)$ commute, we find that $F_2(z)$ transforms $[\hat{A}(z)]$ into $[B(z) = z^r A_0 + \tilde{B}(z)]$. So, if

$$F(z) = F_1(z) F_2(z),$$

then $F(z)$ is of first level (note that $F_1(z)$, $F_2(z)$ are of first level and observe [4] Lemma 1), and takes $[A(z)]$ into $[B(z)]$, and (2.12), (2.13) hold by construction (if (2.11) holds). This completes the proof.

As explained in [5], a *first level formal fundamental solution* of an arbitrarily given meromorphic differential equation (1.2) may be characterized as a pair $H(z) = (F(z), G(z))$, where $F(z)$, $G(z)$ satisfy the following conditions:

(i) *The matrix* $G(z)$ *is analytic and invertible for* $|z| > \tilde{a}$ *(with suitable* $\tilde{a} \geq 0$*) on the Riemann surface of* $\log z$*, its logarithmic derivative*

$$G'(z)G^{-1}(z) = : z^{-1}\tilde{A}(z) \tag{3.2}$$

is meromorphic at $z = \infty$*, and there exists a matrix*

$$Q(z) = p(z)z^{[d]+1}I + d^{-1}\Lambda z^d \tag{3.3}$$

with $d > 0$ *(rational),* $p(z)$ *a polynomial in* z*, and*

$$\Lambda = \operatorname{diag}[\lambda_1 I_{s_1}, \ldots, \lambda_l I_{s_l}],\ l \geq 2,\quad \lambda_i \neq \lambda_j \quad \text{for} \quad i \neq j, \tag{3.4}$$

such that for sufficiently small $\varepsilon > 0$

$$[G(z)\exp\{-Q(z)\}]^{\pm 1} = 0\,(\exp\{|z|^{d-\varepsilon}\}) \tag{3.5}$$

as $z \to \infty$*, uniformly in every sector of finite opening (by* $[d]$ *we denote the largest integer not exceeding* d*).*

(ii) *The matrix* $F(z)$ *is a formal meromorphic transformation from* (1.2) *to*

$$z\tilde{x}' = \tilde{A}(z)\tilde{x} \tag{3.6}$$

with $\tilde{A}(z)$ *as in* (3.2), *and if we write* $F(z) = \sum F_k z^{-k}$*, then for sufficiently large* $c > 0$

$$\|F_k\| \leq c^k \Gamma(k/d), \quad k \geq 1. \tag{3.7}$$

As an application of Proposition 2, we obtain

THEOREM 1. *Every normalized meromorphic equation* $[A(z)]$ *has a first level formal fundamental solution*

$$H(z) = (F(z), G(z)).$$

If for some positive integer p *and some permutation matrix* R *we have*

$$A(z\varepsilon) = R^{-1}A(z)R,$$

then $H(z)$ *can be taken such that*

$$F(z\varepsilon) = R^{-1}F(z)R, \tag{3.8}$$

$$G(z\varepsilon) = R^{-1}G(z)R. \tag{3.9}$$

PROOF. According to Proposition 2, a first level formal meromorphic transformation satisfying (3.8) exists taking $[A(z)]$ into a diagonally blocked equation $[B(z)]$ which terminates. With the same notations as in the proof of Proposition 2, we find by construction of $F(z)$

$$\tilde{B}_j(z\varepsilon) = R_{1,k}^{-1}\tilde{B}_k(z)R_{1,k}, \quad k = \pi(j), \quad 1 \le j \le l.$$

Therefore, with arguments used for the construction of $F_2(z)$ in the proof of Proposition 1, we see that we can select fundamental solutions $\tilde{G}_j(z)$ of $[\tilde{B}_j(z)]$ $(1 \le j \le l)$, such that

$$\tilde{G}_j(z\varepsilon) = R_{1,k}^{-1}\tilde{G}_k(z)R_{1,k}, \quad k = \pi(j), \quad 1 \le j \le l. \tag{3.10}$$

Since the Poincaré rank of $[\tilde{B}_j(z)]$ is at most $r-1$, we find for $\varepsilon \in (0, 1)$

$$\tilde{G}_j^{\pm 1}(z) = 0\,(\exp\{|z|^{r-\varepsilon}\}) \quad \text{as} \quad z \to \infty,$$

uniformly in every sector of finite opening. If we define

$$G(z) = e^{A_0 z^r/r}\tilde{G}(z),$$
$$\tilde{G}(z) = \operatorname{diag}[\tilde{G}_1(z), \ldots, \tilde{G}_l(z)],$$

then $G(z)$ is a fundamental solution of $[B(z)]$ and satisfies (i), if we take

$$\tilde{A}(z) = B(z),\ p(z) \equiv 0,\ d = r,\ \Lambda = A_0$$

(observe that then $Q(z)$ and $\tilde{G}(z)$ commute). Moreover, $F(z)$ as above satisfies (ii), and (3.8), (3.9) hold by construction.

4. First level formal solutions for general equations

A meromorphic equation $[A(z)]$ will be called *essentially irregular singular* (at $z=\infty$), if it is impossible to find a polynomial $p(z)$ for which *the scalar-exponential shift* $x=e^{p(z)}\tilde{x}$ transforms $[A(z)]$ into $[B(z)]$, $B(z)=A(z)-zp'(z)I$, such that $[B(z)]$ has a regular singularity at $z=\infty$. For an equation which is not essentially irregular singular, every formal solution (of the usual kind) converges, hence these cases are trivial and therefore are not considered here.

The eigenvalues of an arbitrary permutation matrix R always are roots of unity, and the Jordan canonical form of R is a diagonal. Hence there exists a constant invertible matrix U and a diagonal U' such that

$$R = U^{-1}e^{2\pi i U'}U, \tag{4.1}$$

and without loss in generality the diagonal elements of U' (which must be rationals) may be taken non-negative and strictly less than one.

PROPOSITION 3. *Consider an arbitrary essentially irregular singular meromorphic equation* $[A(z)]$. *Then there exists a polynomial* $p(z)$, *a* *terminating meromorphic transformation $F_1(z)$ and a permutation matrix R, such that* (*with a natural p for which $R^p = I$, and U, U' as in* (4.1)) *the transformation*

$$x = e^{p(z)} F_1(z) z^{U'} U \hat{x}$$

and the change of variable $z = w^p$ take $[A(z)]$ into a normalized equation $[\hat{A}(w)]$, which then satisfies

$$\hat{A}(w\varepsilon) = R^{-1} \hat{A}(w) R, \quad \varepsilon = e^{2\pi i/p}. \tag{4.2}$$

PROOF. As a consequence of [6], Satz 1, $[A(z)]$ has a formal fundamental solution (of the usual kind) of the form

$$F(z) z^{U'} U z^{J'} e^{Q(z)},$$

where $Q(z)$, J', U, U', and $F(z)$ are as follows:

The matrix $Q(z)$ is diagonal, say

$$Q(z) = \operatorname{diag}[q_1(z), \ldots, q_n(z)],$$

and $q_k(z)$ are polynomials in the variable $z^{1/p}$ (for suitable natural p) with vanishing constant term ($1 \leq k \leq n$), such that for a permutation matrix R with $R^p = I$

$$Q(ze^{2\pi i}) = R^{-1} Q(z) R; \tag{4.3}$$

the matrix J' commutes with $Q(z)$, and U, U' are as in (4.1), whereas $F(z)$ is a formal meromorphic transformation.

According to our assumption, the polynomials $q_1(z), \ldots, q_n(z)$ cannot all be identical, since then $q(z) = q_1(z) = \cdots = q_n(z)$ would be a polynomial in z (observe (4.3)), and the shift $x = e^{q(z)} y$ would lead to a regular singular equation. Hence

$$d = \max_{1 \leq j, k \leq n} \{\deg(q_j(z) - q_k(z))\} > 0 \tag{4.4}$$

(by $\deg q(z)$, for a polynomial in $z^{1/p}$, we mean the rational exponent of its leading term as $z \to \infty$; also observe that $d = 0$ cannot occur, since all $q_k(z)$ vanish at $z = 0$). For a suitable $p(z)$, which according to (4.3) can be taken a polynomial in the variable z, we have

$$\deg\{q_j(z) - p(z)\} \leq d, \quad 1 \leq j \leq n. \tag{4.5}$$

Applying a Proposition in [6], p. 52, to the transposed matrix of $F(z)$ one can show

$$F(z) = F_1(z) F_2(z),$$

where $F_1(z)$ is a terminating meromorphic transformation, and $F_2(z)$ is a formal power series in z^{-1} beginning with I. Applying the transformation

$$x = e^{p(z)} F_1(z) z^{U'} U \hat{x}$$

plus the change of variable $z = w^p$ takes $[A(z)]$ into $[\hat{A}(w)]$, which has a formal fundamental solution of the form

$$\hat{F}(w) w^{pJ'} e^{\hat{Q}(w)},$$
$$\hat{Q}(w) = Q(z) - p(z)I,$$
$$\hat{F}(w) = U^{-1} z^{-U'} F_2(z) z^{U'} U.$$

Since $R^p = I$, we find from (4.1) that the diagonal elements of $e^{2\pi i U'}$ are p-th roots of unity, hence $z^{\pm U'}$ is meromorphic in the variable w, and since the diagonal elements of U' have been taken from the interval [0, 1), we find that $\hat{F}(w)$ is a formal power series in w^{-1} beginning with I. Computing the Laurent expansion of $\hat{A}(w)$ from the formal identity

$$\hat{A}(w) = w[\hat{F}(w) w^{pJ'} e^{\hat{Q}(w)}]' [\hat{F}(w) w^{pJ'} e^{\hat{Q}(w)}]^{-1} \tag{4.6}$$

(observe that J' and $\hat{Q}(w)$ commute), we find

$$\hat{A}(w) = w^{pd} \textstyle\sum_0^\infty \hat{A}_k w^{-k},$$

where $\hat{A}_0$ is the coefficient of w^{pd-1} in $\hat{Q}'(w)$, and therefore $\hat{A}_0$ is diagonal and contains at least two distinct diagonal elements (due to (4.4)). So without loss in generality we may assume $[\hat{A}(w)]$ normalized (since the ordering of the diagonal elements of $Q(z)$ in a formal fundamental solution of the above kind can be arbitrarily prescribed [6]). Using (4.3), (4.1), (4.6), one can verify that (4.2) holds. This completes the proof.

As the main result, we now state

THEOREM 2. *Every essentially irregular singular meromorphic equation $[A(z)]$ has a first level formal fundamental solution.*

PROOF. Let $p(z)$, $F_1(z)$, U', U, p, R and $[\hat{A}(w)]$ be as in Proposition 3. Due to Theorem 1, the normalized equation $[\hat{A}(w)]$ has a first level formal fundamental solution

$$\hat{H}(w) = (\hat{F}(w), \hat{G}(w))$$

for which

$$\hat{F}(w\varepsilon) = R^{-1} \hat{F}(w) R, \tag{4.7}$$

$$\hat{G}(w\varepsilon) = R^{-1} \hat{G}(w) R. \tag{4.8}$$

According to the proof of Theorem 1, if $\hat{r}$ and $\hat{A}_0$ are the Poincaré rank resp. the leading term of $\hat{A}(w)$, then for every ε with $0 < \varepsilon < 1$

$$(4.9) \qquad [\hat{G}(w)e^{-\hat{A}_0 w^{\hat{r}}/\hat{r}}]^{\pm 1} = 0(\exp\{|w|^{d-\varepsilon}\})$$

as $w \to \infty$, uniformly in every sector of finite opening. Due to the definition of first level formal fundamental solutions, $\hat{F}(w) = \sum \hat{F}_k w^{-k}$ is a formal meromorphic transformation from $[\hat{A}(w)]$ to, say $[\tilde{B}(w)]$ (and $\hat{G}(w)$ is a fundamental solution of $[\tilde{B}(w)]$), such that for suitably large $c > 0$

$$(4.10) \qquad \|\hat{F}_k\| \leq c^k \Gamma(k/\hat{r}), \quad k \geq 1.$$

If we define

$$G(z) = z^{U'} U \hat{G}(w) e^{p(z)}$$

then we obtain, using (4.8), (4.1)

$$G(ze^{2\pi i}) = z^{U'} U R \hat{G}(w\varepsilon) e^{p(z)}$$
$$= G(z)R,$$

hence $G(z)$ is a fundamental solution of an equation $[\tilde{A}(z)]$ which is meromorphic in the variable z. With

$$Q(z) = \tilde{p}(z)I + A_0 z^d/d, \; d = \hat{r}/p$$

(where $\tilde{p}(z)$ is obtained from $p(z)$ by dropping terms of degree not larger than d), we find that (4.9) implies (3.5). Defining

$$F(z) = F_1(z) z^{U'} U \hat{F}(w) U^{-1} z^{-U'},$$

we see that $F(z)$ is a formal meromorphic series, at least in the variable w^{-1}; however (4.7) implies

$$F(z\, e^{2\pi i}) = F_1(z) z^{U'} U R \hat{F}(w\varepsilon) R^{-1} U^{-1} z^{-U'}$$
$$= F(z),$$

and therefore $F(z)$ is a series in the variable z^{-1}. If we observe [4], Lemma 1, it is easy to obtain that $F(z)$, when written as

$$F(z) = \textstyle\sum_k \tilde{F}_k w^{-k},$$

has coefficients satisfying (4.10) (with a possibly larger constant $c > 0$), and since $\tilde{F}_k$ must vanish if k is not a multiple of p, we may rewrite

$$F(z) = \textstyle\sum_k \tilde{F}_{kp} z^{-k},$$

and if $F_k = \tilde{F}_{kp}$ for every k, then the coefficients F_k satisfy (3.7) (with $d = \hat{r}/p$ and suitably large $c > 0$). Moreover, it follows from the definition of $F(z)$, $G(z)$ and $\tilde{A}(z)$ that $F(z)$ transforms $[A(z)]$ into $[\tilde{A}(z)]$. Therefore, $H(z) = (F(z), G(z))$ is a first level formal fundamental solution of $[A(z)]$.

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