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Hyperabelian Lie algebras

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Introduction

Every finite-dimensional Lie algebra can be obtained as an extension of a soluble Lie algebra by a semisimple Lie algebra. If we take off the restriction of finite-dimensionality, we are in a different situation and we need a concept of generalized solubility. A Lie algebra is called hyperabelian if it has an ascending series of ideals whose factors are abelian. The class of hyperabelian Lie algebras is a natural generalization of the class of soluble Lie algebras. Moreover every Lie algebra is an extension of a hyperabelian Lie algebra by a semisimple Lie algebra. Here by a semisimple Lie algebra we mean a Lie algebra which has no non-zero soluble ideals. Accordingly it seems to be desirable for us to know the properties of these Lie algebras. In this paper we investigate the class of hyperabelian Lie algebras and present some of their properties. We shall also study the hyper locally nilpotent Lie algebras and semisimple Lie algebras.

In [11] Kawamoto introduced the notion of prime ideals for Lie algebras and investigated a connection between the prime radical and the soluble radical. In Section 2 we shall characterize the hyperabelian Lie algebras by making use of the notion of the prime ideals and certain special sequences. We shall especially show that the prime radical of a Lie algebra coincides with the hyperabelian radical (Theorem 2.7). This generalizes [11, Theorem 7]. Furthermore, concerning the problem of the existence of prime ideals in a Lie algebra we shall show that a Lie algebra is hyperabelian if and only if it has no proper prime ideals (Theorem 2.11).

In Section 3 we shall investigate further properties of hyperabelian Lie algebras and especially of hyper abelian-and-finite Lie algebras. One of the main results of this section is that the ω-th derived algebra of a hyper abelian-and-finite Lie algebra is hypercentral, and in particular if the characteristic of the basic field is zero, its derived algebra is hypercentral (Theorem 3.6). Therefore if a Lie algebra is hyper abelian-and-finite, then its transfinite derived series reaches to zero (Corollary 3.8). This is a generalization of [2, Lemma 8.1.1].

Section 4 is devoted to investigating the class of hyper locally nilpotent Lie algebras. We shall show as a generalization of [2, Theorem 10.1.3] that every infinite-dimensional hyper locally nilpotent Lie algebra has an infinite-dimensional locally nilpotent ideal and further an infinite-dimensional abelian subalgebra

(Corollary 4.3). We shall also show that there exists a locally soluble Lie algebra with trivial Hirsch-Plotkin radical (Theorem 4.6).

In Section 5 we shall discuss about several semisimplicities and their interrelation.

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1. Preliminaries

We shall be concerned with Lie algebras which are not necessarily finite-dimensional over an arbitrary field $\mathfrak{k}$ unless otherwise specified. Throughout this paper L will be a Lie algebra over $\mathfrak{k}$.

We shall mainly use the notation and terminology of [2]. For the sake of convenience we state here some terms which we shall use. By $H\subset L$ (resp. $H\leqq L$, $H\lhd L$, H ch L), we mean that H is a subset (resp. a subalgebra, an ideal, a characteristic ideal) of L. Angular brackets $\langle\ \rangle$ denote the subalgebra generated by their contents. For $X, Y\subset L$, $\langle X^{Y}\rangle$ is the smallest subalgebra of L which contains X and is Y-invariant. For an ordinal α we denote by $L^{(\alpha)}$ (resp. L^{α}, $\zeta_{\alpha}(L)$) the α-th term of the transfinite derived (resp. lower central, upper central) series of L. If $H, K\leqq L$ and $K\lhd H$ then the centralizer of H/K in L is $C_L(H/K)=\{x\in L|[x, H]\subset K\}$.

We shall need some familiar classes of Lie algebras. $\mathfrak{F}$ (resp. $\mathfrak{F}_r$, $\mathfrak{A}$, $\mathfrak{N}$, $\mathfrak{N}_r$, $\mathrm{E}\mathfrak{A}$, $\mathfrak{Z}$, $\mathrm{L}\mathfrak{N}$, $\mathrm{LE}\mathfrak{A}$) is the class of Lie algebras which are finite-dimensional (resp. of dimension $\leqq r$, abelian, nilpotent, nilpotent of class $\leqq r$, soluble, hypercentral, locally nilpotent, locally soluble). We say that L has Min-$\lhd$ (resp. Min-${}^{n}\lhd$) if L satisfies the minimal condition for ideals (resp. n-step subideals). We use the same notation for the classes of Lie algebras satisfying the corresponding conditions. The classes Max-$\lhd$ and Max-${}^{n}\lhd$ are similarly defined. For two classes $\mathfrak{X}$, $\mathfrak{Y}$ of Lie algebras, we denote by $\mathfrak{X}\mathfrak{Y}$ the class of all Lie algebras L with an $\mathfrak{X}$-ideal I such that $L/I\in\mathfrak{Y}$. $\mathfrak{X}$ is I-closed (resp. Q-closed) provided every subideal (resp. quotient) of an $\mathfrak{X}$-algebra is always an $\mathfrak{X}$-algebra. $\mathrm{L}\mathfrak{X}$ consists of all of those algebras L such that every finite subset of L is contained in an $\mathfrak{X}$-subalgebra of L. $\mathrm{D}\mathfrak{X}$ consists of all direct sums of $\mathfrak{X}$-algebras. $\acute{\mathrm{E}}\mathfrak{X}$ consists of all Lie algebras having an ascending $\mathfrak{X}$-series. L is called hyper $\mathfrak{X}$-algebra if it has an ascending $\mathfrak{X}$-series of ideals, that is, there exists an ordinal σ and a family $\{L_{\alpha}\}_{\alpha\leqq\sigma}$ of ideals of L such that

(i) $L_0 = 0,\quad L_{\sigma} = L$,

(ii) $L_{\alpha}\leqq L_{\alpha+1}$ and $L_{\alpha+1}/L_{\alpha}\in\mathfrak{X}$ if $\alpha<\sigma$,

(iii) $L_{\lambda} = \bigcup_{\alpha<\lambda} L_{\alpha}$ if λ is a limit ordinal.

We denote by $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$ the class of all hyper $\mathfrak{X}$-algebras. When we emphasize the role of the ordinal σ in the definition of a hyper $\mathfrak{X}$-algebra L we denote it by

$L \in \acute{\mathrm{E}}(\lhd)_{\sigma}\mathfrak{X}$.

Let $\mathfrak{X}$ be a Q-closed class of Lie algebras. For any Lie algebra L we denote by $\chi(L)$ the sum of all $\mathfrak{X}$-ideals of L. The transfinite upper $\mathfrak{X}$-radicals are defined for all ordinals as follows:

(i) $\chi_0(L) = 0$,

(ii) $\chi_{\alpha+1}(L)/\chi_\alpha(L) = \chi(L/\chi_\alpha(L))$,

(iii) $\chi_\lambda(L) = \cup_{\alpha<\lambda}\chi_\alpha(L)$ for a limit ordinal λ.

$\chi_\alpha(L)$ is called the *α-th upper $\mathfrak{X}$-radical*. This ascending series terminates for some ordinal α in the sense that from that ordinal onwards all terms are equal. We call this terminal $\chi_\alpha(L)$ the *hyper $\mathfrak{X}$-radical* of L and denote it by $\chi_*(L)$. It is clear that $\chi(L/\chi_*(L))=0$.

When $\mathfrak{X}=\mathrm{L}\mathfrak{N}$ (resp. $\mathfrak{A}$, $\mathfrak{N}$, $\mathrm{E}\mathfrak{A}$), we use the sign ρ (resp. α, ν, σ) for χ. It can be seen that $\alpha_*(L)=\nu_*(L)=\sigma_*(L)$. Clearly $\alpha_*(L)\leqq\nu_*(L)\leqq\sigma_*(L)$. For an ordinal β we suppose that $\sigma_\beta(L)\leqq\alpha_\gamma(L)$ for some ordinal γ. Let I be an ideal of L such that $I^{(n)}\leqq\sigma_\beta(L)$ for some positive integer n. Then it is easily seen that $I^{(n-i)}\leqq\alpha_{\gamma+i}(L)$ by induction on i. Thus we have $I\leqq\alpha_{\gamma+n}(L)$, and so $\sigma_{\beta+1}(L)\leqq\alpha_{\gamma+\omega}(L)$. By transfinite induction on β we get $\sigma_\beta(L)\leqq\alpha_*(L)$ for any ordinal β and hence $\sigma_*(L)\leqq\alpha_*(L)$.

We state some characterizations of hyper $\mathfrak{X}$-algebras which will be often used in this paper. First one is well known.

LEMMA 1.1 (cf. [10, Lemma 1.1]). *Let L be a Lie algebra. Then L is a hyper $\mathfrak{X}$-algebra if and only if $\chi(L/I)\neq 0$ for any proper ideal I of L.*

PROPOSITION 1.2. *Let L be a Lie algebra. Then L is a hyper $\mathfrak{X}$-algebra if and only if the hyper $\mathfrak{X}$-radical $\chi_*(L)$ of L coincides with L.*

PROOF. Let $L\in\acute{\mathrm{E}}(\lhd)\mathfrak{X}$. If $L\neq\chi_*(L)$ then by Lemma 1.1 we have $\chi(L/\chi_*(L))\neq 0$. This is a contradiction.

Conversely suppose that $\chi_*(L)=L$. For a proper ideal I of L, there exists a minimal ordinal α such that $\chi_\alpha(L)\nleqq I$. It is clear that α is neither zero nor a limit ordinal. There is an ideal X of L such that $\chi_{\alpha-1}(L)\leqq X\nleqq I$ and $X/\chi_{\alpha-1}(L)\in\mathfrak{X}$. By the minimality of α we have $\chi_{\alpha-1}(L)\leqq I$. Then $(X+I)/I\simeq X/X\cap I$ is a non-zero homomorphic image of $X/\chi_{\alpha-1}(L)$. Hence $(X+I)/I$ is an $\mathfrak{X}$-ideal of L/I and so $\chi(L/I)\neq 0$. Therefore by Lemma 1.1 L is a hyper $\mathfrak{X}$-algebra.

LEMMA 1.3. *Let $\mathfrak{X}$ be a $\{\mathrm{Q}, \mathrm{I}\}$-closed class of Lie algebras. Then L is a hyper $\mathfrak{X}$-algebra if and only if for a non-zero ideal I of a homomorphic image H of L, there exists a non-zero $\mathfrak{X}$-ideal of H which is contained in I.*

PROOF. Since $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$ is Q-closed, we can see that the condition is necessary by [10, Lemma 1.4]. Another implication is clear by Lemma 1.1.

2. Prime ideal and the class $\acute{E}(\lhd)\mathfrak{A}$

In this section we investigate the class of hyperabelian Lie algebras by using the notion of prime ideals and certain sequences of elements of Lie algebras.

Let L be a Lie algebra over any field. An ideal P of L is called *prime* if whenever $[I, J]\leqq P$ with I, J ideals of L then $I\leqq P$ or $J\leqq P$. An ideal Q of L is called *semiprime* if $I^2\leqq Q$ with $I\lhd L$ implies $I\leqq Q$.

LEMMA 2.1 (cf. [11, Theorem 1]). (i) *Let P be an ideal of a Lie algebra L. Then the following conditions are equivalent*:

(1) *P is a prime ideal.*

(2) *If $[\langle x^L\rangle, \langle y^L\rangle]\leqq P$ for x, $y\in L$, then either $x\in P$ or $y\in P$.*

(3) *If $[\langle x^L\rangle, y]\subset P$ for x, $y\in L$, then either $x\in P$ or $y\in P$.*

(ii) *Let Q be an ideal of L. Then the following conditions are equivalent*:

(1) *Q is a semiprime ideal.*

(2) *If $\langle x^L\rangle^2\leqq Q$ for $x\in L$ then $x\in Q$.*

(3) *If $[\langle x^L\rangle, x]\subset Q$ for $x\in L$ then $x\in Q$.*

PROOF. (i) Suppose that $[\langle x^L\rangle, y]\subset P$. Then we have $[\langle x^L\rangle, [y, L]]\subset [\langle x^L\rangle, y, L]+[\langle x^L\rangle, L, y]\subset[P, L]+[\langle x^L\rangle, y]\subset P$. It is easily seen that $[\langle x^L\rangle, [y, {}_iL]]\subset P$ by induction on i. Hence $[\langle x^L\rangle, \langle y^L\rangle]\leqq P$ and therefore the conditions (2) and (3) are equivalent.

Now let I, J be ideals of L such that $[I, J]\leqq P$, $I\nleqq P$ and $J\nleqq P$. Then there exist $x\in I$ and $y\in J$ such that $x, y\notin P$. However $[\langle x^L\rangle, \langle y^L\rangle]\leqq[I, J]\leqq P$, which shows that (2) implies (1). It is clear that (1) implies (2).

The statement (ii) is similarly proved.

The intersection of all prime ideals of L is called the *prime radical* of L and denoted by rad(L) or r_L. In [11, Theorem 7] it is shown that $\sigma(L)\leqq$ rad(L). Hence if L is soluble, then L is the only prime ideal of L. We shall generalize these results in Theorems 2.7 and 2.11. The latter gives a characterization of hyperabelian Lie algebras. For that the following notion plays an important role. We call a sequence x_0, x_1,... of elements of L a *p-sequence* if there exist elements y_1, y_2,... of L and non-negative integers $n(0)$, $n(1)$,... such that $0=n(0)<n(1)<\cdots$ and $x_{i+1}=[x_i, y_{n(i)+1},\dots, y_{n(i+1)}, x_i]$ for $i=0, 1,\dots$. We call a p-sequence $\{x_n\}_{n\geqq 0}$ vanishing if $x_n=0$ for some n, and otherwise we call it non-vanishing. A similar notion in associative rings is considered in [12] and [13].

THEOREM 2.2. *The following conditions are equivalent*:

(i) *L has a proper prime ideal.*

(ii) *L has a proper semiprime ideal.*

(iii) *L has a non-vanishing p-sequence.*

PROOF. Since prime ideals are semiprime it is clear that (i) implies (ii).

Let Q be a proper semiprime ideal of L and let $x_0 \in L \backslash Q$. Then $[\langle x_0^L \rangle, x_0] \not\subset Q$. It is easy to see that there exist $y_1, \dots, y_{n(1)} \in L$ such that $[x_0, y_1, \dots, y_{n(1)}, x_0] \notin Q$. Put $x_1 = [x_0, y_1, \dots, y_{n(1)}, x_0]$. In the same manner we obtain a sequence $x_0, x_1, x_2, \dots$ of elements of L such that $x_{i+1} \in [\langle x_i^L \rangle, x_i] \backslash Q$, which is a non-vanishing p-sequence of L. Thus (ii) implies (iii).

Assume (iii) finally. Let $\{x_n\}_{n \geq 0}$ be a non-vanishing p-sequence of L and let $X = \{x_0, x_1, \dots\}$. Clearly $X \cap \{0\} = \phi$. By Zorn's lemma there exists an ideal P of L maximal with respect to $X \cap P = \phi$. Let $x, y \notin P$ to show that P is a prime ideal. By the maximality of P, we have $(P + \langle x^L \rangle) \cap X \neq \phi$ and $(P + \langle y^L \rangle) \cap X \neq \phi$. Since $x_i \in \langle x_j^L \rangle$ for $j \leq i$, there exists a positive integer i such that $x_i \in (P + \langle x^L \rangle) \cap (P + \langle y^L \rangle)$. Now suppose that $[\langle x^L \rangle, \langle y^L \rangle] \leq P$. Then $x_{i+1} \in [\langle x_i^L \rangle, x_i] \subset [P + \langle x^L \rangle, P + \langle y^L \rangle] \leq P$. This contradicts the fact that $X \cap P = \phi$. Hence $[\langle x^L \rangle, \langle y^L \rangle] \nleq P$ and therefore P is a prime ideal of L. Thus the conditions (i), (ii) and (iii) are equivalent.

REMARK 1. From the proof of this theorem we can get the following equivalent conditions for $x \in L$.

(i) L has a prime ideal not containing x.

(ii) L has a semiprime ideal not containing x.

(iii) L has a non-vanishing p-sequence beginning with x.

REMARK 2. An intersection of semiprime ideals is always semiprime. Hence rad (L) is a semiprime ideal. But it is not necessarily a prime ideal. For instance let L be the direct sum of two non-abelian simple Lie algebras S_1 and S_2. Then clearly each S_i $(i = 1, 2)$ is a prime ideal of L and so rad $(L) = 0$. However 0 is not prime since $[S_1, S_2] = 0$.

So in general a semiprime ideal is not necessarily a prime ideal (cf. [11, Lemma 3(4) and Proposition 4]), but we can show that any proper semiprime ideal Q is contained in a proper prime ideal P. In fact from the proof of the above theorem we can deduce that there exists a non-vanishing p-sequence $\{x_n\}$ such that $X \cap Q = \phi$, where $X = \{x_0, x_1, \dots\}$. Let P be an ideal of L maximal with respect to $X \cap P = \phi$ and $Q \leq P$. Then as in the last paragraph of the same proof we can see that P is a proper prime ideal of L.

Let $\pi(L)$ be the set of elements x of L such that any p-sequence beginning with x is vanishing. By Remark 1 we have the following

COROLLARY 2.3. rad $(L) = \pi(L)$.

COROLLARY 2.4. rad (L) *is the intersection of all the semiprime ideals of* L.

The following lemma is easily seen.

Lemma 2.5. *Let f be a homomorphism from L onto a Lie algebra L'. For an ideal P' of L', P' is prime in L' if and only if $f^{-1}(P')$ is prime in L.*

A Lie algebra L is said to be *semisimple* (or *σ-semisimple*) if $\sigma(L)=0$ or equivalently $\alpha(L)=0$.

Lemma 2.6. *Let L be a semisimple Lie algebra. Then for any non-zero ideal I of L, there exists a prime ideal P of L such that $I \nleqq P$.*

Proof. Let $x_0 \in I \backslash \{0\}$. It is easy to see that $[\langle x_0^L \rangle, x_0] \neq 0$ since $\langle x_0^L \rangle \notin \mathfrak{A}$. So we can find $x_1 \in I$ and $y_1, \dots, y_{n(1)} \in L$ such that $x_1 = [x_0, y_1, \dots, y_{n(1)}, x_0] \neq 0$. Repeating this procedure we can get a non-vanishing p-sequence beginning with x_0. By Remark 1 L has a prime ideal P of L such that $x_0 \notin P$. This completes the proof.

Now we can show one of the main results of this section.

Theorem 2.7. $\mathrm{rad}\,(L) = \alpha_*(L)$.

Proof. First we show by induction on λ that $\alpha_\lambda(L) \leqq \mathrm{rad}\,(L)$. This is obvious for $\lambda = 0$. So let $\lambda > 0$ and assume that the assertion is true for all ordinals less than λ. If λ is a limit ordinal, evidently we have $\alpha_\lambda(L) = \cup_{\mu < \lambda} \alpha_\mu(L) \leqq \mathrm{rad}\,(L)$. Suppose that λ is not a limit ordinal and so $\lambda - 1$ exists. Let $x \in \alpha_\lambda(L)$ and $\{x_n\}_{n \geqq 0}$ be a p-sequence beginning with x. Since $\langle x^L \rangle \leqq \alpha_\lambda(L)$ and $(\langle x^L \rangle + \alpha_{\lambda-1}(L))/\alpha_{\lambda-1}(L) \in \mathrm{E}\mathfrak{A}$, we have $\langle x^L \rangle^{(j)} \leqq \alpha_{\lambda-1}(L)$ for some positive integer j. Therefore $x_j \in \alpha_{\lambda-1}(L) \leqq \mathrm{rad}\,(L)$. By Corollary 2.3 $\{x_n\}_{n \geqq j}$ is a vanishing p-sequence and so is $\{x_n\}_{n \geqq 0}$. By Corollary 2.3 again we have $x \in \mathrm{rad}\,(L)$, whence $\alpha_\lambda(L) \leqq \mathrm{rad}\,(L)$. Thus $\alpha_*(L) \leqq \mathrm{rad}\,(L)$.

Now assume that $\alpha_*(L) \lneqq \mathrm{rad}\,(L)$. Clearly $L/\alpha_*(L)$ is semisimple. By Lemma 2.6 there exists a prime ideal $P/\alpha_*(L)$ of $L/\alpha_*(L)$ such that $\mathrm{rad}\,(L) \nleqq P$. By Lemma 2.5 P is a prime ideal of L and so $\mathrm{rad}\,(L) \leqq P$. This is a contradiction. Thus $\alpha_*(L) = \mathrm{rad}\,(L)$.

The following corollary is [11, Theorem 7].

Corollary 2.8. (i) $\sigma(L) \leqq \mathrm{rad}\,(L)$.
(ii) *If $L \in \mathrm{Max}\text{-}\lhd$ then $\sigma(L) = \mathrm{rad}\,(L)$.*

Proof. In Section 1 we have shown that $\alpha_*(L) = \sigma_*(L)$. Hence by Theorem 2.7 we have $\sigma(L) \leqq \mathrm{rad}\,(L)$. If L satisfies the maximal condition for ideals, then $\alpha_*(L) = \alpha_n(L)$ for some positive integer n and $\alpha_n(L)$ is soluble. Thus $\alpha_n(L) \leqq \sigma(L)$ and therefore $\mathrm{rad}\,(L) = \alpha_*(L) = \sigma(L)$.

Remark 3. (i) $\mathrm{rad}\,(L/\mathrm{rad}\,(L)) = 0$ over any field.
(ii) $\mathrm{rad}\,(L)$ ch L over any field of characteristic zero.

In fact it is known that $\nu(L)$ ch L over any field of characteristic zero (cf. [2, Corollary 6.3.2]). By transfinite induction on β we can see that $\nu_\beta(L)$ ch L. Thus we have $\alpha_*(L)=\nu_*(L)$ ch L.

Owing to Corollary 2.3 we have the following characterizations of hyperabelian Lie algebras. In group theory similar results are known (cf. [3, Satz 4.1] and [4, p.360]).

COROLLARY 2.9. *A Lie algebra L is hyperabelian if and only if any p-sequence of L is vanishing.*

COROLLARY 2.10. *A Lie algebra is hyperabelian if and only if every countable dimensional subalgebra is hyperabelian.*

As another consequence of Theorem 2.7 we show the following theorem as already announced.

THEOREM 2.11. *Let L be a Lie algebra. Then L is hyperabelian if and only if L has no proper prime ideals.*

PROOF. By Proposition 1.2 and Theorem 2.7, L is hyperabelian if and only if $L=\mathrm{rad}\,(L)$. On the other hand, L has no proper prime ideals if and only if $L=\mathrm{rad}\,(L)$.

3. The classes $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ and $\acute{\mathrm{E}}(\lhd)\,(\mathfrak{A}\cap\mathfrak{F})$

In this section we investigate further properties of hyperabelian Lie algebras and those with finite-dimensional factors.

The following is a Lie analogue of [5, Lemma 4.1].

PROPOSITION 3.1. *The following conditions for L are equivalent:*

(i) *L is hyperabelian.*

(ii) *If I is a non-zero ideal of an epimorphic image H of L, there is an abelian ideal A of H with* $0\neq A\leqq I$.

(iii) *For any epimorphic image H of L, there exists an* $\mathfrak{N}_2$*-ideal N of H such that* $C_H(N)=\zeta_1(N)$.

PROOF. By Lemma 1.3 the equivalence of (i) and (ii) is clear.

Assume (ii). Let H be an epimorphic image of L and A be a maximal abelian ideal of H. If $C_H(A)=A$, then (iii) is obviously satisfied. So we may suppose that $A\lneqq C_H(A)$. Then by the assumption and Zorn's lemma there exists a maximal abelian ideal N/A of H/A such that $N/A\leqq C_H(A)/A$. Then $A\leqq N\cap C_H(N)=\zeta_1(N)\lhd H$. By the maximality of A we have $A=\zeta_1(N)$. Now assume that $A\lneqq C_H(N)$. By the assumption again there is a non-zero abelian ideal B/A of H/A such that $B \leqq C_H(N)$. Then $(B+N)/A$ is an abelian ideal of H/A and contained in $C_H(A)/A$. By the maximality of N/A we have $B \leqq N \cap C_H(N) = \zeta_1(N) = A$. This contradicts the fact $B/A \neq 0$. Therefore $C_H(N) = A = \zeta_1(N)$. Since $N^2 \leqq A$, N is an $\mathfrak{N}_2$-ideal of H. Thus (ii) implies (iii).

Assume (iii) finally. Let H be a non-zero epimorphic image of L. By the assumption there exists an ideal N of H such that $N^2 \leqq \zeta_1(N) = C_H(N)$. Suppose that $\zeta_1(N) = 0$. Since $N \in \mathfrak{N}_2$ we have $N = 0$ and so $H = C_H(0) = \zeta_1(0) = 0$. This is a contradiction and therefore $\zeta_1(N)$ is a non-zero abelian ideal of H. By Lemma 1.1 it follows that $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$. Thus (iii) implies (i). This completes the proof.

As a consequence of this proposition we have the following corollary, which can be also deduced from [2, Lemma 9.1.2].

COROLLARY 3.2. *Let L be a hyperabelian Lie algebra. Then $\nu(L) \in \mathfrak{F}$ if and only if $L \in \mathfrak{F} \cap \mathrm{E}\mathfrak{A}$.*

PROOF. Let L be hyperabelian and let $\nu(L) \in \mathfrak{F}$. Then by Proposition 3.1 there exists an $\mathfrak{N}_2$-ideal N of L such that $C_L(N) = \zeta_1(N)$. Since $\nu(L) \in \mathfrak{F}$, we have $N \in \mathfrak{F}$. Thus $L/C_L(N)$ and $C_L(N)$ are finite-dimensional. It follows that $L \in \mathfrak{F} \cap \mathrm{E}\mathfrak{A}$. The converse is clear.

Now we construct a Gruenberg algebra which shows that the class $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ is not L-closed and so forth. Another such example is seen in [15]. Let $\mathfrak{k}$ be any field and let F be the free Lie algebra over $\mathfrak{k}$ with free generators $x_0, x_1, \dots$. For each non-negative integer n, the subalgebra $\langle x_0, x_1, \dots, x_n \rangle$ of F is a free Lie algebra on $x_0, x_1, \dots, x_n$ and we denote it by F_n. F and F_n are graded with $\deg(x_i) = 1$ for $i = 0, 1, \dots$. Let R be the homogeneous ideal of F generated by $\sum_{i=1}^{\infty} F_i^{i+1}$ and let Q_n (resp. R_n) be the homogeneous ideal of F_n generated by $F_1^2 + \cdots + F_{n-1}^n$ (resp. $F_1^2 + \cdots + F_n^{n+1}$) for $n \in \mathbf{N}$. Furthermore, let $L = F/R$, $Y_n = F_n/Q_n$ and $L_n = Y_n/Y_n^{n+1} \simeq F_n/R_n$ for $n \in \mathbf{N}$. Then L_n is a finite-dimensional and nilpotent graded Lie algebra, whose every non-zero element has degree less than $n+1$. For a non-zero element x of a graded Lie algebra, the degree of the leading component of x is called the degree of x.

For each $n \in \mathbf{N}$, the natural injection $F_n \to F$ induces a graded homomorphism $\phi_n: L_n \to L$ such that $\phi_n(x_i + R_n) = x_i + R$ (for $0 \leqq i \leqq n$). We claim that ϕ_n is injective. Suppose that $\operatorname{Ker} \phi_n \neq 0$ and let $x + R_n \in \operatorname{Ker} \phi_n$ $(x \in F_n \backslash R_n)$. Then we have $x \in R \cap F_n$. Since $\operatorname{Ker} \phi_n$ is homogeneous, we may assume that x is homogeneous and $\deg(x) \leqq n$. It follows that $x \in Q_n^F \cap F_n$. It is easy to see that $Q_n^F \cap F_n = Q_n$. Then $x \in Q_n \leqq R_n$. This is a contradiction. Thus ϕ_n is injective as claimed. We identify L_n with $\phi_n(L_n) = (F_n + R)/R$ and regard L as $\bigcup_{i=0}^{\infty} L_i$.

The algebra L is countable dimensional and locally nilpotent. Therefore L is a Gruenberg algebra and by [1, Theorem 4.6] $L \in \acute{\mathrm{E}}\mathfrak{A}$.

Now we show that L has no non-zero bounded left Engel elements, which implies that $\beta(L)=0$ and $L\notin \acute{\mathrm{E}}(\lhd)\mathfrak{A}$, where $\beta(L)$ is the Baer radical of L. Assume that L has a non-zero bounded left Engel element and let $[F, {}_m x]\subset R$ for some $x\in F\backslash R$ and $m\in \mathbf{N}$. Then there exists $n\in\mathbf{N}$ such that $x\in F_n$. Put $s=mn+1$.

We note that $Y_s=F_s/Q_s$ is the free product $L_{s-1}*\langle x_s\rangle$ of L_{s-1} and $\langle x_s\rangle$. In fact, since $Q_s\geqq R_{s-1}$, the natural injection $F_{s-1}\to F_s$ induces a graded homomorphism f: $L_{s-1}\to Y_s$ such that $f(x_i+R_{s-1})=x_i+Q_s$ $(0\leqq i\leqq s-1)$. Then there exists a graded homomorphism p: $L_{s-1}*\langle x_s\rangle\to Y_s$ such that $p|_{L_{s-1}}=f$ and $p(x_s)=x_s+Q_s$ (cf. [6, Proposition 2.6.1]). Conversely let g: $F_s\to L_{s-1}*\langle x_s\rangle$ be a homomorphism such that $g(x_i)=x_i+R_{s-1}$ $(0\leqq i\leqq s-1)$ and $g(x_s)=x_s$. Since $Q_s=R_{s-1}^{F_s}$, we have $Q_s\leqq \operatorname{Ker} g$. Hence g induces a graded homomorphism q: $Y_s\to L_{s-1}*\langle x_s\rangle$ such that $q(x_i+Q_s)=x_i+R_{s-1}$ $(0\leqq i\leqq s-1)$ and $q(x_s+Q_s)=x_s$. Then clearly we can see that pq (resp. qp) is the identity mapping of Y_s (resp. $L_{s-1}*\langle x_s\rangle$). It follows that $Y_s\simeq L_{s-1}*\langle x_s\rangle$ as graded Lie algebras.

$[x_{s,m}(x+R_{s-1})]$ is a special basic monomial of x_s and $(x+R_{s-1})$ in $L_{s-1}*\langle x_s\rangle$ (cf. [6, p. 37]). By [6, Theorem 2.6.8], $[\bar{x}_{s,m}\bar{x}]\neq 0$ in Y_s where $\bar{x}_s=x_s+Q_s$, $\bar{x}=x+Q_s$. Since $x+R\in L_n$, we may have $\deg(x)\leqq n$ and so $\deg([\bar{x}_{s,m}\bar{x}])\leqq 1+mn=s$ in Y_s. It follows that $[\bar{x}_{s,m}\bar{x}]\notin Y_s^{s+1}$ and hence $[x_{s,m}x]\notin R_s$. But by the assumption $[x_{s,m}x]\in R\cap F_s$ and so $[x_{s,m}x]\in Q_s^F\cap F_s=Q_s\leqq R_s$. This is a contradiction.

We have proved the following

THEOREM 3.3. *Over any field there exists a Gruenberg algebra with trivial Baer radical.*

COROLLARY 3.4. *Over any field*

(i) $\acute{\mathrm{E}}(\lhd)\mathfrak{A}\lneqq\acute{\mathrm{E}}\mathfrak{A}$,

(ii) *The class* $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ *is not* L-*closed.*

By Theorem 3.3 we have $\mathrm{LE}\mathfrak{A}\nleqq\acute{\mathrm{E}}(\lhd)\mathfrak{A}$. But it is outstanding whether every hyperabelian Lie algebra is locally soluble. However we can show the following

PROPOSITION 3.5. *Over any field*

$$\acute{\mathrm{E}}_{\omega+1}(\lhd)\mathfrak{A}\leqq \mathrm{LE}\mathfrak{A}.$$

PROOF. Let $L\in\acute{\mathrm{E}}_{\omega+1}(\lhd)\mathfrak{A}$ and let $\{L_\alpha\}_{\alpha\leqq\omega+1}$ be an ascending $\mathfrak{A}$-series of ideals of L. For any finitely generated subalgebra $X=\langle x_1, x_2,\dots, x_n\rangle$ of L, we have $X^2\leqq L_\omega$. There exists $n\in\mathbf{N}$ such that $[x_i, x_j]\in L_n$ for any $i, j\in\{1, 2,\dots, n\}$. X^2 is spanned by elements of the form $[x_{i(1)}, x_{i(2)},\dots, x_{i(k)}]$ $(i(j)\in\{1, 2,\dots, n\}$, $k\geqq 2)$. Since $L_n\lhd L$ we have $X^2\leqq L_n$ and so X is soluble. Thus $L\in\mathrm{LE}\mathfrak{A}$.

REMARK 1. In the category of restricted Lie algebras we have $\acute{\mathrm{E}}\mathfrak{A}\leqq\mathrm{LE}\mathfrak{A}$.

In fact let $L \in \acute{\mathrm{E}}\mathfrak{A}$ and $\{L_\alpha\}$ be an ascending $\mathfrak{A}$-series of restricted subalgebras of L. We assume transfinite inductively that $L_\alpha \in \mathrm{LE}\mathfrak{A}$ for $\alpha<\lambda$. If λ is a limit ordinal or zero then evidently we have $L_\lambda \in \mathrm{LE}\mathfrak{A}$. Suppose that $\lambda-1$ exists and $L_{\lambda-1} \in \mathrm{LE}\mathfrak{A}$. Let X be a finitely generated restricted subalgebra of L_λ. Then $X/X \cap L_{\lambda-1} \simeq (X+L_{\lambda-1})/L_{\lambda-1} \in \mathfrak{F} \cap \mathfrak{A}$. By [6, Corollary 2.5.2] we have $X \cap L_{\lambda-1} \in \mathfrak{G}$. Hence $X \cap L_{\lambda-1} \in \mathrm{E}\mathfrak{A}$ and so $X \in \mathrm{E}\mathfrak{A}$. Thus $L \in \mathrm{LE}\mathfrak{A}$.

Now we shall show some properties of the class of hyper abelian-and-finite Lie algebras, which generalize [10, §2].

THEOREM 3.6. *Let L be a Lie algebra over a field $\mathfrak{k}$.*
(i) *If $L \in \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F})$ then $L^{(\omega)} \leqq \rho(L) \in \mathfrak{Z}$.*
(ii) *If $L \in \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}_r)$ for some $r \in \mathbf{N}$ then $L^{(d)} \in \mathfrak{Z}$ for some $d \leqq r^2$.*
(iii) *If $L \in \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F})$ and* char $\mathfrak{k}=0$ *then $L^2 \in \mathfrak{Z}$.*

PROOF. Let $L \in \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F})$. By [8, Proposition 6], we have $\acute{\mathrm{E}}(\lhd)\mathfrak{F} \cap \mathrm{L}\mathfrak{N} = \mathfrak{Z}$. Hence $\rho(L)$ is hypercentral. Let $\{L_\alpha\}_{\alpha \leqq \sigma}$ be an ascending $(\mathfrak{A} \cap \mathfrak{F})$-series of ideals of L. Then for each α, $L/C_L(L_{\alpha+1}/L_\alpha)$ is embedded in $\mathrm{End}_{\mathfrak{k}}(L_{\alpha+1}/L_\alpha)$ and so $L/C_L(L_{\alpha+1}/L_\alpha) \in \mathfrak{F} \cap \mathrm{E}\mathfrak{A}$. It follows that $L^{(\omega)} \leqq \bigcap_{\alpha<\sigma} C_L(L_{\alpha+1}/L_\alpha)$. Therefore $\{L^{(\omega)} \cap L_\alpha\}_{\alpha \leqq \sigma}$ is an ascending central series of $L^{(\omega)}$ and so $L^{(\omega)}$ is hypercentral. If $L_{\alpha+1}/L_\alpha \in \mathfrak{F}_r$ for every α then $L^{(d)} \leqq \bigcap_\alpha C_L(L_{\alpha+1}/L_\alpha)$ for some $d \leqq r^2$. Similarly $L^{(d)}$ has an ascending central series and hence $L^{(d)}$ is hypercentral. We have now proved (i) and (ii).

Next we suppose char $\mathfrak{k}=0$ to show (iii). Let $n(\alpha)=\dim(L_{\alpha+1}/L_\alpha)$. By Lie's theorem, L induces simultaneously triangularizable endomorphisms of $L_{\alpha+1}/L_\alpha$. Hence L^2 induces simultaneously nilpotent endomorphisms of $L_{\alpha+1}/L_\alpha$ and therefore $[L_{\alpha+1}, {}_{n(\alpha)-1}L^2] \leqq L_\alpha$. Refining each factor $L^2 \cap L_{\alpha+1}/L^2 \cap L_\alpha$ by $[L_{\alpha+1}, {}_iL^2]+(L^2 \cap L_\alpha)$ $(1 \leqq i < n(\alpha))$ we obtain an ascending central series of L^2. Thus L^2 is hypercentral. This completes the proof.

COROLLARY 3.7. *Over any field of characteristic zero,*

$$\mathfrak{Z} \leqq \acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \leqq \mathfrak{Z}\mathfrak{A} \leqq \acute{\mathrm{E}}(\lhd)\mathfrak{A}.$$

PROOF. The first inequality follows from [8, Proposition 6]. The second one is a straight consequence of Theorem 3.6 (iii). Let $L \in \mathfrak{Z}\mathfrak{A}$. Then $L^2 \in \mathfrak{Z}$ and so $\{\zeta_\alpha(L^2), L\}_{\alpha \geqq 0}$ is an ascending $\mathfrak{A}$-series of ideals of L. Thus $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$.

REMARK 2. In Corollary 3.7 the first inequality and the last one hold for any field. But the second inequality does not always hold for a field of non-zero characteristic. For example let char $\mathfrak{k}=p>0$ and V be a vector space over $\mathfrak{k}$ with basis $\{e_0, e_1, \dots, e_{p-1}\}$. Considered as an abelian Lie algebra V has derivations x, y such that $e_ix=e_{i+1}$, $e_iy=ie_i$ $(0 \leqq i \leqq p-1)$ where $e_p=e_0$. Then

$[x, y]=x$ and so $\langle x, y\rangle$ is a two-dimensional non-abelian subalgebra of Der(V). Let L be the split extension $V\dotplus\langle x\ y\rangle$. Then $L\in \acute{\mathrm{E}}(\lhd)(\mathfrak{A}\cap\mathfrak{F}_p)$ and it is easy to see that $L^2=V\dotplus\langle x\rangle\notin\mathfrak{Z}$.

REMARK 3. The two-dimensional non-abelian Lie algebra distinguishes the class $\acute{\mathrm{E}}(\lhd)(\mathfrak{A}\cap\mathfrak{F})$ from $\mathfrak{Z}$.

Let A be an infinite extension field of $\mathfrak{k}$ and f be the regular representation of A. Consider A as an abelian Lie algebra over $\mathfrak{k}$. Then we can form the split extension $L=A\dotplus f(A)$, where $A\lhd L$ and $[a, f(b)]=ab$ for $a, b\in A$. This is considered in [7, §2], where it is shown that every non-zero ideal of L contains A. Therefore L has no non-zero finite-dimensional ideals of L and clearly L is metabelian. Thus $L\in\mathfrak{Z}\mathfrak{A}\backslash\acute{\mathrm{E}}(\lhd)(\mathfrak{A}\cap\mathfrak{F})$.

There exists a perfect Fitting algebra (cf. [16, p. 96]) and so $\mathfrak{Z}\mathfrak{A}\lneqq\acute{\mathrm{E}}(\lhd)\mathfrak{A}$. Moreover this Lie algebra shows that the classes $\acute{\mathrm{E}}(\lhd)(\mathfrak{A}\cap\mathfrak{F})$, $\acute{\mathrm{E}}(\lhd)(\mathfrak{A}\cap\mathfrak{F}_r)$ are not L-closed.

COROLLARY 3.8. *If $L\in\acute{\mathrm{E}}(\lhd)(\mathfrak{A}\cap\mathfrak{F})$ then $L^{(\alpha)}=0$ for some ordinal α.*

PROOF. By Theorem 3.6 (i) $L^{(\omega)}$ is hypercentral. It is known that the transfinite derived series of a $\mathfrak{Z}$-algebra reaches to zero (cf. [2, Lemma 8.1.1]). Hence $L^{(\alpha)}=0$ for some ordinal α.

The following is [10, Lemma 3.2 and Theorem 3.6].

LEMMA 3.9. *Let L be a locally finite Lie algebra over $\mathfrak{k}$ and let $\bar{\mathfrak{k}}$ be an algebraic closure of $\mathfrak{k}$. Then the following are equivalent:*

(i) *L is locally nilpotent.*
(ii) *Every subalgebra of L is serial.*
(iii) *Every 1-dimensional subalgebra of L is serial.*
(iv) *Every 2-dimensional $\bar{\mathfrak{k}}$-subalgebra of $L\otimes_{\mathfrak{k}}\bar{\mathfrak{k}}$ is abelian.*

As in the case of supersoluble Lie algebras, we can state the following criterion for a hyperfinite Lie algebra (in particular a hyper abelian-and-finite Lie algebra) to be hypercentral (cf. [10, Theorem 3.3]).

PROPOSITION 3.10. *Let L be a hyperfinite Lie algebra over a field $\mathfrak{k}$ and let $\bar{\mathfrak{k}}$ be an algebraic closure of $\mathfrak{k}$. Then the following are equivalent:*

(i) *L is hypercentral.*
(ii) *Every subalgebra of L is ascendant.*
(iii) *Every subalgebra of L is serial.*
(iv) *Every 1-dimensional subalgebra of L is ascendant.*
(v) *Every 1-dimensional subalgebra of L is serial.*
(vi) *Every 2-dimensional $\bar{\mathfrak{k}}$-subalgebra of $L\otimes_{\mathfrak{k}}\bar{\mathfrak{k}}$ is abelian.*

PROOF. The statement follows from [8, Proposition 6], [9, Corollary 3.3] and Lemma 3.9.

4. The class $\acute{E}(\lhd)L\mathfrak{N}$

In this section we shall observe the class of hyper locally nilpotent Lie algebras which contains the class of hyperabelian Lie algebras.

LEMMA 4.1. *Let L be a hyper locally nilpotent Lie algebra and let $\{H_\beta\}_{\beta\leqq\sigma}$ be an ascending series of $\rho(L)$ such that $H_\beta\lhd L$ for any β. Then $\bigcap_{\beta<\sigma}C_L(H_{\beta+1}/H_\beta)\leqq\rho(L)$.*

PROOF. Put $C=\bigcap_{\beta<\sigma}C_L(H_{\beta+1}/H_\beta)$ and let $\{L_\alpha\}$ be an ascending $L\mathfrak{N}$-series of ideals of L. Assume that $C\nleqq\rho(L)$ and let α be the minimal ordinal with respect to $C\cap L_\alpha\nleqq\rho(L)$. Then α is a non-limit ordinal and $\alpha-1$ exists. Now we prove that $C\cap L_\alpha\in L\mathfrak{N}$. Let X be a finitely generated subalgebra of $C\cap L_\alpha$. Since $L_\alpha/L_{\alpha-1}\in L\mathfrak{N}$, there exists $n\in\mathbf{N}$ such that $X^n\leqq L_{\alpha-1}\cap C$. Hence $X^n\leqq\rho(L)$ by the minimality of α. For any sequence $x_1, x_2,\dots$ of elements of X and $s\in\mathbf{N}$, there exists an ordinal $\beta(s)$ minimal with respect to $[x_1,\dots, x_n, x_{n+1},\dots, x_{n+s}]\in H_{\beta(s)}$. Since each x_i belongs to C we have $\beta(s+1)<\beta(s)$ for any $s\in\mathbf{N}$. Therefore $\beta(t)=0$ for some $t\in\mathbf{N}$. Then $[x_1,\dots, x_{n+t}]=0$. By [8, Proposition 5] $X\in\mathfrak{Z}\cap\mathfrak{G}\leqq\mathfrak{N}$. Thus $C\cap L_\alpha$ is an $L\mathfrak{N}$-ideal of L and so $C\cap L_\alpha\leqq\rho(L)$. This is a contradiction.

From Lemma 4.1 we can deduce the following

THEOREM 4.2. *Let L be a hyper locally nilpotent Lie algebra with finite-dimensional Hirsch-Plotkin radical $\rho(L)$. Then L is finite-dimensional and soluble.*

PROOF. By Lemma 4.1 we have $C_L(\rho(L))\leqq\rho(L)$ and so $C_L(\rho(L))$ is finite-dimensional. Since $L/C_L(\rho(L))$ is embedded in $\mathrm{Der}\,(\rho(L))$ it is finite-dimensional. Therefore $L\in\mathfrak{F}\cap\acute{E}(\lhd)L\mathfrak{N}=\mathfrak{F}\cap E\mathfrak{A}$.

COROLLARY 4.3. *Every infinite-dimensional $\acute{E}(\lhd)L\mathfrak{N}$-algebra has an infinite-dimensional $L\mathfrak{N}$-ideal and further an infinite-dimensional abelian subalgebra.*

PROOF. The statement follows from Theorem 4.2 and [2, Theorem 10.1.3].

COROLLARY 4.4. $\acute{E}(\lhd)L\mathfrak{N}\cap\mathrm{Min}\text{-}\lhd^2\leqq\mathfrak{F}\cap E\mathfrak{A}$.

PROOF. By [2, Lemma 8.1.3], if L satisfies the condition $\mathrm{Min}\text{-}\lhd^2$ then $\rho(L)$ is finite-dimensional. Hence the assertion follows from Theorem 4.2.

In Corollary 4.4 we cannot relax the condition Min-$\lhd^2$. In fact, there exists an infinite-dimensional soluble Lie algebra which satisfies the condition Min-$\lhd$. For instance let V be a vector space with basis $\{e_0, e_1,...\}$ and let x be the downward shift on V, that is, $e_0x=0$ and $e_ix=e_{i-1}$ for all $i>0$. Considering V as an abelian Lie algebra we can form the split extension $L=V\dotplus\langle x\rangle$. Then it is easy to see that L satisfies the condition Min-$\lhd$.

But by using Lemma 4.1 we can show the following proposition which is a generalization of Corollary 4.4.

PROPOSITION 4.5. *Let* $L\in \acute{\mathrm{E}}(\lhd)\mathrm{L}\mathfrak{N}\cap$ Min-$\lhd$. *If* $\rho(L)$ *has an ascending* $\mathfrak{F}$-*series of ideals of* L *then* $L/\rho(L)\in\mathfrak{F}\cap\mathrm{E}\mathfrak{A}$.

PROOF. Since $\rho(L)\in\mathrm{L}\mathfrak{N}\cap\acute{\mathrm{E}}(\lhd)\mathfrak{F}$ it is hypercentral by [8, Proposition 6]. Refining the given ascending $\mathfrak{F}$-series of $\rho(L)$ by the upper central series of $\rho(L)$. we obtain an ascending central series $\{H_\alpha\}_{\alpha\leqq\sigma}$ of $\rho(L)$ such that $H_\alpha\lhd L$ and $H_{\alpha+1}/H_\alpha\in\mathfrak{F}$ for all $\alpha<\sigma$. Put $C=\bigcap_\alpha C_L(H_{\alpha+1}/H_\alpha)$. Then we have $\rho(L)\leqq C$ and so $\rho(L)=C$ by Lemma 4.1. Since $L\in$ Min-$\lhd$ there exist finite number of ordinals $\alpha(1),..., \alpha(n)$ such that $C=\bigcap_{i=1}^n C_L(H_{\alpha(i)+1}/H_{\alpha(i)})$. Then $\oplus_{i=1}^n(H_{\alpha(i)+1}/H_{\alpha(i)})$ is a finite-dimensional faithful $L/\rho(L)$-module. Thus $L/\rho(L)\in\mathfrak{F}\cap\acute{\mathrm{E}}(\lhd)\mathrm{L}\mathfrak{N}=\mathfrak{F}\cap\mathrm{E}\mathfrak{A}$.

Every non-zero soluble Lie algebra has a non-zero nilpotent ideal. Furthermore, if L $(\neq 0)$ is an $\mathrm{L}(\mathfrak{F}\cap\mathrm{E}\mathfrak{A})$-algebra over a field of characteristic zero then L^2 is locally nilpotent by [2, Lemma 13.3.10] and so $\rho(L)\neq 0$. It is reasonable to ask whether every non-zero $\mathrm{LE}\mathfrak{A}$-algebra has non-zero Hirsch-Plotkin radical. We give the negative answer to this problem in the following, which also shows that the class $\acute{\mathrm{E}}(\lhd)\mathrm{L}\mathfrak{N}$ is not L-closed.

THEOREM 4.6. *Over any field there exists a non-zero locally soluble Lie algebra with trivial Hirsch-Plotkin radical.*

PROOF. Let L_1 be any soluble Lie algebra. We construct an ascending chain of soluble Lie algebras inductively. Now suppose that we have constructed a soluble algebra L_n. Let $U(L_n)$ be the universal enveloping algebra of L_n. $U(L_n)$ is a faithful L_n-module under right regular representation of $U(L_n)$. We form the split extension $U(L_n)\dotplus L_n$ of $U(L_n)$ by L_n and call this new Lie algebra L_{n+1}. Clearly L_{n+1} is soluble. Thus we obtain an ascending chain of soluble algebras $L_1\leqq L_2\leqq\cdots$ as desired.

Now let L be the direct limit of $\{L_n\}$. Then we have $L\in\mathrm{LE}\mathfrak{A}$ since each L_n is soluble. Assume that $\rho(L)\neq 0$ and let $0\neq a\in\rho(L)$. Then there exists $n\in\mathbf{N}$ such that $a\in L_n$. For the unit element 1_n of $U(L_n)$, we have $[1_n, a]=\bar{a}\in\rho(L)$ where $\bar{a}$ is the element of $U(L_n)$ corresponding to a. Then we have $\langle\bar{a}, a\rangle\in\mathfrak{N}$. But $[\bar{a}, {}_ka]=\bar{a}^{k+1}\neq 0$ for any $k\in\mathbf{N}$. This is a contradiction.

REMARKS. In general $\acute{\mathrm{E}}(\lhd)\mathrm{L}\mathfrak{N} \nleq \acute{\mathrm{E}}\mathfrak{A}$. But countable dimensional $\acute{\mathrm{E}}\mathrm{L}\mathfrak{N}$-algebras are contained in $\acute{\mathrm{E}}\mathfrak{A}$.

By [2, Corollary 16.3.11] $\mathfrak{E} \cap \acute{\mathrm{E}}(\lhd)\mathrm{L}\mathfrak{N} = \mathrm{L}\mathfrak{N}$, where $\mathfrak{E}$ is the class of Engel algebras.

In groups theory it is known that $\acute{\mathrm{E}}\mathrm{L}\mathfrak{N} = \acute{\mathrm{E}}(\lhd)\mathrm{L}\mathfrak{N}$ (cf. [14, p. 59]). But for Lie algebras it is an open question. We can only know that $(\mathrm{L}\mathfrak{F} \cup \mathfrak{E}) \cap \acute{\mathrm{E}}\mathrm{L}\mathfrak{N} \leqq \acute{\mathrm{E}}(\lhd)\mathrm{L}\mathfrak{N}$ over any field of characteristic zero (cf. [2, Theorem 13.3.7 and 16.3.13]).

5. Semisimplicities

This final section is devoted to discussing several semisimplicities. It is clear that $\sigma(L/\alpha_*(L)) = 0$ for any Lie algebra L. So in a sense the theory of Lie algebras is reduced to the theory of the hyperabelian Lie algebras and the semisimple Lie algebras. But it is awkward that there is a semisimple and locally nilpotent Lie algebra (see Theorem 3.3) unlike the theory of finite-dimensional Lie algebras. Here we need to recall and define some classes of semisimple Lie algebras. $L \in \mathfrak{S}$ if and only if $L = 0$ or L is non-abelian simple. When $0 \neq L \in \mathfrak{S}$ we call briefly L simple. $\mathrm{D}\mathfrak{S}$ consists of all direct sums of simple Lie algebras. $L \in (\mathrm{SS})$ (resp. $(\rho\mathrm{SS})$, (RSS)) if and only if L has no non-zero $\mathfrak{A}$ (resp. $\mathrm{L}\mathfrak{N}$, $\mathrm{LE}\mathfrak{A}$)-ideals. $L \in (\mathrm{CSS})$ if and only if $C_L(s(L)) = 0$, where $s(L)$ is the sum of all simple ideals of L.

Concerning these classes we have the following

PROPOSITION 5.1. *Over any field*

$$\mathrm{D}\mathfrak{S} < \acute{\mathrm{E}}(\lhd)\mathfrak{S} = \acute{\mathrm{E}}\mathfrak{S} < (\mathrm{CSS}) < (\mathrm{RSS}) < (\rho\mathrm{SS}) < (\mathrm{SS}).$$

PROOF. If is clear that $(\rho\mathrm{SS}) \leqq (\mathrm{SS})$. By Theorem 3.3 there exists an $\mathrm{L}\mathfrak{N}$-algebra L such that $\alpha(L) = 0$.

Evidently we have $(\mathrm{RSS}) \leqq (\rho\mathrm{SS})$. By Theorem 4.6 there exists an $\mathrm{LE}\mathfrak{A}$-algebra L with $\rho(L) = 0$. Hence $(\mathrm{RSS}) < (\rho\mathrm{SS})$.

Let $L \in (\mathrm{CSS})$ and H be an $\mathrm{LE}\mathfrak{A}$-ideal of L. Since a minimal ideal of an $\mathrm{LE}\mathfrak{A}$-algebra is abelian we have $s(L) \cap H = 0$. Hence $H \leqq C_L(s(L)) = 0$ and so $L \in (\mathrm{RSS})$. Thus we have $(\mathrm{CSS}) \leqq (\mathrm{RSS})$. Now let F be a non-abelian free Lie algebra. It is well known that every proper non-zero ideal of F is an infinitely generated free Lie algebra (cf. [6, Corollary to Theorem 2.3.5]). Therefore $s(F) = 0$ and F has no non-zero $\mathrm{LE}\mathfrak{A}$-ideals. Thus $F \in (\mathrm{RSS}) \setminus (\mathrm{CSS})$.

It is shown in [9, Corollary 1.6] that $\acute{\mathrm{E}}\mathfrak{S} = \acute{\mathrm{E}}(\lhd)\mathfrak{S}$. Let $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{S}$ and suppose that $C_L(s(L)) \neq 0$. By Lemma 1.3 there exists a simple ideal I such that $0 \neq I \leqq C_L(s(L))$. Then $0 \neq I^2 \leqq [s(L), C_L(s(L))] = 0$, which is a contradiction. Hence $C_L(s(L)) = 0$ and so $L \in (\mathrm{CSS})$. Thus we have $\acute{\mathrm{E}}\mathfrak{S} \leqq (\mathrm{CSS})$. Now let $K = \mathrm{Dr}_{n \in \mathbf{N}} S_n$ where each S_n is a simple Lie algebra. Then every element of $\mathrm{Cr}_{n \in \mathbf{N}} S_n$

induces a derivation of K. Let $\delta=(\delta_n)\in \mathrm{Cr}_{n\in\mathbf{N}}S_n$ with $\delta_n\neq 0$ for any $n\in\mathbf{N}$. We can form the split extension $M=K\dotplus\langle\delta\rangle$. Then it is easy to see that $\mathrm{s}(M)=K$ and $C_M(K)=0$. But M/K is one-dimensional. Hence by Lemma 1.1 we have $M\notin \acute{\mathrm{E}}(\lhd)\mathfrak{S}$. Thus $M\in(\mathrm{CSS})\backslash\acute{\mathrm{E}}\mathfrak{S}$.

By making use of Lemma 1.1 we have $\mathrm{D}\mathfrak{S}\leqq\acute{\mathrm{E}}(\lhd)\mathfrak{S}$. It is known that there exists an $\acute{\mathrm{E}}\mathfrak{S}$-algebra L such that $L\notin\mathrm{D}\mathfrak{S}$ (cf. [2, p. 266]). The proof is completed.

REMARK 1. Over any field of characteristic zero we have $\mathfrak{H}\cap(\rho\mathrm{SS})=\mathfrak{H}\cap \mathrm{D}\mathfrak{S}=\mathrm{D}(\mathfrak{S}\cap\mathfrak{F})$ where $\mathfrak{H}$ is the class of neo-classical algebras (cf. [2, §13]). Furthermore it can be seen that $\mathrm{L}(\lhd)\mathfrak{F}\cap(\mathrm{SS})=\mathrm{D}(\mathfrak{S}\cap\mathfrak{F})$ where $\mathrm{L}(\lhd)\mathfrak{F}$ is the class of Lie algebras generated by finite-dimensional ideals.

REMARK 2. A Lie algebra L is said to be prime if 0 is a prime ideal of L. If L is prime then L is semisimple. On the other hand every semisimple Lie algebra is a subdirect sum of prime algebras.

If L is a prime algebra with $\mathrm{s}(L)\neq 0$ then $L\in(\mathrm{CSS})$, while L is not necessarily an $\acute{\mathrm{E}}\mathfrak{S}$-algebra. For example let W be a Lie algebra over the real numbers field $\mathbf{R}$ with basis $\{w_r|r\in\mathbf{R}\}$ and multiplication $[w_r, w_s]=(r-s)w_{r+s}$, which is a generalized Witt algebra and simple (cf. [2, p. 206]). Let T be a complementary $\mathbf{Q}$-subspace of the rational numbers field $\mathbf{Q}$ in $\mathbf{R}$ and δ be a derivation of W such that $w_r\delta=qw_r$ for $r=q+t$ $(q\in\mathbf{Q},\ t\in T)$. Now we can form the split extension $L=W\dotplus\langle\delta\rangle$. Then it is easy to see that W is the unique non-trivial ideal of L and $\mathrm{s}(L)=W$. Hence L is prime, but $L\notin\acute{\mathrm{E}}\mathfrak{S}$ since L/W is one-dimensional.

From now on we treat the class (CSS). It is characterized by the following

PROPOSITION 5.2. *Let L be a Lie algebra over any field. Then L is a* (CSS)-*algebra if and only if every non-zero ideal of L contains a simple ideal of L.*

Proof. Let $L\in(\mathrm{CSS})$ and assume that there exists a non-zero ideal I of L which contains no simple ideals of L. Then $I\cap\mathrm{s}(L)=0$. Hence we have $0\neq I\leqq C_L(\mathrm{s}(L))$, which is a contradiction.

Conversely suppose that every non-zero ideal of L contains a simple ideal of L. Then $\mathrm{s}(L)\neq 0$. If $C_L(\mathrm{s}(L))\neq 0$ then $C_L(\mathrm{s}(L))$ contains a simple ideal I of L. Since $I\leqq\mathrm{s}(L)$ we have $I^2\leqq[\mathrm{s}(L), C_L(\mathrm{s}(L))]=0$. This is a contradiction. Thus $C_L(\mathrm{s}(L))=0$ and so $L\in(\mathrm{CSS})$.

COROLLARY 5.3. $\acute{\mathrm{E}}\mathfrak{S}$ *is the largest* Q-*closed subclass of* (CSS).

PROOF. The result follows from Proposition 5.2 and Lemma 1.3.

REMARK 3. By Proposition 5.1 the class (CSS) is not Q-closed. But it can be seen that (CSS) is I-closed. In fact let $L\in(\mathrm{CSS})$ and $I\lhd L$. Then $\mathrm{s}(I)=I\cap$

s(L) and s(L)=s(I)$\oplus S_1$ where $I \cap S_1=0$. If $x \in C_I$(s(I)), we have [x, s(L)]= [x, S_1]$\subset I \cap S_1=0$. Hence $x \in C_L$(s(L))=0 and so C_I(s(I))=0.

The following is a striking characterization of the class (CSS).

PROPOSITION 5.4. *$L \in$ (CSS) if and only if there exist a* D$\mathfrak{S}$*-algebra S and a Lie algebra L' such that*

$$\mathrm{Inn}(S) \leqq L' \leqq \mathrm{Der}(S) \quad \text{and} \quad L \simeq L'$$

where Inn (S) *is the set of inner derivations of S.*

PROOF. Let Inn (S)$\leqq L \leqq$Der (S) for some $S \in$ D$\mathfrak{S}$. Since Inn (S) is isomorphic to S and Inn (S)$\lhd$Der (S), we have Inn (S)$\leqq$s(L). Let $\delta \in C_L$(s(L)). Then for any $x \in S$ we have ad ($x\delta$)=[ad (x), δ]=0. Since $\zeta_1(S)=0$ it follows that $x\delta=0$ and so $\delta=0$. Thus we have C_L(s(L))=0.

Now suppose that $L \in$ (CSS). We denote by $\mathrm{ad}_{s(L)}(L)$ the set of derivations of s(L) induced by the elements of L. Then Inn (s(L))$\leqq \mathrm{ad}_{s(L)}(L) \leqq$ Der (s(L)) and $\mathrm{ad}_{s(L)}(L)$ is isomorphic to L since C_L(s(L))=0. This completes the proof.

COROLLARY 5.5. *Let L be a* (CSS)*-algebra over a field of characteristic zero and let* s(L)$=\oplus_{i=1}^n S_i$ *where each S_i is an* ($\mathfrak{F} \cap \mathfrak{S}$)*-algebra or the Witt algebra. Then* L=s(L) $\in$ D$\mathfrak{S}$.

PROOF. It is not hard to show that Der (s(L))=Inn(s(L)). Since C_L(s(L))=0, we have Inn (s(L))=$\mathrm{ad}_{s(L)}(L)$ and so s(L)=L.

For two classes $\mathfrak{X}$, $\mathfrak{Y}$ of Lie algebras, $L \in$ (É(L)$\mathfrak{X}$)$\mathfrak{Y}$ means that L has an ascending series of ideals in which all the factors from the beginning are $\mathfrak{X}$-algebras and only the last factor is a $\mathfrak{Y}$-algebra.

PROPOSITION 5.6. *Let $\mathfrak{X}$ be a* {Q, I}*-closed class of Lie algebras. If $L \in$* É($\lhd$)($\mathfrak{X} \cup \mathfrak{S}$) *then $L \in$* (É(L)$\mathfrak{X}$)(CSS).

PROOF. Let $\chi_*(L)$ be the hyper $\mathfrak{X}$-radical of L and put $\bar{L}=L/\chi_*(L)$. To show that $\bar{L} \in$ (CSS) suppose that $C_{\bar{L}}$(s($\bar{L}$))$\neq 0$. Put $\bar{C}=C_{\bar{L}}$(s($\bar{L}$)). $\bar{L}$ as well as L itself has an ascending series $\{\bar{L}_\alpha\}$ of ideals with $\bar{L}_{\alpha+1}/\bar{L}_\alpha \in \mathfrak{X} \cup \mathfrak{S}$. Let β be the least ordinal such that $\bar{C} \cap \bar{L}_\beta \neq 0$. Clearly β is a non-limit ordinal and $\beta-1$ exists. Put $\bar{I}=\bar{C} \cap \bar{L}_\beta$. Since $\chi(\bar{L})=0$ and $\bar{I} \simeq (\bar{I}+\bar{L}_{\beta-1})/\bar{L}_{\beta-1} \lhd \bar{L}_\beta/\bar{L}_{\beta-1} \in \mathfrak{X} \cup \mathfrak{S}$, $\bar{I}$ is a simple ideal of $\bar{L}$. Then $0 \neq \bar{I}^2 \leqq$[s($\bar{L}$), $\bar{C}$]=0. This is a contradiction. Thus $\bar{C}=0$ and therefore $\bar{L} \in$ (CSS).

COROLLARY 5.7. *Over any field of characteristic zero, if $L \in$* Min-$\lhd^2$ *then $L \in$* (É(L)$\mathfrak{A}$)(CSS).

Proof. The statement follows from [2, Theorem 8.2.3] and Proposition 5.6.

Corollary 5.8. *Over any field of characteristic zero*

$$\mathrm{Min}\text{-}\lhd^2 \cap (\mathrm{SS}) \leqq (\mathrm{CSS}).$$

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