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Zeta functions of Selberg's type associated with homogeneous vector bundles

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0. Introduction

Let G be a connected noncompact semisimple Lie group with finite center, and let K be a maximal compact subgroup of G. Let $\tilde{M}$ be the symmetric space G/K. We endow $\tilde{M}$ with a G-invariant metric. We assume throughout this paper that $\mathrm{rank}(\tilde{M})=1$.

Let Γ be a discrete torsion-free subgroup of G such that the quotient $\Gamma\backslash G$ is compact. Γ acts on the symmetric space $\tilde{M}$ by left translations and the quotient space $\Gamma\backslash\tilde{M}$ is also compact. We give to the quotient manifold $\Gamma\backslash\tilde{M}$ which we will call $\bar{M}$, the push down Riemannian metric. Then $\bar{M}$ is the most general compact locally symmetric space of negative curvature. Also, the simply connected covering manifold of $\bar{M}$ is $\tilde{M}$, and we have $\pi_1(\bar{M})=\Gamma$.

Let T be a finite dimensional unitary representation of Γ on a vector space E_T with character χ_T. Since Γ is unimodular, there exists a G-invariant measure $d\dot{x}$ on the quotient space $\Gamma\backslash G$. We denote by $L^2(\Gamma\backslash G, T)$ the space of E_T valued measurable functions f on G such that (i) $f(\gamma x)=T(\gamma)f(x)$ for $\gamma\in\Gamma$, $x\in G$ and (ii) $\int_{\Gamma\backslash G}\|f(\dot{x})\|^2 d\dot{x}<\infty$. Since Γ is cocompact, the right regular representation $\pi_{\Gamma,T}$ of G on $L^2(\Gamma\backslash G, T)$ decomposes

$$\pi_{\Gamma,T} = \sum\nolimits_{\pi\in\hat{G}} n_{\Gamma,T}(\pi)\pi$$

and $n_{\Gamma,T}(\pi)<\infty$ for any $\pi\in\hat{G}$. Here $\hat{G}$ stands for the set of all equivalence classes of irreducible unitary representations of G. Suppose that a function f is a C^∞ element of $L^2(\Gamma\backslash G, T)$ with compact support on G. Then the operator $\pi_{\Gamma,T}(f)=\int_G f(x)\pi_{\Gamma,T}(x)dx$ on $L^2(\Gamma\backslash G, T)$ is well defined and is of trace class. Therefore $\mathrm{tr}\,\pi_{\Gamma,T}(f)=\sum_{\pi\in\hat{G}} n_{\Gamma,T}(\pi)\Theta_\pi(f)$, where Θ_π denotes the character of the class π. On the other hand, we may compute a trace of $\pi_{\Gamma,T}(f)$ in a different manner by using the Selberg trace formula.

In this paper, applying a suitable function in $\mathscr{C}^1(G)$ to the trace formula, we will consider the generalization of the following results.

Let X be a compact Riemann surface of genus bigger than 2. Then $X=\Gamma\backslash H$ where $H=SL(2, \boldsymbol{R})/SO(2)$ is the upper half plane, and Γ is a discrete subgroup of

$SL(2, \boldsymbol{R})$, acting freely on H via fractional linear transformations. Let χ be a character of a finite dimensional unitary representation of Γ. In an important paper [25], A. Selberg constructed a function of complex variable $Z_\Gamma(s, \chi)$, which is called Selberg's zeta function attached to the data (Γ, χ), and showed how the location and the order of the zeros of $Z_\Gamma(s, \chi)$ give us information about the spectrum of the Laplace-Beltrami operator of X on the one hand and about the topology of X on the other hand. Furthermore, in a well known paper [7], R. Gangolli constructed a certain zeta function for the general compact locally symmetric space of negative curvature, and also he showed that this zeta function has all of the properties possessed by Selberg's zeta function.

In essence our object is as follows.

Let (τ, V_τ) be an irreducible unitary representation of K. We consider the homogeneous vector bundle $E_\tau = G \times_\tau V_\tau$ over $\tilde{M} = G/K$. There is a unique G-invariant connection ∇ on E_τ such that if s is a C^∞ cross-section, $Y \in T_{eK}(\tilde{M})$ (e is the identity element of G), π: $G \to \tilde{M}$ is the canonical projection and $d\pi$ designates the differential of π at e, then $\nabla_{d\pi(Y)}(s) = d/dt(s(\exp(tY)K))|_{t=0}$. We denote the connection Laplacian on E_τ by ∇^2, and we put $D = -\nabla^2$. Now, let $\hat{G}_{\tau,T}$ be a subset of $\hat{G}$ defined by $\{\pi \in L^2(\Gamma\backslash G, T); \pi|_K \ni \tau\}$. The operator D induces an operator D_T on $L^2(\Gamma\backslash G, T)$. Hence one can consider about the spectrum of D_T on $L^2(\Gamma\backslash G, T)$. The principal aim of this paper is to investigate a certain zeta function $Z_{\tau,T}(s)$ (of a complex variable s) attached to the data (G, K, Γ, T, τ), which provides information about the spectrum of D_T. This means that our zeta function gives us information related to the determination of the subset $\hat{G}_{\tau,T}$ of $\hat{G}$ (see Remark 1 in Section 7). In particular, if τ is a trivial one dimensional representation of K, then our zeta functions are nothing but the zeta functions constructed by Selberg and Gangolli.

We will show that $Z_{\tau,T}$ is holomorphic in a half plane $\operatorname{Re} s > 2\rho_o$ where ρ_o is a positive real constant depending only on (G, K), and that $Z_{\tau,T}$ has a meromorphic continuation to the whole complex plane. In addition to this property of $Z_{\tau,T}$, we will determine the location and the order of zeros and poles of $Z_{\tau,T}$ in connection with the distribution of the series of representations which belongs to $\hat{G}_{\tau,T}$ (Theorem 7.1).

We shall prove that $Z_{\tau,T}$ satisfies a functional equation (Theorem 7.2), and that $Z_{\tau,T}$ has an Euler product expansion (Theorem 7.3). Moreover, we shall show that if $Z_{\tau,T}$ is an entire function then the order of $Z_{\tau,T}$ as an entire function can be related to the structure of (G, K) and it equals to $\dim(G/K)$ (Theorem 7.4).

In the previous paper [27], we have dealt with the case when $G = SU(n, 1)$ and τ is the one dimensional unitary representation of $K = S(U(n) \times U(1))$.

The problem that we will treat in this paper has been studied in the case when the zeta function is associated with the group $G = SL(2, \boldsymbol{C})$ (for detail, see [24]).

We use the standard notation $\boldsymbol{Z}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ for the ring of integers, the field of real numbers, and the field of complex numbers, respectively. We denote by $\boldsymbol{R}^+$ the set of nonnegative real numbers. Furthermore, for any finite set F, we denote the number of elements of F by the notation either $[F]$ or $\sharp F$.

1. Preliminaries

Let G be a connected noncompact semisimple Lie group with finite center, K a maximal compact subgroup of G. Let $\mathfrak{g}$, $\mathfrak{k}$ be their respective Lie algebras and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ with respect to the involution θ determined by $\mathfrak{k}$. Let $\mathfrak{a}_{\mathfrak{p}}$ be a maximal abelian subspace of $\mathfrak{p}$. Let M and M^* be the centralizer and the normalizer of $A_{\mathfrak{p}}$ in K respectively, where of course $A_{\mathfrak{p}}=\exp\mathfrak{a}_{\mathfrak{p}}$. Let $W=M^*/M$ be the Weyl group of $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$. Throughout this paper we will assume that G has real rank one, that is, $\dim\mathfrak{a}_{\mathfrak{p}}=1$. Extend $\mathfrak{a}_{\mathfrak{p}}$ to a maximal abelian θ-stable subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, so that $\mathfrak{a}=\mathfrak{a}_{\mathfrak{k}}+\mathfrak{a}_{\mathfrak{p}}$, with $\mathfrak{a}_{\mathfrak{k}}=\mathfrak{a}\cap\mathfrak{k}$, $\mathfrak{a}_{\mathfrak{p}}=\mathfrak{a}\cap\mathfrak{p}$. Then $\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$.

For any subsapce $\mathfrak{l}$ of $\mathfrak{g}$, we denote by $\mathfrak{l}_c$ and $\mathfrak{l}^*$ the complexification and the real dual of $\mathfrak{l}$ respectively. Furthermore denote by $\mathfrak{l}^*_c$, the complexification of $\mathfrak{l}^*$. Let $\Delta=\Delta(\mathfrak{g}_c, \mathfrak{a}_c)$ denote the set of roots of $(\mathfrak{g}_c, \mathfrak{a}_c)$. Order the dual spaces of $\mathfrak{a}_{\mathfrak{p}}$ and $\mathfrak{a}_{\mathfrak{p}}+i\mathfrak{a}_k$ compatibly, as usual (cf. [13]), and let Δ^+ be the set of positive roots under this order. Let

$$P_+ = \{\alpha\in\Delta_+;\, \alpha\not\equiv 0 \quad \text{on} \quad \mathfrak{a}_{\mathfrak{p}}\},$$
$$P_- = \{\alpha\in\Delta_+;\, \alpha\equiv 0 \quad \text{on} \quad \mathfrak{a}_{\mathfrak{p}}\}.$$

Put $\rho=(1/2)\sum_{\alpha\in p_+}\alpha$. For $\alpha\in\Delta^+$, let X_α be a root vector belonging to α, and put $\mathfrak{n}=\sum_{\alpha\in p_+} \boldsymbol{C}X_\alpha\cap\mathfrak{g}$. Let N be an analytic subgroup of G corresponding to $\mathfrak{n}$. Then we have the Iwasawa decompositions $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}_{\mathfrak{p}}+\mathfrak{n}$, $G=KA_{\mathfrak{p}}N$. Since $\dim\mathfrak{a}_{\mathfrak{p}}=1$, there is an element λ in $\mathfrak{a}^*_{\mathfrak{p}}$ such that $\mathfrak{n}=\mathfrak{n}_\lambda\oplus\mathfrak{n}_{2\lambda}$ with $\mathfrak{n}_{j\lambda}=\{X\in\mathfrak{g};\ \mathrm{ad}\,H(X)=j\cdot\lambda(H)X$, for any $H\in\mathfrak{a}_{\mathfrak{p}}\}$ $(j=1, 2)$. Namely if Σ is the set of restrictions to $\mathfrak{a}_{\mathfrak{p}}$ of elements of P_+ then $\lambda\in\Sigma$ and 2λ is the only other possible element in Σ. Let p be the number of roots in P_+ whose restriction to $\mathfrak{a}_{\mathfrak{p}}$ is λ, and let q be the number of the remaining elements of P_+. Choose $H_o\in\mathfrak{a}_{\mathfrak{p}}$ so that $\lambda(H_o)=1$.

Let $\langle\ ,\ \rangle$ denote the Killing form of $\mathfrak{g}$, that is,

$$\langle X, Y\rangle = \mathrm{tr}\,(\mathrm{ad}\,X\,\mathrm{ad}\,Y) \qquad \text{for} \quad X, Y\in\mathfrak{g}.$$

Put $|X|^2=-\langle X, \theta X\rangle$, then $|\cdot|$ is a norm on $\mathfrak{g}$. Also, the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{a}_{\mathfrak{p}}\times\mathfrak{a}_{\mathfrak{p}}$ puts in duality $\mathfrak{a}_{\mathfrak{p}}$ with itself. Given $\mu\in\mathfrak{a}^*_{\mathfrak{p}}$ there is a unique element $H_\mu\in\mathfrak{a}_{\mathfrak{p}}$ so that $\mu(H)=\langle H_\mu, H\rangle$ for all $H\in\mathfrak{a}_{\mathfrak{p}}$. On $\mathfrak{a}^*_{\mathfrak{p}}$ we use the dual inner product, that we also denote by $\langle\ ,\ \rangle$. With respect to these inner products, one knows that

$$\langle H_o, H_o\rangle = 2p + 8q, \quad \rho(H_o) = (1/2)(p+2q),$$

$$H_\lambda = (2p+8q)^{-1}H_o \quad \text{and} \quad \langle\rho, \rho\rangle = (1/4)(p+2q)^2(2p+8q)^{-1}.$$

Throught this paper, we will denote by ρ_o the number $\rho(H_o)$.

For any $x \in G$, let $H(x) \in \mathfrak{a}_\mathfrak{p}$ be defined by $x = k \exp H(x)n$, $k \in K$, $n \in N$.

For any $h \in A_\mathfrak{p}$, we put $u(h) = \lambda(\log h)$. Then $u = u(h)$ may be regarded as a parameter on the group $A_\mathfrak{p}$. By this parametrization $A_\mathfrak{p}$ can be identified with $\boldsymbol{R}$. Let du be the standard Lebesgue measure on $\boldsymbol{R}$. Via the identification of $A_\mathfrak{p}$ with $\boldsymbol{R}$, we get a Haar measure dh on $A_\mathfrak{p}$ which we fix from now on. On the other hand, for any $\nu \in \mathfrak{a}_\mathfrak{p}^*$, we put $r = r(\nu) = \nu(H_o)$. Then r is a parameter on $\mathfrak{a}_\mathfrak{p}^*$, and maps $\mathfrak{a}_\mathfrak{p}^*$ isomorphically onto $\boldsymbol{R}$. In these parameters, $\nu(\log h) = u(h)r(\nu)$ for $\nu \in \mathfrak{a}_\mathfrak{p}^*$, $h \in A_\mathfrak{p}$. Let dr be the Lebesgue measure on $\boldsymbol{R}$. Then $dr/2\pi$ is the measure on $\boldsymbol{R}$ dual to the measure du on $\boldsymbol{R}$ in the sense of Fourier transforms. We denote by $d\nu$ the measure on $\mathfrak{a}_\mathfrak{p}^*$ that we obtain from $dr/2\pi$. Then dh, $d\nu$ are dual in the sense of Fourier transforms.

Let dk and dm denote the normalized Haar measures on K and M respectively. On N we fix a Haar measure normalized by the following condition: Let $\bar{n} = \theta(n^{-1})$ for each n in N. The measure dn is to satisfy the condition $\int_N \exp(-2\rho(H(\bar{n})))dn = 1$. Having fixed the above measures on K, $A_\mathfrak{p}$, N, we normalize the Haar measure dx on G so that

$$\int_G f(x)dx = \int_{KA_\mathfrak{p}N} f(khn) \exp 2\rho(\log h) dk\, dh\, dn.$$

These normalization will be adhered to throughout in this paper.

For any subgroup L of G, let $\hat{L}$ be the set of all equivalence classes of irreducible unitary representations of L. If $\pi \in \hat{L}$ is a finite-dimensional representation, then we put $\chi_\pi = \operatorname{tr} \pi$ and $d_\pi = \dim \pi$.

If $\nu \in \mathfrak{a}_{\mathfrak{p}c}^*$ and $(\sigma, H_\sigma) \in \hat{M}$ put

$$H^{\sigma,\nu} = \left\{ \begin{array}{ll} f\colon G \to H_\sigma; f(gmhn) = \sigma(m)^{-1} \exp(-(i\nu+\rho)(\log h))f(g) & \\ \qquad\qquad (g \in G,\ m \in M,\ h \in A_\mathfrak{p},\ n \in N) & \\ \text{and} \qquad \int_{K\backslash M} f(k)\|^2 dk < \infty & \end{array} \right\}$$

If $g \in G, f \in H^{\sigma,\nu}$ define

$$(\pi_{\sigma,\nu}(g)f)(x) = f(g^{-1}x) \qquad (x \in G).$$

Then $\pi_{\sigma,\nu}$ defines a represntation of G on $H^{\sigma,\nu}$. If $r = r(\nu) \in \boldsymbol{R}$ then this representation is called a (unitary) principal series representation of G. We denote by $\hat{G}_u$ the set of all equivalence classes of irreducible unitary principal series representations. On the other hand, for $r=r(\nu)\in i\boldsymbol{R}$, the representation $\pi_{\sigma,\nu}$ is called a complementary series representation of G whenever it is unitarizable. We denote by $\hat{G}_c$ the set of all equivalence classes of the complementary series representations.

If $f\in C_c(G)$, we define the Abel transform F_f by

$$F_f(mh) = \exp\rho(\log h)\int_{K\times N} f(kmhnk^{-1})dn\, dk$$

for $m\in M$, $h\in A_{\mathfrak{p}}$. Let $\Theta_{\sigma,\nu}=\Theta_{\pi_{\sigma,\nu}}$ $(\sigma\in\hat{M},\ \nu\in\mathfrak{a}_{\mathfrak{p}c}^*)$ denote the character of $\pi_{\sigma,\nu}$. Then it is known that

$$\Theta_{\sigma,\nu}(f) = \int_M\int_{A_{\mathfrak{p}}} F_f(mh)\chi_\sigma(m)\exp i\nu(\log h)dh\, dm. \tag{1.1}$$

Applying the Fourier inversion formula and the Peter-Weyl theorem we see that

$$F_f(mh) = \sum_{\sigma\in\hat{M}}\int_{\mathfrak{a}_{\mathfrak{p}}^*}\Theta_{\sigma,\nu}(f)\exp(-i\nu(\log h))\overline{\chi_\sigma(m)}d\nu. \tag{1.2}$$

Define for $m\in M$, $\nu\in\mathfrak{a}_{\mathfrak{p}}^*$,

$$D(mh) = \exp(\rho(\log h))|\det(\mathrm{Ad}\,(mh)^{-1}-\mathrm{I})|_{\mathfrak{n}}|. \tag{1.3}$$

Clearly $D(mh)\neq 0$ if $h\neq e$. Moreover, if $h\neq e$ then it is known that

$$F_f(mh) = D(mh)\int_{G/A_{\mathfrak{p}}} f(gmhg^{-1})d\dot{g}. \tag{1.4}$$

Here the measure $d\dot{g}$ on $G/A_{\mathfrak{p}}$ is defined by

$$\int_G \phi(g)dg = \int_{G/A_{\mathfrak{p}}}\int_{A_{\mathfrak{p}}}\phi(gh)dh\, d\dot{g}.$$

Now let $\hat{G}_d$ be the set of all equivalence classes of the discrete series representations of G, that is, those classes $\omega\in\hat{G}$ that contain a square integrable representation of G. Let Θ_ω denote the character of $\omega\in\hat{G}_d$. Then we state the version of the Plancherel formula for G (see [11]):

$$f(e) = \sum_{\omega\in\hat{G}_d} d(\omega)\Theta_\omega(f) + [W]^{-1}\sum_{\sigma\in\hat{M}}\int_{\mathfrak{a}_{\mathfrak{p}}^*}\Theta_{\sigma,\nu}(f)\mu_\sigma(\nu)d\nu \tag{1.5}$$

for any K-finite (on both sides) function f in $C_c^\infty(G)$. Here $d(\omega)$ stands for the formal degree of $\omega\in\hat{G}_d$ and $\mu_\sigma(\nu)$ is the Plancherel measure corresponding to $\sigma\in\hat{M}$. Of course, the quantity $d(\omega)$ and the function $\mu_\sigma(\nu)$ depend on the choice of the Haar measure on G.

Throughout this paper, for simplicity, if $f(\nu)$ is a function on $\mathfrak{a}_{\mathfrak{p}c}^*$, then we will write

$$f(r) = f(\nu) \qquad (r = r(\nu) = \nu(H_o),\ r\lambda = \nu).$$

Then $\mu_\sigma(r)$ ($\sigma \in \hat{M}$) is a meromorphic function on $\boldsymbol{C}$ that restricts to an even, nonnegative, analytic function on $\boldsymbol{R}$ that has polynomial growth.

Let $\mathfrak{m}$ be the Lie algebra of M. Then $\mathfrak{a}_{\mathfrak{k}}$ is a maximal abelian subalgebra of $\mathfrak{m}$. Let Δ_M be the root system of $(\mathfrak{m}_c, \mathfrak{a}_{\mathfrak{k},c})$. We define an order on Δ_M so that the set P_- is the set of all positive roots in Δ_M.

If $\sigma \in \hat{M}$, we denote by Λ_σ its highest weight and we put $\rho_M = 1/2\sum_{\alpha\in p_-}\alpha$. Both Λ_σ and ρ_M are trivially extended to $\mathfrak{a}_c$. Put

$$q_\sigma(r) = q_\sigma(\nu) = \textstyle\prod_{\alpha\in\Delta^+}\langle\alpha,\ i\nu + \Lambda_\sigma + \rho_M\rangle .$$

Let b denote the number of different positive restricted roots of $\mathfrak{a}_{\mathfrak{p}}$. Then it is known that, up to a constant factor depending only on G, $\mu_\sigma(r)$ is given by the formula (see [15])

$$q_\sigma(r/b)\phi_\sigma(r/b).$$

Here $\phi_\sigma(r) = 1$, if $\mathfrak{g} = \mathfrak{so}(2n+1, 1)$, and otherwise $\phi_\sigma(r) = \tanh r$ or $\coth r$, depending on σ. The choice of tanh or coth is done roughly as follows. There exists a distinguished element $\gamma \in \exp \mathfrak{a}_{\mathfrak{k}}$ of order at most two. If $H^* \in \mathfrak{a}_{\mathfrak{k}}$ satisfies $\exp(H^*) = \gamma$, then $\exp(\Lambda_\sigma + \rho_M)(H^*) = \pm 1$. The tanh is used when the sign is $-$ and the coth when the sign is $+$ (see [23]).

Since

$$d_\sigma = \dim\sigma = \prod_{\alpha\in P_-}\frac{\langle\alpha,\ \Lambda_\sigma + \rho_M\rangle}{\langle\alpha,\ \alpha\rangle}$$

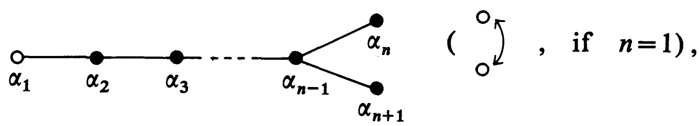
we see that $q_\sigma(r)$ can be written

$$q_\sigma(r) = c\cdot d_\sigma\cdot p_\sigma(r)$$

where c is a constant depending only on G and on the normalization of the Haar measure on G, and $p_\sigma(r)$ is a monic polynomial of degree $\dim(G/K) - 1 = p + q$.

We will need a very explicit formula of $\mu_\sigma(r)$. Miatello has computed it for each particular group, but he uses a different normalization from our Haar measure on G. Now we list them below in our normalization of the Haar measure:

(I) $G = SO_o(2n+1, 1)$ $(n \geq 1)$, $\mathfrak{g}_c \simeq \mathfrak{d}_{n+1}$.
The Satake diagram:



$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, 2,\ldots, n)$, $\alpha_{n+1} = \varepsilon_n + \varepsilon_{n+1}$ (see [2]).

$\Delta^+ = \{\varepsilon_i \pm \varepsilon_j;\ 1 \leqslant i < j \leqslant n+1\}$.

$P_+ : \varepsilon_1 \pm \varepsilon_j$ $(1 < j \leqslant n+1)$ (They all restrict to λ).

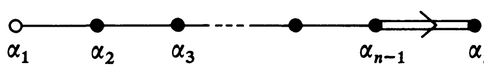
The general form for a highest weight:

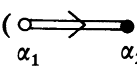
$$\Lambda_\sigma = \textstyle\sum_{i=2}^{n+1} s_i \cdot \varepsilon_i \ (s_2 \geqslant s_3 \geqslant \cdots \geqslant s_n \geqslant |s_{n+1}|,\ s_i \in \boldsymbol{Z}).$$

Then

$$\mu_\sigma(r) = \frac{\pi}{2^{4n-2}\Gamma(n+1/2)^2}\, d_\sigma \prod_{j=1}^{n} (r^2 + (s_{j+1} + n - j)^2).$$

(II) $G = SO_0(2n, 1)$ $(n \geqslant 2)$, $\mathfrak{g}_c \simeq \mathfrak{b}_n$.
The Satake diagram:



(

, if $n=2$)

$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, 2,\ldots, n-1)$, $\alpha_n = \varepsilon_n$.

$\Delta^+ = \{\varepsilon_i;\ 1 \leqslant i \leqslant n\} \cup \{\varepsilon_i \pm \varepsilon_j;\ 1 \leqslant i < j \leqslant n\}$.

$P_+ : \varepsilon_1, \varepsilon_1 \pm \varepsilon_j$ $(1 < j \leqslant n)$ (They all restrict to λ).

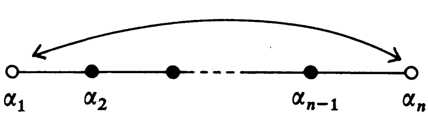
The general form for a highest weight:

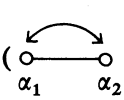
$$\Lambda_\sigma = \textstyle\sum_{i=2}^{n} s_i \cdot \varepsilon_i \ (s_2 \geqslant s_3 \geqslant \cdots \geqslant s_n \geqslant 0,\ s_i \in \boldsymbol{Z}).$$

Then

$$\mu_\sigma(r) = \frac{\pi}{2^{4n-4}\Gamma(n)^2}\, d_\sigma r \prod_{j=2}^{n} (r^2 + (s_j + n - j + 1/2)^2) \tanh \pi r.$$

(III) $G = SU(n, 1)$ $(n \geqslant 2)$, $\mathfrak{g}_c \simeq \mathfrak{a}_n$.
The Satake diagram:



(

, if $n=2$)

$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, 2,\ldots, n)$.

$\Delta^+ = \{\varepsilon_i - \varepsilon_{j+1};\ 1 \leqslant i \leqslant j \leqslant n\}$.

$P_+ : \varepsilon_1 - \varepsilon_{j+1}$ $(1 \leqslant j < n)$ restrict to λ,

$\varepsilon_i - \varepsilon_{n+1}$ $(1 < i \leqslant n)$ restrict to λ,

$\varepsilon_1 - \varepsilon_{n+1}$ restricts to 2λ.

The general form for a highest weight:

$$\Lambda_\sigma = \sum_{i=2}^{n-1} s_i\cdot\varepsilon_i + (s/2)\sum_{i=2}^{n}\varepsilon_i \ (s_2 \geqslant s_3 \geqslant \cdots \geqslant s_{n-1} \geqslant 0,\ s_i,\ s \in \boldsymbol{Z}).$$

If we set $s_n=0$, then

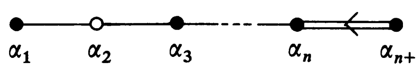
$$\mu_\sigma(r) = \frac{\pi}{2^{2n-2}\Gamma(n)^2}\, d_\sigma \frac{r}{2} \prod_{j=1}^{n-1}\left\{\left(\frac{r}{2}\right)^2 + (2s_{j+1}+s+n-2j)^2/4\right\}\phi_\sigma\!\left(\frac{r}{2}\right),$$

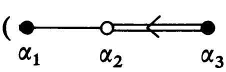
where $\phi_\sigma(r)=\tanh \pi r$ or $\coth \pi r$, and $\phi_\sigma(r)=\tanh \pi r$ if and only if $s+n$ is an odd integer.

For $SL(2, \boldsymbol{R})$ there are two representations of M, one trivial and one not. The Plancherel measures in the two caces are of $\pi r\cdot\tanh \pi r$ and $\pi r\cdot\coth \pi r$, respectively.

(IV) $G=Sp(n, 1)$ $(n\geqslant 2)$ $\mathfrak{g}_c \simeq \mathfrak{c}_{n+1}$.

The Satake diagram:



(

, if $n=2$)

$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(i = 1, 2,\dots, n)$, $\alpha_{n+1} = 2\varepsilon_{n+1}$.

$\Delta^+ = \{2\varepsilon_i;\ 1 \leqslant i \leqslant n+1\} \cup \{\varepsilon_i \pm \varepsilon_j;\ 1 \leqslant i < j \leqslant n+1\}$.

P_+: $\varepsilon_2 \pm \varepsilon_j$ $(3 \leqslant j \leqslant n+1)$ restrict to λ,

$\varepsilon_1 \pm \varepsilon_j$ $(3 \leqslant j \leqslant n+1)$ restrict to λ,

$\varepsilon_1 + \varepsilon_2$, $2\varepsilon_1$ and $2\varepsilon_2$ restrict to 2λ.

The general form for a highest weight:

$$\Lambda_\sigma = \sum_{i=3}^{n+1} s_i\cdot\varepsilon_i + s\alpha_1 \ (s_3 \geqslant \cdots \geqslant s_{n+1} \geqslant 0,\ s \geqslant 0,\ s_i,\ 2s \in \boldsymbol{Z}).$$

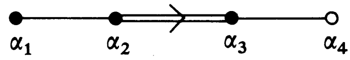
Then

$$\mu_\sigma(r) = \frac{\pi}{2^{4n}\Gamma(2n)^2}\, d_\sigma \frac{r}{2} \prod_{j=3}^{n+1}\left\{\left(\left(\frac{r}{2}\right)^2 + \left(s_j - s + n - j + \frac{3}{2}\right)^2\right)\right.$$
$$\left.\cdot\left(\left(\frac{r}{2}\right)^2 + \left(s_j + s + n - j + \frac{5}{2}\right)^2\right)\left(\left(\frac{r}{2}\right)^2 + \left(s + \frac{1}{2}\right)^2\right)\right\}\phi_\sigma\!\left(\frac{r}{2}\right),$$

where $\phi_\sigma(r)=\tanh \pi r$ or $\coth \pi r$ and $\phi_\sigma(r)=\tanh \pi r$ if and only if $s \in \boldsymbol{Z}$.

(V) $G=F_{4(-20)}$, $\mathfrak{g}_c \simeq f_4$.

The Satake diagram:



$$\alpha_1 = \varepsilon_2 - \varepsilon_3,\ \alpha_2 = \varepsilon_3 - \varepsilon_4,\ \alpha_3 = \varepsilon_4,\ \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4).$$

$$\Delta^+ = \{\varepsilon_i;\ 1 \leqslant i \leqslant 4\} \cup \{\varepsilon_i \pm \varepsilon_j;\ 1 \leqslant i < j \leqslant 4\} \cup \left\{\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\right\}.$$

P_+: $\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ restrict to λ,

ε_1 and $\varepsilon_1 \pm \varepsilon_j$ $(2 \leqslant j \leqslant 4)$ restrict to 2λ.

The general form for a highest weight:

$$\Lambda_\sigma = s_1 \cdot \varepsilon_2 + s_2 \cdot \varepsilon_3 + s_3 \cdot \varepsilon_4\ (s_1 \geqslant s_2 \geqslant s_3 \geqslant 0,\ 2s_i \in \boldsymbol{Z},\ s_i - s_j \in \boldsymbol{Z}).$$

Then

$$\begin{aligned}\mu_\sigma(r) = \frac{\pi}{2^{20}\Gamma(8)^2}\, d_\sigma\left(\frac{r}{2}\right)&\left(\left(\frac{r}{2}\right)^2 + \left(\frac{s_3+1}{2}\right)^2\right)\left(\left(\frac{r}{2}\right)^2 + \left(\frac{s_2+3}{2}\right)^2\right)\\ &\cdot\left(\left(\frac{r}{2}\right)^2 + \left(\frac{s_1+5}{2}\right)^2\right)\left(\left(\frac{r}{2}\right)^2 + \left(s_1 - s_2 - s_2 + \frac{1}{2}\right)^2\right)\\ &\cdot\left(\left(\frac{r}{2}\right)^2 + \left(s_1 - s_2 + s_3 + \frac{3}{2}\right)^2\right)\left(\left(\frac{r}{2}\right)^2 + \left(s_1 + s_2 - s_3 + \frac{7}{2}\right)^2\right)\\ &\cdot\left(\left(\frac{r}{2}\right)^2 + \left(s_1 + s_2 + s_3 + \frac{9}{2}\right)^2\right)\phi_\sigma\left(\frac{r}{2}\right),\end{aligned}$$

where $\phi_\sigma(r) = \tanh \pi r$ or $\coth \pi r$ and $\phi_\sigma(r) = \tanh \pi r$ if and only if $s_i \in \boldsymbol{Z}$.

Suppose that $(\pi, H) \in \hat{G}$. Then (π, H) is K-finite, that is, as a representation of K, it is the unitary direct sum $H = \sum_{\tau\in\hat{K}} H_\tau = \sum_{\tau\in\hat{K}} m_\tau V_\tau$, where $\pi|_{H_\tau} \simeq m_\tau \tau$ $(m_\tau < \infty)$ for any $\tau \in \hat{K}$.

For $f \in C_c(G)$ and $h \in C(K)$, we define the convolution products of them by

$$(f*h)(x) = \int_K f(xh)h(k^{-1})dk, \quad (h*f)(x) = \int_K h(k)f(k^{-1}x)dk.$$

Let E_τ: $H \to H_\tau$ be the orthogonal projection onto H_τ. For any $\tau \in \hat{K}$, we observe that if $d_\tau f * \chi_\tau = f$ then

$$\Theta_\pi(f) = \int_G \operatorname{tr}(E_\tau \pi(x) E_\tau) f(x)dx. \tag{1.6}$$

If $(\pi, H) = (\pi_{\sigma,\nu}, H^{\sigma,\nu})$ $(\sigma \in \hat{M},\ \nu \in \mathfrak{a}_{\mathfrak{p}c}^*)$ then the τ-primary component $H_\tau = H_\tau^{\sigma,\nu}$ is identified with $V_\tau \otimes \mathrm{Hom}_M(V_\tau, H_\tau)$ via the map A given by

$$A(v\otimes T)(khn) = \exp(-(\mathrm{i}\nu+\rho)(\log h))T(\tau(k^{-1})v),$$

where $v \in V_\tau$ and $T \in \mathrm{Hom}_M(V_\tau, H_\sigma)$.

Let $\tau, \gamma \in \hat{K}$, $T \in \mathrm{Hom}_M(V_\tau, V_\gamma)$, $\nu \in \mathfrak{a}^*_{\mathfrak{p}c}$ and $x \in G$. We define an Eisenstein integral by

$$E_{\gamma,\tau}(T: \mathrm{i}\nu: x) = \int_K \exp(-(\mathrm{i}\nu+\rho)H(xk)\gamma(\kappa(xk))T\tau(k^{-1})dk.$$

These integrals are essentially the matrix coefficients of the principal series. That is, it can be easily seen that

$$\langle \pi_{\sigma,\nu}(x)A(w\otimes p^j_\sigma),\ A(v\otimes p^k_\sigma)\rangle = \langle E_{\tau,\tau}(p^{j*}_\sigma p^k_\sigma:\ -\mathrm{i}\nu: x)w,\ v\rangle$$

where $\{p^j_\sigma; j=1,\dots,\alpha_\sigma\}$ is the basis of $\mathrm{Hom}_M(V_\tau, V_\sigma)$ and p^{j*}_σ is the adjoint operator of p^j_σ [28].

Let us fix $\tau \in \hat{K}$. Put $\hat{M}_\tau = \{\sigma \in \hat{M};\ [\sigma: \tau|_M] \neq 0\}$. Then we have

$$V_\tau = \textstyle\sum_{\alpha\in\hat{M}_\tau}\sum_{j=1}^{\alpha_\sigma} H^j_\sigma \quad \text{with } (\tau|_M,\ H^j_\sigma)\in\sigma.$$

Let $q^j_\sigma = p^{*j}_\sigma p^j_\sigma$ $(j=1,\dots,\alpha_\sigma)$. The following lemma will be used below.

LEMMA 1.1 [11]. *For each $\sigma \in \hat{M}_\tau$, let $a_\sigma(\nu)$ be a C^∞ function on $\mathfrak{a}^*_{\mathfrak{p}c}$. Let*

$$a_\tau(\nu) = \textstyle\sum_{\sigma\in\hat{M}_\tau}\sum_{j=1}^{\alpha_\sigma} a_\sigma(\nu)\cdot q^j_\sigma.$$

Then we have

$$\textstyle\sum_{\sigma\in\hat{M}_\tau} \mathrm{tr}\,(E_\tau\pi_{\sigma,\nu}(x)E_\tau)a_\sigma(\nu) = \mathrm{tr}\,(a_\tau(\nu):\ -\mathrm{i}\nu: x)$$

for any $\nu \in \mathfrak{a}^*_{\mathfrak{p}c}$.

PROOF. By the matrix expression for $E_\tau\pi_{\sigma,\nu}(x)E_\tau$ we get

$$\mathrm{tr}\,(E_\tau\pi_{\sigma,\nu}(x)E_\tau) = \textstyle\sum_{j=1}^{\alpha_\sigma} \mathrm{tr}\,E_{\tau,\tau}(q^j_\sigma:\ -\mathrm{i}\nu: x)$$

for each $\sigma \in \hat{M}_\tau$. Hence by the very definition of $a_\tau(\nu)$, we have the desired result.

We now refer to the important property of Eisenstein integrals.

THEOREM 1.2 [11] [32]. *If G is a real rank one semisimple Lie group with finite center, then there exist a meromorphic* $\mathrm{End}_M(V_\tau)$*-valued function $c_\tau(r) = c_\tau(\nu)$ $(r=r(\nu))$ on $\mathfrak{a}^*_{\mathfrak{p}c}$ and a meromorphic* $\mathrm{End}_C(\mathrm{End}_M(v_\tau))$*-valued function $\sigma(r: u) = \sigma(\nu: h)$ $(u=u(h),\ h\in A_\mathfrak{p})$ on $\mathfrak{a}^*_{\mathfrak{p}c}$ so that for any $T \in \mathrm{End}_M(V_\tau)$ and $u>0$, we have*

$$(1.7)\qquad E_{\tau,\tau}(T: \mathrm{i}\,r: h)$$
$$= e^{-\rho_0 u}\{\exp(\mathrm{i}\,ru)\sigma(r: u)Tc_\tau(r)$$

$$+\exp(-\mathrm{i}\, ru)\sigma(-r\colon u)\tau(m^*)^{-1}c_\tau(\bar{r})^* T\tau(m^*)\}$$

for each $\tau \in \hat{K}$. *Here* $m^* \in M^* - M$.

In connection with the function $c_\tau(r)$, the following results are known.

PROPOSITION 1.3 [11]. *For any* $\tau \in \hat{K}$, *define*

$$\mu_\tau(r) = \sum_{\sigma\in\hat{M}_\tau} \sum_{j=1}^{\alpha_\sigma} \mu_\sigma(r)\cdot q_\sigma^j.$$

There exist constants $b_\sigma \in \boldsymbol{R}$ *depending only on* σ *so that, if*

$$b_\tau = \sum_{\sigma\in\hat{M}_\tau} \sum_{j=1}^{\alpha_\sigma} b_\sigma q_\sigma^j$$

then

$$c_\tau(r)\cdot c_\tau(\bar{r})^* = \mu_\tau(r)^{-1} b_\tau\,. \tag{1.8}$$

PROPOSITION 1.4 [26]. *There is a rational function* $q_1(r)$ *such that if* $\operatorname{Im} r<0$,

$$\|c_\tau(r)^{-1}\| \leqslant |q_1(r)|\,. \tag{1.9}$$

Here $\|\cdot\|$ *stands for the operator norm in* $\operatorname{End}_M(V_\tau)$.

We turn our attention to the function $\sigma(r\colon u)$.

PROPOSITION 1.5 [26] [32]. *There exist* $\operatorname{End}_{\boldsymbol{C}}(\operatorname{End}_M(V_\tau))$*-valued rational functions* $\Gamma_k(r)$ *on* $\mathfrak{a}_{\mathfrak{p}c}^*$ *such that* $\sigma(r\colon u)$ *is given by a series*

$$\sigma(r\colon u) = \sum_{k=0}^{\infty} \Gamma_k(\mathrm{i}\, r-\rho_o)\, e^{-ku}. \tag{1.10}$$

Furthermore, for any k $(k=0, 1,\ldots)$, *there exist a rational function* $q_2(r)$ *and a constant* c *such that if* $\operatorname{Im} r>0$ *then we have the estimates*

$$\|\Gamma_k(\mathrm{i}\, r-\rho_o)\| \leqslant c^k |q_2(r)|\,. \tag{1.11}$$

Let $X_1,\ldots, X_m$ be a basis of $\mathfrak{m}$ such that $\langle X_i, X_j\rangle = -\delta_{ij}$. Put

$$\omega_M = -\sum_{j=1}^m X_j^2\,.$$

Then, since ω_M lies in the center $Z(\mathfrak{m})$ of the universal enveloping algebra $U(\mathfrak{m})$ of $\mathfrak{m}$, for a given $\sigma \in \hat{M}$, there is λ_σ such that

$$\sigma(\omega_M) = (2p+8q)\lambda_\sigma \mathrm{I}.$$

The functions $\Gamma_k(r)$ are defined by means of complicated recurrence relations. But, by the very definition of that relations, one can find the following proposition.

PROPOSITION 1.6 [26]. *The poles of* $\Gamma_k(\mathrm{i}\, r-\rho_o)$ *lie in the set*

$$P_{k,\tau} = \left\{ r = \frac{-\mathrm{i}}{2}\left(j + \frac{\lambda_\sigma - \lambda_\xi}{j}\right) \in \boldsymbol{C}; \quad \begin{array}{l} j = 1, 2, \dots, k, \\ \sigma, \xi \in \hat{M}_\tau \end{array} \right\}.$$

Now let $H = \{r \in \boldsymbol{C};\ \mathrm{Im}\, r > 0\}$. Let $\bar{H}$ be its closure. Set

$$P^+_{k,\tau} = H \cap P_{k,\tau}.$$

Then we get

$$\emptyset = P^+_{0,\tau} \subset P^+_{1,\tau} \subset \cdots \subset P^+_{k,\tau} \subset \cdots.$$

Also it is seen that this chain is stationary if k is sufficiently large. Accordingly we put

$$P^+_\tau = \bigcup_{k=1}^{\infty} P^+_{k,\tau}.$$

On the other hand, it is known that the matrix entries of $c_\tau(r)$ are expressible as linear combinations of products of beta functions [29]. Therefore, as a consequence of Proposition 1.3 and Proposition 1.4, we know the fact that the poles of $c_\tau(\bar{r})^{*-1}$ form a discrete subset of the imaginary axis. Furthermore we see that only finitely many of those poles lie in $\bar{H}$. Let

$$Z^+_\tau = \{r \in \bar{H};\ c_\tau(\bar{r}) = 0\}$$

and put

$$\tilde{P}_\tau = P^+_\tau \cup Z^+_\tau.$$

Then one finds the fact that the set of poles of $\mathrm{tr}\,(\sigma(r\colon u)b_\tau c_\tau(\bar{r})^{*-1})$ in $\bar{H}$ is contained in the finite set $\tilde{P}_\tau$. So, let $r = \mathrm{i}\, z_1, \mathrm{i}\, z_2, \dots, \mathrm{i}\, z_p$ $(z_j \geqslant 0)$ be the poles of $\mathrm{tr}\,\sigma(r\colon u)b_\tau c_\tau(\bar{r})^{*-1})$ that occur as a pole of $\mathrm{tr}(\Gamma_k(\mathrm{i}\, r - \rho_o)b_\tau c_\tau(\bar{r})^{*-1})$ for some k so that $k + z_j \leqslant \rho_o$, and let N_j be the order of $\mathrm{i}\, z_j$ as a pole of $\mathrm{tr}(\sigma(r\colon u)b_\tau c_\tau(\bar{r})^{*-1})$.

Let now $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$. Let $Y_1, \dots, Y_n$ be defined by $\langle X_i, Y_j \rangle = \delta_{ij}$. Set $\Omega = \sum X_i Y_i$, the Casimir element of $\mathfrak{g}$. Suppose that $\tilde{\lambda}_{\xi,r} = \tilde{\lambda}_{\xi,\nu}$ is the eigenvalue of Ω on the class $\pi_{\xi,\nu}$. Then it is known that

$$\tilde{\lambda}_{\xi,r} = -(2p + 8q)^{-1}(r^2 + \rho_o^2 + \lambda_\xi).$$

Now we define a polynomial which will play an important role in our argument as follows:

$$P_\tau(r) = \prod_{\xi \in \hat{M}_\tau j = 1, \dots, p} (r - \lambda_{\xi, \mathrm{i} z_j})^{N_j}. \tag{1.12}$$

Here we put

$$\lambda_{\xi,r} = (2p + 8q)\tilde{\lambda}_{\xi,r} = -(r^2 + \rho_o^2 + \lambda_\xi). \tag{1.13}$$

2. The trace formula

Let Γ be a discrete, torsion free subgroup of G such that $\Gamma\backslash G$ is compact. Fix a G-invariant measure $d\dot{x}$ on $\Gamma\backslash G$ by requiring that for each $f \in C_c(G)$ we have

$$\int_G f(x)dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(\gamma x)d\dot{x}.$$

We denote the volume of $\Gamma\backslash G$ in the invariant measure $d\dot{x}$ by $\mathrm{vol}(\Gamma\backslash G)$.

Let (T, E_T) be a finite dimensional unitary representation of Γ with character χ_T. Let $L^2(\Gamma\backslash G, T)$ denote the set of functions $f\colon G\to E_T$ such that

$$f(\gamma x) = T(\gamma)f(x) \qquad \text{for all} \quad x\in G,\ \gamma\in\Gamma$$

and

$$\int_{\Gamma\backslash G} \|f(x)\|_T^2 d\dot{x} < \infty$$

where $\|\cdot\|_T$ is the norm on E_T.

Because $\Gamma\backslash G$ is compact the right regular representation $\pi_{\Gamma,T}$ of G on $L^2(\Gamma\backslash G, T)$ splits into a direct sum of irreducible unitary representations of G and we can write

$$\pi_{\Gamma,T} = \sum_{\pi\in\hat{G}} n_{\Gamma,T}(\pi)\cdot\pi, \quad n_{\Gamma,T}(\pi) < \infty.$$

Here $n_{\Gamma,T}(\pi)$ is the number of summands of $\pi_{\Gamma,T}$ which lie in the class $\pi\in\hat{G}$.

We now discribe the trace formula on $L^2(\Gamma\backslash G, T)$. Let $f\in L^2(\Gamma\backslash G, T)$. If $\phi\in C_c^\infty(G)$ then we get

$$\begin{aligned}(\pi_{\Gamma,T}(\phi)f)(x) &= \int_G f(xg)\phi(g)dg \qquad (x\in G)\\ &= \int_G f(g)\phi(x^{-1}g)dg\\ &= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(\gamma g)\phi(x^{-1}\gamma g)dg\\ &= \int_{\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma} \phi(x^{-1}\gamma g)T(\gamma)\right)f(g)dg.\end{aligned}$$

General theory implies

$$\mathrm{tr}\,\pi_{\Gamma,T}(\phi) = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \phi(g^{-1}\gamma g)\chi_T(\gamma)dg.$$

For $\gamma\in\Gamma$, let C_Γ denote the set of representatives in Γ for the Γ-conjugacy class of elements of Γ and let G_γ be the centralizer of γ in G. We put $\Gamma_\gamma = \Gamma\cap G_\gamma$.

Also we normalize the measure on G_γ and $G_\gamma\backslash G$ so that for $f,\ h \in C_c(G)$

$$\int_G f(g)dg = \int_{G_\gamma\backslash G}\int_{G_\gamma} f(xg)dx\, d\dot{g},$$

$$\int_{G_\gamma} h(g)dg = \int_{\Gamma_\gamma\backslash G_\gamma} \sum_{\delta\in\Gamma_\gamma} h(\delta g)d\dot{g}.$$

With these normalizations,

$$\begin{aligned}
\operatorname{tr}\pi_{\Gamma,T}(\phi) &= \int_{\Gamma\backslash G} \sum_{\gamma\in C_\Gamma}\sum_{\delta\in\Gamma_\gamma\backslash\Gamma} \phi(g^{-1}\delta^{-1}\gamma\delta g)\chi_T(\delta^{-1}\gamma\delta)\,d\dot{g}\\
&= \sum_{\gamma\in C_\Gamma}\chi_T(\gamma)\int_{\Gamma\backslash G}\sum_{\delta\in\Gamma_\gamma\backslash\Gamma}\phi((\delta g)^{-1}\gamma(\delta g))\,d\dot{g}\\
&= \sum_{\gamma\in C_\Gamma}\chi_T(\gamma)\int_{\Gamma_\gamma\backslash G}\phi(g^{-1}\gamma g)\,d\dot{g}\\
&= \sum_{\gamma\in C_\Gamma}\chi_T(\gamma)\int_{G_\gamma\backslash G}\int_{\Gamma_\gamma\backslash G_\gamma}\phi(g^{-1}x^{-1}\gamma xg)\,d\dot{x}\,d\dot{g}\\
&= \sum_{\gamma\in C_\Gamma}\chi_T(\gamma)\int_{G_\gamma\backslash G}\int_{\Gamma_\gamma\backslash G_\gamma}\phi(g^{-1}\gamma g)\,d\dot{x}\,d\dot{g}\\
&= \sum_{\gamma\in C_\Gamma}\chi_T(\gamma)\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G}\phi(g^{-1}\gamma g)\,d\dot{g}.
\end{aligned}$$

Since all elements $\gamma\in\Gamma$ are semisimple and Γ has no elements of finite order, it follows that every element $\gamma\in\Gamma$ is conjugate in G to an element of the Cartan subgroup $A=A_{\mathfrak{t}}A_{\mathfrak{p}}$. Choose an element $h(\gamma)$ of A to which γ is conjugate, and let $h(\gamma)=m_\gamma h_{\mathfrak{p}}(\gamma)$ $(m_\gamma\in A_{\mathfrak{t}},\ h_{\mathfrak{p}}(\gamma)\in A_{\mathfrak{p}})$. We now further demand that $h(\gamma)$ be chosen so that $h_{\mathfrak{p}}(\gamma)$ lies in $A_{\mathfrak{p}}^+=\exp\mathfrak{a}_{\mathfrak{p}}^+$, where $\mathfrak{a}_{\mathfrak{p}}^+$ is the positive Weyl chamber in $\mathfrak{a}_{\mathfrak{p}}$. We then define $u_\gamma=\lambda(\log h_{\mathfrak{p}}(\gamma))=u(h_{\mathfrak{p}}(\gamma))$. Of course u_γ depends only on γ. Also, m_γ is determined up to conjugacy in M. Therefore, the following are well defined:

$$C(\gamma) = D(\gamma)^{-1} = D(h(\gamma))^{-1},$$

$$V(\gamma) = \operatorname{vol}(A_{\mathfrak{p}}\backslash G_{h(\gamma)}) \qquad (A_{\mathfrak{p}}\backslash G_{h(\gamma)} \text{ is compact}),$$

$$\chi_\sigma(m_\gamma) = \operatorname{tr}\sigma(m_\gamma) \quad\text{and}\quad F_\phi(\gamma) = F_\phi(h(\gamma)) = F_\phi(m_\gamma h_{\mathfrak{p}}(\gamma)).$$

Now since

$$\operatorname{vol}(A_{\mathfrak{p}}\backslash G_{h(\gamma)})\int_{G_\gamma\backslash G}\phi(g^{-1}h(\gamma)g)d\dot{g} = \int_{G\backslash A_{\mathfrak{p}}}\phi(gh(\gamma)g^{-1})d\dot{g},$$

using (1.4) we get

$$\int_{G_\gamma \backslash G} \phi(g^{-1}\gamma g)d\dot{g} = V(\gamma)^{-1}C(\gamma)F_\phi(\gamma).$$

Hence we have the trace formula,

$$\begin{aligned}(2.1)\qquad \operatorname{tr}\pi_{\Gamma,T}(\phi) &= \textstyle\sum_{\pi\in\hat{G}} n_{\Gamma,T}(\pi)\Theta_\pi(\phi)\\ &= \chi_T(e)\operatorname{vol}(\Gamma\backslash G)\phi(e)\\ &\quad+ \textstyle\sum_{\gamma\in C_\Gamma\backslash\{e\}} \chi_T(\gamma)\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)V(\gamma)^{-1}C(\gamma)F_\phi(\gamma).\end{aligned}$$

An element $\gamma\in\Gamma$ $(\gamma\neq e)$ is called primitive if it can not be expressed as δ^j for some $j>1$, $\delta\in\Gamma$ We denote the set of all primitive elements of Γ by P_Γ. It is well known that every γ $(\neq e)$ is equal to a positive power of a unique primitive element δ. The integer $j(\gamma)$ is defined by $\gamma=\delta^{j(\gamma)}$. Then we have $u_\gamma= j(\gamma)u_\delta$. Moreover, it is known that

$$\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)V(\gamma)^{-1} = j(\gamma)^{-1}u_\gamma.$$

Hence the trace formula is rewritten as follows:

$$\begin{aligned}(2.2)\qquad \operatorname{tr}\pi_{\Gamma,T}(\phi) &= \chi_T(e)\operatorname{vol}(\Gamma\backslash G)\phi(e)\\ &\quad+ \textstyle\sum_{\gamma\in C_\Gamma\backslash\{e\}} \chi_T(\gamma)j(\gamma)^{-1}u_\gamma C(\gamma)F_\phi(\gamma).\end{aligned}$$

If $\phi\in L^1(G)$ insted of $\phi\in C_c^\infty(G)$, the operator $\pi_{\Gamma,T}(\phi)$ is still defined but it need not be true that $\pi_{\Gamma,T}(\phi)$ is of trace class. Now we refer to a sufficient condition for $\phi\in L^1(G)$ to be put into the trace formula. That is to say, it is a sufficient condition that ϕ is an admissible function (cf. [6] [8]). The fact that ϕ is admissible means that both sides of the trace formula converge absolutely.

At first, we prepare several notations.

Let $\Xi(x)$ be the spherical function of weight zero, that is,

$$\Xi(x) = \int_K \exp - \rho(H(xk))dk.$$

Let $\mathscr{C}^1(G)$ be the set of all C^∞-functions on G such that for any positive integer m and each D, a product of a left invariant differential operator and a right invariant differential operator on G, there exists a constant $K(m, D)$ such that

$$\sup_{x\in G}|Df(x)| < K(m, D)\Xi(x)^2(1+\sigma(x))^{-m}.$$

Here $\sigma(x)=X$ if $x=k\cdot\exp X$ $(k\in K,\ X\in\mathfrak{p})$ is a polar decomposition of $x\in G$.

Then $\mathscr{C}^1(G)$ with the topology defined by the seminorms

$$\nu_{D,m}(f) = \sup_{x\in G}|(1+\sigma(x))^m\Xi(x)^{-2}Df(x)|$$

is a Fréchet space. It is clear from the definitions that $f \in \mathscr{C}^1(G)$ implies that $\Omega^k f \in \mathscr{C}^1(G)$ for all k and that $\mathscr{C}^1(G) \subset L^1(G)$.

PROPOSITION 2.1 [8] [21]. *Let $f \in \mathscr{C}^1(G)$. Suppose that f is K-finite and K-central (i.e. $f(kxk^{-1})=f(x)$ for all $k \in K$). Then f is admissible.*

3. The series $\eta_{\tau,T}(r)$

In this section, we shall define a series $\eta_{\tau,T}(r)=\eta_{\Gamma,T,\tau}(r)$ $(\tau \in \hat{K})$ by means of applying a suitable admissible function to the trace formula. This series is the most important one for defining the zeta function of Selberg's type. The first half of this section, we devote ourselves to studying the function which we will need for the sake of defining $\eta_{\tau,T}(r)$.

In the first place, we define a new polynomial $P^\sigma_\tau(r)$ for each $\sigma \in \hat{M}_\tau$ through the use of the polynomial $P_\tau(r)$ which is defined at the end of Section 1 as follows:

$$(3.1)\qquad P^\sigma_\tau(r) = P_\tau(\lambda_{\sigma,r}) = \prod_{\xi \in \hat{M}_\tau j=1,\dots,p} (\lambda_{\sigma,r} - \lambda_{\xi,iz_j})^{N_j}$$
$$= \prod_{\xi \in \hat{M}_\tau j=1,\dots,p} (-r^2 - z_j^2 - \lambda_\sigma + \lambda_\xi)^{N_j}.$$

Let D^σ_τ be the differential operator on $\boldsymbol{R}(\simeq A_{\mathfrak{p}})$ whose Fourier transform is P^σ_τ.

Let ε_o be a fixed positive real number and let g be a real valued function in $C^\infty(\boldsymbol{R})$ such that: (i) g is even, (ii) g vanishes in some neighborhood of zero, (iii) g is constant, equal to κ, say, in $\{x \in \boldsymbol{R};\ |x| \geqslant \varepsilon_o\}$ and (iv) $0 \leqslant g \leqslant \kappa$. Such functions surely exist. The value of κ and of ε_o will be chosen conveniently later on.

Let $\tau_M = \tau|_M$. For any complex number s, define a function ${}_\tau\mathscr{G}_s$ on $MA_{\mathfrak{p}}$ by

$$(3.2)\qquad {}_\tau\mathscr{G}_s(mh) = \textstyle\sum_{\sigma \in \hat{M}_\tau} [\sigma : \tau_M] \overline{\chi_\sigma(m)} D^\sigma_\tau(g(u(h))) \exp(-(s-\rho_o)|u(h)|),$$

$m \in M$, $h \in A_{\mathfrak{p}}$. Since g vanishes in a neighborhood of zero, ${}_\tau\mathscr{G}_s$ is a smooth function on $MA_{\mathfrak{p}}$ for fixed s.

Let

$$H(r) = \int_0^\infty g'(x) \exp(\mathrm{i}\, rx) dx \quad (r \in \boldsymbol{C}).$$

Because of the properties of g, we see that g' is in $C_c^\infty(\boldsymbol{R})$ and $g'(x)=0$ if $|x| \geqslant \varepsilon_o$. Hence $H(0)=\kappa=g(\varepsilon_o)-g(0)$. Moreover $H(r)$ can be viewed as the Fourier transform of the function $G(x)$ defined by

$$G(x) = \begin{cases} g'(x) & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leqslant 0. \end{cases}$$

Hence an application of the Paley-Wiener theorem gives us the following lemma.

LEMMA 3.1. *H is an entire function of r. Furthermore, for any integers $n\geqslant 1$ and $m\geqslant 0$, we can find the positive constant $C_{m,n}$ such that we have the estimates*

$$|d^m H(r)/dr^m| \leqslant \begin{cases} C_{m,n}(1+|r|)^{-n} & \text{if } \operatorname{Im} r \geqslant 0 \\ C_{m,n}(1+|r|)^{-n}\exp(\varepsilon_o|\operatorname{Im} r|) & \text{if } \operatorname{Im} r < 0. \end{cases}$$

Using this function $H(r)$, we can calculate the Fourier transform ${}_\tau\hat{\mathscr{G}}_s(\sigma, v)$ of ${}_\tau\mathscr{G}_s$ at the character (χ_σ, v) of $MA_\mathfrak{p}$.

LEMMA 3.2. *Let*

$${}_\tau\hat{\mathscr{G}}_s(\sigma, v) = \int_{A\mathfrak{p}}\int_M \chi_\sigma(m)\exp(\mathrm{i}\, v(\log h))_\tau\mathscr{G}_s(mh)dmdh.$$

Then we have

$$(3.3) \qquad {}_\tau\hat{\mathscr{G}}_s(\sigma, r) = [\sigma:\tau_M]P_\tau^\sigma(r)\left\{\frac{H(\mathrm{i}(s-\rho_o)-r)}{s-\rho_o+\mathrm{i}r} + \frac{H(\mathrm{i}(s-\rho_o)+r}{s-\rho_o-\mathrm{i}r}\right\}$$

for any $s\in\boldsymbol{C}$ satisfing $\operatorname{Re}(s-\rho_o\pm\mathrm{i}\, r)>0$ and $\sigma\in\hat{M}$. Here of course we put $r=r(v)$.

PROOF. By the Peter-Weyl theorem we get

$$\int_M \chi_\sigma(m)\chi_\xi(m)dm = \begin{cases} 0 & \text{if } \sigma \not\simeq \xi \\ 1 & \text{if } \sigma \simeq \xi. \end{cases}$$

Hence,

$$\begin{aligned} {}_\tau\hat{\mathscr{G}}_s(\sigma, r) &= [\sigma:\tau_M]\int_{-\infty}^{\infty} D_\tau^\sigma(g(u))\exp(-(s-\rho_o)|u|)\exp(\mathrm{i}\, ru)du \\ &= [\sigma:\tau_M]P_\tau^\sigma(r)\int_{-\infty}^{\infty} g(u)\exp(-(s-\rho_o)|u|+\mathrm{i}\, ru)du \\ &= [\sigma:\tau_M]P_\tau^\sigma(r)\left\{\int_0^\infty g(u)\exp(-(s-\rho_o)+\mathrm{i}\, r)u\, du\right. \\ &\quad \left.+\int_{-\infty}^0 g(u)\exp(s-\rho_o+\mathrm{i}\, r)u\, du\right\} \\ &= [\sigma:\tau_M]P_\tau^\sigma(r)\left\{\int_0^\infty g(u)\exp(-s+\rho_o+\mathrm{i}\, r)u\, du\right. \\ &\quad \left.+\int_0^\infty g(u)\exp(-s+\rho_o-\mathrm{i}\, r)u\, du\right\}, \end{aligned}$$

because g is an even function. Integration by parts yields

$$\int_0^\infty g(u)\exp(-s+\rho_o+\mathrm{i}\,r)u\,du$$

$$=\left[g(u)\,\frac{\exp(-s+\rho_o+\mathrm{i}\,r)u}{-s+\rho_o+\mathrm{i}\,r}\right]_0^\infty-\frac{1}{-s+\rho_o+\mathrm{i}\,r}\int_0^\infty g'(u)\exp\mathrm{i}(\mathrm{i}(s-\rho_o)+r)u\,du$$

$$=\frac{H(\mathrm{i}(s-\rho_o)+r)}{s-\rho_o-\mathrm{i}\,r}$$

for $\mathrm{Re}(s-\rho_o-\mathrm{i}\,r)>0$. Similarly we obtain

$$\int_0^\infty g(u)\exp(-s+\rho_o-\mathrm{i}\,r)u\,du=\frac{H(\mathrm{i}(s-\rho_o)-r)}{s-\rho_o+\mathrm{i}\,r}$$

for $\mathrm{Re}(s-\rho_o+\mathrm{i}\,r)>0$. The assertion now follows.

Now let

$$h_s(x)=\frac{H(\mathrm{i}(s-\rho_o)-x)}{s-\rho_o+\mathrm{i}\,x}+\frac{H(\mathrm{i}(s-\rho_o)+x)}{s-\rho_o-\mathrm{i}\,x}$$

We now prove an estimate that we will need lator on.

PROPOSITION 3.3. *Let f be a meromorphic function on $\bar{H}=\{z\in \boldsymbol{C};\ \mathrm{Im}\,z\geqslant 0\}$ such that $|f(z)|\leqslant|q(z)|$ for any $z\in\bar{H}$, $q(z)$ being a rational function. Suppose also that the polse $z_1,\dots, z_k$ of f in $\bar{H}$ all lie in the upper half plane H and that N_j is the order of the pole z_j $(j=1,\dots, k)$. We put, for a complex s with* $\mathrm{Re}\,s>\rho_o$

$$I_s(t)=\int_{-\infty}^\infty h_s(x)\exp(\mathrm{i}\,tx)f(x)dx.$$

Let a be a positive number. Then we have the following estimates:

If $s-\rho_o+\mathrm{i}\,z_j\neq 0$ for all j $(j=1,\dots, k)$, then there exist polynomials $p_j(t)$, depending on s, of degree N_j-1 and a constant c_s, depending only on s such that

$$\exp at^2\Big(I_s(t)-\textstyle\sum_{j=1}^k p_j(t)\exp(\mathrm{i}\,z_jt)-c_s\exp(-t(s-\rho_o))\Big)$$
$$=O(\exp(\varepsilon_0|\mathrm{Re}\,s-\rho_0-at|))\quad as\quad t\longrightarrow\infty.$$

If $s-\rho_o+\mathrm{i}\,z_m=0$ for some m $(1\leqslant m\leqslant k)$, then there exist polynomials $p_j(t)$ $(j\neq m)$ of degree N_j-1 and a polynomial $\tilde{p}_m(t)$ of degree N_m, all of them depending only on s, such that

$$\exp at^2\Big(I_s(t)-\textstyle\sum_{\substack{j=1\\ j\neq m}}^k p_j(t)\exp(\mathrm{i}\,z_jt)-\tilde{p}_m(t)\exp(-t(s-\rho_o))\Big)$$
$$=O(\exp(\varepsilon_o|\mathrm{Re}\,s-\rho_o-at|))\quad as\quad t\longrightarrow\infty.$$

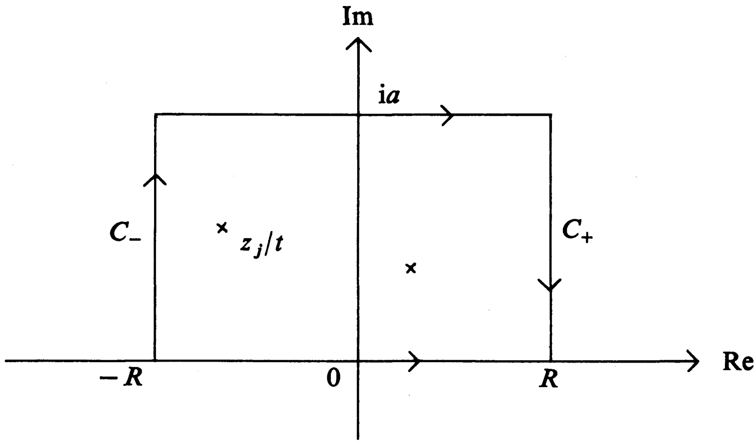
PROOF. Since the latter half of assertions can be proved same as the proof of first one, we will prove only the first assertion.

Since we are interested in large values of t, we may assume $t > \operatorname{Im} z_j$ for all j.

Let $x = y \cdot t$. Then we get

$$I_s(t) = t \int_{-\infty}^{\infty} h_s(yt) \exp(\mathrm{i}\, t^2 y) f(yt) dy. \tag{3.4}$$

Also we may assume $ta > \operatorname{Im} z_j$ $(j = 1, \ldots, k)$ and $ta > \operatorname{Re}(s - \rho_o)$. Now we will shift the contour of integration up to the line $\{y \in \boldsymbol{C};\ \operatorname{Im} y = a\}$. Let $C_{\pm} = \{y = \pm R + \mathrm{i}\, r;\ 0 \leqslant r \leqslant a\}$. We assume that the integrand of (3.4) have no poles on the lines $C_{\pm}$. We consider the following rectangular contour integration.



All the poles of the integrand of (3, 4) lie in the interior of this rectangle and these are at z_j/t $(j = 1, \ldots, k)$ and $\mathrm{i}(s - \rho_o)/t$. Therefore, by the residue theorem we obtain

$$\begin{aligned} I_s(t) = {} & t \int_{-\infty}^{\infty} h_s(ut + \mathrm{i}\, at) \exp(\mathrm{i}\, t^2(u + \mathrm{i}\, a)) f(ut + \mathrm{i}\, at) du \\ & + 2\pi \mathrm{i}\, t \textstyle\sum_{j=1}^{k} \operatorname{Res}_{y = z_j/t} (h_s(yt) \exp(\mathrm{i}\, t^2 y) f(yt)) \\ & + 2\pi \mathrm{i}\, t \operatorname{Res}_{y = \mathrm{i}(s - \rho_o)/t} (h_s(yt) \exp(\mathrm{i}\, t^2 y) f(yt)) \\ & + t \lim_{R \to +\infty} \int_{C_+ + C_-} h_s(yt) \exp(\mathrm{i}\, t^2 y) f(yt) dy. \end{aligned}$$

Let $y = R + \mathrm{i}\, r$ $(0 \leqslant r \leqslant a)$ on C_+, then

$$\begin{aligned} & |h_s((R + \mathrm{i}\, r)t) \exp(\mathrm{i}\, t^2(R + \mathrm{i}\, r)) f((R + \mathrm{i}\, r)t)| \\ & \quad \leqslant \left\{ \left| \frac{H(\mathrm{i}(s - \rho_o) - (R + \mathrm{i}\, r)t)}{s - \rho_o + \mathrm{i}(R + \mathrm{i}\, r)t} \right| + \left| \frac{H(\mathrm{i}(s - \rho_o) + (R + \mathrm{i}\, r)t)}{s - \rho_o - \mathrm{i}(R + \mathrm{i}\, r)t} \right| \right\} \\ & \qquad \cdot \exp(-t^2 r) |f((R + \mathrm{i}\, r)t)| \end{aligned}$$

By Lemma 3.1, for any integer n, there exists a constant $C_{2n,0}$ such that the right side of this inequality is dominated by

$$\leqslant K|q((R+\mathrm{i}\,r)t)|C_{2n,0}\{(\mathrm{Re}\,s-\rho_o-rt)^2+(Rt-|\mathrm{Im}\,s|)^2\}^{-n-\frac{1}{2}}\cdot$$
$$\cdot\exp(\varepsilon_o|\mathrm{Re}\,s-\rho_o-rt|)\exp(-t^2r)$$

for some constant K (>0), if R is sufficiently large. Since n is arbitrary we get

$$\lim_{R\to+\infty}\int_{C_+} h_s(yt)\exp(\mathrm{i}\,t^2y)f(yt)dy=0.$$

Similarly we have

$$\lim_{R\to+\infty}\int_{C_-} h_s(yt)\exp(\mathrm{i}\,t^2y)f(yt)dy=0.$$

Therefore we see that

$$\begin{aligned}I_s(t)&=t\exp(-t^2a)\int_{-\infty}^{\infty}h_s(ut+\mathrm{i}\,at)\exp(\mathrm{i}\,t^2u)f(ut+\mathrm{i}\,at)du\\&\quad+2\pi\,\mathrm{i}\,t\textstyle\sum_{j=1}^{k}\mathrm{Res}_{y=z_j/t}\,h_s(yt)\exp(\mathrm{i}\,t^2y)f(yt)\\&\quad+2\pi\,\mathrm{i}\,t\,\mathrm{Res}_{y=\mathrm{i}(s-\rho_o)/t}\,h_s(yt)\exp(\mathrm{i}\,t^2y)f(yt).\end{aligned}\tag{3.5}$$

In this place, if we put $ut=x$ in the first term of (3.5), then the following equality holds.

$$\begin{aligned}&t\exp(-t^2a)\int_{-\infty}^{\infty}h_s(ut+\mathrm{i}\,at)\exp(\mathrm{i}\,t^2u)f(ut+\mathrm{i}\,at)du\\&=\exp(-t^2a)\int_{-\infty}^{\infty}h_s(x+\mathrm{i}\,at)\exp(\mathrm{i}\,tx)f(x+\mathrm{i}\,at)dx.\end{aligned}$$

By means of the assumption on f, if we let $|z|$ be sufficiently large, then there exists a non-negative integer N such that

$$|f(z)|\leqslant K'|z|^N,$$

where K' is a certain constant. Accordingly, if t is so large then

$$|h_s(x+\mathrm{i}\,at)\exp(\mathrm{i}\,tx)f(x+\mathrm{i}\,at)|\leqslant K'|h_s(x+\mathrm{i}\,at)|\,|x+\mathrm{i}\,at|^N.$$

Therefore, if we use Lemma 3.1 again then, for each n, we can find a constant C'_n such that the above expression is dominated by

$$|x+\mathrm{i}\,at|^N\left\{\frac{1}{|s-\rho_o-at+\mathrm{i}\,x|}+\frac{1}{|s-\rho_o+at-\mathrm{i}\,x|}\right\}$$
$$\cdot\{(\mathrm{Re}\,s-\rho_o-at)^2+(|x|-|\,\mathrm{Im}\,s|)^2\}^{-n}\exp(\varepsilon_o|\mathrm{Re}\,s-\rho_o-at|).$$

Hence we have

$$(3.6)\qquad \int_{-\infty}^{\infty} h_s(x+\mathrm{i}\,at)\exp(\mathrm{i}\,tx)f(x+\mathrm{i}\,at)dx = O(\exp(\varepsilon_o|\mathrm{Re}\,s-\rho_o-at|)).$$

On the other hand, since

$$\mathrm{Res}_{y=\mathrm{i}(s-\rho_o)/t}\, h_s(yt) = -(i/t)H(0),$$

we have

$$(3.7)\qquad 2\pi\,\mathrm{i}\,t\,\mathrm{Res}_{y=\mathrm{i}(s-\rho_o)/t}(h_s(yt)\exp(\mathrm{i}\,t^2y)f(yt))$$
$$= 2H(0)\exp(-(s-\rho_o)t)f(\mathrm{i}\,(s-\rho_o)) = c_s\exp(-(s-\rho_o)t).$$

Here we put $c_s = 2\pi H(0)f(\mathrm{i}\,(s-\rho_o))$.

Suppose that

$$f(z) = \textstyle\sum_{i=-N_j}^{\infty} b_{ij}(z-z_j)^i.$$

Then

$$2\pi\,\mathrm{i}\,t\,\mathrm{Res}_{y=z_j/t}(h_s(yt)\exp(\mathrm{i}\,t^2y)f(yt))$$
$$= \frac{2\pi\mathrm{i}t}{(N_j-1)!}\lim_{y\to z_j/t}\frac{d^{N_j-1}}{dy^{N_j-1}}\{\textstyle\sum_{i=-N_j}^{N_j-1} b_{ij}t^i(y-z_j/t)^{i+N_j}\cdot h_s(yt)\exp(\mathrm{i}t^2y)\}$$
$$= \frac{2\pi\mathrm{i}t^{1-N_j}}{(N_j-1)!}\lim_{y\to z_j/t}\frac{d^{N_j-1}}{dy^{N_j-1}}\{\textstyle\sum_{i=0}^{2N_j-1} b_{i-N_j j}t^i(y-z_j/t)^i\cdot h_s(yt)\exp(\mathrm{i}t^2y)\}$$

If we apply the Leibniz rule to the last expression, then we see that there exists a polynomial $p_j(t)$ such that

$$(3.8)\qquad 2\pi\,\mathrm{i}\,t\,\mathrm{Res}_{y=z_j/t}(h_s(yt)\exp(\mathrm{i}\,t^2y)f(yt)) = p_j(t)\exp(iz_jt),$$

and

$$\deg p_j(t) = 2(N_j-1)+(1-N_j) = N_j-1.$$

The assertion of our proposition follows from the equalities (3.5), (3.7), (3.8) and the estimation (3.6).

We next use a wave packet to define a function ${}_\tau g_s(x)$ on G closely related to the series $\eta_{\tau,T}(s)$.

PROPOSITION 3.4. *For each $\tau\in\hat{K}$ and $s\in\boldsymbol{C}$, with* $\mathrm{Re}\,s>2\rho_o$, *put*

$$(3.9)\qquad {}_\tau g_s(x) = d_\tau^{-1}[W]^{-1}\textstyle\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]^{-1}\displaystyle\int_{\mathfrak{a}_\mathfrak{p}^*}\mathrm{tr}\,(E_\tau\pi_{\sigma,\nu}(x)E_\tau)_\tau\hat{\mathscr{G}}_s(\sigma,\nu)\mu_\sigma(\nu)d\nu.$$

Then ${}_\tau g_s$ *has the following properties*:
(i) ${}_\tau g_s$ *is K-central and* $d_\tau \chi_\tau * {}_\tau g_s = d_\tau \cdot {}_\tau g_s * \chi_\tau = {}_\tau g_s$, *in particular* ${}_\tau g_s$ *is K-finite.*
(ii) ${}_\tau g_s \in \mathscr{C}^1(G)$.
(iii) $\Theta_{\sigma,\nu}({}_\tau g_s) = {}_\tau\hat{\mathscr{G}}_s(\sigma, \nu)$.

PROOF. (i) This is immediate from the definition of ${}_\tau g_s$. (iii) Frobenius reciprocity says that

$$tr\,(E_\tau \pi_{\sigma,\nu}(e) E_\tau) = d_\tau[\tau\colon \pi_{\sigma,\nu}|_K] = d_\tau[\sigma\colon \tau_M].$$

Hence, by definition,

$${}_\tau g_s(e) = [W]^{-1} \sum_{\sigma \in \hat{M}_\tau} \int_{\mathfrak{a}_{\mathfrak{p}}^*} {}_\tau\hat{\mathscr{G}}_s(\sigma, \nu)\mu_\sigma(\nu)d\nu.$$

Therefore, by means of the Plancherel formula, we have the desired result.
(ii) For the sake of simplicity, we put

$$h_s(r) = h_s(r(\nu)) = \frac{H(\mathrm{i}(s-\rho_o)-r)}{s-\rho_o+\mathrm{i}\,r} + \frac{H(\mathrm{i}(s-\rho_o)+r)}{s-\rho_o-\mathrm{i}\,r}$$

Then, by Lemma 3.2,

$${}_\tau g_s(x) = (1/4\pi d_\tau) \int_{-\infty}^{\infty} \sum_{\sigma \in \hat{M}_\tau} \mathrm{tr}\,(E_\tau \pi_{\sigma,r}(x) E_\tau) P_\tau^\sigma(r) h_s(r) \mu_\sigma(r) dr.$$

If we put (see, Section 1)

$$q_\tau(r) = \sum_{\sigma \in \hat{M}_\tau} \sum_{j=1}^{\alpha_\sigma} P_\tau^\sigma(r) q_\sigma^j \quad (\alpha_\sigma = [\sigma\colon \tau_M]),$$

then

$${}_\tau g_s(x) = (1/4\pi d_\tau) \int_{-\infty}^{\infty} \mathrm{tr}\, E_{\tau,\tau}(q_\tau(r)\mu_\tau(r)\colon -\mathrm{i}\,r\colon x) h_s(r) dr,$$

by Lemma 1.1. Let $u = u(h)$ $(h \in A_p)$. Using (1.7), ${}_\tau g_s(h)$ can be put into the following form.

$$\begin{aligned} {}_\tau g_s(h) = (1/4\pi d_\tau)\, e^{-\rho_o u} \int_{-\infty}^{\infty} & h_s(r)\, \mathrm{tr}\,(e^{\mathrm{i}ru} \sigma(r\colon u) q_\tau(r) \mu_\tau(r) c_\tau(r) \\ & + e^{-\mathrm{i}ru} \sigma(-r\colon u) \tau(m^*)^{-1} c_\tau(\bar{r})^* q_\tau(r) \mu_\tau(r) \tau(m^*)) dr. \end{aligned}$$

Thanks to the relation (1.8), we have

$$\begin{aligned} (3.10) \qquad {}_\tau g_s(h) = \frac{e^{-\rho_o u}}{4\pi d_\tau} \int_{-\infty}^{\infty} & h_s(r) \{ e^{\mathrm{i}ru}\, \mathrm{tr}\,(\sigma(r\colon u) b_\tau q_\tau(r) c_\tau(\bar{r})^{*-1}) \\ & + e^{-\mathrm{i}ru}\, \mathrm{tr}\,(\sigma(-r\colon u) \tau(m^*)^{-1} q_\tau(r) c_\tau(r)^{-1} b_\tau \tau(m^*)\} dr. \end{aligned}$$

We now consider the above integral breaking into two pieces. By (1.10), we have

$$\int_{-\infty}^{\infty} h_s(r)\, e^{iru}\, \mathrm{tr}(\sigma(r: u)b_\tau q_\tau(r)c_\tau(\bar{r})^{*-1})dr$$

$$= \int_{-\infty}^{\infty} h_s(r)\, e^{iru}\, \mathrm{tr}\,(\textstyle\sum_{k=0}^{\infty} \Gamma_k(\mathrm{i}\, r-\rho_o)\, e^{-ku}\, b_\tau q_\tau(r)c_\tau(\bar{r})^{*-1})dr.$$

By the definition of $q_\tau(r)$, the integrand has no poles on the real line. Therefore, by (1.9) and (1.11), there is a rational function $q(r)$ and a constant c so that

$$|\mathrm{tr}\,(\Gamma_k(\mathrm{i}\, r-\rho_o)b_\tau q_\tau(r)c_\tau(\bar{r})^{*-1})| \leqslant c^k|q(r)|,$$

if $r \in \bar{H}$. Hence the integrand is dominated by

$$|h_s(r)\, e^{iru}\, q(r)/(1-ce^{-u})| \leqslant |h_s(r)q(r)|,$$

if $u>1$. Therefore, since $h_s(r)$ is a rapidly decreasing function by Lemma 3.1, the dominated convergence theorem implies that the above integral is

$$= \textstyle\sum_{k=0}^{\infty} \displaystyle\int_{-\infty}^{\infty} h_s(r)\, e^{iru}\, \mathrm{tr}\,(\Gamma_k(\mathrm{i}\, r-\rho_o)b_\tau q_\tau(r)c_\tau(\bar{r})^{*-1})dr\, e^{-ku}.$$

The poles of $\mathrm{tr}\,(\Gamma_k(\mathrm{i}\, r-\rho_o)b_\tau q_\tau(r)c_\tau(\bar{r})^{*-1})$ in $\bar{H}$ are all pure imaginaly numbers. Let $\mathrm{i}\, r_1,\dots, \mathrm{i}\, r_{p_k}$ be the poles of this function in $\bar{H}$. Then, by the definition of $q_\tau(r)$, we have

$$r_j > \rho_o \qquad (j = 1, 2,\dots, p_k).$$

Now let $N_{j,k}$ denote the order of pole at $\mathrm{i}\, r_j$. Then, thanks to the estimate of Proposition 3.3, there exist polynomials $p_{j,k}(u)$ of degree $N_{j,k}-1$ or $N_{j,k}$ and a constant c_s so that

$$\textstyle\sum_{k=0}^{\infty} \displaystyle\int_{-\infty}^{\infty} h_s(r)e^{iru}\, \mathrm{tr}\,(\Gamma_k(\mathrm{i}\, r-\rho_o)b_\tau(r)c_\tau(r)^{*-1})dre^{-ku}$$
$$= \textstyle\sum_{k=0}^{\infty} \sum_{j=1}^{p_k} p_{j,k}(u)e^{-(r_j+k)}+c_s e^{-(s-\rho_o)u}/(1-e^{-u})$$
$$+ \exp\,(-au^2) \textstyle\sum_{k=0}^{\infty} g_k(u)e^{-ku}$$

Here $g_k(u)$ is a function which satisfies

$$|g_k(u)| \leqslant c^k O(\exp \varepsilon_o|\mathrm{Re}\, s-\rho_o-au|) \qquad \text{as} \quad u \longrightarrow \infty,$$

for some constant c. Hence we have

$$|\textstyle\sum_{k=0}^{\infty} g_k(u)e^{-ku}| \leqslant (1/(1-ce^{-u}))O(\exp \varepsilon_o|\mathrm{Re}\, s-\rho_o-au|)$$
$$= O(\exp \varepsilon_o|\mathrm{Re}\, s-\rho_o-au|).$$

Next, we consider the other half of the integral in (3.10). Since the poles of $\sigma(-r: u)$ and $c_\tau(r)^{-1}$ are the negative of the poles of $\sigma(r: u)$ and $c_\tau(\bar{r})^{*-1}$ respectively, and the function $h_s(r)$ and the polynomial $q_\tau(r)$ are even, we have the similar estimates of the first one.

Therefore, we easily see that there are polynomials $q_{j,k}(u)$ and a constant c_s such that

$$\begin{aligned}{}_\tau g_s(h) = \frac{e^{-\rho_o u}}{4\pi d_\tau}\{&\textstyle\sum_{k=0}^{\infty}\sum_{j=1}^{p_k} q_{j,k}(u)e^{-(r_j+k)u} + \tilde{c}_s e^{-(s-\rho_o)u}/(1-e^{-u})\\ &+\exp(-au^2)O(\exp\varepsilon_0|\mathrm{Re}\, s-\rho_o-au|)\} \quad \text{as} \quad u\to\infty.\end{aligned}$$

Since $r_j>\rho_o=(p+2q)/2$, $e^{2\rho_o u}{}_\tau g_s(h)$ decays exponentially as $u\to\infty$. Let us consider the Cartan decomposition $G=KA_{\mathfrak{p}}K$. Then, since the Haar measure dg on G can be written by

$$dg = (constant)(\sinh u)^p(\sinh 2u)^q dk\ du\ dk',$$

we see that

$$(1+\sigma(x))^m{}_\tau g_s(x)\in L^1(G)$$

for all non-negative $m\in \boldsymbol{Z}$. On the other hand,

$$\begin{aligned}(\Omega_\tau^k g_s)(x) &= \frac{1}{4\pi d_\tau}\mathrm{tr}\int_{-\infty}^{\infty}\Omega_k E_{\tau,\tau}(q_\tau(r)\mu(r)\colon -\mathrm{i}\,r\colon x)h_s(r)dr\\ &= \frac{1}{4\pi d_\tau}\mathrm{tr}\int_{-\infty}^{\infty}\frac{(-1)^k}{(2p+8q)^k}E_{\tau,\tau}(\textstyle\sum_{\sigma,j}\mu_\tau(r)P_\tau^\sigma(r)\\ &\qquad\cdot(r^2+\rho_o^2+\lambda_\sigma)q_\sigma^j\colon -\mathrm{i}\,r\colon x)h_s(r)dr.\end{aligned}$$

Therefore, arguing as in the case of ${}_\tau g_s$, we obtain a similar estimate to show that

$$(1+\sigma(x))^m(\Omega^k{}_\tau g_s)(x)\in L^1(G)$$

for all $m, k\in \boldsymbol{Z}$ ($\geqslant 0$). It follows that ${}_\tau g_s\in\mathscr{C}^1(G)$ and Proposition 3.4 is proved (see [21]).

Suppose that $\mathrm{Re}\, s>2\rho_o$. Then, if we apply the results (i) and (ii) of Proposition 3.4 to Proposition 2.1, we see that the function ${}_\tau g_s$ is admissible. Hence, $\mathrm{tr}\,\pi_{\Gamma,T}({}_\tau g_s)$ can be evaluated by the right hand of the trace formula (2.2). It implies

$$\begin{aligned}\mathrm{tr}\,\pi_{\Gamma,T}({}_\tau g_s) &= \textstyle\sum_{\pi\in\hat{G}} n_{\Gamma,T}(\pi)\Theta_\pi({}_\tau g_s)\\ &= \chi_T(e)\,\mathrm{vol}(\Gamma\backslash G){}_\tau g_s(e)\\ &\quad+\textstyle\sum_{\gamma\in C_\Gamma\backslash\{e\}}\chi_T(\gamma)j(\gamma)^{-1}u_\gamma C(\gamma)F_{{}_\tau g_s}(\gamma),\end{aligned}$$

where both sides converge absolutely for $\operatorname{Re} s > 2\rho_o$.

In order to parametrize the elements of $\hat{G}$ which appear in the left side of trace formula, we now refer to the result of Langlands concerning the classification of representations.

PROPOSITION 3.5 [16] [19]. *If $\pi \in \hat{G}$ then π satisfies one of the following conditions*:

(a) $\pi \in \hat{G}_d$ *(the discrete series).*

(b) $\pi \in \hat{G}_u$ *(the irreducible unitary principal series).*

(c) $\pi \in \{\pi^+_{\sigma,0}, \pi^-_{\sigma,0}\}$ $(\sigma \in \hat{M})$ *(the limit of discrete series).*

(d) $\pi \in \hat{G}_c$, *the complementary series, that is,* $\pi \simeq \pi_{\sigma,\nu} (\simeq \pi_{\sigma,-\nu})$ *for some* $\nu \in \mathrm{i}\,\mathfrak{a}_\mathfrak{p}^*$ *with* $0 < |r(\nu)| \leqslant \rho_o$.

(e) π *is infinitesmally equivalent with* $L_{\sigma,\nu} = \pi_{\sigma,\nu}/\operatorname{Ker} A(\nu)$ *with* $0 < -\mathrm{i}\, r(\nu) \leqslant \rho_o$, *where* $A(\nu)\colon H^{\sigma,\nu} \to H^{\sigma^s,-\nu}$ $(e \neq s \in W)$ *is the canonical intertwining operator* (cf. Wallach [28]). *Moreover* $L_{\sigma,\nu} \simeq L_{\sigma',\nu'}$ *iff* $\sigma \simeq \sigma'$ *and* $\nu = \nu'$.

We now put

$$Q_\tau = \{\pi \in \hat{G};\ \pi \subset \pi_{\Gamma,T},\ \Theta_\pi({}_\tau g_s) \neq 0\}.$$

For $\sigma \in \hat{M}_\tau$ we set

$$^1Q_\tau^\sigma = \{r = r(\nu) \in \boldsymbol{R}^+;\ \pi_{\sigma,\nu} \in \hat{G},\ \pi_{\sigma,\nu} \subset \pi_{\Gamma,T}\},$$

$$^2Q_\tau^\sigma = \{r = r(\nu) \in \mathrm{i}\,\boldsymbol{R}^+ \backslash \{0\};\ \pi_{\sigma,\nu} \in \hat{G},\ \pi_{\sigma,\nu} \subset \pi_{\Gamma,T}\}.$$

By the definition of ${}_\tau g_s$, it is clear that

$$\hat{G}_d \cap Q_\tau = \emptyset;$$

For convenience' sake, we set up the following agreements: 1°. Suppose that the representation $\pi_{\sigma,o}$ is reducible. Then it is known the fact that at the least one of $\Theta_{\pi^+_{\sigma},o}({}_\tau g_s)$ and $\Theta_{\pi^-_{\sigma},o}({}_\tau g_s)$ is zero. Hence we make a change in the definition of $\pi_{\sigma,o}$ to the following effect:

$$\pi_{\sigma,o} = \begin{cases} \pi^+_{\sigma,o} & \text{if} \quad \Theta_{\pi^+_{\sigma},o}({}_\tau g_s) \neq 0 \\ \pi^-_{\sigma,o} & \text{if} \quad \Theta_{\pi^-_{\sigma},o}({}_\tau g_s) \neq 0. \end{cases}$$

2°. Let $\pi \simeq L_{\sigma,\nu} = \pi_{\sigma,\nu}/\operatorname{Ker} A(\nu)$ as an infinitesimal representation, where $A(\nu)$: $H^{\sigma,\nu} \to H^{\sigma^s,-\nu}$. If $\operatorname{Ker} A(\nu) \cap (H^{\sigma,\nu})_\tau = 0$, then one finds that $\Theta_{L_{\sigma,\nu}}({}_\tau g_s) = \Theta_{\sigma,\nu}({}_\tau g_s)$. If $\operatorname{Ker} A(\nu) \cap (H^{\sigma,\nu})_\tau \neq 0$, then it turns out, by the definition of $P_\tau(\nu)$, that $P_\tau(\lambda_{\sigma,\nu}) = P_\tau^\sigma(\nu) = 0$. Thus $\Theta_{\sigma,\nu}({}_\tau g_s) = 0$. In either case, we define π by $\pi_{\sigma,\nu}$.

Under these agreements, we let

$$Q_\tau^\sigma = {}^1Q_\tau^\sigma \cup {}^2Q_\tau^\sigma.$$

Then we have

$$Q_\tau = \cup_{\sigma\in\hat{M}_\tau} Q_\tau^\sigma .$$

Also we put $\tilde{Q}_\tau^\sigma=\{r=r(v)\in Q_\tau^\sigma;\ P_\tau^\sigma(v)\neq 0\}$. Set

$$\tilde{Q}_\tau = \cup_{\sigma\in\hat{M}_\tau} \tilde{Q}_\tau^\sigma .$$

Recall that $\Theta_{\sigma,v}({}_\tau g_s)={}_\tau\hat{\mathscr{G}}_s(\sigma,v)$. Therefore, by (1.2), we have

$$\begin{aligned} F_{{}_\tau g_s}(\gamma) &= {}_\tau\mathscr{G}_s(h(\gamma)) \\ &= \textstyle\sum_{\sigma\in\hat{M}_\tau} [\sigma\colon \tau_M]\overline{\chi_\sigma(m_\gamma)}D_\tau^\sigma(g(u_\gamma)\exp(-(s-\rho_o)u_\gamma)). \end{aligned}$$

Moreover, it is well known that the numbers u_γ $(\gamma\in C_\Gamma\backslash\{e\})$ are bounded away from zero. If we choose and fix ε_o so small that it is smaller than the smallest of these numbers, namely $0<\varepsilon_o<u_\gamma$ for all $\gamma(\neq e)\in\Gamma$, then we have

$$D_\tau^\sigma(g(u_\gamma)\exp(-(s-\rho_o)u_\gamma)) = \kappa P_\tau^\sigma(\mathrm{i}(s-\rho_o))\exp(-(s-\rho_o)u_\gamma),$$

by the definition of the function g. Hence we can rewrite the trace formula as follows:

$$\begin{aligned} (3.11)\quad \textstyle\sum_{\pi\in\hat{G}} n_{\Gamma,T}(\pi)\Theta_\pi({}_\tau g_s) &= \chi_T(e)\,\mathrm{vol}(\Gamma\backslash G)_\tau g_s(e) \\ &+ \kappa \textstyle\sum_{\sigma\in\hat{M}_\tau} [\sigma\colon \tau_M]P_\tau^\sigma(\mathrm{i}(s-\rho_o))\sum_{\gamma\in C_\Gamma\backslash\{e\}} \chi_T(\gamma)\overline{\chi_\sigma(m_\gamma)}j(\gamma)^{-1} \\ &\qquad\cdot u_\gamma C(\gamma)\exp(-(s-\rho_o)u_\gamma). \end{aligned}$$

PROPOSITION 3.6. *For* $\mathrm{Re}\,s>2\rho_o$, *set*

$$\begin{aligned} (3.12)\quad \eta_{\tau,T}(s) &= \kappa\textstyle\sum_{\sigma\in\hat{M}_\tau} [\sigma\colon \tau_M]P_\tau^\sigma(\mathrm{i}(s-\rho_o)) \\ &\quad\cdot\textstyle\sum_{\gamma\in C_\Gamma\backslash\{e\}} \chi_T(\gamma)\overline{\chi_\sigma(m_\gamma)}j(\gamma)^{-1}u_\gamma C(\gamma)\exp(-(s-\rho_o)u_\gamma). \end{aligned}$$

Then $\eta_{\tau,T}(s)$ *is holomorphic in the half plane* $\mathrm{Re}\,s>2\rho_o$. *Moreover, the series* (3.12) *converges uniformly with respect to* χ_T *for each s in the half plane* $\mathrm{Re}\,s>2\rho_o$.

PROOF. Since ${}_\tau g_s$ is admissible, the series is absolutely convergent and uniformly convergent in any half plane $\mathrm{Re}\,s>2\rho_o+\delta$ with $\delta>0$. Therefore the series defines a holomorphic function in the half plane $\mathrm{Re}\,s>2\rho_o$. The uniformly statement with respect to χ_T comes from observing that $|\chi_T(\gamma)|\leqslant\chi_T(e)=\dim T$, and that $C(\gamma)>0$ for every γ. Thus the series (3.12) is dominated by a multiple of $\eta_{\tau,\chi}(s)$ where χ is the trivial character of Γ.

Because of (3.11), we can easily see that

$$\eta_{\tau,T}(s) = \textstyle\sum_{\pi\in\hat{G}} n_{\Gamma,T}(\pi)\Theta_\pi({}_\tau g_s) - \chi_T(e)\,\mathrm{vol}(\Gamma\backslash G)_\tau g_s(e).$$

Next, we will show that each term on the right side of the above expression has a meromorphic continuation to the whole complex plane. This gives us a meromorphic continuation of $\eta_{\tau,T}(s)$.

In the first place, we note that

$$\begin{aligned}&\sum_{\pi\in\hat{G}} n_{\Gamma,T}(\pi)\Theta_\pi({}_\tau g_s)\\ &\quad= \sum_{\sigma\in\hat{M}_\tau}\sum_{\nu\in\tilde{Q}_\tau^\sigma} n_{\Gamma,T}(\pi_{\sigma,\nu})\Theta_{\sigma,\nu}({}_\tau g_s)\\ &\quad= \sum_{\sigma\in\hat{M}_\tau}\sum_{\nu\in\tilde{Q}_\tau^\sigma} n_{\Gamma,T}(\pi_{\sigma,\nu})_\tau\hat{\mathscr{G}}_s(\sigma,\nu).\end{aligned}$$

PROPOSITION 3.7. *Let*

$$A_\tau(s) = \sum_{\sigma\in\hat{M}_\tau}\sum_{\nu\in\tilde{Q}_\tau^\sigma} n_T(\sigma,\nu)_\tau\hat{\mathscr{G}}_s(\sigma,\nu),$$

where we put $n_T(\sigma,\nu)=n_{\Gamma,T}(\pi_{\sigma,\nu})$ for the sake of simplicity. Then the function $A_\tau(s)$ is holomorphic in the half plane $\mathrm{Re}\, s>2\rho_o$, and has a meromorphic continuation to the whole complex plane. The poles of $A_\tau(s)$ occur at the points $\rho_o\pm\mathrm{i}\, r_\sigma$, where $r_\sigma\in\tilde{Q}_\tau^\sigma(\sigma\in\hat{M}_\tau)$. These poles are all simple. The residues at $\rho_o+\mathrm{i}\, r_\sigma$ and $\rho_o-\mathrm{i}\, r_\sigma$ both equal $n_T(\sigma,r_\sigma)P_\tau^\sigma(r_\sigma)[\sigma:\tau_M]$ $(\sigma\in\hat{M}_\tau)$ if $r_\sigma\neq0$. Finally, if $r_\sigma=0$ for some $\sigma\in\hat{M}_\tau$, the residue of $A_\tau(s)$ at ρ_o is $2\kappa n_T(\sigma,0)P_\tau^\sigma(0)[\sigma:\tau_M]$.

In order to prove this proposition, we need the following result due to Wallach.

PROPOSITION 3.8 [30]. *There exists a positive number x_o such that for any $\tau\in\hat{K}$ and all x with $x>x_o$ we have*

$$\sum_{\pi\in\hat{G}}[\tau:\pi|_K]n_{\Gamma,T}(\pi)(1+|\pi(\Omega)|)^{-x}<\infty.$$

PROOF OF PROPOSITION 3.7. Note the fact that $\mathrm{Re}(s-\rho_o-\mathrm{i}\, r_\sigma)>0$ for all $\sigma\in\hat{M}_\tau$ $(\tau\in\hat{K})$. By (3.3) we have

$$\begin{aligned}(3.13)\quad A_\tau(s) &= \sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\sum_{r_\sigma\in\tilde{Q}_\tau^\sigma} P_\tau^\sigma(r_\sigma)n_T(\sigma,r_\sigma)\\ &\qquad\cdot\left\{\frac{H(\mathrm{i}(s-\rho_o)+r_\sigma)}{s-\rho_o-\mathrm{i}r_\sigma}+\frac{H(\mathrm{i}(s-\rho_o)-r_\sigma)}{s-\rho_o+\mathrm{i}r_\sigma}\right\},\end{aligned}$$

if $\mathrm{Re}\, s>2\rho_o$. Each term in the series is a meromorphic function of s.

A consequence of Proposition 3.8 says that $\{\rho_o\pm\mathrm{i}\, r_\sigma;\ r_\sigma\in\tilde{Q}_\tau^\sigma\}$ contains no finite accumulation points. If $\mathcal{O}$ is a compact set that is disjoint from $\{\rho_o\pm\mathrm{i}\, r_\sigma;\ r_\sigma\in\tilde{Q}_\tau^\sigma\}$ then the distance between them is positive, so there is a positive constant C so that for all $s\in\mathcal{O}$,

$$\begin{aligned}&\sum_{r_\sigma\in\tilde{Q}_\tau^\sigma}\left|n_T(\sigma,r_\sigma)P_\tau^\sigma(r_\sigma)\frac{H(\mathrm{i}(s-\rho_o)-r_\sigma)}{s-\rho_o+\mathrm{i}r_\sigma}\right|\\ &\quad\leqslant C\sum_{r_\sigma\in\tilde{Q}_\tau^\sigma} n_T(\sigma,r_\sigma)|P_\tau^\sigma(r_\sigma)H(\mathrm{i}(s-\rho_o)-r_\sigma)|.\end{aligned}$$

Since $\pi_{\sigma,\nu}(\Omega)$ is a polynomial in ν and $\mathcal{O}$ is compact, using Lemma 3.1 we see that

$$\sup_{s\in\mathcal{O},r_\sigma\in\tilde{Q}^\sigma_\tau}|P^\sigma_\tau(r_\sigma)\,(1+|\pi_{\sigma,r_\sigma}(\Omega)|)^x H(\mathrm{i}(s-\rho_o)-r_\sigma)|<\infty\,,$$

where x is an arbitrary real number. Hence, if $x>x_o$, then there exists a positive constant M such that

$$\begin{aligned}&\textstyle\sum_{r_\sigma\in\tilde{Q}^\sigma_\tau} n_T(\sigma, r_\sigma)|P^\sigma_\tau(r_\sigma)H(\mathrm{i}\,(s-\rho_o)-r_\sigma)|\\ &\leqslant M\textstyle\sum_{r_\sigma\in\tilde{Q}^\sigma_\tau} n_T(\sigma, r_\sigma)(1+|\pi_{\sigma,r_\sigma}(\Omega)|)^{-x}<\infty,\end{aligned}$$

by Proposition 3.8. Similar analysis is valid for the term involving $H(\mathrm{i}\,(s-\rho_o)+r)$. Hence the series

$$\textstyle\sum_{r_\sigma\in\tilde{Q}^\sigma_\tau} P^\sigma_\tau\,(r_\sigma)n_T(\sigma, r_\sigma)\left\{\dfrac{H(\mathrm{i}(s-\rho_o)+r_\sigma)}{s-\rho_o-\mathrm{i}r_\sigma}+\dfrac{H(\mathrm{i}(s-\rho_o)-r_\sigma)}{s-\rho_o+\mathrm{i}r_\sigma}\right\}$$

converges uniformly on any compact subsets that are disjoint from $\{\rho_o\pm\mathrm{i}r_\sigma;\ r_\sigma\in\tilde{Q}^\sigma_\tau\}$. If we turn our attention to the fact that $\sharp\{\hat{M}_\tau\}$ is finite, then the series gives us the meromorphic continuation of $A_\tau(s)$. The assertion about the poles of $A_\tau(s)$ follows from the direct calculations. This completes the proof of the proposition.

We next investigate the analytic continuation of the second term of (3.3). By definition,

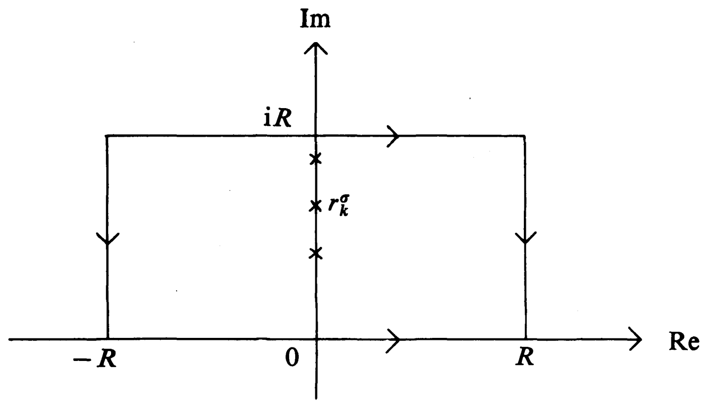
$$\begin{aligned}{}_\tau g_s(e) &= (1/2\pi[W])\textstyle\sum_{\sigma\in\hat{M}_\tau}\displaystyle\int_{-\infty}^{\infty}{}_\tau\hat{\mathscr{G}}_s(\sigma, r)\mu_\sigma(r)\,dr\\ &= (1/4\pi)\textstyle\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\displaystyle\int_{-\infty}^{\infty}P^\sigma_\tau(r)\left\{\frac{H(\mathrm{i}(s-\rho_o)-r)}{s-\rho_o+\mathrm{i}r}\right.\\ &\qquad\qquad\left.+\frac{H(\mathrm{i}(s-\rho_o)+r)}{s-\rho_o-\mathrm{i}r}\right\}\mu_\sigma(r)dr\,.\end{aligned}$$

Since $P^\sigma_\tau(r)$ and $\mu_\sigma(r)$ are even, we see that

$${}_\tau g_s(e) = (1/2\pi)\textstyle\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\displaystyle\int_{-\infty}^{\infty}P^\sigma_\tau(r)\frac{H(\mathrm{i}(s-\rho_o)+r)}{s-\rho_o-\mathrm{i}r}\,\mu_\sigma(r)\,dr\,.$$

The function $r\to\mu_\sigma(r)$ is meromorphic in the upper half plane, and can only have simple poles (see, Section 1). Let r^σ_k ($k\geqslant 0$, $\sigma\in\hat{M}_\tau$) be the poles, if any, and let d^σ_k be the residue of $\mu_\sigma(r)$ at the pole r^σ_k.

We now shift the integration into the complex plane by using a rectangular contour with vertices at $-R$, $+R$, $R+\mathrm{i}\,R$, $-R+\mathrm{i}\,R$ as in the figure below. Of course, we assume that there is no poles on the rectangular contour.



Using the residue theorem we see that

$$ {}_\tau g_s(e)=\mathrm{i}\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\sum_{\{k;\mathrm{Im}\,r_k<R\}}P_\tau^\sigma(r_k^\sigma)\,\frac{H(\mathrm{i}(s-\rho_o)+r_k^\sigma)}{s-\rho_o-\mathrm{i}r_k^\sigma}\,d_k^\sigma+I_+^R+I_-^R+J^R. $$

Here

$$ I_\pm^R = (\mathrm{i}/2\pi)\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\int_0^R P_\tau^\sigma(\pm R+\mathrm{i}\,r)\,\frac{H(\mathrm{i}(s-\rho_o)+(\pm R+\mathrm{i}\,r))}{s-\rho_o-\mathrm{i}(\pm R+\mathrm{i}\,r)} \cdot\mu_\sigma(\pm R+\mathrm{i}r)\,dr $$

and

$$ J^R = (1/2\pi)\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\int_{-R}^R P_\tau^\sigma(r+\mathrm{i}\,R)\,\frac{H(\mathrm{i}(s-\rho_o)+(r+\mathrm{i}\,R))}{s-\rho_o-\mathrm{i}(r+\mathrm{i}\,R)} \cdot\mu_\sigma(r+\mathrm{i}\,R)dr. $$

Let $0\leqslant r\leqslant R$. Then, since $\mathrm{Im}\,(\mathrm{i}\,(s-\rho_o)+(\pm R+\mathrm{i}\,r))=\mathrm{Re}\,(s-\rho_o+r)>0$, Lemma 3.1 implies that for any n ($\geqslant 0$) there is a constant $C_{n,0}$ such that

$$ |H(\mathrm{i}\,(s-\rho_o)+(\pm R+\mathrm{i}\,r))|\leqslant C_{n,0}(1+|\mathrm{i}\,(s-\rho_o+r)\pm R|)^{-n}. $$

Also, since $\mu_\sigma(\pm R+\mathrm{i}\,r)$ is a polynomial growth function, there exists a polynomial $Q(R)$ so that the integrand of $I_\pm^R$ is dominated by

$$ Q(R)\,(1+|\mathrm{i}\,(s-\rho_o+r)\pm R|)^{-n-1} $$

Since n is arbitrary, it can be easily seen that

$$ \lim_{R\to+\infty} I_\pm^R = 0. $$

Similarly one finds that $\lim_{R\to+\infty} J^R=0$. Hence we get

$$ (3.14)\qquad {}_\tau g_s(e)=\mathrm{i}\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\sum_{k=0}^\infty P_\tau^\sigma(r_k^\sigma)\,\frac{H(\mathrm{i}\,(s-\rho_o)+r_\tau^\rho)}{s-\rho_o-\mathrm{i}\,r_k^\sigma}\,d_k^\sigma. $$

Of course, if $\mu_\sigma(r)$ $(\sigma \in \hat{M}_\tau)$ has no poles in the upper half plane, that is, if $\mathfrak{g} \simeq \mathfrak{so}(2n+1, 1)$ then this sum is to be interpreted as zero.

If $\mathfrak{g} \not\simeq \mathfrak{so}(2n+1, 1)$ then, as we have seen in Section 1,

$$\mu_\sigma(r) = c \cdot d_\sigma p_\sigma(r) \times (\tanh(\pi r/b) \quad \text{or} \quad \coth(\pi r/b)),$$

where $p_\sigma(r)$ is a polynomial and b is the number of distinct positive restricted roots. Hence we see that r_k^σ is purely imaginary, $r_k^\sigma = O(k)$, and $d_k^\sigma = O(k^a)$ where a is a positive integer depending only on σ $(\in \hat{M}_\tau)$ and G.

We now claim that the series on the right side of (3.14) converges absolutely, uniformly with respect to s varying over a compact subset $\mathcal{O}$ of the complex plane, provided that $\mathcal{O}$ is disjoint from the points $\{\rho_o + \mathrm{i}\, r_k;\ k \geqslant 0,\ \sigma \in \hat{M}_\tau\}$. Indeed, for $s \in \mathcal{O}$, we see that

$$\mathrm{Im}\,(\mathrm{i}\,(s-\rho_o) + r_k^\sigma) > 0$$

for large enough k. For such k, the estimate

$$|H(\mathrm{i}\,(s-\rho_o) + r_k^\sigma)| \leqslant C_{n,0}(1 + |\mathrm{i}\,(s-\rho_o) + r_k^\sigma|)^{-n}$$

of Lemma 3.1 is available. Since s is confined to which misses $\rho_o + \mathrm{i}\, r_k^\sigma$, we get

$$|H(\mathrm{i}\,(s-\rho_o) + r_k^\sigma)| \leqslant C(\sigma, n)|r_k^\sigma|^{-n}$$

for large k, with $C(\sigma, n)$ independent of k. Using the facts on r_k^σ and d_k^σ, as we have mentioned above, we conclude by choosing n large that the series on the right side does indeed converge.

It follows that the series defines a meromorphic function of s with simple poles at the points $\rho_o + \mathrm{i}\, r_k^\sigma$ $(k \geqslant 0,\ \sigma \in \hat{M}_\tau)$, and the residue of this function at the pole $\rho_o + \mathrm{i}\, r_k^\sigma$ is equal to $\mathrm{i}\,\kappa[\sigma\colon \tau_M] P_\tau^\sigma(r_k^\sigma) d_k^\sigma$. We summarize these observations.

PROPOSITION 3.9. *For $\mathrm{Re}\, s > 2\rho_o$, we have*

$$\begin{aligned}(3.15)\quad & \chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G)_\tau g_s(e) \\ & = \mathrm{i}\chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G) \textstyle\sum_{\sigma \in \hat{M}_\tau} [\sigma\colon \tau_M] \sum_{k=0}^{\infty} P_\tau^\sigma(r_k^\sigma) \dfrac{H(\mathrm{i}(s-\rho_o)+r_k^\sigma)}{s-\rho_o - \mathrm{i} r_k^\sigma}\, d_k^\sigma .\end{aligned}$$

Here $\{r_k^\sigma;\ k \geqslant 0,\ \sigma \in \hat{M}_\tau\}$ are the poles of the function $\mu_\sigma(r)$ in the upper half plane and d_k^σ is the residue of that function at the pole r_k^σ. The series converges absolutely and uniformly for s in any compact set disjoint from $\{\rho_o + \mathrm{i}\, r_k;\ k \geqslant 0,$ $\sigma \in \hat{M}_\tau\}$, and define a meromorphic function of s in the whole complex plane. Thus it gives us a meromorphic continuation of the left side of (3.15). *This function has simple poles at the points $\rho_o + \mathrm{i}\, r_k^\sigma$, $k \geqslant 0$, $\sigma \in \hat{M}_\tau$, and has the residue $\mathrm{i}\,\kappa\chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G)[\sigma\colon \tau_M] P_\tau^\sigma(r_k^\sigma) d_k^\sigma$ at the pole $\rho_o + \mathrm{i}\, r_k^\sigma$.*

We now have the following proposition on account of the studies that we have seen up to the present.

PROPOSITION 3.10. *For* $\mathrm{Re}\, s > 2\rho_o$, *the function*

$$\eta_{\tau,T}(s) = \kappa \sum_{\sigma\in\hat{M}_\tau} [\sigma: \tau_M] P^\sigma_\tau(\mathrm{i}\,(s-\rho_o)) \cdot \sum_{\gamma\in C_\Gamma\setminus\{e\}} \chi_T(\gamma)\overline{\chi_\sigma(m_\gamma)}\, j(\gamma)^{-1} u_\gamma C(\gamma) \exp(-(s-\rho_o)u_\gamma)$$

is holomorphic. Moreover, this function has meromorphic continuation to the whole complex plane, via the relation

$$\eta_{\tau,T}(s) = A_\tau(s) - \chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G)_\tau g_s(e). \tag{3.16}$$

The poles of $\eta_{\tau,T}$ *are all simple, and are as follows*:

Pole	Residue	
$\rho_o \pm \mathrm{i}\, r_\sigma$	$\kappa n_T(\sigma, r_\sigma) P^\sigma_\tau(r_\sigma)[\sigma: \tau_M]$	$r_\sigma \in \tilde{Q}^\sigma_\tau(\sigma \in \hat{M}_\tau)$
$\rho_o + \mathrm{i}\, r^\sigma_k$	$-\mathrm{i}\,\kappa\chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G)[\sigma: \tau_M] P^\sigma_\tau(r^\sigma_k) d^\sigma_k$	$k \geqslant 0,\ \sigma \in \hat{M}_\tau$.

Here, of course, if $P^\sigma_\tau(r^\sigma_k)=0$ *for some* k, σ, *then we understand that there is no pole at* $\rho_o+\mathrm{i}\, r^\sigma_k$, *and if* $r^\sigma_k = r^\xi_j$ *for some* (σ, k), (ξ, j), *then the residue of* $\eta_{\tau,T}(s)$ *at* $s=\rho_o+\mathrm{i}\, r^\sigma_k$ *is* $-\mathrm{i}\,\kappa\chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G)([\sigma: \tau_M] P^\sigma_\tau(r^\sigma_k) d^\sigma_k + [\xi: \tau_M] P^\xi_\tau(r^\xi_j) d^\xi_j)$. *Moreover, if* $0 \in \tilde{Q}^\sigma_\tau$ *then the residue of* $\eta_{\tau,T}(s)$ *at* $s=\rho_o$ *is* $2\kappa n_T(\sigma, 0) P^\sigma_\tau(0)[\sigma: \tau_M]$, *and if* $r^\sigma_k = r_\xi$ *for some* (σ, k), ξ, *then the residue at* $s=\rho_o+\mathrm{i}\, r^\sigma_k$ *is* $\kappa(n_T(\xi, r_\xi) P^\xi_\tau(r_\xi)[\xi: \tau_M] - \mathrm{i}\,\chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G)[\sigma: \tau_M] P^\sigma_\tau(r^\sigma_k) d^\sigma_k)$.

4. Another expression of $\eta_{\tau,T}$

In order to prove the functional equation of $\eta_{\tau,T}$ we now define a new function $L_{\tau,T}$, which we can call a sort of modified theta function. For the purpose of defining this function we need the following proposition.

PROPOSITION 4.1. *Let* $\tau \in \hat{K}$. *For each* $t>0$, *define the function* ${}_\tau h_t(x)$ *on* G *by*

$${}_\tau h_t(x) = d_\tau^{-1}[W]^{-1} \sum_{\sigma\in\hat{M}_\tau} \exp\lambda_\sigma t \int_{\mathfrak{a}^*_{\mathfrak{p}}} \mathrm{tr}\,(E_\tau \pi_{\sigma,\nu}(x) E_\tau) P^\sigma_\tau(\nu) \exp\lambda_{\sigma,\nu} t \mu_\sigma(\nu) d\nu.$$

Then ${}_\tau h_t$ *possesses the following properties*:

(i) $d_\tau \cdot \chi_\tau * {}_\tau h_t = d_\tau \cdot {}_\tau h_t * \chi_\tau = {}_\tau h_t$.

(ii) ${}_\tau h_t \in \mathscr{C}^1(G)$.

(iii) $\Theta_{\sigma,\nu}({}_\tau h_t) = [\sigma: \tau_M] P^\sigma_\tau(r(\nu)) \exp(-(r(\nu)^2 + \rho_o^2)t)$.

This proposition follows immediately from the result in [21, Theorem 4.12], so we omitt the proof.

This proposition implies that the function ${}_\tau h_t$ is admissible. We now put the function ${}_\tau h_t$ into the trace formula.

By means of (iii) of Proposition 4.1 and (1.2), we have

$$F_{{}_\tau h_t}(\gamma) = (1/2\pi)(\textstyle\sum_{\sigma\in\hat{M}_\tau}\overline{\chi_\sigma(m_\gamma)}\displaystyle\int_{-\infty}^{\infty}\Theta_{\sigma,r}({}_\tau h_t)\exp(-iru_\gamma)dr.$$

Hence we get the trace formula

$$\begin{aligned}(4.1)\qquad &\textstyle\sum_{\pi\in\hat{G}} n_{\Gamma,T}(\pi)\Theta_\pi({}_\tau h_t) - \chi_T(e)\operatorname{vol}(\Gamma\backslash G)_\tau h_t(e)\\ &= (1/2\pi)\textstyle\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\sum_{\gamma\in C_\Gamma\setminus\{e\}}\chi_T(\gamma)\overline{\chi_\sigma(m_\gamma)}j(\gamma)^{-1}u_\gamma C(\gamma)\\ &\cdot\int_{-\infty}^{\infty}\exp(-(r^2+\rho_o^2)t)P_\tau^\sigma(r)\exp(-\mathrm{i}\,ru_\gamma)\end{aligned}$$

Since

$$(1/2\pi)\int_{-\infty}^{\infty}\exp(-(r^2+\rho_o^2)t)\exp(-\mathrm{i}\,ru_\gamma)\,dr = \frac{1}{\sqrt{4\pi t}}\exp(-(\rho_o^2 t+u_\gamma^2/4t)),$$

according to the definition of D_τ^σ we see that

$$(1/2\pi)\int_{-\infty}^{\infty}\exp(-(r^2+\rho_o^2)t)P_\tau^\sigma(r)\exp(-\mathrm{i}\,ru_\gamma)\,dr = \frac{1}{\sqrt{4\pi t}}D_\tau^\sigma(\exp(-(\rho_o^2 t+u_\gamma^2/4t))).$$

Therefore, if we define $L_{\tau,T}$ by the left side of (4.1), we see that

$$\begin{aligned}(4.2)\quad L_{\tau,T}(t) = &\textstyle\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\sum_{\gamma\in C_\Gamma-\{e\}}\chi_T(\gamma)\chi_\sigma(m_\gamma)j(\gamma)^{-1}\\ &\cdot u_\gamma C(\gamma)\frac{1}{\sqrt{4\pi t}}D_\tau^\sigma(\exp(-(\rho_o^2 t+u_\gamma^2/4t))).\end{aligned}$$

Since ${}_\tau h_t$ is admissible, we note the fact that the series (4.2) converges absolutely for $t>0$.

By the very definition of D_τ^σ, D_τ^σ can be written as the form

$$D_\tau^\sigma = C\textstyle\prod_{\sigma\in\hat{M}_\tau}\prod_{j=1}^m\left(\dfrac{d^2}{du^2}+c_{j,\sigma}\right)^{N_j},$$

where C and $c_{j,\sigma}$ are certain constants. But, since

$$\left(\frac{d^2}{du^2}+c\right)\exp(-u^2/4t) = \left(\frac{u^2}{4t^2}-\frac{1}{2t}+c\right)\exp(-u^2/4t),$$

we can easily see that

$$(4.3)\qquad D_\tau^\sigma(\exp(-(\rho_o^2 t+u_\gamma^2/4t))) = P(u_\gamma, t^{-1})\exp(-(\rho_o^2 t+u_\gamma^2/4t)),$$

where $P(u, x)$ is a polynomial in two variable u and x, with the property that if we write

$$P(u, x) = \textstyle\sum_{i,j}a_{i,j}u^i x^j,$$

then $a_{i,j}=0$ in case of $i>j$.

Recall the fact that the constant ε_o is defined to be satisfied $0<\varepsilon_o\leqslant u_\gamma$ for all $\gamma\in C_\Gamma\backslash\{e\}$. We now find the following lemma.

LEMMA 4.2. *There exists a constant M such that*

$$\sup_{t>0,\gamma\in C_\Gamma\backslash\{e\}}|P(u_\gamma, t)\exp(-(u_\gamma-\varepsilon_o/4)t)|\leqslant M.$$

PROOF. Since $u_\gamma\geqslant\varepsilon_o$, we have $t\leqslant\varepsilon_o^{-1}u_\gamma t$. Hence, if $i\leqslant j$ then

$$\begin{aligned} u_\gamma^i t^j &= (u_\gamma t)^i t^{j-i} \\ &\leqslant (u_\gamma t)^i(\varepsilon_o^{-1}u_\gamma t)^{j-i} = \varepsilon_o^{i-j}(u_\gamma t)^j\,. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} |P(u_\gamma, t)| &\leqslant \textstyle\sum_{i,j}|a_{i,j}|u_\gamma^i t^j \\ &\leqslant \textstyle\sum_{i,j}|a_{i,j}|\varepsilon_o^{i-j}(u_\gamma t)^j, \end{aligned}$$

where we used the property that $a_{i,j}=0$ if $i>j$. Now we put

$$\tilde{P}(x) = \textstyle\sum_{i,j}|a_{i,j}|\varepsilon_o^{j-i}x^j.$$

On the other hand, one finds that

$$\exp(-(u_\gamma-\varepsilon_o/4)t)\leqslant\exp(-u_\gamma t)\exp(u_\gamma t/4) = \exp(-(3u_\gamma t/4))\,.$$

Hence

$$\begin{aligned} &\sup_{t>0,\,\gamma\in C_\Gamma\backslash\{e\}}|P(u_\gamma, t)\exp(-(u_\gamma-\varepsilon_o/4)t)| \\ &\quad\leqslant\sup_{t>0,\,\gamma\in C_\Gamma\backslash\{e\}}|\tilde{P}(u_\gamma t)\exp(-(3u_\gamma t/4))| \\ &\quad\leqslant\sup_{x>0}\tilde{P}(x)\exp\Big(-\frac{3}{4}x\Big)<+\infty. \end{aligned}$$

The lemma now follows.

Now we have the following theorem, which asserts the fact that the function $\eta_{\tau,T}(s)$ is related via an integral transform to a sort of modified theta function $L_{\tau,T}(t)$.

THEOREM 4.3. *Suppose that* $\mathrm{Re}\,s>2\rho_o$. *Then we have*

$$\eta_{\tau,T}(s) = 2\kappa(s-\rho_o)\int_0^\infty\exp(-s(s-2\rho_o)t)L_{\tau,T}(t)dt.$$

PROOF. First we assume that s is real and $s>2\rho_o$. Note that $u_\gamma^2=(u_\gamma-\varepsilon_o/2)^2+(u_\gamma-\varepsilon_o/4)\varepsilon_o$. By (4.3) we get

$$\begin{aligned} &|D_\tau^\sigma(\exp(-(\rho_o^2 t+u_\gamma^2/4t)))| \\ &\leqslant\Big|P\Big(u_\gamma,\frac{1}{t}\Big)\exp\Big(-\Big(u_\gamma-\frac{\varepsilon_o}{4}\Big)\varepsilon_o/4t\Big)\Big|\exp\Big(-\Big(\rho_o^2 t+\Big(u_\gamma-\frac{\varepsilon_o}{2}\Big)^2/4t\Big)\Big). \end{aligned}$$

The assertion of Lemma 4.2 implies that one can find a constant $\tilde{M}$ such that

$$\sup_{t>0,\gamma\in C_\Gamma\setminus\{e\}}\left|P\left(u_\gamma, \frac{1}{t}\right)\exp\left(-\left(u_\gamma-\frac{\varepsilon_o}{4}\right)\varepsilon_o/4t\right)\right|\leqslant \tilde{M}.$$

Hence we obtain

$$\begin{aligned}&\int_0^\infty |\exp(-s(s-2\rho_o)t)\frac{1}{\sqrt{4\pi t}}D_\tau^\sigma(\exp(-(\rho_o^2t+u_\gamma^2/4t)))|dt\\ &\quad\leqslant \hat{M}\int_0^\infty \frac{1}{\sqrt{4\pi t}}\exp\left(-\left(\left(s-\rho_o\right)^2 t+\left(u_\gamma-\frac{\varepsilon_o}{2}\right)^2/4t\right)\right)dt\\ &\quad= \tilde{M}(2(s-\rho_o))^{-1}\exp\left(-(s-\rho_o)\left(u_\gamma-\frac{\varepsilon_o}{2}\right)\right)\\ &\quad= \tilde{M}(2(s-\rho_o))^{-1}\exp((s-\rho_o)\varepsilon_o/2)\exp(-(s-\rho_o)u_\gamma).\end{aligned}$$

Here we used the well known formula

$$\int_0^\infty \frac{1}{\sqrt{4\pi t}}\exp(-(x^2t+y^2/4t))dt=(2x)^{-1}\exp(-xy), \tag{4.4}$$

valid for $x>0$, $y>0$.

Therefore we have an estimate

$$\begin{aligned}&2\kappa(s-\rho_o)\int_0^\infty \exp(-s(s-\rho_o)t)\textstyle\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\sum_{\gamma\in C_\Gamma\setminus\{e\}}\chi_T(\gamma)\\ &\qquad\qquad\cdot\overline{\chi_\sigma(m_\gamma)}j(\gamma)^{-1}u_\gamma C(\gamma)\frac{1}{\sqrt{4\pi t}}D_\tau^\sigma(\exp(-(\rho_o^2t+u_\gamma^2/4t)))\\ &\leqslant \kappa\tilde{M}\exp(s-\rho_o)\varepsilon_o/2\cdot\textstyle\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\sum_{\gamma\in C_\Gamma\setminus\{e\}}\chi_T(\gamma)\overline{\chi_\sigma(m_\gamma)}j(\gamma)^{-1}\cdot\\ &\qquad\qquad\cdot u_\gamma C(\gamma)\exp(-(s-\rho_o)u_\gamma),\end{aligned}$$

for s real and $s>2\rho_o$. But, since the series $\eta_{\tau,T}(s)$ converges absolutely and uniformly on compacts of $\operatorname{Re}s>2\rho_o$ as we have seen in Proposition 3.5, it is clear that the series of the above expression converges on compacts with respect to s $(>2\rho_o)$. Hence the dominated convergence theorem implies that

$$\begin{aligned}&2\kappa(s-\rho_o)\int_0^\infty \exp(-(s-\rho_o)t)L_{\tau,T}(t)dt\\ &\quad=2\kappa(s-\rho_o)\textstyle\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\sum_{\gamma\in C_\Gamma\setminus\{e\}}\chi_T(\gamma)\overline{\chi_\sigma(m_\gamma)}j(\gamma)^{-1}\\ &\quad\cdot u_\gamma C(\gamma)\int_0^\infty \exp(-s(s-\rho_o)t)\frac{1}{\sqrt{4\pi t}}D_\tau^\sigma(\exp(-(\rho_o^2t+u_\gamma^2/4t)))dt.\end{aligned} \tag{4.5}$$

Since

$$\frac{1}{\sqrt{4\pi t}}\exp(-s(s-\rho_o)t)\exp(-(\rho_o^2t+u_\gamma^2/4t))\leqslant\frac{1}{\sqrt{4\pi t}}\exp(-(s-\rho_o)^2t+\varepsilon_o^2/4t),$$

and the right side of this inequality is integrable by means of the formula (4.4), the same argumentation that we have used in the above says that we can change the order of the differentiation and the integration in (4.5). If we use the formula (4.4) again, then

$$\begin{aligned}
&\int_0^\infty \exp(-s(s-\rho_o)t)\frac{1}{\sqrt{4\pi t}}D^\sigma_\tau(\exp(-(\rho_o^2 t+u^2/4t)))dt \\
&= D^\sigma_\tau\Big(\int_0^\infty \frac{1}{\sqrt{4\pi t}}\exp(-(s-\rho_o)^2 t + u^2/4t)dt\Big) \\
&= (2(s-\rho_o))^{-1}D^\sigma_\tau(\exp(-(s-\rho_o)u)) \\
&= (2(s-\rho_o))^{-1}P^\sigma_\tau(\mathrm{i}(s-\rho_o))\exp(-(s-\rho_o)u),
\end{aligned}$$

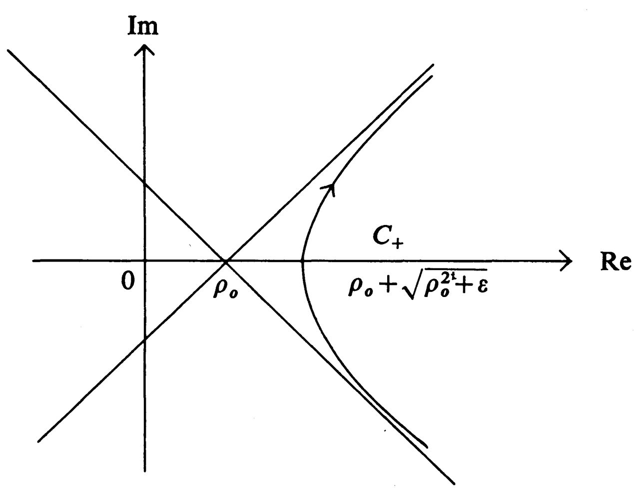
for $u \geqslant \varepsilon_o$. Hence, by (4.5) we obtain for s real and $s > 2\rho_o$,

$$2\kappa(s-\rho_o)\int_0^\infty \exp(-s(s-\rho_o)t)L_{\tau,T}(t) = \eta_{\tau,T}(s).$$

The procedure can be justified easily by the analytic continuation, since the function $\eta_{\tau,T}(s)$ is holomorphic in the half plane $\operatorname{Re} s > 2\rho_o$. This completes the proof of Theorem 4.3.

We now have the following inversion formula.

COROLLARY 4.4. *Let ε be a positive number. Let us consider the part of hyperbolic curve $C_+ = \{s = \sigma + \mathrm{i}\mu;\ (\sigma-\rho_o)^2 - \mu^2 = \rho_o^2 + \varepsilon,\ \sigma > \rho_o\}$, as in the figure below.*



Then we have

(4.6) $$L_{\tau,T}(t) = \frac{1}{2\pi i\kappa}\int_{C_+} \exp(s(s-2\rho_o)t)\eta_{\tau,T}(s)ds,$$

for $t>0$.

PROOF. Note the fact that $\mathrm{Re}\, s(s-2\rho_o)=(\sigma-\rho_o)^2-\rho_o^2-\mu^2=\varepsilon>0$ and $\mathrm{Re}\, s>2\rho_o$ for $s=\sigma+i\mu\in C_+$. Hence the formula of Theorem 4.3 is available for any s in C_+. For s in C_+, we put $s=\rho_o+\sqrt{\rho_o^2+p}$ $(s(s-2\rho_o)=p)$ on that formula. Then one finds that

$$\frac{\eta_{\tau,T}(\rho_o+\sqrt{\rho_o^2+p})}{2\kappa\sqrt{\rho_o^2+p}} = \int_0^\infty \exp(-pt)L_{\tau,T}(t)dt,$$

for $\mathrm{Re}\, p>0$. This says that the function $\eta_{\tau,T}(\rho_o+\sqrt{\rho_o^2+p})/2\kappa\sqrt{\rho_o^2+p}$ is a Laplace transform of the function $L_{\tau,T}(t)$. Hence, by the Laplace inversion formula we see that

$$L_{\tau,T}(t) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \exp(pt)\ \frac{\eta_{\tau,T}(\rho_o+\sqrt{\rho_o^2+p})}{2\kappa\sqrt{\rho_o^2+p}}\,dp.$$

Since $p=s(s-2\rho_o)$, we see that $ds=dp/2\sqrt{\rho_o^2+p}$. Hence we obtain the desired formula.

REMARK. Let Ω_K be the Casimir operator of K. Then there exists a scalar λ_τ such that $\tau(\Omega_K)=(2p+8q)\lambda_\tau \mathrm{I}$. Now let $D=-\Omega+\lambda_\tau \mathrm{I}$ (see §0). Consider the vector bundle $\Gamma\backslash E_\tau=\Gamma\backslash G\times_\tau V_\tau\to\overline{M}=\Gamma\backslash G/K$. Then D is a second order, elliptic, formally selfadjoint differential operator on $\Gamma\backslash E_\tau$. The spectrum of D is the sequence of eigenvalues $0=\lambda_0<\lambda_1<\lambda_2<\cdots$, and $\lim_{i\to\infty}\lambda_i=\infty$. Let $C^\infty(\Gamma\backslash E_\tau\colon \overline{M})$ denote the space of C^∞ cross-sections of $\Gamma\backslash E_\tau$. Put $C_\lambda^\infty(\Gamma\backslash E_\tau\colon\overline{M})=\{f\in C^\infty(\Gamma\backslash E_\tau\colon\overline{M});\ Df=\lambda f\}$. As it is well known, e^{-tD} exists and is of trace class for $t>0$. Moreover, if we put $m_i=\dim C_{\lambda_i}^\infty(\Gamma\backslash E_\tau\colon\overline{M})$ then

$$\mathrm{tr}(e^{-tD}) = \textstyle\sum_{i=0}^\infty m_i e^{-\lambda_i t}.$$

N. Wallach shows in [30] that this can be written

$$= \textstyle\sum_{\pi\in\hat{G}} n_{\Gamma,1}(\pi)\,[\tau\colon\pi|_K]e^{(\lambda_\pi-\lambda_\tau)t}.$$

Here $(2p+8q)^{-1}\lambda_\pi$ is the eigenvalue of Ω on the class $\pi\in\hat{G}$.

If we define the function ${}_\tau\tilde{h}_t(x)$ $(t>0)$ on G, which is similar to the function ${}_\tau h_t$, by

$${}_\tau\tilde{h}_t(x) = d_\tau^{-1}[W]^{-1}e^{-\lambda_\tau t}\textstyle\sum_{\sigma\in\hat{M}_\tau}\int_{\mathfrak{a}_\mathfrak{p}^*}\mathrm{tr}(E_\tau\pi_{\sigma,\nu}(x)E_\tau)e^{\lambda_{\sigma,\nu}t}\mu_\sigma(\nu)d\nu,$$

then we find that $\Theta_{\sigma,\nu}({}_\tau\tilde{h}_t)=[\sigma\colon\tau_M]e^{(\lambda_{\sigma,\nu}-\lambda_\tau)t}$. In general, ${}_\tau\tilde{h}_t$ does not belong to $\mathscr{C}^1(G)$, so it is not admissible. But, at least formally, one finds that

$$\operatorname{tr}(e^{-tD}) = \sum_{\pi\in\hat{G}} n_{\Gamma,1}(\pi)\Theta_\pi({}_\tau\tilde{h}_t).$$

For this reason, we called $I_{\tau,T}$ a sort of modified theta function. One can find details in [5], [21] and [30] on these matters.

5. Functional equation of $\eta_{\tau,T}$

In the first place, we prove several estimates of $\eta_{\tau,T}$ that we will need later on.

LEMMA 5.1. *Put*

$$A_\tau^\sigma(s) = \sum_{r_\sigma\in\tilde{Q}_\tau^\sigma} P_\tau^\sigma(r_\sigma) n_T(\sigma, r_\sigma)\left\{\frac{H(\mathrm{i}(s-\rho_o)+r_\sigma)}{s-\rho_o-\mathrm{i}r_\sigma} + \frac{H(\mathrm{i}(s-\rho_o)-r_\sigma)}{s-\rho_o+\mathrm{i}r_\sigma}\right\}.$$

Then by (3.13) *we have*

$$A_\tau(s) = \sum_{\sigma\in\hat{M}_\tau} [\sigma:\tau_M] A_\tau^\sigma(s).$$

For any real numbers a, b $(a<b)$, *we take a subset* $\mathcal{O}_1$ *of* $\{s\in \boldsymbol{C};\ a\leqslant \operatorname{Re} s\leqslant b\}$ *which satisfies a condition that* $\mathcal{O}_1\cap\{\rho_o\pm \mathrm{i}\, r_\sigma;\ r_\sigma\in\tilde{Q}_\tau^\sigma\}=\emptyset$. *Then there exists a polynomial* P_1 *such that*

$$|A_\tau^\sigma(s)| \leqslant |P_1(\operatorname{Im} s)|,$$

for any s *in* $\mathcal{O}_1$.

PROOF. Since the set $\{\rho_o+\mathrm{i}\, r_\sigma;\ r_\sigma\in\tilde{Q}_\tau^\sigma\}(\subset\{\operatorname{Re} s=\rho_o\}\cup\{s=\mathrm{i}x;\ 0\leqslant x\leqslant 2\rho_o\})$ has no finite points of accumulation, there exists a positive constant δ such that

$$\operatorname{Inf}_{s\in\mathcal{O}_1, r_\sigma\in\tilde{Q}_\tau^\sigma} |s-\rho_o+\mathrm{i}\, r_\sigma| \geqslant \delta\ (>0),$$

by means of the definition of $\mathcal{O}_1$. Also, by the same reason, there exists a polynomial P and a constant C so that

$$\sum_{r_\sigma\in[n,n+1]\cap\tilde{Q}_\sigma^\tau} |n_T(\sigma, r_\sigma)P_\tau^\sigma(r_\sigma)| \leqslant P(n),$$

for any $n\in \boldsymbol{Z}$, and

$$\sum_{r_\sigma\in\mathrm{i}[0,2\rho_o]\cap\tilde{Q}_\tau^\sigma} |n_T(\sigma, r_\sigma)P_\tau^\sigma(r_\sigma)| \leqslant C.$$

Now we have

$$|A_\tau^\sigma(s)| \leqslant \delta^{-1}\sum_{r_\sigma\in\tilde{Q}_\tau^\sigma} n_T(\sigma, r_\sigma)\,|P_\alpha^\sigma(r_\sigma)|\cdot\{|H(\mathrm{i}\,(s-\rho_o)+r_\sigma)| + |H(\mathrm{i}\,(s-\rho_o)-r_\sigma)|\}$$

$$= \delta^{-1}[\sum_{n\in\boldsymbol{Z}}\sum_{r_\sigma\in[n,n+1]\cap\tilde{Q}_\tau^\sigma} n_T(\sigma, r_\sigma)|P_\tau^\sigma(r_\sigma)| \cdot\{|H(\mathrm{i}\,(s-\rho_o)+r_\sigma)| + |H(\mathrm{i}\,(s-\rho_o)-r_\sigma)|\} \tag{5.1}$$

$$+\sum_{r_\sigma\in \mathrm{i}[0,2\rho_o]\cap \tilde{Q}_\tau^\sigma} n_T(\sigma, r_\sigma)|P_\tau^\sigma(n_\sigma)|$$
$$\cdot\{|H(\mathrm{i}\,(s-\rho_o)+r_\sigma)|+|H(\mathrm{i}\,(s-\rho_o)-r_\sigma)|\}].$$

If we put

$$K(z) = \sup_{x\in[-1,1]}|H(z+x)|,$$

then, since the function H is rapidly decreasing, K is also rapidly decreasing. Hence we see that

$$\begin{aligned}&\sum_{n\in Z}\sum_{r_\sigma\in[n,n+1]\cap \tilde{Q}_\tau^\sigma} n_T(\sigma, r_\sigma)|P_\tau^\sigma(r_\sigma)H(\mathrm{i}\,(s-\rho_o)+r_\sigma)|\\ &\quad\leqslant \sum_{n\in Z} K(\mathrm{i}\,(s-\rho_o)+n)\sum_{r_\sigma\in[n,n+1]\cap \tilde{Q}_\tau^\sigma} n_T(\sigma, r_\sigma)|P_\tau^\sigma(r_\sigma)|\\ &\quad\leqslant \sum_{n\in Z} K(\mathrm{i}\,(s-\rho_o)+n)P(n).\end{aligned}$$

The fact that K is rapidly decreasing implies that there exists a polynomial Q, with $\deg Q$ as large as we please, so that

$$K(\mathrm{i}\,(s-\rho_0)+n) \leqslant Q(|n-\mathrm{Im}\,s|)^{-1},$$

for any s in $\mathcal{O}_1$ (see, Lemma 3.1). Hence the last expression of the above inequality is dominated by

$$\sum_{n\in Z} Q(|n-\mathrm{Im}\,s|)^{-1}P(n).$$

If $\deg Q\geqslant 2+\deg P$ then it is easy to show that $\sum_{n\in Z} Q(|n-\mathrm{Im}\,s|)^{-1}P(n)$ is of polynomial growth as a function of s.

On the other hand, since the set $\{n_\sigma\in \mathrm{i}z[0, 2\rho_o]\cap \tilde{Q}_\tau^\sigma\}$ is finite, the function defined by the sum with respect to $r_\sigma\in \mathrm{i}\,[0, 2\rho_o]\cap \tilde{Q}_\tau^\sigma$ is rapidly decreasing relative to $\mathrm{Im}\,s$ on account of Lemma 3.1.

Finally, similar analysis on the term involving $H(\mathrm{i}\,(s-\rho_0)-r_\sigma)$ shows that $|A_\tau^\sigma(s)|$ is of at most polynomial growth in $s\in\mathcal{O}_1$. This proves the assertion of our Lemma.

LEMMA 5.2. *Set*

$${}_\tau G_\tau^\sigma = \sum_{k>0} P_\tau^\sigma\,(r_k^\sigma)\,\frac{H(\mathrm{i}\,(s-\rho_o)+r_k^\sigma)}{s-\rho_o-\mathrm{i}\,r_k^\sigma}\,d_k^\sigma.$$

Then by (3.15) *we have*

$${}_\tau g_s(e) = \sum_{\sigma\in\hat{M}_\tau}[\sigma;\tau_M]_\tau G_s^\sigma.$$

Let δ be a positive number. Let a and b be real numbers so that $a<b$. Put $\mathcal{O}_2=\{s;\ |\mathrm{Im}\,s|\geqslant\delta,\ a\leqslant\mathrm{Re}\,s\leqslant b\}$. Is $s\in\mathcal{O}_2$ then there exists a polynomial P_2 such that

$$|{}_\tau G_s^\sigma| \leqslant |P_2(\operatorname{Im} s)|.$$

PROOF. Since r_k^σ is pure imaginary, it is easy to see that

$$|s-\rho_o-\mathrm{i}\, r_k^\sigma| \geqslant |\operatorname{Im} s| \geqslant \delta,$$

for any s in $\mathcal{O}_2$. Also, since $|r_k^\sigma|=O(k)$ and $|d_k^\sigma|=O(k^c)$ for some positive c (cf. Section 3), there is a polynomial P so that $|P_\tau^\sigma(r_k^\sigma)d_k^\sigma|\leqslant P(k)$. On the other hand, we see that $\operatorname{Im}(\mathrm{i}(s-\rho_o)+r_k^\sigma)>0$ for large enough k. For such k, we find that for any integer n, one can find a constant C_n such that

$$|H(\mathrm{i}(s-\rho_o)+r_k^\sigma)| \leqslant C_n(1+|\operatorname{Im} s+k|)^{-n},$$

because of Lemma 3.1. Hence,

$$|{}_\tau G_s^\sigma| \leqslant C_n \textstyle\sum_{k>0} P(k)(1+|\operatorname{Im} s+k|)^{-n}.$$

Since n is arbitrary, we conclude by choosing n large that $|{}_\tau G_s^\sigma|$ is of at most polynomial growth in $\operatorname{Im} s$ ($s\in\mathcal{O}_2$) as claimed.

The following lemma is an immediate consequence from the relation

$$\eta_{\tau,T}(s) = \textstyle\sum_{\sigma\in\hat{M}_\tau}[\sigma\colon \tau_M]\{A_\tau^\sigma(s) - \mathrm{i}\,\chi_T(e)\operatorname{vol}(\Gamma\backslash G)_\tau G_\tau^\sigma\},$$

and above two lemmas.

LEMMA 5.3. *There is a polynomial P_3 such that*

$$|\eta_{\tau,T}(s)| \leqslant |P_3(\operatorname{Im} s)|,$$

for all s in $\mathcal{O}_1\cap\mathcal{O}_2$.

The following lemma is a immediate result of [6, Theorem 4.4].

LEMMA 5.4. *Define for any $j\geqslant 0$,*

$$Q(j) = \#\{\gamma\in C_\Gamma\backslash\{e\};\, u_\gamma\leqslant j\}.$$

Then we have

$$2\rho_o j\cdot\exp(-2\rho_0 j)Q(j)\longrightarrow 1 \quad \text{as} \quad j\longrightarrow\infty.$$

Using Lemma 5.4, we get the following estimate.

LEMMA 5.5. *Suppose that* $\operatorname{Re} s>2\rho_o$. *Then there exists a polynomial M such that the following estimate holds*:

$$|\eta_{\tau,T}(s)| \leqslant |M(\operatorname{Im} s)\exp(-(s-\rho_o)\varepsilon_o/2)|.$$

PROOF. If we put

$$\tilde{Q}(j) = \sharp\{\gamma \in C_\Gamma \backslash \{e\}\,;\, j \leqslant u_\gamma < j+1\},$$

then, by Lemma 5.4, one can find a positive integer j_o such that the following estimate holds: If $j \geqslant j_o$ then there is a positive constant c_o such that

$$\tilde{Q}(j) \leqslant c_o j^{-1} \exp 2\rho_o j.$$

Let

$$F^\sigma_{\tau,T}(s) = \sum_{\gamma \in C_\Gamma \backslash \{e\}} \chi_T(\gamma)\overline{\chi_\sigma(m_\gamma)}\, j(\gamma)^{-1} u_\gamma C(\gamma) \exp(-(s-\rho_0)u_\gamma).$$

Then it is clear that

$$\eta_{\tau,T}(s) = \sum_{\sigma \in \hat{M}_\tau} [\sigma\colon \tau_M] P^\sigma_\tau(\mathrm{i}\,(s-\rho_o)) F^\sigma_{\tau,T}(s).$$

We now split up sums of $F^\sigma_{\tau,T}(s)$ on $C_\Gamma \backslash \{e\}$ into two sums over $\{\gamma \in C_\Gamma \backslash \{e\};\ u_\gamma \leqslant j_o\}$ and $\{\gamma \in C_\Gamma \backslash \{e\};\ u > j_o\}$ and denote them by $\sum_1$ and $\sum_2$ respectively.

Since the set $\{\gamma \in C_\Gamma \backslash \{e\};\ u_\gamma \leqslant j_o\}$ is finite, if we note the fact that $u_\gamma \geqslant \varepsilon_o$, then we can find a constant c_1 such that

$$|\textstyle\sum_1 \chi_T(\gamma)\overline{\chi_\sigma(m_j)}\, j(\gamma)^{-1} u_\gamma C(\gamma) \exp(-(s-\rho_o)u_\gamma)| \leqslant c_1 |\exp(-(s-\rho_o)\varepsilon_o)|$$

for $\mathrm{Re}\, s > 2\rho_o$.

On the other hand, we obtain the following estimates:

$$|\chi_T(\gamma)\chi_\sigma(m_\gamma)| \leqslant c_2, \quad |j(\gamma)^{-1}| \leqslant 1 \quad \text{and}$$

$$C(\gamma) = \exp(-u_\gamma \rho_o) |\det(\mathrm{Ad}\,(h(\gamma))^{-1} - \mathrm{I})|_n|^{-1} \leqslant c_3 \exp(-u_\gamma \rho_o).$$

Here c_2 and c_3 are certain constants.

Therefore, we see that

$$\begin{aligned}
&|\textstyle\sum_2 \chi_T(\gamma)\chi_\sigma(m_\gamma) j(\gamma)^{-1} u_\gamma C(\gamma) \exp(-(s-\rho_o)u_\gamma)| \\
&\quad \leqslant c_2 c_3 \textstyle\sum_2 u_\gamma |\exp(-(s-\rho_o)u_\gamma)| \exp(-u_\gamma \rho_o) \\
&\quad \leqslant c_2 c_3 \textstyle\sum_{j > j_o} (j+1) |\exp(-sj)| \tilde{Q}(j) \\
&\quad \leqslant c_o c_2 c_3 \textstyle\sum_{j > j_o} (j+1) j^{-1} |\exp(-(s-2\rho_o)j)| \\
&\quad \leqslant c_o c_2 c_3 (j_o+1) j_o^{-1} \textstyle\sum_{j > j_o} |\exp(-(s-2\rho_o)j)| \\
&\quad = c_o c_2 c_3 (j_o+1) j_o^{-1} \dfrac{|\exp(-(s-2\rho_o)j_o)|}{1 - |\exp(-(s-2\rho_o)|},
\end{aligned}$$

if $\mathrm{Re}\, s > 2\rho_o$. Since $-\mathrm{Re}\,(s-2\rho_o) j_o = -\mathrm{Re}\,(s-\rho_o)\, j_o + \rho_o j_o \leqslant -\mathrm{Re}\,(s-\rho_o)\varepsilon_o + \rho_o j_o$, the last expression of the above inequality is dominated by

$$c_o c_2 c_3 (j_o+1) j_o^{-1} \exp(\rho_o j_o) |\exp(-(s-\rho_o)\varepsilon_o)|.$$

Therefore, if $\mathrm{Re}\, s > 2\rho_o$ then there is a constant K such that

$$|F^{\sigma}_{\tau,T}(s)| \leqslant K|\exp(-(s-\rho_o)\varepsilon_o)|.$$

Hence we obtain

$$|\eta_{\tau,T}(s)| \leqslant \kappa K\{\textstyle\sum_{\sigma\in\hat{M}_\tau}|f^{\sigma}_{\tau}(s)|\}|\exp(-(s-\rho_o)\varepsilon_o/2)|,$$

where we put

$$f^{\sigma}_{\tau}(s) = [\sigma\colon \tau_M]P^{\sigma}_{\tau}(\mathrm{i}(s-\rho_o))\exp(-(s-\rho_o)\varepsilon_o/2),\ (\sigma\in\hat{M}_\tau).$$

But, since the function $f^{\sigma}_{\tau}(s)$ is rapidly decreasing with respect to the variable $\mathrm{Re}\, s$ and is of polynomial growth with respect to the variable $\mathrm{Im}\, s$, there exists a polynomial R^{σ}_{τ} such that

$$|f^{\sigma}_{\tau}(s)| \leqslant |R^{\sigma}_{\tau}(\mathrm{Im}\, s)|,$$

for each $\sigma\in\hat{M}_\tau$. Hence, if we define the polynomial M by

$$M(s) = \kappa K \textstyle\sum_{\sigma\in\hat{M}_\tau} R^{\sigma}_{\tau}(s),$$

then we get the desired result.

Recall the fact that all the poles of the meromorphic function ${}_{\tau}g_s(e)$ lie below ρ_o on the real line discretely. Let $\tilde{r}_\tau=\max\{r<0;\ r\in\{\text{the poles of } {}_{\tau}g_s(e)\}\}$. We now select a small positive number $\tilde{\varepsilon}$ which satisfies $\tilde{r}_\tau<-\tilde{\varepsilon}<0$. Then we get the following result from Corollary 4.4.

Proposition 5.6. *Suppose that $t>0$. Then the following relation hold*:

(i) $$L_{\tau,T}(t) = \frac{1}{2\pi \mathrm{i}k}\int_{-\tilde{\varepsilon}-\mathrm{i}\infty}^{-\tilde{\varepsilon}+\mathrm{i}\infty}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(s)ds$$
$$+\,2\textstyle\sum_{\sigma\in\hat{M}_\tau}\sum_{r_\sigma\in\tilde{Q}^{\sigma}_{\tau}} n_T(\sigma, r_\sigma)\ \Theta_{\sigma,r_\sigma}({}_{\tau}h_t)$$
$$+\,(1/\kappa)\textstyle\sum_{\sigma\in\hat{M}_\tau}\sum_{\rho_o+\mathrm{i}r^{\sigma}_k\in[0,\rho_o]}(\mathrm{Res}_{s=\rho_o+\mathrm{i}r^{\sigma}_k}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(s).$$

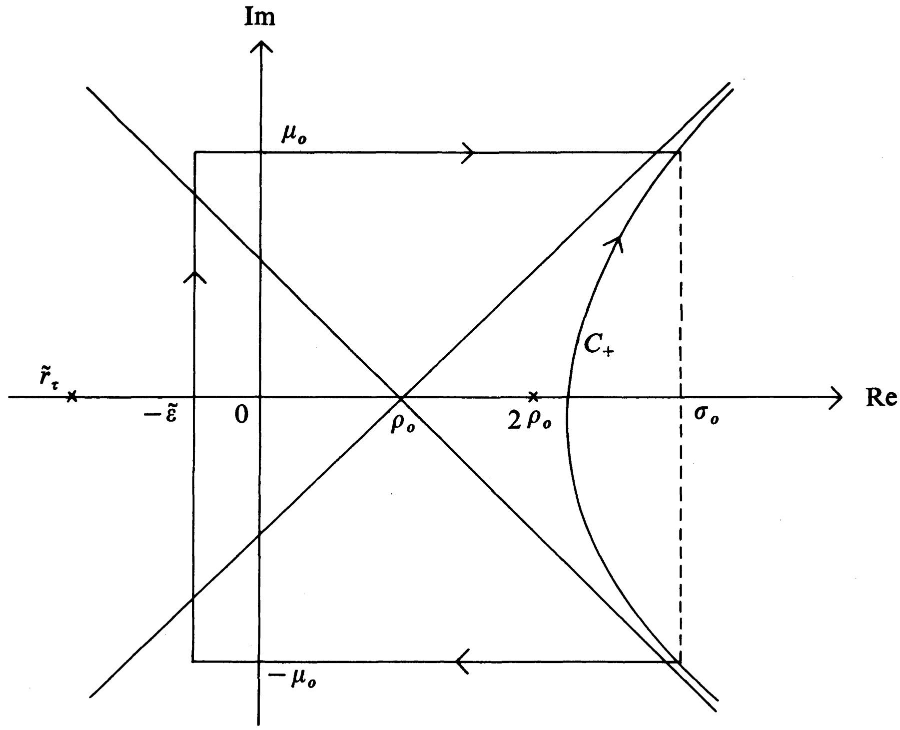
(ii) $$L_{\tau,T}(t) = \frac{1}{2\pi \mathrm{i}\kappa}\int_{-\tilde{\varepsilon}-\mathrm{i}\infty}^{-\tilde{\varepsilon}+\mathrm{i}\infty}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(2\rho_o-s)\ ds.$$

Proof. (i) By Lemma 4.4,

$$L_{\tau,T}(t) = \frac{1}{2\pi \mathrm{i}\kappa}\int_{C_+}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(s)ds,$$

where $C_+=\{s=\sigma+\mathrm{i}\mu;\ (\sigma-\rho_o)^2-\mu^2=\rho_o^2+\varepsilon,\ \sigma>\rho_o\}$.

Let μ_o be a sucifficiently large positive number. Let $\sigma_o+\mathrm{i}\mu_o$ be a point of intersection of C_+ with the line $\{s\in \boldsymbol{C};\ \mathrm{Im}\, s=\mathrm{i}\mu_o\}$, that is, $(\sigma_o-\rho_o)^2-\mu_o^2=\rho_o^2+\varepsilon$ holds. We note the fact that $\sigma_0>2\rho_o$. We now shift the integration by using the following contour, as in the figure below.



Using the residue theorem one finds

$$
\begin{aligned}
L_{\tau,T}(t) &= \frac{1}{2\pi i\kappa}\int_{-\tilde{\varepsilon}-i\infty}^{-\tilde{\varepsilon}+i\infty}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(s)ds \\
&+ \frac{1}{2\pi i\kappa}\lim\nolimits_{\mu_o\to\infty}\int_{-\tilde{\varepsilon}+i\mu_o}^{\sigma_o+i\mu_o}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(s)ds \\
&+ \frac{1}{2\pi i\kappa}\lim\nolimits_{\mu_o\to\infty}\int_{-\tilde{\varepsilon}-i\mu_o}^{\sigma_o-i\mu_o}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(s)ds \qquad (5.2)\\
&+ (1/\kappa)\sum\nolimits_{r_\sigma\in\tilde{Q}_\tau}\mathrm{Res}_{s=\rho_o\pm ir_\sigma}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(s) \\
&+ (1/\kappa)\sum\nolimits_{\sigma\in\hat{M}_\tau}\sum\nolimits_{\rho_o+ir_k^\sigma\in[0,\rho_o]}\mathrm{Res}_{s=\rho_o+ir_k^\sigma}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(s)
\end{aligned}
$$

where $\tilde{Q}_\tau=\bigcup_{\sigma\in\hat{M}_\tau}\tilde{Q}_\tau^\sigma$.

Combining Proposition 3.10 with Proposition 4.1, we see that

$$
\begin{aligned}
&\mathrm{Res}_{s=\rho_o\pm ir_\sigma}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(s) \\
&\quad= \kappa[\sigma\colon \tau_M]n_T(\sigma, r_\sigma)P_\tau^\sigma(r_\sigma)\exp\,(-(\rho_o^2+r_o^2)t) \\
&\quad= \kappa\Theta_{\sigma,r_\sigma}({}_\tau h_t),
\end{aligned}
$$

so the fourth term of (5.2) equals

$$2\sum_{r_\sigma\in\tilde{Q}_\tau} n_T(\sigma, r_\sigma)\,\Theta_{\sigma,r_\sigma}({}_\tau h_t).$$

Therefore, to prove (i), it is sufficient to show that the second term and the third term are equal to zero. To begin with, we shall prove the assertion concerning the second term. It is clear that

$$\int_{-\tilde{\varepsilon}+i\mu_o}^{\sigma_o+i\mu_o}\exp(s(s-2\rho_o)t)\eta_{\tau,T}(s)ds$$
$$=\exp(-(\mu_o^2+\rho_o^2)t)\int_{-\tilde{\varepsilon}}^{\sigma_o}\exp\{(\sigma-\rho_o)^2t+2\,i\,(\sigma-\rho_0)\mu_o t\}\cdot\eta_{\tau,T}(\sigma+i\,\mu_o)d\sigma.$$

Let α be a positive real number such that $\alpha>2\rho_o$. Since we are interested in large values of μ_o, we may assume that $\sigma_o>\alpha$. Then by Lemma 5.3 and 5.5, we can find polynomials P_3 and M such that

$$\left|\int_{-\tilde{\varepsilon}+i\mu_o}^{\sigma_o+i\mu_o}\exp(s(s-2\rho_o)t)\eta_{\tau,T}(s)\,ds\right|$$
$$\leqslant\exp(-(\mu_o^2+\rho_o^2)t)\int_{-\tilde{\varepsilon}}^{\alpha}\exp((\sigma-\rho_o)^2t)\,|P_3(\mu_o)|d\sigma$$
$$+\exp(-(\mu_o^2+\rho_o^2)t)\int_{\alpha}^{\sigma_o}\exp((\sigma-\rho_o)^2t)\,|M(\mu_0)|\exp(-(\sigma-\rho_o)\varepsilon_o/2)d\sigma.$$

We can see easily that

$$\lim_{\mu_o\to\infty}\exp(-(\mu_o^2+\rho_o^2)t)\,|P_3(\mu_o)|\int_{-\tilde{\varepsilon}}^{\alpha}\exp((\sigma-\rho_o)^2t)d\sigma=0.$$

We now put

$$I(\mu_o\colon t)=\exp(-(\mu_o^2+\rho_o^2)t)\,|M(\mu_o)|\int_{\alpha}^{\sigma_o}\exp((\sigma-\rho_o)^2t)\cdot\exp(-(\sigma-\rho_o)\varepsilon_o/2)d\sigma.$$

It suffices to show that $\lim_{\mu_o\to\infty}I(\mu_o\colon t)=0$ for our aim. Integration by parts yields

$$I(\mu_o\colon t)$$
$$=\exp(-(\mu_o^2+\rho_o^2)t)|M(\mu_o)|\left\{\left[\exp((\sigma-\rho_o)^2t)\frac{\exp(-(\sigma-\rho_o)\varepsilon_0/2)}{2(\sigma-\rho_o)t}\right]_{\alpha}^{\sigma_o}\right.$$
$$\left.+(1/2t)\int_{\alpha}^{\sigma_o}\exp((\sigma-\rho_o)^2t)\frac{(1+(\sigma-\rho_o)\varepsilon_o/2)}{(\sigma-\rho_o)^2}\exp(-(\sigma-\rho_o)\,\varepsilon_o/2)d\sigma\right\}.$$

Because $(\sigma_o-\rho_o)^2=\mu_o^2+\rho_o^2+\varepsilon$, we have

$$\left[\exp((\sigma-\rho_o)^2t)\frac{\exp(-(\sigma-\rho_o)\varepsilon_o/2)}{2(\sigma-\rho_o)t}\right]_{\alpha}^{\sigma_o}\times\exp(-(\mu_o^2+\rho_o^2)t)|M(\mu_o)|$$

$$(5.3) \qquad = \frac{\exp(\varepsilon t)|M(\mu_o)|}{2t\sqrt{\mu_o^2+\rho_o^2+\varepsilon}}\exp(-\sqrt{\mu_o^2+\rho_o^2+\varepsilon}\cdot\varepsilon_o/2)$$

$$-\frac{1}{2t}\exp(-(\mu_o^2+\rho_o^2)t(|M(\mu_o)|\exp\{(\alpha-\rho_o)^2t-(\alpha-\rho_o)\varepsilon_o/2\}/(\alpha-\rho_o).$$

On the other hand, if $\alpha \leqslant \sigma \leqslant \sigma_o$ then we have

$$0 \leqslant \frac{1+(\sigma-\rho_o)\varepsilon_o/2}{(\sigma-\rho_o)^2} \leqslant \frac{1+(\alpha-\rho_o)\varepsilon_o/2}{(\alpha-\rho_o)^2}.$$

Therefore, we get

$$\left|I(\mu_o\colon t)-(1/2t)\int_\alpha^{\sigma_o}\exp((\sigma-\rho_o)^2t)\,\frac{(1+(\sigma-\rho_o)\varepsilon_o/2)}{(\sigma-\rho_o)^2}\exp(-(\sigma-\rho_o)\varepsilon_o/2)d\sigma\right|$$

$$\geqslant \left|1-\frac{(1+(\alpha-\rho_o)\varepsilon_o/2)}{2t(\alpha-\rho_o)^2}\right|\cdot I(\mu_o\colon t).$$

Hence we have an estimate

$$0 \leqslant I(\mu_o\colon t)\left|1-\frac{(1+(\alpha-\rho_o)\varepsilon_o/2)}{2t(\alpha-\rho_o)^2}\right|$$

$$(5.4) \qquad \leqslant \frac{e^{\varepsilon t}|M(\mu_o)|}{2t\sqrt{\mu_o^2+\rho_o^2+\varepsilon}}\exp(-\sqrt{\mu_o^2+\rho_o^2+\varepsilon}\cdot\varepsilon_o/2)$$

$$+\frac{1}{2t}\exp(-(\mu_o^2+\rho_o^2)t)(|M(\mu_o)|\exp\{(\alpha-\rho_o)^2t-(\alpha-\rho_o)\varepsilon_o/2\}/(\alpha-\rho_o),$$

by means of (5.3).

Since the functions $\mu_o \to \exp(-\sqrt{\mu_o^2+\rho_o^2+\varepsilon}\cdot\varepsilon_o/2)$ and $\mu_o \to \exp(-(\mu_o^2+\rho_o^2)t)$ are rapidly decreasing with respect to μ_o, it is clear that $\lim_{\mu_o\to\infty} I(\mu_0\colon t)=0$, if $t \neq (1+(\alpha-\rho_0)\varepsilon_o/2)/2(\alpha-\rho_o)^2$, on account of the inequality (5.4). But, since the function $I(\mu_0\colon t)$ is continuous with respect to the variable t, we have $\lim_{\mu_o\to\infty} I(\mu_o\colon t)=0$ for all t (>0).

Similar analysis shows that the third term of (5.2) equals zero. This proves the assertion (i).

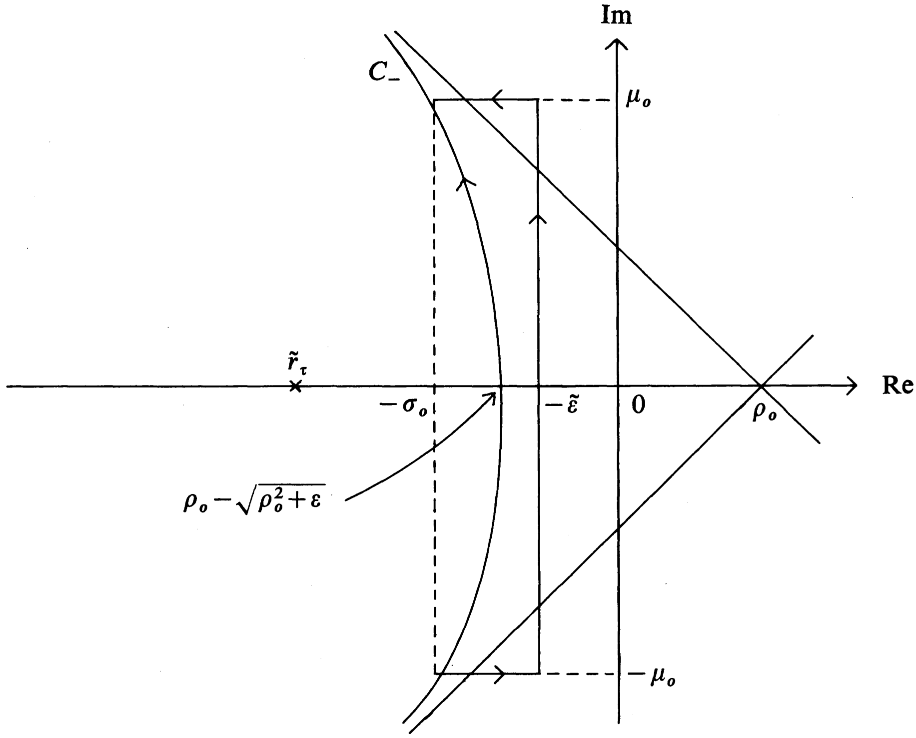
(ii) If we change the variable $s \to 2\rho_o - s$ on the formula in Corollary 4.4, then we see that

$$L_{\tau,T}(t) = \frac{1}{2\pi i\kappa}\int_{C_-}\exp(s(s-2\rho_o)t)\eta_{\tau,T}(2\rho_o-s)ds,$$

where $C_-=\{s=\sigma+i\mu;\ (\sigma-\rho_o)^2-\mu^2=\rho_o^2+\varepsilon,\ \sigma<0\}$.

We may assume that

$$\tilde{r}_\tau < \rho_o - \sqrt{\rho_o^2+\varepsilon} < -\tilde{\varepsilon}.$$



We shift the integration by using the contour, as in the figure above. Since the function $s \to \eta_{\tau,T}(2\rho_o - s)$ does not have a pole in the region which is surrounded by the above contour, one finds that

$$
\begin{aligned}
L_{\tau,T}(t) &= \frac{1}{2\pi i\kappa}\int_{-\tilde{\varepsilon}-i\infty}^{-\tilde{\varepsilon}+i\infty} \exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(2\rho_o-s)\,ds \\
&+ \frac{1}{2\pi i\kappa}\lim_{\mu_o\to\infty}\int_{-\tilde{\varepsilon}+i\mu_o}^{-\sigma_o+i\mu_o} \exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(2\rho_o-s)\,ds \qquad (5.5)\\
&+ \frac{1}{2\pi i\kappa}\lim_{\mu_o\to\infty}\int_{-\tilde{\varepsilon}-i\mu_o}^{-\sigma_o-i\mu_o} \exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(2\rho_o-s)\,ds.
\end{aligned}
$$

If we put $s' = 2\rho_o - s$ in the second and third terms of (5.5), then it is easy to see that these terms have the form of the third and second terms of (5.2) respectively. Hence one finds that both of them are equal to zero. This shows the formula of (ii).

We now define the function $\Phi_{\tau,T}$ by

$$\Phi_{\tau,T}(t) = \kappa\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]P_\tau^\sigma(it)\,\mathrm{vol}\,(\Gamma\backslash G)\chi_T(e)\mu_\sigma(it).$$

This function contributes the functional equation of $\eta_{\tau,T}$ as follows.

THEOREM 5.7. *The function* $\eta_{\tau,T}$ *satisfies the functional equation*

$$\eta_{\tau,T}(s)+\eta_{\tau,T}(2\rho_o-s)+\Phi_{\tau,T}(s-\rho_o)=0, \quad s\in \boldsymbol{C}. \tag{5.6}$$

PROOF. These terms are all meromorphic with simple poles, by Proposition 3.10. Moreover, the poles of $\eta_{\tau,T}(s)+\eta_{\tau,T}(2\rho_o-s)$ are at $\rho_0\pm\mathrm{i}\, r_k^\sigma$ $(k\geqslant 0,\ \sigma\in\hat{M}_\tau)$ with residues $\mp\mathrm{i}\kappa[\sigma\colon \tau_M]P_\tau^\sigma(r_k^\sigma)\chi_T(e)\operatorname{vol}(\Gamma\backslash G)d_k^\sigma$ respectively. On the other hand the poles of the function

$$\Phi_{\tau,T}(s-\rho_o)=\kappa\sum\nolimits_{\sigma\in\hat{M}_\tau}[\sigma\colon\tau_M]P_\tau^\sigma(\mathrm{i}\,(s-\rho_o))\operatorname{vol}(\Gamma\backslash G)\chi_T(e)\mu_\sigma(\mathrm{i}\,(s-\rho_o))$$

are at $s=\rho_0\pm\mathrm{i}\, r_k^\sigma$ $(k\geqslant 0,\ \sigma\in\hat{M}_\tau)$ and the residues are $\pm\mathrm{i}\,\kappa[\sigma\colon\tau_M]P_\tau^\sigma(r_k^\sigma)\chi_T(e)\cdot\operatorname{vol}(\Gamma\backslash G)d_k^\sigma$ respectively. It follows that the function $\eta_{\tau,T}(s)+\eta_{\tau,T}(2\rho_o-s)+\Phi_{\tau,T}(s-\rho_o)$ is an entire function of s.

Add (i) to (ii) of the preceding proposition and divide it by 2. Then

$$\begin{aligned} L_{\tau,T}(t) &= \frac{1}{4\pi\mathrm{i}\kappa}\int_{-\bar{\varepsilon}-\mathrm{i}\infty}^{-\bar{\varepsilon}+\mathrm{i}\infty}\exp\,(s(s-2\rho_o)t)\{\eta_{\tau,T}(s)+\eta_{\tau,T}(2\rho_o-s)\}\,ds \\ &\quad+\sum\nolimits_{\sigma\in\hat{M}_\tau}\sum\nolimits_{r_\sigma\in\tilde{Q}_\tau^\sigma}n_T(\sigma,r_\sigma)\Theta_{\sigma,r_\sigma}({}_\tau h_t) \\ &\quad+\frac{1}{2\kappa}\sum\nolimits_{\sigma\in\hat{M}_\tau}\sum\nolimits_{\rho_o+\mathrm{i}r_k^\sigma\in[0,\rho_o]}\operatorname{Res}_{s=\rho_o+\mathrm{i}r_k^\sigma}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(s)\,. \end{aligned} \tag{5.7}$$

On the other hand, by definition

$$L_{\tau,T}(t)=\sum\nolimits_{\pi\in\hat{G}}n_{\Gamma,T}(\pi)\Theta_\pi({}_\tau h_t)-\chi_\Gamma(e)\operatorname{vol}(\Gamma\backslash G)_\tau h_t(e)\,, \tag{5.8}$$

for $t>0$. The argument of the same kind that we have accomplished in Section 3 shows that

$$\sum\nolimits_{\pi\in\hat{G}}n_{\Gamma,T}(\pi)\Theta_\pi({}_\tau h_t)=\sum\nolimits_{\sigma\in\hat{M}_\tau}\sum\nolimits_{r_\sigma\in\tilde{Q}_\tau^\sigma}n_T(\sigma,r_\sigma)\Theta_{\sigma,r_\sigma}({}_\tau h_t)\,. \tag{5.9}$$

Furthermore, the Plancherel theorem implies that

$$\begin{aligned} &\chi_T(e)\operatorname{vol}(\Gamma\backslash G)_\tau h_t(e) \\ &=\frac{1}{4\pi}\chi_T(e)\operatorname{vol}(\Gamma\backslash G)\sum\nolimits_{\sigma\in\hat{M}_\tau}[\sigma\colon\tau_M]\int_{-\infty}^{\infty}P_\tau^\sigma(r)\exp\,(-(r^2+\rho_o^2)t)\mu_\sigma(r)dr \\ &=\frac{1}{4\pi\kappa}\int_{-\infty}^{\infty}\exp\,(-(r^2+\rho_o^2)t)\Phi_{\tau,T}(-\mathrm{i}r)dr. \end{aligned}$$

Now we put $-\mathrm{i}\,r=s-\rho_o$. Then, since $-(r^2+\rho_o^2)=s(s-2\rho_o)$, we see that the last expression of the above equality is equal to

$$\frac{1}{4\pi\mathrm{i}\kappa}\int_{\rho_o-\mathrm{i}\infty}^{\rho_o+\mathrm{i}\infty}\exp\,(s(s-2\rho_o)t)\Phi_{\tau,T}(s-\rho_o)ds.$$

We now shift the integration into the complex plane by using a rectangular contour with vertices at $\rho_o - i\mu$, $\rho_o + i\mu$, $-\tilde{\varepsilon} + i\mu$, $-\tilde{\varepsilon} - i\mu$, where μ is a positive real number. Using the residue theorem one finds

$$\begin{aligned}
&\chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G)_\tau h_t(e) \\
&= \frac{1}{4\pi i\kappa}\int_{-\tilde{\varepsilon}-i\infty}^{-\tilde{\varepsilon}+i\infty} \exp\,(s(s-2\rho_o)t)\Phi_{\tau,T}(s-\rho_o)ds \\
&\quad + \frac{1}{2\kappa}\sum\nolimits_{\sigma\in\hat{M}_\tau}\sum\nolimits_{\rho_o+ir_k^\sigma\in[0,\rho_o]}\mathrm{Res}_{s=\rho_o+ir_k^\sigma}(s(s-2\rho_o)t)\Phi_{\tau,T}(s-\rho_o) \\
&\quad + \frac{1}{4\pi i\kappa}\lim\nolimits_{\mu\to\infty}\int_{-\tilde{\varepsilon}+i\mu}^{\rho_o+i\mu}\exp\,(s(s-2\rho_o)t)\Phi_{\tau,T}(s-\rho_o)ds \\
&\quad + \frac{1}{4\pi i\kappa}\lim\nolimits_{\mu\to\infty}\int_{\rho_o-i\mu}^{-\tilde{\varepsilon}-i\mu}\exp\,(s(s-2\rho_o)t)\Phi_{\tau,T}(s-\rho_o)ds.
\end{aligned}$$

Let $s=\sigma\pm i\mu$ $(-\tilde{\varepsilon}\leqslant\sigma\leqslant\rho_o)$. Since

$$\exp\,(s(s-2\rho_o)t) = \exp\,(-(\mu^2+\rho_o^2)t)\exp\,\{(\sigma-\rho_o)^2\pm 2i(\sigma-\rho_o)\mu\}t$$

and $\Phi_{\tau,T}(s-\rho_o)$ is a polynomial growth function with respect to μ, we can easily see that the third and fourth terms of the above equality are equal to zero. Therefore we see that

$$\begin{aligned}
L_{\tau,T}(t) &= \sum\nolimits_{\sigma\in\hat{M}_\tau}\sum\nolimits_{r_\sigma\in\hat{Q}_\tau^\sigma} n_T(\sigma, r_\sigma)\Theta_{\sigma,r_\sigma}({}_\tau h_t) \\
&\quad - \frac{1}{4\pi i\kappa}\int_{-\tilde{\varepsilon}-i\infty}^{-\tilde{\varepsilon}+i\infty}\exp\,(s(s-2\rho_o)t)\Phi_{\tau,T}(s-\rho_o)ds \\
&\quad - \frac{1}{2\kappa}\sum\nolimits_{\sigma\in\hat{M}_\tau}\sum\nolimits_{\rho_o+ir_k^\sigma\in[0,\rho_o]}\mathrm{Res}_{s=\rho_o+ir_k^\sigma}(s(s-2\rho_o)\Phi_{\tau,T}(s-\rho_o),
\end{aligned} \tag{5.10}$$

by means of (5.8) and (5.9).

Note the fact that

$$\mathrm{Res}_{s=\rho_o+ir_k^\sigma}\exp\,(s(s-2\rho_o)t)\eta_{\tau,T}(s) + \mathrm{Res}_{s=\rho_o+ir_k^\sigma}(s(s-2\rho_o)t)\Phi_{\tau,T}(s-\rho_o) = 0.$$

Combining (5.7) with (5.10) we get

$$\int_{-\tilde{\varepsilon}-i\infty}^{-\tilde{\varepsilon}+i\infty}\exp\,(s(s-2\rho_o)t)q_{\tau,T}(s)ds=0,$$

Here we put

$$q_{\tau,T}(s) = \eta_{\tau,T}(s) + \eta_{\tau,T}(2\rho_o-s) + \Phi_{\tau,T}(s-\rho_o).$$

Since $q_{\tau,T}(s)$ is an entire function, by the same argument that we carried out before, we can show that if we shift the integration then we obtain

$$\int_{\rho_o-i\infty}^{\rho_o+i\infty}\exp\,(s(s-2\rho_o)t)q_{\tau,T}(s)ds = 0.$$

Namely, if we put $s=\rho_o+\mathrm{i}\,r$ then we have

$$\int_{-\infty}^{\infty}\exp(-(r^2+\rho_o^2)t)q_{\tau,T}(\rho_o+\mathrm{i}\,r)dr=0.$$

It is clear that

$$\int_{-\infty}^{\infty}|(d^n/dt^n)\{\exp(-(r^2+\rho_o^2)t)\}q_{\tau,T}(\rho_o+\mathrm{i}\,r)|dr<\infty.$$

Therefore we can change the order of the integration with differentiation. Hence, for any non-negative integer n, we have

$$(5.11)\qquad \int_{-\infty}^{\infty}(r^2+\rho_o^2)^n\exp(-(r^2+\rho_o^2)t)q_{\tau,T}(\rho_o+\mathrm{i}\,r)dr=0.$$

Since $q_{\tau,T}(\rho_o+\mathrm{i}\,r)$ is an even function of r, one deduces from (5.11) that

$$q_{\tau,T}(\rho_o+\mathrm{i}\,r)=0$$

for all $r\in\boldsymbol{R}$. But $q_{\tau,T}$ is entire, hence $q_{\tau,T}(s)\equiv 0$. This completes the proof of Theorem 5.7.

For each $\sigma\in\hat{M}_\tau$, we put

$$\eta_T^\sigma(s)=\kappa P_\tau^\sigma(\mathrm{i}\,(s-\rho_o))\sum_{\gamma\in C_\Gamma\setminus\{e\}}\chi_T(\gamma)\overline{\chi_\sigma(m_\gamma)}j(\gamma)^{-1}u_\gamma C(\gamma)\cdot\exp(-(s-\rho_o)u_\gamma).$$

Accordingly we have

$$\eta_{\tau,T}(s)=\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\eta_T^\sigma(s).$$

Moreover, if we define

$$\Phi_T^\sigma(r)=\kappa P_\tau^\sigma(\mathrm{i}r)\operatorname{vol}(\Gamma\backslash G)\chi_T(e)\mu_\sigma(\mathrm{i}r),$$

then it is obvious to see that

$$\Phi_{\tau,T}(r)=\sum_{\sigma\in\hat{M}_\tau}[\sigma:\tau_M]\Phi_T^\tau(r).$$

Now we have the following result as a corollary of Theorem 5.7.

COROLLARY 5.8. *The function $\eta_T^\sigma(s)$ is holomorphic in the half plane* $\operatorname{Re}s>2\rho_o$, *and it has the following properties*:
(i) *$\eta_T^\sigma(s)$ has meromorphic continuation to the whole complex plane, via the relation*

$$\eta_T^\sigma(s)=\sum_{r_\sigma\in\tilde{Q}_\tau^\sigma}n_T(\sigma,r_\sigma)P_\tau^\sigma(r_\sigma)\left\{\frac{H(\mathrm{i}\,(s-\rho_o)+\mathrm{i}\,r_\sigma)}{s-\rho_o-\mathrm{i}r_\sigma}+\frac{H(\mathrm{i}(s-\rho_o)-r_\sigma)}{s-\rho_o-\mathrm{i}r_\sigma}\right\}$$
$$-\mathrm{i}\chi_T(e)\operatorname{vol}(\Gamma\backslash G)\sum_{k\geq 0}P_\tau^\sigma(r_k^\sigma)\frac{H(\mathrm{i}(s-\rho_o)+r_k^\sigma)}{s-\rho_o-\mathrm{i}\,r_k^\sigma}d_k^\sigma.$$

The poles of η_T^σ are all simple, and are as follows:

Pole	*Residue*	
$\rho_o \pm \mathrm{i}\, r_\sigma$	$\kappa n_T(\sigma, r_\sigma) P_\tau^\sigma(r_\sigma)$	$r_\sigma \in \tilde{Q}_\tau^\sigma$
$\rho_o + \mathrm{i}\, r_k^\sigma$	$-\mathrm{i}\, \kappa \chi_T(e) \operatorname{vol}(\Gamma\backslash G) P_\tau^\sigma(r_k^\sigma) d_k^\sigma$	$k \geqslant 0.$

If some two of these poles coincide with each other, then we interpret the residue of that pole is the sum of each residue.

(ii) *η_T^σ satisfies functional equation*

$$\eta_T^\sigma(s) + \eta_T^\sigma(2\rho_o - s) + \Phi_T^\sigma(s-\rho_o) = 0, \quad s \in \boldsymbol{C}.$$

PROOF. For each $\sigma \in \hat{M}_\tau$, we define

$$\tilde{P}_\tau^\xi(r) = \begin{cases} P_\tau^\xi(r) & \text{if} \quad \xi \neq \sigma \\ 2P_\tau^\sigma(r) & \text{if} \quad \xi \simeq \sigma. \end{cases}$$

Using these polynomials, we now define the functions $\tilde{\eta}_{\tau,T}$ and $\tilde{\Phi}_{\tau,T}$ as the same kind that we defined $\eta_{\tau,T}$ and $\Phi_{\tau,T}$ by using $P_\tau^\sigma(\sigma \in \hat{M}_\tau)$ before. Then it is easy to see that

$$\tilde{\eta}_{\tau,T}(s) + \tilde{\eta}_{\tau,T}(2\rho_o - s) + \tilde{\Phi}_{\tau,T}(s-\rho_o) = 0, \quad s \in \boldsymbol{C}.$$

Also, by definition we have

$$\eta_T^\sigma(s) = \tilde{\eta}_{\tau,T}(s) - \eta_{\tau,T}(s).$$

Hence the assertions (i) and (ii) follow immediately from the results of Proposition 3.10 and Theorem 5.7 respectively.

6. Definition of zeta function

For the purpose of defining the zeta function, we have to improve on the definition of $\eta_{\tau,T}$-function. At the first half of this section, we devote ourselves to investigation about the analytical properties of that function.

Above all things, for each $\sigma \in \hat{M}_\tau$, let us define

$$\begin{aligned} (6.1) \quad \tilde{\eta}_T^\sigma(s) &= P_\tau^\sigma(\mathrm{i}\,(s-\rho_o))^{-1} \eta_T^\sigma(s) \\ &= \textstyle\sum_{\gamma \in C_\Gamma \backslash \{e\}} \chi_T(\gamma) \overline{\chi_\sigma(m_\gamma)} j(\gamma)^{-1} u_\gamma C(\gamma) \exp\left(-(s-\rho_o) u_\gamma\right), \end{aligned}$$

for $\operatorname{Re} s > 2\rho_0$, and

$$\begin{aligned} (6.2) \qquad \tilde{\Phi}_T^\sigma(r) &= P_\tau^\sigma(\mathrm{i}\, r)^{-1} \Phi_T^\sigma(r) \\ &= \kappa \chi_T(e) \operatorname{vol}(\Gamma\backslash G) \mu_\sigma(\mathrm{i}\, r). \end{aligned}$$

Then it is clear that

$$\tilde{\eta}_T^\sigma(s) + \tilde{\eta}_T^\sigma(2\rho_o - s) + \tilde{\Phi}_T^\sigma(s-\rho_o) = 0, \quad s \in \boldsymbol{C}.$$

Let E_σ denote the set of all zeros of the polynomial $P_\tau^\sigma(r)$. Define

$$E_\sigma^1 = \{r_k^\sigma \in H;\ P_\tau^\sigma(r_k^\sigma)=0\}.$$

Since the set E_σ^1 is finite, we can write it as

$$E_\sigma^1 = \{r(\sigma, 1), r(\sigma, 2), \dots, r(\sigma, j_\sigma)\}.$$

Note the fact that the function defined by the series

$$\sum_{k=0}^\infty P_\tau^\sigma(r_k^\sigma) \frac{H(\mathrm{i}(s-\rho_o)+r_k^\sigma)}{s-\rho_o-\mathrm{i}r_k^\sigma} d_k^\sigma$$

has no poles at $s=\rho_o+\mathrm{i}\, r(\sigma, j), j=1, 2, \dots, j_\sigma$. Hence the function $\tilde{\eta}_T^\sigma(s)$ does not have a pole at the point $s=\rho_o+\mathrm{i}\, r(\sigma, j)$ $(j=1, 2, \dots, j_\sigma)$.

Since the function $\tilde{\eta}_T^\sigma$ has meromorphic continuation, via the relation

$$\begin{aligned}(6.3)\quad \tilde{\eta}_T^\sigma(s) = {} & P_\tau^\sigma(\mathrm{i}(s-\rho_o))^{-1} \sum_{r_\sigma \in \tilde{Q}_\tau^\sigma} P_\tau^\sigma(r_\sigma) n_T(\sigma, r_\sigma) \\ & \cdot \left\{\frac{H(\mathrm{i}(s-\rho_o)+r_\sigma)}{s-\rho_o-\mathrm{i}r_\sigma} + \frac{H(\mathrm{i}(s-\rho_o)-r_\sigma)}{s-\rho_o+\mathrm{i}r_\sigma}\right\} \\ & - \mathrm{i}\chi_T(e) \operatorname{vol}(\Gamma\backslash G) P_\tau^\sigma(\mathrm{i}(s-\rho_o))^{-1} \sum_{k=0}^\infty P_\tau^\sigma(r_k^\sigma) \frac{H(\mathrm{i}(s-\rho_o)+r_k^\sigma)}{s-\rho_o-\mathrm{i}r_k^\sigma} d_k^\sigma,\end{aligned}$$

$\tilde{\eta}_T^\sigma(s)$ has poles possibly at $s=\rho_0 \pm \mathrm{i}\, r$ $(r \in E_\sigma)$ on account of the fact that P_τ^σ is an even polynomial.

Let $P_{\sigma,j}^\pm(s)$ $(j=1,\dots,j_\sigma)$ denote the principal part of Laurent expansion of $\tilde{\eta}_T^\sigma(s)$ at $s=\rho_o \pm \mathrm{i}\, r(\sigma, j)$:

$$(6.4)\qquad P_{\sigma,j}^\pm(s) = \sum_{m=1}^{m_j} \frac{d_\pm(\sigma, j\colon m)}{(s-\rho_o \mp \mathrm{i}r(\sigma, j))^m}.$$

Then we have

$$P_{\sigma,j}^\pm(2\rho_o - s) = \sum_{m=1}^{m_j} \frac{d_\pm(\sigma, j\colon m)}{(-1)^m(s-\rho_o \pm \mathrm{i}\, r(\sigma, j))^m}.$$

Hence the functional equation of $\tilde{\eta}_T^\sigma$ says that

$$P_{\sigma,j}^+(s) + P_{\sigma,j}^-(2\rho_0 - s) + \frac{d(\sigma, j)}{s-\rho_o-\mathrm{i}\, r(\sigma, j)} = 0,$$

where we put

$$d(\sigma, j) = \operatorname{Res}_{s=\rho_o+\mathrm{i}r(\sigma,j)} \tilde{\Phi}_T^\sigma(s-\rho_0).$$

Therefore, one finds that

$$\begin{aligned} &d_+(\sigma, j\colon 1) - d_-(\sigma, j\colon 1) + d(\sigma, j) = 0, \\ &d_+(\sigma, j\colon 2k+1) - d_-(\sigma, j\colon 2k+1) = 0 \qquad (3 \leqslant 2k+1 \leqslant m_j) \\ \text{and}\quad &d_+(\sigma, j\colon 2k) + d_-(\sigma, j\colon 2k) = 0 \qquad (2 \leqslant 2k \leqslant m_j). \end{aligned}$$

We now put

$$\varphi_T^\sigma(r) = \textstyle\sum_{j=1}^{j_\sigma} d(\sigma, j) \left\{\dfrac{1}{r - \mathrm{i} r(\sigma, j)} - \dfrac{1}{r + \mathrm{i}\, r(\sigma, j)}\right\}. \tag{6.5}$$

Also we put

$$F_T^1(\sigma, s) = \textstyle\sum_{j=1}^{j_\sigma} \{P_{\sigma,j}^+(s) + P_{\sigma,j}^-(s)\}. \tag{6.6}$$

Note the fact that

$$F_T^1(\sigma, s) + F_T^1(\sigma, 2\rho_o - s) + \varphi_T^\sigma(s - \rho_o) = 0. \tag{6.7}$$

Let

$$\tilde{\eta}_T^{\sigma,1}(s) = \tilde{\eta}_T^\sigma(s) - F_T^1(\sigma, s), \tag{6.8}$$

and let

$$\Delta_T^\sigma(r) = \tilde{\Phi}_T^\sigma(r) - \varphi_T^\sigma(r). \tag{6.9}$$

According to the relation (6.7), we can easily see that the functional equation

$$\tilde{\eta}_T^{\sigma,1}(s) + \tilde{\eta}_T^{\sigma,1}(2\rho_o - s) + \Delta_T^\sigma(s - \rho_o) = 0$$

holds for any $s \in \boldsymbol{C}$.

Let $E_\sigma^2 = E_\sigma \backslash \{E_\sigma^1 \cup (-E_\sigma^1)\}$. The set E_σ^2 is also finite, hence this can be either written by

$$\{\pm \tilde{r}_1, \dots, \pm \tilde{r}_{t_\sigma}\} \quad \text{or} \quad \{0, \pm \tilde{r}_1, \dots, \pm \tilde{r}_{t_\sigma}\};$$

where $\tilde{r}_j \neq 0$ $(j = 1, \dots, t_\sigma)$.

Let $R_{\sigma,j}^\pm(s)$ be the principal part of Laurent expansion of $\tilde{\eta}_T^{\sigma,1}(s)$ at $s = \rho_o \pm \mathrm{i}\, \tilde{r}_j$:

$$R_{\sigma,j}^\pm(s) = \textstyle\sum_{n=1}^{n_j} \dfrac{d^\pm(\sigma, j\colon n)}{(s - \rho_o \mp \mathrm{i}\, \tilde{r}_j)^n}. \tag{6.10}$$

Therefore, we have

$$R_{\sigma,j}^\pm(2\rho_o - s) = \textstyle\sum_{n=1}^{n_j} \dfrac{d^\pm(\sigma, j\colon n)}{(-1)^n (s - \rho_o \pm \mathrm{i}\, \tilde{r}_j)^n}.$$

Because the function $\Delta_T^\sigma(s-\rho_o)$ has no poles at $s=\rho_o\pm i\tilde{r}_j$ $(j=1,\dots,t_\sigma)$, the functional equation of $\tilde{\eta}_T^{\sigma,1}$ implies that

$$R^{\pm}_{\sigma,j}(s)+R^{\mp}_{\sigma,j}(2\rho_o-s)=0.$$

This means that

$$d^+(\sigma,j\colon 2k)+d^-(\sigma,j\colon 2k)=0 \qquad (2\leqslant 2k\leqslant n_j)$$

and

$$d^+(\sigma,j\colon 2k-1)-d^-(\sigma,j\colon 2k-1)=0 \qquad (1\leqslant 2k-1\leqslant n_j).$$

In particular, if $s=\rho_o+i\tilde{r}_j$ is a pole of $\tilde{\eta}_T^{\sigma,1}(s)$ then $s=\rho_o-i\tilde{r}_j$ is also the pole of $\tilde{\eta}_T^{\sigma,1}(s)$, and vice versa.

On the other hand, if $0\in E_\sigma^2$, let R_σ be the principal part of Laurent expansion of $\tilde{\eta}_T^{\sigma,1}(s)$ at $s=\rho_o$:

$$R_\sigma(s)=\sum_{n=1}^{n_o}\frac{d^o(n)}{(s-\rho_o)^n}. \tag{6.11}$$

The same argument as we have proceed above shows that

$$R_\sigma(s)+R_\sigma(2\rho_o-s)=0,$$

hence we get

$$d^o(2k)=0 \qquad (1\leqslant 2k\leqslant n_0).$$

We put

$$F_T^2(\sigma,s)=\begin{cases}\sum_{j=1}^{t_\sigma}\{R^+_{\sigma,j}(s)+R^-_{\sigma,j}(s)\} & \text{if}\quad 0\notin E_\sigma\\ \sum_{j=1}^{t_\sigma}\{R^+_{\sigma,j}(s)+R^-_{\sigma,j}(s)\{+R_\sigma(s) & \text{if}\quad 0\in E_\sigma.\end{cases} \tag{6.12}$$

We now define the function $H_T^\sigma(s)$ by

$$H_T^\sigma(s)=\tilde{\eta}_T^{\sigma,1}(s)-F_T^2(\sigma,s). \tag{6.13}$$

Summing up these observations, we have the following proposition.

PROPOSITION 6.1. *The function $H_T^\sigma(s)$ is holomorphic in the half plane* $\mathrm{Re}\,s>2\rho_o$, *and is a meromorphic function of s in the whole complex plane. The poles of $H_T^\sigma(s)$ are all simple, and are as follows:*

Pole	*Residue*	
$\rho_o\pm i\,r_\sigma$	$\kappa n_T(\sigma,r_\sigma)$	$r_\sigma\in\tilde{Q}_\tau^\sigma$
$\rho_o+i\,r_k^\sigma$	$-i\,\kappa\chi_T(e)\,\mathrm{vol}(\Gamma\backslash G)d_k^\sigma$	$k\geqslant 0,\quad r_k^\sigma\notin E_\sigma^1.$

Furthermore, the following functional equation holds:

$$H_T^\sigma(s) + H_T^\sigma(2\rho_o - s) + \Delta_T^\sigma(s-\rho_o) = 0, \quad s \in \boldsymbol{C}.$$

Note the fact that $0 \in \tilde{Q}_\tau^\sigma$ implies that $0 \notin E_\sigma^2$.

It is easy to see that the residue d_k^σ of the Plancherel measure $\mu_\sigma(r)$ is pure imaginary in all cases. Hence the residue $-\mathrm{i}\,\kappa\chi_T(e)\operatorname{vol}(\Gamma\backslash G)d_k^\sigma$ is real (see, Section 1).

Also it is known [7] that the number $\operatorname{vol}(\Gamma\backslash G)$ is a rational number, for our normalization of Haar measure. Furthermore, we can find the fact that $\mathrm{i}\,d_k^\sigma$ is a rational number, whose denominator depends only on $\sigma \in \hat{M}_\tau$, and not on k. Let κ_σ denote this denominator. We now choose $\kappa = H(0)$ to be equal to the least common multiple of the integers κ_σ $(\sigma \in \hat{M}_\tau)$. It turns out that the function $H_{\tau,T}$ defined by

$$H_{\tau,T}(s) = \textstyle\sum_{\sigma\in\hat{M}_\tau} H_T^\sigma(s)\alpha_\sigma \qquad (\alpha_\sigma = [\sigma\colon \tau_M]) \tag{6.14}$$

has only simple poles with integer residues. Therefore, we can find a meromorphic function $Z_{\tau,T}(s)$ such that

$$(d/ds)(\log Z_{\tau,T}(s)) = H_{\tau,T}(s). \tag{6.15}$$

The function $Z_{\tau,T}$ will be defined up to a multiplicative constant, which we will now fix. As we have seen, if $0 \in \tilde{Q}_\tau^\sigma$, for some $\sigma \in \hat{M}_\tau$, then $H_{\tau,T}$ has a pole at ρ_o with a residue $2\kappa\sum_{\sigma\in\hat{M}_\tau, 0\in\tilde{Q}_\tau^\sigma} n_T(\sigma, 0)\alpha_\sigma$. Hence, $Z_{\tau,T}$ will have a zero at ρ_o of order $2\kappa\sum_{\sigma\in\hat{M}_\tau, 0\in Q_\tau^\sigma} n_T(\sigma, 0)\alpha_\sigma$. We will denote this even integer by m_o. Of course, $m_o = 0$ if $0 \notin \tilde{Q}_\tau$. We now normalize $Z_{\tau,T}$ by requiring that

$$\lim_{s\to\rho_o}(s-\rho_o)^{-m_o} Z_{\tau,T}(s) = 1.$$

This determines $Z_{\tau,T}$ completely. We shall call this the zeta function attached to the data $(G\colon K, \tau\colon \Gamma, T)$.

In the forthcoming section, we shall study on the various properties of the zeta function $Z_{\tau,T}$.

7. Zeta function $Z_{\tau,T}$

In this section we shall describe the main results of this paper, that is, the fact that the zeta function $Z_{\tau,T}$ defined in the preceding section has the same kind of the properties possessed by Selberg's and Gangolli's ones.

At the first place, we state the following theorem concerning the location of the zeros and the poles of $Z_{\tau,T}$.

THEOREM 7.1. *The function $Z_{\tau,T}(s)$ is holomorphic in the half plane* $\operatorname{Re} s >$

$2\rho_o$, and it has a meromorphic continuation to the whole complex plane. The zeros and the poles described below are the only zeros and the poles of $Z_{\tau,T}$:
(i) $Z_{\tau,T}(s)$ always has certain zeros that we can call spectral zeros. These are located at the points $\rho_o \pm \mathrm{i}\, r_\sigma$ ($r_\sigma \in \tilde{Q}_\tau^\sigma$, $\sigma \in \hat{M}_\tau$) with at most finite exceptional points (see (iii) below). The order of the zero at $\rho_o \pm \mathrm{i}\, r_\sigma$ equals $\kappa n_T(\sigma, r_\sigma)\alpha_\sigma$ where α_σ stands for the number $[\sigma: \tau_M]$, and κ is the positive integer mentioned in Section 6. Of course, if $r_\sigma = r_\xi$ for some σ and ξ in $\hat{M}_\tau$, then we understand the order of the zero at the point $\rho_o \pm \mathrm{i}\, r_\sigma$ is equal to $\kappa(n_T(\sigma, r_\sigma)\alpha_\sigma + n_T(\xi, r_\xi)\alpha_\xi)$. Moreover, the spectral zeros lie on the line $\mathrm{Re}\, s = \rho_o$ except for a finite number of r_σ. Thus $Z_{\tau,T}(s)$ satisfies a sort of modified Riemann hypothesis. The representations π_{σ,r_σ} which correspond to the $r_\sigma \in \boldsymbol{R}\backslash\{0\}$ (that is, $\mathrm{Re}\,(\rho_o + \mathrm{i}\, r_\sigma) = \rho_o$) are all in the unitary principal series. Those $\rho_o \pm \mathrm{i}\, r_\sigma$ which are off the line $\mathrm{Re}\, s = \rho_o$ are all real, and lie in the interval $[0, 2\rho_o]$, symmetrically about ρ_o. The corresponding representations π_{σ,r_σ} are almost all in the complementary series.
(ii) Apart from the spectral zeros of $Z_{\tau,T}$, there may exist a certain series of zeros and poles of $Z_{\tau,T}$. These exist only when $\dim(G/K)$ is even. These are located at the points $\rho_o + \mathrm{i}\, r_k^\sigma$ ($r_k^\sigma \notin E_k^\sigma$, see Section 6) where r_k^σ is a pole of the Plancherel measure $\mu_\sigma(r)$ in the upper half plane $\mathrm{Im}\, r \geqslant 0$. Whether we have zeros or poles depends on the sign of the number $\mathrm{i}\, d_k^\sigma$, where d_k^σ is the residue of $\mu_\sigma(r)$ at r_k^σ. If this sign is positive, then $Z_{\tau,T}$ has poles at the points $\rho_o + i\, r_k^\sigma$ ($k \geqslant 0$, $\sigma \in \hat{M}_\tau$, $r_k^\sigma \notin E_\sigma^1$). In the opposite case, $Z_{\tau,T}$ has zeros at $\rho_o + \mathrm{i}\, r_k^\sigma$. In any case, the order of the zero or the pole is always equal to $\kappa\chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G)|d_k^\sigma|\alpha_\sigma$. Furthermore, if $r_k^\sigma = r_j^\xi \notin E_\sigma^1 \cup E_\xi^1$ (σ, $\xi \in \hat{M}_\tau$) for some σ, ξ, k and j, then we must obviously change the above statement. Namely, if the sign of $\mathrm{i}\, d_k^\sigma\alpha_\sigma + \mathrm{i}\, d_j^\xi\alpha_\xi$ is positive (resp. negative) then $Z_{\tau,T}$ has a pole (resp. zero) at the point $\rho_o + \mathrm{i}\, r_k^\sigma$ with the order $\kappa\chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G)|d_k^\sigma\alpha_\sigma + d_j^\xi\alpha_\xi|$. Of course, if $d_k^\sigma\alpha_\sigma + d_j^\xi\alpha_\xi = 0$ then there exists neither a pole nor a zero at this point.
(iii) Suppose that $r_\xi = r_k^\sigma \notin E_\sigma^1$ for some ξ, σ in $\hat{M}_\tau$ and k. In this case, if the sign of $\kappa n_T(\xi, r_\xi)\alpha_\xi - \mathrm{i}\,\kappa\chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G)d_k^\sigma\alpha_\sigma$ is positive (resp. negative), then $Z_{\tau,T}$ has a zero (resp. pole) at the point $\rho_o + \mathrm{i}\, r_\xi$. At any rate, the order of the zero or the pole equals $\kappa|n_T(\xi, r_\xi)\alpha_\xi - \mathrm{i}\,\chi_T(e)\,\mathrm{vol}\,(\Gamma\backslash G)d_k^\sigma\alpha_\sigma|$.

PROOF. It is well known that the r_σ which are real correspond to representations of the principal series, and the purely imaginary r_σ correspond to representations either in the complementary series or the series of representations described in (e) of Proposition 3.5. Hence the assertion of this theorem now follows from Proposition 6.1, (6.14) and (6.15).

REMARK 1. The zeros described in the statements of (i) are called spectral because their location and order provides us spectral information, in the following sense: Let $\pi_{\Gamma,T}$ be the representation of G induced from the representation T of Γ (see Section 2). Then the certain representations $\pi_{\sigma,r}$ $(\sigma\in\hat{M}_\tau,\ r\in\boldsymbol{C})$ of G occur as summands in $\pi_{\Gamma,T}$. The assertion of (i) implies that the order of the zero at $\rho_o\pm i\,r_\sigma$ is essentially equal to the multiplicity $n_T(\sigma, r_\sigma)$ with which the representation π_{σ,r_σ} occurs in $\pi_{\Gamma,T}$.

REMARK 2. Note the fact that if $r_\sigma\in\tilde{Q}_\tau$ satisfies $r_\sigma\in\boldsymbol{R}\backslash\{0\}$ then the points $\rho_0\pm i\,r_\sigma$ belong to the set of spectral zeros.

REMARK 3. The point described in the statement of (iii) is somewhat special in that the behavior of $Z_{\tau,T}$ at this point has both spectral and structural aspects of G/K.

REMARK 4. Suppose that $r_\xi=r_\sigma=\cdots=r_\mu=r_k^{\xi'}=r_l^{\sigma'}=\cdots=r_m^{\mu'}$ $(r_k^{\xi'}\notin E_{\xi'}^1,$ $r_l^{\sigma'}\notin E_{\sigma'}^1,\dots, r_m^{\mu'}\notin E_{\mu'}^1)$ for some $\xi, \sigma,\dots, \mu, \xi', \sigma',\dots, \mu'$ in $\hat{M}_\tau$ and non-negative integers $k, l,\dots, m$. We denote this by $\tilde{r}$. Then the judgment of whether the point $\rho_o+i\tilde{r}$ is a zero or a pole is carried out by the same procedure as we mentioned above.

The proof of the following theorem is the same as that of [7, Theorem 2.9]. But we include its proof for completeness.

THEOREM 7.2. *$Z_{\tau,T}$ satisfies the following functional equation*:

$$Z_{\tau,T}(2\rho_o-s) = Z_{\tau,T}(s)\exp\left(\int_0^{s-\rho_o} \Delta_{\tau,T}(r)dr\right),\quad s\in\boldsymbol{C}.$$

Here we put $\Delta_{\tau,T}(r)=\sum_{\sigma\in\hat{M}_\tau}\Delta_T^\sigma(r)\alpha_\sigma$.

PROOF. It is obvious to see that

$$H_{\tau,T}(s) + H_{\tau,T}(2\rho_o-s) + \Delta_{\tau,T}(s-\rho_o) = 0$$

for all $s\in\boldsymbol{C}$. Since

$$H_{\tau,T}(s) = \frac{d}{ds}\log Z_{\tau,T}(s),$$

it is evident that the functional equation of $H_{\tau,T}$ leads by integration to

$$Z_{\tau,T}(2\rho_o-s) = c\cdot Z_{\tau,T}(s)\exp\left(\int_0^{s-\rho_o} \Delta_{\tau,T}(r)dr\right), \tag{7.1}$$

where c is a nonzero constant. Note that the expression $\exp\left(\int_0^{s-\rho_o}\Delta_{\tau,T}(r)dr\right)$ is well defined, when $\int_0^{s-\rho_o}\Delta_{\tau,T}(r)dr$ is interpreted as a contour integral. Indeed, $\Delta_{\tau,T}(r)$ is meromorphic and its residues at the poles are always integral. It follows that two different contours from 0 to $s-\rho_o$ will lead to values for $\int_0^{s-\rho_o}\Delta_{\tau,T}(r)dr$

that differ by an integral multiple of $2\pi i$. Therefore $\exp\left(\int_0^{s-\rho_o} \Delta_{\tau,T}(r)\,dr\right)$ is clearly well defined.

We now claim that $c=1$.

Recall that the multiplicity m_o of zero of $Z_{\tau,T}$ at ρ_o ($m_o=0$ if ρ_o is not a zero) is an even integer. Hence we have $(s-\rho_o)^{m_o}=(\rho_o-s)^{m_o}$. Thus from (7.1) we obtain

$$(7.2)\qquad \begin{aligned}&(\rho_o-s)^{-m_o}Z_{\tau,T}(2\rho_o-s)\\ &\quad = c(s-\rho_o)^{-m_o}Z_{\tau,T}(s)\exp\left(\int_0^{s-\rho_o}\Delta_{\tau,T}(r)dr\right).\end{aligned}$$

Let $F(s)=(s-\rho_o)^{-m_o}Z_{\tau,T}(s)$ in a neighborhood of ρ_o. Then the definition of the normalization of $Z_{\tau,T}$ implies that $F(\rho_o)=1$. On the other hand $Z_{\tau,T}(2\rho_o-s)=(\rho_o-s)^{m_o}F(2\rho_o-s)$ in a neighborhood of ρ_o, so $(\rho_o-s)^{-m_o}Z_{\tau,T}(2\rho_o-s)\to 1$ as $s\to\rho_o$. Thus letting $s\to\rho_o$ in (7.2), we see that $c=1$. This completes the proof of Theorem 7.2.

For any linear form λ on $\mathfrak{a}_{\mathfrak{p},c}$, let ξ_λ denote the character of the Cartan subgroup $A=A_t A_{\mathfrak{p}}$ defined by $\xi_\lambda(h)=\exp\lambda(\log h)$ $(h\in A)$.

We now enumerate the roots in P_+ as $\alpha_1,\dots,\alpha_t$. Let L be the semi lattice in $\mathfrak{a}^*_{\mathfrak{p},c}$ defined by $L=\{\sum_{i=1}^t m_i\alpha_i;\ m_i\geqslant 0,\ m_i\in \boldsymbol{Z}\}$. For $\lambda\in L$, define m_λ to be the number of distinct orderd t-tuples $(m_1,\dots,m_t)$ such that $\lambda=\sum_{i=1}^t m_i\alpha_i$.

Let $F_T^i(\sigma, s)$ $(i=1, 2, \sigma\in\hat{M}_\tau)$ be the meromorphic functions defined at Section 6. Recall the fact that all of the poles of $F_T^i(\sigma, s)$ lie in the half plane $\operatorname{Re} s\leqslant 2\rho_o$, because the function $\tilde{\eta}_T^\sigma(s)$ is holomorphic in $\operatorname{Re} s>2\rho_o$. Choose a point $s_o\in \boldsymbol{C}$ such that $\operatorname{Re} s_o>2\rho_o$. We now put

$$(7.3)\qquad f_T(\sigma, s) = \exp\int_{s_o}^{s-\rho_o}\{F_T^1(\sigma, r)+F_T^2(\sigma, r)\}dr$$

for each $\sigma\in\hat{M}_\tau$, and further we put

$$(7.4)\qquad f_{\sigma,T}(s) = \textstyle\prod_{\sigma\in\hat{M}_\tau} f_T(\sigma, s).$$

Here, of course, we demand that the contour of the expression (7.3) should be chosen so that the poles of $F_T^i(\sigma, r)$ $(i=1, 2, \sigma\in\hat{M}_\tau)$ do not lie on it. Since the residue of the pole of $F_T^i(\sigma, r)$ need not be an integer, $f_T(\sigma, s)$ is not well defined for a general choice of contour from s_o to $s-\rho_o$. Thus, we take a particular path.

With these understood, we have the product representation of $Z_{\tau,T}$ as follows.

THEOREM 7.3. *$Z_{\tau,T}$ has an infinite product representation in the half plane $\operatorname{Re} s>2\rho_o$. That is, if the point $s-\rho_o$ $(\operatorname{Re} s>2\rho_o)$ is not a pole of $F_T^i(\sigma, r)$ $(i=1, 2, \sigma\in\hat{M}_\tau)$, then there exists a non zero constant $C_\tau(s_o)$ depending on s_o such that*

$$Z_{\tau,T}(s) = C_\tau(s_o)\, f_{\tau,T}(s) \prod_{\sigma\in\hat{M}_\tau} \prod_{\delta\in P_\Gamma} \prod_{\lambda\in L} \cdot(\det\,(\mathrm{I} - T(\delta)\chi_\sigma(m_\delta)^{-1}\xi_\lambda(h(\delta))^{-1}\exp\,(-su_\delta)))^{\kappa m_\lambda \alpha_\sigma}$$

Here I denotes the identity matrix of degree$=\dim T$, *and* det *means determinant.*

PROOF. Let us consider first of all the series

$$\tilde{\eta}_T^\sigma(s) = \sum_{\gamma\in C\setminus\{e\}} \chi_T(\gamma)\chi_\sigma(m_\gamma)^{-1} j(\gamma)^{-1} u_\gamma C(\gamma)\exp\,(-(s-\rho_o)u_\gamma).$$

valid for $\mathrm{Re}\, s > 2\rho_o$. Now, it is easy to see that

$$C(\gamma) = \xi_\rho(h_\mathfrak{p}(\gamma))^{-1} \prod_{\alpha\in P_+} (1-\xi_\alpha(h(\gamma))^{-1})^{-1}.$$

Recall the fact that

$$C_\Gamma\setminus\{e\} = \cup_{\delta\in P_\Gamma} \{\delta^j;\, j\geqslant 1\}.$$

Combining with these facts we have

$$\text{(7.5)}\quad \tilde{\eta}_T^\sigma(s) = \kappa \sum_{\delta\in P_\Gamma} \sum_{j\geqslant 1} \chi_T(\delta^j)\chi_\sigma(m_\delta)^{-j} u_\delta \cdot \prod_{\alpha\in P_+} (1-\xi_\alpha(h(\delta))^{-j})^{-1}\exp\,(-sju_\delta)$$

Now expand $(1-\xi_\alpha(h(\delta))^{-j})^{-1}$ as a power series,

$$\sum_{m\geqslant 0} \xi_\alpha(h(\delta))^{-jm}.$$

This series converges because $\xi_\alpha(h_\mathfrak{p}(\delta))^{-1} < 1$ by our choice of $h(\delta)$, namely $h_\mathfrak{p}(\delta)\in A_\mathfrak{p}^+$.

Next, multiply together these series for the various $\alpha\in P_+$. Then we see that

$$\prod_{\alpha\in P_+} (1-\xi_\alpha(h(\delta))^{-j})^{-1} = \sum_{\lambda\in L} m_\lambda \xi_\lambda(h(\delta))^{-j}.$$

Therefore (7.5) becomes, with a rearrangement,

$$\tilde{\eta}_T^\sigma(s) = \kappa \sum_{\delta\in P_\Gamma} \sum_{\lambda\in L} \sum_{j\geqslant 1} u_\delta m_\lambda \chi_T(\delta^j)\xi_\lambda(h(\delta))^{-j}\chi_\sigma(m_\delta)^{-j}\exp\,(-sju_\delta).$$

If $\varepsilon_1(\delta), \varepsilon_2(\delta),\ldots, \varepsilon_d(\delta)$ are the eigenvalues of $T(\delta)$, then we get

$$\chi_T(\delta^j) = \sum_{i=1}^d (\varepsilon_i(\delta))^j.$$

Hence we can easily see that

$$\begin{aligned}
\tilde{\eta}_T^\sigma(s) &= \kappa \sum_{i=1}^d \sum_{\delta\in P_\Gamma} \sum_{\lambda\in L} m_\lambda u_\delta \sum_{j\geqslant 1} \varepsilon_i(\delta)^j \xi_\lambda(h(\delta))^{-j}\chi_\sigma(m_\delta)^{-j}\exp\,(-sju_\delta) \\
\text{(7.6)}\qquad & \\
&= \kappa \sum_{i=1}^d \sum_{\delta\in P_\Gamma} \sum_{\lambda\in L} m_\lambda u_\delta \frac{\varepsilon_j(\delta)\xi_\lambda(h(\delta))^{-1}\chi_\sigma(m_\delta)^{-1}\exp\,(-su_\delta)}{1-\varepsilon_i(\delta)\xi_\lambda(h(\delta))^{-1}\chi_\sigma(m_\delta)^{-1}\exp\,(-su_\delta)}.
\end{aligned}$$

These manipulations are valid for $\mathrm{Re}\, s > 2\rho_o$ because of the absolute convergence of the series which defines the function $\tilde{\eta}_T^\sigma(s)$. Since

$$H_T^\sigma(s) = \tilde{\eta}_T^\sigma(s) - (F_T^1(\sigma, s) + F_T^2(\sigma, s)),$$

integrating this along the particular path mentioned above, we can find a non zero constant $C_\sigma(s_o)$ such that

$$\begin{aligned}&\exp \int_{s_o}^{s-\rho_o} H_T^\sigma(r)dr \\ &= C_\sigma(s_o) f_T(\sigma, s) \prod_{i=1}^d \prod_{\delta \in P_\Gamma} \prod_{\lambda \in L} (1 - \varepsilon_i(\delta)\xi_\lambda(h(\delta))^{-1}\chi_\sigma(m_\delta)^{-1} \exp(-su_\delta))^{\kappa m_\lambda} \\ &= C_\sigma(s_o) f_T(\sigma, s) \prod_{\delta \in P_\Gamma} \prod_{\lambda \in L} (\det(\mathrm{I} - T(\delta)\xi_\lambda(h(\delta))^{-1}\chi_\sigma(m_\delta)^{-1} \exp(-su_\delta)))^{\kappa m_\lambda},\end{aligned}$$

by means of the expression (7.6). If we put $C_\tau(s_o) = c \prod_{\sigma \in \hat{M}_\tau} C_\sigma(s_o)$ for a certain non zero constant c, then the formula

$$\frac{d}{ds} \log Z_{\tau,T}(s) = \sum_{\sigma \in \hat{M}_\tau} H_T^\sigma(s)\alpha_\sigma$$

implies that the assertion of this theorem.

We finally come to the assertion concerning the order of $Z_{\tau,T}$ when it is an entire function. That is to say, we have the following theorem.

THEOREM 7.4. *If $Z_{\tau,T}$ is an entire function, then the order of it is finite and equals* $\dim(G/K)$.

PROOF. Let δ be a fixed positive real number. If $\mathrm{Re}\, s > 2\rho_o + \delta$, then the same argument as in the proof of Lemma 5.5, in particular, says that the function $\tilde{\eta}_T^\sigma$ is bounded. Also, it is easy to see that the function $F_T^i(\sigma, s)$ $(i = 1, 2, \sigma \in \hat{M}_\tau)$ is bounded in the half plane $\mathrm{Re}\, s > 2\rho_o + \delta$. Hence, in the half plane $\mathrm{Re}\ s > 2\rho_o + \delta$, $H_{\tau,T}(s)$ is bound. Accordingly, we see that $|Z_{\tau,T}(s)| \leqslant \exp A_1|s|$ for some constant A_1.

Now let $n = \dim(G/K) = p + q - 1$. Then, it is easy to see that there is a constant C such that $|\mu_\sigma(r)| \leqslant C(1 + |r|)^{n-1}$. Therefore, in absolute value, the function $\int_0^{s-\rho_o} \Delta_{\tau,T}(r)dr$ which appears in the functional equation for $Z_{\tau,T}$ is less then or equal to $A_2|s|^n$ for some constant A_2. It follows from the functional equation for $Z_{\tau,T}$ that

$$|Z_{\tau,T}(s)| \leqslant \exp A_1|s| \exp A_2|s|^n \leqslant \exp A_3|s|^n$$

for some constant A_3, whenever s is in the half plane $\mathrm{Re}\, s < -\delta$.

On the other hand, since $Z_{\tau,T}(s)$ is holmorphic in $-\delta \leqslant \mathrm{Re}\, s \leqslant 2\rho_o + \delta$, using the maximum modulus principle, one can easily prove that

$$|Z_{\tau,T}(s)| \leqslant \exp(B_1|s|^N + B_2), \quad -\delta \leqslant \mathrm{Re}\, s \leqslant 2\rho_o + \delta$$

for $|\mathrm{Im}\, s|$ sufficiently large, on account of Lemma 5.1 and 5.2. Here B_1 and B_2

are certain constants, and N is some integer (cf. [7]). This verifies the hypothesis of Phragmén–Lindelöf theorem and we conclude that, as an entire function, $Z_{\tau,T}$ has finite order which is less then or equal to n.

We shall show that the order of $Z_{\tau,T}$ is more than or equal to n by means of a different point of view. For any $r>0$, let

$$N_{\tau,T}(r) = \sum_{\sigma\in\hat{M}_\tau} \sum_{|\lambda_{\sigma,r_\sigma}|<r,\,\pi_{\sigma,r_\sigma}\in\hat{G}_u} n_T(\sigma, r_\sigma)\alpha_\delta.$$

Furthermore we put

$$\begin{aligned}\tilde{L}_{\tau,T}(t) &= \int_0^\infty \exp(-tr)dN_{\tau,T}(r)\\ &= \sum_{\sigma\in\hat{M}_\tau} \sum_{\pi_{\sigma,r_\sigma}\in\hat{G}_u} \exp(t\tilde{\lambda}_{\sigma,r_\sigma})n_T(\sigma, r_\sigma)\alpha_\sigma,\end{aligned}$$

for any $t>0$. We now set $\hat{G}_{\tau,T}=\{\pi\in\hat{G};\ \pi|_K\ni\tau,\ n_T(\pi)\neq 0\}$. Then, since $\sharp\{\hat{G}_{\tau,T}\backslash(\hat{G}_u\cap\hat{G}_{\tau,T})\}$ is finite, the same argument as in [30] implies that there exists a constant C_G such that the equality

$$\lim_{t\downarrow 0} t^{n/2}\,\tilde{L}_{\tau,T}(t) = C_G d_\tau \operatorname{vol}(\Gamma\backslash G)\chi_T(e)$$

holds. Hence, applying the theorem of Karamatata one gets

$$N_{\tau,T}(r)/r^{n/2} \longrightarrow C_G d_\tau \operatorname{vol}(\Gamma\backslash G)\chi_T(e)/\Gamma(n/2+1)$$

as $r\to\infty$.

Since $\tilde{\lambda}_{\sigma,r_\sigma}=-(2p+8q)^{-1}(r_\sigma^2+\rho_o^2+\lambda_\sigma)$, this fact leads without difficulty to the following: The series $\sum_\sigma \sum_{\rho_o+\mathrm{i}r_\sigma\neq 0} n_T(\sigma, r_\sigma)\alpha_\sigma/|\rho_o+\mathrm{i}\,r_\sigma|^k$ converges if $k>n$, and diverges if $k\leqslant n$. It follows that the exponent of convergence of the zeros of the entire function $Z_{\tau,T}$ is at least n. This says that the order of $Z_{\tau,T}$ is more than or equal to n. Hence, together with what we showed above, this implies the assertion.

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