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On the asymptotic behavior of solutions of certain semilinear parabolic equations in $\mathbf{R}^N$

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This paper is concerned with the asymptotic behavior of solutions of the Cauchy problems for semilinear parabolic equations of the form

(0.1) $$\partial u/\partial t - \Delta u + f(u) = 0, \quad (x, t) \in \mathbf{R}^N \times (0, +\infty),$$

(0.2) $$u(x, 0) = u_0(x), \quad x \in \mathbf{R}^N,$$

where u_0 is a given initial-function, Δ is the N-dimensional Laplace operator, and f is a continuous nondecreasing function on $\mathbf{R}$ satisfying the sign condition

(0.3) $$sf(s) > 0 \qquad \text{for} \quad s \neq 0.$$

Recently Gmira and Veron [4] have treated the asymptotic behavior as $t \to \infty$ of solutions of (0.1) in the canonical cases $f(s) = s|s|^{p-1}$, $p > 1$. They studied the asymptotic behavior of solutions of such typical semilinear equations by applying a comparison theorem to the solutions of the associated ordinary differential equation

(0.4) $$dz/dt + f(z) = 0, \quad t \in (0, +\infty).$$

Further, in order to investigate more precise behaviors of the solutions, they employed suitable scaling transformations and well-known results concerning the order-preserving semigroup of nonlinear contractions which provides solutions of (0.1) with $f(s) = s|s|^{p-1}$.

The purpose of this paper is to make an attempt to extend their results so as to cover a much wider class of equations of the form (0.1). Here we assume that f has the same order as the function $s|s|^{p-1}$ near $s = 0$ in the sense that $|s^{-p}f(s)|$ is bounded and bounded away from 0 as $s \to 0$. Under this condition we discuss the asymptotic behavior of solutions of (0.1) by applying appropriate comparison theorems to the solutions of (0.1) with $f(s) = s|s|^{p-1}$. We shall employ the scaling transformations as well as basic properties of nonlinear semigroup providing generalized solutions of (0.1) in a way similar to [4]. In this sense our argument relies heavily upon the work of Gmira and Veron, although our results extend considerably those of [4] as illustrated via some examples in the last section of this paper.

Solutions of (0.1) converge to zero in general, but solutions multiplied by an increasing factor $t^{N/2}$ might converge to some nontrivial functions. Here we show that if the nonlinear function $f(s)$ satisfies certain growth conditions and if initial functions are nontrivial, then the solutions converge to $C_0(4\pi)^{-N/2}\exp(-|x|^2/4t)$, where C_0 is a constant depending only upon the initial function u_0. Further, we give some sufficient conditions for the constant C_0 to be nonzero; and these conditions have not been known even for the case $f(s)=s|s|^{p-1}$.

This paper consists of four sections. In Section 1 we discuss the existence of solutions of the Cauchy problem (0.1)–(0.2) and give some basic estimates concerning their asymptotic properties. Our main results are stated in Section 2 along with some comments. The full statements of the results and their proofs are given in Section 3. Finally, in Section 4, several remarks on the main theorems are given and concrete examples of nonlinearities $f(s)$ are given to illustrate the main results.

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1. Existence and basic properties of solutions

We begin by introducing some function spaces. By X_q $(1\leqq q<\infty)$ we denote the usual Lebesgue spaces $L^q(\mathbf{R}^N)$; for $q=\infty$, we set

$$X_\infty = \{w\in C(\mathbf{R}^N);\ \lim_{|x|\to\infty} w(x)=0\}. \tag{1.1}$$

The norm of the Banach space X_q is denoted by $|\cdot|_q$. Further, $W^{m,q}(\mathbf{R}^N)$ and $W^{m,q}_{\mathrm{loc}}(\mathbf{R}^N)$ denote the usual Sobolev spaces.

In what follows f is a continuous nondecreasing function on $\mathbf{R}$ satisfying (0.3). For each $q\in[1,\infty]$ we consider the operator

$$A_q w = -\Delta w + f(w) \quad \text{in} \quad X_q$$

with domain $D(A_q)$ specified as follows:

$$D(A_q) = \{w\in W^{2,q}(\mathbf{R}^N);\ f(w)\in X_q\} \quad \text{if} \quad 1<q<\infty,$$

$$D(A_1) = \{w\in X_1;\ \Delta w, f(w)\in X_1,\ w\in W^{1,r}(\mathbf{R}^N) \text{ for } 1\leqq r<N/(N-1)\},$$

and

$$D(A_\infty) = \{w\in X_\infty;\ \Delta w\in X_\infty,\ w\in W^{2,q}_{\mathrm{loc}}(\mathbf{R}^N) \text{ for some } q>N\}.$$

Notice that if $w\in D(A_\infty)$, then $w\in W^{2,q}_{\mathrm{loc}}(\mathbf{R}^N)$ for any $q>N$; see [6]. Let $1\leqq q\leqq\infty$. Since A_q is known to be a densely defined m-accretive operator in X_q, a standard generation theorem for nonlinear semigroups [3] can be applied to get for a given $u_0\in X_q$ a solution of (0.1) represented by the exponential formula

$$u(\cdot, t) = S_q(t)u_0 = \lim_{n\to\infty}(I+(t/n)A_q)^{-n}u_0,$$

where $\{S_q(t);\ t\geqq 0\}$ denotes the nonlinear contraction semigroup in the Banach space X_q generated by $-A_q$. It is well-known that for each t, $S_q(t)$ is order-preserving, namely: If $\geqq$ denotes the natural order relation in X_q, then $S_q(t)u_0\geqq S_q(t)v_0$ whenever $u_0\geqq v_0$. In what follows we call the function $S_q(t)u_0$ the *semigroup solution* of the problem (0.1)–(0.2) with $u_0\in X_q$.

We now give a few basic properties of the semigroup solutions.

PROPOSITION 1.1 ([4], [5], [7]). *Assume that* $1\leqq q<\infty$.

(i) *Let* $u_0\in D(A_q)$ *and* $u(t)=S_q(t)u_0$; *then*

(a) u: $[0, \infty)\to X_q$ *is Lipschitz continuous and* $u(0)=u_0$;

(b) *for a.e.* $t>0$, $u(t)\in D(A_q)$, $du(t)/dt\in X_q$ *and* $du/dt+A_qu=0$.

(ii) *If* $u_0\in X_q$, *then* $u(t)\in X_q\cap L^\infty(\mathbf{R}^N)$ *for all* $t>0$, *and there is a constant* $C=C(N)>0$ *for which*

$$|u(t)|_r \leqq Ct^{-N(1/q-1/r)/2}|u_0|_q \tag{1.2}$$

holds for $t>0$ *and* $r\in[q, \infty]$.

(iii) *If* $u_0\in X_q$ *and* $1\leqq q\leqq 2$, *then*:

(c) *For each* $r\in[2, \infty]$, u: $(0, \infty)\to L^r(\mathbf{R}^N)$ *is locally Lipschitz continuous.*

(d) *For a.e.* $t>0$, $du(t)/dt$ *lies in* $L^r(\mathbf{R}^N)$ *for* $2\leqq r<\infty$ *and there is a constant* $C=C(N)>0$ *such that*

$$|du(t)/dt|_r \leqq Ct^{-1-N(1/q-1/r)/2}|u_0|_q. \tag{1.3}$$

For the proof we refer to [4], [5] and [7]. We also use the next result in the proof of Theorem 3.4.

PROPOSITION 1.2. *Let* $1\leqq q<\infty$, u_0, $v_0\in X_q$, $u(t)=S_q(t)u_0$ *and let* $v(t)=S_q(t)v_0$. *Then we have*:

(i) $0\leqq \int_0^\infty\int_{\mathbf{R}^N}[f(u)-f(v)][u-v]|u-v|^{q-2}\,dxdt\leqq(1/q)|u_0-v_0|_q^q$;

(ii) $0\leqq \int_0^\infty\int_{\mathbf{R}^N} f(u)u|u|^{q-2}\,dxdt\leqq(1/q)|u_0|_q^q$;

(iii) *If* $u_0\in X_1$, *then the limit* $\lim_{t\to\infty}\int_{\mathbf{R}^N}u(x, t)\,dx$ *exists and equals*

$$\int_{\mathbf{R}^N}u_0(x)dx - \int_0^\infty\int_{\mathbf{R}^N}f(u(x, t))dxdt.$$

PROOF. Since $f(0)=0$, we easily see that $v=0$ if $v_0=0$. So assertion (ii) immediately follows from (i) by setting $v_0=0$. We therefore prove assertions (i) and (iii). Suppose first that u_0 and v_0 belong to $C_0^\infty(\mathbf{R}^N)$; then we have

$$\partial(u-v)/\partial t - \Delta(u-v) + f(u) - f(v) = 0 \quad \text{a.e. in} \quad \mathbf{R}^N\times(0, \infty).$$

Multiplying both sides by $(u-v)|u-v|^{q-2}$, integrating the terms on the resultant equality over $\mathbf{R}^N$ and then using the inequality (see [2])

$$-\int_{\mathbf{R}^N}[\Delta(u-v)](u-v)\,|u-v|^{q-2}\,dx \geqq 0,$$

we obtain

$$(d/dt)\,|u(t)-v(t)|_q^q + q\int_{\mathbf{R}^N}[f(u)-f(v)](u-v)\,|u-v|^{q-2}\,dx \leqq 0.$$

Integrating both sides of this inequality over $(0, T)$ yields

$$(1.4)\quad |u(T)-v(T)|_q^q + q\int_0^T\int_{\mathbf{R}^N}[f(u)-f(v)]\,(u-v)\,|u-v|^{q-2}dxdt \leqq |u_0-v_0|_q^q.$$

Since $C_0^\infty(\mathbf{R}^N)$ is dense in X_q and $S_q(t)$ is a contraction semigroup on X_q, Fatou's lemma implies that the estimate (1.4) also holds for u_0, $v_0 \in X_q$, $u=S_q(t)u_0$, and $v=S_q(t)v_0$. The second inequality in (i) is now obtained by letting $T\to\infty$ in (1.4) and noting that f is nondecreasing.

We then prove assertion (iii). First assume that $u_0 \in D(A_1)$. Then, by Proposition 1.1 (i), du/dt, Δu and $f(u)$ belong to X_1 for a.e. $t>0$. Hence

$$\int_{\mathbf{R}^N}\Delta u(x, t)dx = 0 \quad \text{for a.e.} \quad t > 0$$

so that (0.1) gives

$$(d/dt)\int_{\mathbf{R}^N}u(x, t)dx + \int_{\mathbf{R}^N}f(u(x, t))dx = 0 \quad \text{for a.e.} \quad t > 0.$$

Integrating this over $(0, T)$ yields

$$(1.5)\qquad \int_{\mathbf{R}^N}u(x, T)dx = \int_{\mathbf{R}^N}u_0(x)dx - \int_0^T\int_{\mathbf{R}^N}f(u(x, t))dxdt.$$

We next consider the general case. Let $u_0 \in X_1$ and choose a sequence $\{u_{0n}\} \subset D(A_1)$ so that $u_{0n}\to u_0$ in X_1 as $n\to\infty$. Assertion (i) with $q=1$ and $v_0=u_{0n}$ implies that

$$\left|\int_0^T\int_{\mathbf{R}^N}f(u)dxdt - \int_0^T\int_{\mathbf{R}^N}f(u_n)dxdt\right| \leqq \int_0^\infty\int_{\mathbf{R}^N}|f(u)-f(u_n)|\,dxdt$$
$$\leqq |u_0-u_{0n}|_1 \to 0 \quad \text{as} \quad n\to\infty$$

for any $T>0$, where $u_n(t)\equiv S_1(t)u_{0n}$. Since $u_n(t)\to u(t)$ in X_1 uniformly for $t\geqq 0$, we see that (1.5) is valid for any $u_0 \in X_1$ and any $T>0$. Since $f(u)$ is integrable over $\mathbf{R}^N\times(0, \infty)$ by (ii), assertion (iii) follows from (1.5) by letting $T\to\infty$. The proof is complete.

Finally we prepare a comparison theorem which is useful to investigate the asymptotic behavior of solutions.

We denote by $z(t;\beta)$ the unique solution of the initial value problem:

$$\text{(ODE)} \qquad dz/dt + f(z) = 0 \quad (t>0), \quad z(0) = \beta.$$

LEMMA 1.1 (Comparison Theorem). *Suppose both f and g are continuous, nondecreasing functions satisfying condition* (0.3), *and that $f(s) \geqq g(s)$ on some interval* $[0, \alpha]$. *Let u and v be the solutions of the problems*

$$\partial u/\partial t - \Delta u + f(u) = 0, \quad u(x, 0) = u_0(x) \in \cup_{1\leqq q\leqq\infty} X_q, \tag{1.6}$$

$$\partial v/\partial t - \Delta v + g(v) = 0, \quad v(x, 0) = v_0(x) \in \cup_{1\leqq q\leqq\infty} X_q, \tag{1.7}$$

respectively. Then;

(i) $z(t;\beta_1) \leqq u(\cdot, t) \leqq z(t;\beta_2)$ *if* $\beta_1 \leqq u_0 \leqq \beta_2$,

(ii) $0 \leqq u(\cdot, t) \leqq v(\cdot, t) \leqq \alpha$ *if* $0 \leqq u_0 \leqq v_0 \leqq \alpha$.

PROOF. Since assertion (i) is a well-known result, we prove only assertion (ii). Without loss of generality we may assume that both u_0 and v_0 are in $C_0^\infty(\mathbf{R}^N)$, and so $u(t)$ and $v(t)$ belong to $\cap_{1\leqq q\leqq\infty} X_q$ for all $t \geqq 0$. Using the order-preserving property and L^∞-nonexpansiveness of the semigroup $\{S(t)\}$, we see that $0 \leqq u(t) \leqq \alpha$ and $0 \leqq v(t) \leqq \alpha$ if $0 \leqq u_0 \leqq \alpha$ and $0 \leqq v_0 \leqq \alpha$. Hence, by assumption, $f(u(t)) \geqq g(u(t)) \geqq 0$. But $f(u(t)) \in X_2$ since $u(t) \in D(A_2)$ and so $g(u(t)) \in X_2$. Now multiply both sides of the equation

$$\partial(u-v)/\partial t - \Delta(u-v) + f(u) - g(u) + g(u) - g(v) = 0$$

by $(u-v)^+$ and integrate the resultant terms over $\mathbf{R}^N$ to get

$$(d/dt)|(u-v)^+|_2^2 + 2\int_{\mathbf{R}^N} [f(u)-g(u)](u-v)^+ dx$$
$$+ 2\int_{\mathbf{R}^N} [g(u)-g(v)](u-v)^+ dx \leqq 0 \qquad \text{for a.e. } t.$$

Since the second and the third terms are nonnegative, we get

$$|(u(t)-v(t))^+|_2^2 \leqq |(u_0-v_0)^+|_2^2 = 0 \qquad \text{for all } t \geqq 0.$$

Hence $u \leqq v$ and the proof is complete.

REMARK. Fix any $t_0>0$ and set $\beta = |u(\cdot, t_0)|_\infty$; then $0 \leqq \beta < \infty$ by Proposition 1.1. Let $z(t; t_0, \beta)$ $(t>t_0)$ denote the solution of (ODE) with $z(t_0; t_0, \beta) = \beta$. Then Lemma 1.1 implies

$$z(t; t_0, -\beta) \leqq u(x, t) \leqq z(t; t_0, \beta) \tag{1.8}$$

for all $t \geqq t_0$ and $x \in \mathbf{R}^N$.

2. Asymptotic behavior of solutions

In this section we present a summarized exposition of our results on the asymptotic behavior of solutions of the problem (0.1)–(0.2). The precise statements of the results and their proofs will be given in the next section.

In what follows we set $X = \bigcup_{1 \leqq q \leqq \infty} X_q$ and, for each $u_0 \in X_q$, we write $u = S(t)u_0 = S_q(t)u_0$ for simplicity in notation.

First we prove that any solution vanishes in a finite time if $\int_{-1}^{1} |f(s)|^{-1} ds < \infty$.

Before we go into the detailed studies on the asymptotic behavior of solutions, we observe that

$$\lim_{t\to\infty} |u(t)|_\infty = 0. \tag{2.1}$$

for every $u_0 \in X$. In fact, the above result follows from the inequality (1.2) if $u_0 \in X_q$ for some $1 \leqq q < \infty$; and it follows from the inequality (1.8) if $u_0 \in X_\infty$. Since the solutions always converge to zero, we are interested in the order of decay of the solutions. We shall see that it depends strongly upon the order of $f(s)$ as s tends to zero.

Secondly, we prove that the solutions exponentially decay if $0 < \liminf_{s\to 0} |f(s)/s|$.

As mentioned in Section 1, each $S(t)$ is order-preserving; hence $u(t) \geqq 0$ whenever $u_0 \geqq 0$. So, as far as nonnegative initial data are concerned, the above conditions may be modified into the following: $\int_0^1 f(s)^{-1} ds < \infty$ in the first result; and $\liminf_{s\downarrow 0} f(s)/s > 0$ in the second result.

We also consider the solutions multiplied by suitable increasing factors (for example $t^{1/(p-1)}$, $t^{N/2}$), and investigate their asymptotic behaviors. Here we discuss the problems in the case where $f(s)$ is comparable with the function $s|s|^{p-1}$ near $s=0$. More precisely, we consider the following three types of order conditions for $f(s)$:

$$0 < K_1 = \liminf_{s\downarrow 0} f(s)/s^p \leqq \limsup_{s\downarrow 0} f(s)/s^p = K_2 < \infty \tag{C.1}$$

for some constants K_1, K_2, and some $p \in (1, \infty)$,

$$0 \leqq \limsup_{s\to 0} |f(s)|/|s|^{1+2/N} < \infty, \tag{C.2}$$

and

$$0 < \liminf_{s\to 0} |f(s)|/|s|^{1+2/N} \leqq \infty. \tag{C.3}$$

Assume condition (C.1) and that $f(s)/s$ is nondecreasing on some interval

$(0, \eta)$. If $u_0 \geqq 0$, and ess. $\lim_{|x|\to\infty} |x|^{2/(p-1)} u_0(x) = \infty$, then

$$(K_2/\gamma)^{-\gamma} \leqq \liminf_{t\to\infty} t^\gamma u(x, t) \leqq \limsup_{t\to\infty} t^\gamma u(x, t) \leqq (K_1/\gamma)^{-\gamma}$$

where $\gamma = 1/(p-1)$.

Suppose condition (C.2) and $u_0 \in X_1$; then

$$\{t^{N/2} u(x, t) - C_0 (4\pi)^{-N/2} \exp(-|x|^2/4t)\} \to 0 \qquad \text{as} \quad t \to \infty$$

where $C_0 = \lim_{t\to\infty} \int_{\mathbf{R}^N} u(x, t)dx = \int_{\mathbf{R}^N} u_0(x)dx - \int_0^\infty \int_{\mathbf{R}^N} f(u(x, t))\,dxdt$. In other words the solution is similar to the solution of the linear heat equation as t tends to ∞. We note that in this case, the asymptotic behavior is determined by the diffusion term Δu and does not reflect the nonlinearity of f. Moreover if f satisfies a stronger condition

$$\limsup_{s\downarrow 0} f(s)/s^p < \infty \qquad \text{for some} \quad p > 1 + 2/N,$$

then $C_0 = C_0(u_0)$ is positive for any $u_0 \in X_1$ with $u_0 \geqq 0$ and $|u_0|_1 > 0$.

Suppose condition (C.3) and $u_0 \in X_1$; then

$$t^{N/2} u(x, t) \to 0 \qquad \text{as} \quad t \to \infty.$$

It should be noted that in this case the asymptotic behavior of the solutions is determined by the nonlinear term $f(u)$ rather than the diffusion term Δu.

However if f satisfies both (C.2) and (C.3), the asymptotic behavior is specified by not only the nonlinear term but also the diffusion term.

In each case, it will be shown that the convergence is uniform on $E_C = \{x \in \mathbf{R}^N: |x| \leqq Ct^{1/2}\}$ for any $C > 0$.

3. Main results and their proofs

In this section we state main results and their proofs. We begin with the following theorem.

THEOREM 3.1. *Suppose* $\int_{-1}^1 |f(s)|^{-1} ds < \infty$ *and* $u_0 \in X$. *Then there exists a positive number* T *such that*

$$u(x, t) = 0 \qquad \textit{for} \quad t \geqq T \quad \textit{and} \quad x \in \mathbf{R}^N.$$

PROOF. We consider the function $z(t; t_0, \pm\beta)$ as defined in the Remark after Lemma 1.1. It is well-known that the assumption on f is equivalent to the existence of $T = T(t_0, \beta) > 0$ such that $z(t; t_0, \pm\beta) = 0$ for all $t \geqq T$. Hence the conclusion follows from the inequality (1.8).

The next theorem asserts that the solutions decay exponentially if $|s^{-1}f(s)|$ is bounded away from 0 as $s \to 0$.

THEOREM 3.2. *If* $\liminf_{s\to 0} |f(s)/s| = K > 0$ *and if* $u_0 \in X_q$ *for some* $1 \leqq q \leqq \infty$, *then*

$$\lim_{t\to\infty} \exp((K-\varepsilon)t)|u(t)|_r = 0$$

for any $\varepsilon > 0$ *and any* $r \in [q, \infty]$. *If the limit inferior is* $+\infty$, *then* $K-\varepsilon$ *may be replaced by any positive number.*

PROOF. In view of the usual decomposition $u_0 = u_0^+ - u_0^-$, $u_0^+ = u_0 \vee 0$, $u_0^- = -(u_0 \wedge 0)$, and the order-preserving property of the semigroup $\{S(t);\ t \geqq 0\}$, it is sufficient to consider the case where $u_0 \geqq 0$ or else $u_0 \leqq 0$. Here we assume that $u_0 \geqq 0$, since another case is similarly treated by using the dual statement of Lemma 1.1. We show that for any $K_1 \in (0, K)$ and any $r \in [q, \infty]$,

$$\limsup_{t\to\infty} \exp(K_1 t)\,|u(t)|_r < \infty.$$

By assumption there exists for any K_1 a constant $\delta > 0$ such that $K_1 s \leqq f(s)$ on $[0, \delta]$. On the other hand, it follows from (2.1) that there exists a $T > 0$ such that $0 \leqq u(x, t) \leqq \delta$ for all $t \geqq T$. Let v be the solution of

$$\partial v/\partial t - \Delta v + K_1 v = 0, \quad t > T, \quad v(x, T) = u(x, T).$$

We see in a way similar to Proposition 1.1 that

$$|v(\cdot, t)|_r \leqq (t-T)^{-N(1/q-1/r)/2} \exp(-K_1(t-T))\,|u(\cdot, T)|_q$$

for all $t \geqq T$, $r \in [q, \infty]$. Applying Lemma 1.1 with $g(v) = K_1 v$, we get $0 \leqq u(x, t) \leqq v(x, t)$ for $t \geqq T$, which proves Theorem 3.2.

In the next theorem, we state an asymptotic property of the solutions multiplied by an increasing factor t^γ, $\gamma = 1/(p-1)$.

THEOREM 3.3. *Let* $\gamma = 1/(p-1)$. *Let* f *satisfy condition* (C.1) *and* $f(s)/s$ *be nondecreasing on* $(0, \eta)$ *for some* $\eta > 0$. *Suppose* $u_0 \in X$, $u_0 \geqq 0$ *and* $\operatorname{ess.lim}_{|x|\to\infty} |x|^{2\gamma} u_0(x) = \infty$. *Then, for any* $C > 0$,

$$\begin{aligned}(K_2/\gamma)^{-\gamma} &\leqq \liminf_{t\to\infty} [\inf_{E_C} t^\gamma u(x, t)] \\ &\leqq \limsup_{t\to\infty} [\sup_{E_C} t^\gamma u(x, t)] \leqq (K_1/\gamma)^{-\gamma},\end{aligned}$$

where $E_C = \{x \in \mathbf{R}^N : |x| \leqq C t^{1/2}\}$.

The next corollary follows immediately from Theorem 3.3.

COROLLARY 3.1. *If in particular* $K_1 = K_2 = K$ *in Theorem* 3.3, *then*

$$\lim_{t\to\infty}\{t^\gamma u(x, t)-(K/\gamma)^{-\gamma}\} = 0$$

and the convergence is uniform on $E_C=\{x\in \mathbf{R}^N;\ |x|\leqq Ct^{1/2}\}$ *for each* $C>0$.

PROOF OF THEOREM 3.3. Using the same argument as in [4, Proposition 2.1] together with Lemma 1.1, we get

$$\text{ess.}\lim_{|x|\to\infty}|x|^{2\gamma}u(x, t) = \infty \tag{3.1}$$

for any $t>0$.

Choose any pair of numbers C_1, C_2 such that $0<C_1<K_1\leqq K_2<C_2$. Then by assumption one finds a constant $\delta>0$ such that $C_1s^p\leqq f(s)\leqq C_2s^p$ on $[0, \delta]$. Hence it follows from (2.1) that there is a constant $T>0$ such that $0\leqq u(x, t)\leqq\delta$ for $t\geqq T$ and $x\in\mathbf{R}^N$. Let $v_i(x, t)$ $(i=1, 2)$ be the solution of the equation

$$\partial v_i/\partial t - \Delta v_i + C_iv_i^p = 0 \quad (t>T), \quad v_i(x, T) = u(x, T).$$

Then Lemma 1.1 implies that $v_2(x, t)\leqq u(x, t)\leqq v_1(x, t)$ on $\mathbf{R}^N\times(T, \infty)$. The application of [4, Theorem 2.1] and (3.1) implies that

$$\lim_{t\to\infty}\{t^\gamma v_i(x, t)-(C_i/\gamma)^{-\gamma}\} = 0$$

holds uniformly on E_C for each $C>0$. Thus we obtain

$$\begin{aligned}(C_2/\gamma)^{-\gamma} &\leqq \liminf_{t\to\infty}[\inf_{E_C} t^\gamma u(x, t)]\\ &\leqq \limsup_{t\to\infty}[\sup_{E_C} t^\gamma u(x, t)] \leqq (C_1/\gamma)^{-\gamma}\end{aligned}$$

for each $C>0$. Since C_1 and C_2 are arbitrary in so far as $0<C_1<K_1\leqq K_2<C_2$, the assertion follows.

Next, we consider the solutions multiplied by another increasing factor t^ν, $\nu=N/2$.

THEOREM 3.4. *Let* $\nu=N/2$. *Suppose condition* (C.2) *and* $u_0\in X_1$. *Then we have*

(i) $\lim_{t\to\infty}\{t^\nu u(x, t)-C_0(4\pi)^{-\nu}\exp(-|x|^2/4t)\} = 0$,

and the convergence is uniform on $E_C=\{x\in\mathbf{R}^N:\ |x|\leqq Ct^{1/2}\}$ *for each* $C>0$, *where*

$$C_0 = \lim_{t\to\infty}\int_{\mathbf{R}^N}u(x, t)dx = \int_{\mathbf{R}^N}u_0(x)dx - \int_0^\infty\int_{\mathbf{R}^N}f(u(x, t))dxdt.$$

(ii) *If in addition* $u_0\geqq 0$ *and* $|u_0|_1>0$, *then for any compact subset* G *of* $\mathbf{R}^N$ *and any* $t_0>1$, *there exists a constant* $M>0$ *such that*

$$t^\nu(\log t)^\nu u(x, t)\geqq M \qquad \text{on} \quad G\times[t_0, \infty).$$

In order to prove Theorem 3.4, we employ the following scaling transformations (see [4]): Fix any sequence $\{t_k\}$ with $t_k \uparrow \infty$ and define

$$u_k(x, t) = t_k^\nu u(t_k^{1/2}x, t_k t), \quad k = 1, 2,\dots,$$

where $u(x, t)$ is the solution of (0.1)–(0.2). Each u_k satisfies

$$\partial u_k/\partial t - \Delta u_k + t_k^{1+\nu} f(t_k^{-\nu}u_k) = 0 \quad \text{in} \quad \mathbf{R}^N \times (0, \infty)$$
$$u_k(x, 0) = t_k^\nu u_0(t_k^{1/2}x) \quad \text{in} \quad \mathbf{R}^N.$$

We can then give a proof of Theorem 3.4 by using the following lemma which is parallel to [4, Proposition 3.1].

LEMMA 3.1. *Under the hypotheses of Theorem 3.4, the sequence $\{u_k\}$ converges uniformly on any compact subset of $\mathbf{R}^N \times (0, \infty)$ to the function*

$$w(x, t) = C_0(4\pi t)^{-\nu} \exp(-|x|^2/4t). \tag{3.2}$$

PROOF. By Proposition 1.1 we get

$$|u_k(\cdot, t)|_q \leqq Ct^{-\nu(1-1/q)}|u_0|_1 \quad \text{for} \quad 1 \leqq q \leqq \infty, \tag{3.3}$$

$$|(\partial/\partial t)u_k|_q \leqq Ct^{-\nu(1-1/q)-1}|u_0|_1 \quad \text{for} \quad 2 \leqq q < \infty, \quad \text{and} \tag{3.4}$$

$$|u_k(\cdot, 0)|_1 = |u_0|_1. \tag{3.5}$$

Also, by assumption there exist constants ε, $K>0$ such that

$$|f(s)| \leqq K|s|^{1+2/N} \quad \text{on} \quad [-\varepsilon, +\varepsilon]. \tag{3.6}$$

For any $\delta>0$, (3.3) implies that there exists a constant $C_\delta>0$ such that

$$|u_k(\cdot, t)|_\infty \leqq C_\delta \quad \text{for any} \quad t \geqq \delta \quad \text{and} \quad k = 1, 2,\dots.$$

From this inequality and (3.6) we see that there exists a constant $k_0 = k_0(\delta)$ such that

$$t_k^{1+\nu}|f(t_k^{-\nu}u_k(x, t)| \leqq K|u_k(x, t)|^{1+2/N} \tag{3.7}$$

for $k \geqq k_0$ and $t \geqq \delta$. Let $2 \leqq q < \infty$. Then using (3.3), (3.4), (3.7), and the relations $\Delta u_k = \partial u_k/\partial t + t_k^{1+\nu} f(t_k^{-\nu}u_k)$, one finds a constant $C(\delta, q)>0$ such that

$$|\Delta u_k(\cdot, t)|_q \leqq C(\delta, q) \quad \text{for} \quad t \geqq \delta \quad \text{and} \quad k \geqq k_0.$$

Fix a q with $N+1<q<\infty$. Using the regularity theory for elliptic operators, compact imbedding results in Sobolev spaces and then applying the diagonal process, we can choose a subsequence $\{u_{k_n}\}$ and a function $w \in C(\mathbf{R}^N \times (0, \infty))$

such that $\{u_{k_n}\}$ converges to w uniformly on any compact subset of $\mathbf{R}^N \times (0, \infty)$. We can now employ the same method as in [4, Proposition 3.1] to show that w coincides with the function (3.2). The proof is complete.

PROOF OF THEOREM 3.4 (i). Lemma 3.1 implies that for each $C>0$,

$$t^{\nu}u(t^{1/2}y, ts) - C_0(4\pi s)^{-\nu}\exp(-|y|^2/4s)$$

converges to 0 as t tends to ∞, uniformly for y and s with $|y| \leqq C$, $1/C \leqq s \leqq C$. Setting $s=1$ and $y=x/t^{1/2}$, we obtain Theorem 3.4 (i).

To prove the second assertion of Theorem 3.4, we need the following lemma which is verified in [4, Theorems 4.1 and 4.2].

LEMMA 3.2. *Let $w(x, t)$ be the solution of*

$$\partial w/\partial t - \Delta w + |w|^{2/N}w = 0, \quad w(x, 0) = w_0(x). \tag{3.8}$$

If $w_0 \in X_1$, then $\lim_{t\to\infty} t^{\nu}w(x, t)=0$ holds uniformly on E_C for each $C>0$. If $w_0 \in X_1$, $w_0 \geqq 0$ and $|w_0|_1>0$, then for each compact subset G of $\mathbf{R}^N$ and each $t_0>1$, there exists a constant $M>0$ such that

$$t^{\nu}(\log t)^{\nu}w(x, t) \geqq M \quad \text{on} \quad G\times[t_0, \infty).$$

PROOF OF THEOREM 3.4 (ii). By assumption there exist constants K, $\delta>0$ such that

$$f(s) \leqq Ks^{1+2/N} \quad \text{on} \quad [0, \delta]. \tag{3.9}$$

Set $v_0(x)=\min[u_0(x), \delta]$. Then $v_0 \in X_1$, $v_0 \geqq 0$ and $|v_0|_1>0$. Let $v(x, t)$ and $w(x, t)$ be the solutions of (0.1) and (3.8) with the initial conditions $v(x, 0)=v_0(x)$ and $w(x, 0)=v_0(x/K^{1/2})$, respectively. If we define $w_*(x, t)=w(K^{1/2}x, Kt)$, then w_* satisfies

$$\partial w_*/\partial t - \Delta w_* + Kw_*^{1+2/N} = 0, \quad w_*(x, 0) = v_0(x).$$

Lemma 1.1 then implies $v(x, t) \geqq w_*(x, t)=w(K^{1/2}x, Kt)$. Moreover we get $u(x, t) \geqq v(x, t)$ since $u_0 \geqq v_0$, so that we obtain $u(x, t) \geqq w(K^{1/2}x, Kt)$. We can now apply Lemma 3.2 to obtain the assertion.

Under condition (C.3), we consider the asymptotic behavior of the solutions multiplied by t^{ν}.

THEOREM 3.5. *Suppose condition* (C.3) *and $u_0 \in X_1$. Then*

$$\lim_{t\to\infty} t^{\nu}u(x, t) = 0$$

holds uniformly on E_C for any $C>0$.

PROOF. We assume that $u_0 \geqq 0$ by the same reason as in the proof of Theorem 3.2. By assumption there exist $K, \delta > 0$ such that $f(s) \geqq K s^{1+2/N}$ on $[0, \delta]$. On the other hand (2.1) implies that there exists a $T>0$ such that $0 \leqq u(x, t) \leqq \delta$ for all $t \geqq T$. Let $w(x, t)$ be the solution of (3.8) with initial condition $w(x, 0) = u(x/K^{1/2}, T)$. Set $w_*(x, t) = w(K^{1/2}x, K(t-T))$. Then

$$\partial w_*/\partial t - \Delta w_* + K w_*^{1+2/N} = 0 \quad (t > T), \quad w_*(x, T) = u(x, T).$$

Further, Lemma 1.1 implies $0 \leqq u(x, t) \leqq w_*(x, t) = w(K^{1/2}x, K(t-T))$ for any $t \geqq T$. Therefore the assertion follows from Lemma 3.2.

Finally, we consider the solutions under conditions (C.2) and (C.3).

COROLLARY 3.2. *Suppose conditions* (C.2), (C.3) *and* $u_0 \in X_1$. *Then*:

(i) $\lim_{t\to\infty} t^\nu u(x, t) = 0$ *uniformly on* E_C *for any* $C > 0$.

(ii) $\int_0^\infty \int_{\mathbf{R}^N} f(u(x, t))dxdt = \int_{\mathbf{R}^N} u_0(x)dx$.

(iii) $\lim_{t\to\infty} |u(\cdot, t)|_1 = 0$.

PROOF. Theorems 3.4 and 3.5 together imply that $\lim_{t\to\infty} t^\nu u(x, t) = 0$ holds uniformly on E_C for each $C > 0$, and that $C_0 = 0$. Therefore we get

$$\int_0^\infty \int_{\mathbf{R}^N} f(u(x, t))dxdt = \int_{\mathbf{R}^N} u_0(x)dx \quad \text{and}$$

$$\lim_{t\to\infty} \int_{\mathbf{R}^N} u(x, t)dx = 0.$$

This means that $\lim_{t\to\infty} |u(\cdot, t)|_1 = 0$ if $u_0 \geqq 0$ or else $u_0 \leqq 0$. For the general case we consider u_0^+ and u_0^- defined as in the proof of Theorem 3.2. Then the assertion is obtained via the order-preserving property of the semigroup $\{S(t): t \geqq 0\}$.

4. Remarks and examples

In this section, we make some remarks on the main theorems and give some examples to illustrate our results.

In Theorem 3.4, it is important to know whether or not C_0 is nonzero. Recall that $C_0 = C_0(u_0)$ is a real number defined for initial function $u_0 \in X_1$ by $C_0 = \lim_{t\to\infty} \int_{\mathbf{R}^N} S(t)u_0(x)dx$. We here give a sufficient condition for C_0 to be nonzero.

PROPOSITION 4.1. *Suppose there exists a nondecreasing function* $h(s)$ *satisfying*

(i) $f(s) \leqq sh(s)$ on $[0, \eta]$, *and*

(ii) $\int_0^\eta h(s)/s^{1+2/N}ds < \infty$ *for some* $\eta > 0$.

If $u_0 \in X_1$, $u_0 \geqq 0$ *and* $|u_0|_1 > 0$, *then* $C_0 > 0$.

PROOF. Without loss of generality we may suppose that $h(s)$ is defined on all of $[0, \infty)$, nondecreasing, and satisfies

(i)′ $f(s) \leqq sh(s)$ on $[0, \infty)$, and

(ii)′ $\int_0^T h(s)/s^{1+2/N}ds < \infty$ for any $T > 0$.

Suppose $u_0 \in D(A_1)$, $u_0 \geqq 0$ and $|u_0|_1 > 0$. Set $y(t) = \int_{\mathbf{R}^N} u(x, t)dx$. Proposition 1.1 implies that y is a Lipschitz continuous, nonnegative, and nonincreasing function. On the other hand, in the proof of Proposition 1.2, we showed the relation

$$(d/dt)\int_{\mathbf{R}^N} u(x, t)dx + \int_{\mathbf{R}^N} f(u(x, t))dx = 0.$$

Applying condition (i)′ and then Proposition 1.1, one finds a positive constant $K = K(N)$ such that $0 \leqq f(u(x, t)) \leqq g(t)u(x, t)$, where $g(t) = h(Kt^{-N/2}|u_0|_1)$. Therefore we have

$$dy/dt + g(t)y(t) \geqq 0.$$

Since $y(0)$ is positive by assumption on u_0, there exists a constant $\delta > 0$ such that $y(t) > 0$ on $[0, \delta]$. So Gronwall's inequality implies

$$y(t) \geqq y(\delta)\exp\left(-\int_\delta^t g(s)ds\right) \quad \text{for any} \quad t \geqq \delta.$$

At this point condition (ii)′ is applied to get

$$0 < \int_\delta^\infty g(s)ds = L\int_0^T h(s)/s^{1+2/N}ds < \infty,$$

where $L = (2/N)(K|u_0|_1)^{2/N}$ and $T = \delta^{-N/2}K|u_0|_1$. Hence we have the estimate

$$y(t) \geqq y(\delta)\exp\left(-L\int_0^T h(s)/s^{1+2/N}ds\right) \quad \text{for any} \quad t \geqq \delta.$$

Using the continuous dependence on initial data u_0, we see that the above inequality holds for any $u_0 \in X_1$ with $u_0 \geqq 0$ and $|u_0|_1 > 0$. Thus we have

$$C_0(u_0) = \lim_{t\to\infty} y(t) > 0.$$

COROLLARY 4.1. *Suppose* $\limsup_{s\downarrow 0} f(s)/s^p<\infty$ *for some* $p>1+2/N$. *If* $u_0\in X_1$, $u_0\geqq 0$, *and* $|u_0|_1>0$, *then* $C_0(u_0)$ *is positive.*

PROOF. Set $h(s)=Cs^{p-1}$, where C is a constant satisfying $\limsup_{s\downarrow 0} f(s)/s^p<C$. Then the assertion follows directly from Proposition 4.1.

Finally, we give some examples of f and investigate the asymptotic behavior of the associated solutions. Consider the following function.

(1) $f(s) = \sum_{i=1}^{k} a_i s^{p_i}$.

Suppose f is nondecreasing on $[0, \infty)$, $0<p_1<p_2<\cdots<p_k$, and $a_i\neq 0$ for all i. Hence we have $a_1>0$. Assume $u_0\in X$ and $u_0\geqq 0$.

(i) For $0<p_1<1$, Theorem 3.1 implies that there exists a constant $T>0$ such that

$$u(x, t) = 0 \qquad \text{for all} \quad t \geqq T.$$

(ii) When $p_1=1$ and $u_0\in X_q$ for some $1\leqq q\leqq\infty$, Theorem 3.2 can be applied to conclude that

$$\lim_{t\to\infty} \exp((a_1-\varepsilon)t)\,|u(t)|_r = 0$$

for any $\varepsilon>0$ and any $r\in[q, \infty]$.

(iii) If $p_1>1$ and $\text{ess.}\lim_{|x|\to\infty} |x|^{2\gamma}u_0(x)=\infty$, then, by Theorem 3.3,

$$\lim_{t\to\infty} [t^\gamma u(x, t)-(a_1/\gamma)^{-\gamma}] = 0,$$

and the convergence is uniform on $E_C=\{x\in\mathbf{R}^N\colon |x|\leqq Ct^{1/2}\}$ for any $C>0$, where $\gamma=1/(p_1-1)$.

(iv) If $1<p_1<1+2/N$ and $u_0\in X_1$, then Theorem 3.5 states that

$$\lim_{t\to\infty} t^\nu u(x, t) = 0 \quad \text{uniformly on} \quad E_C \quad \text{for any} \quad C>0,$$

where $\nu=N/2$.

(v) If $1+2/N<p_1$ and $u_0\in X_1$, then it follows from Theorem 3.4 that

$$\lim_{t\to\infty} [t^\nu u(x, t)-C_0(4\pi)^{-\nu}\exp(-|x|^2/4t)] = 0$$

uniformly on E_C for any $C>0$, where $C_0=\lim_{t\to\infty}\int_{\mathbf{R}^N} u(x, t)dx$. Moreover if $|u_0|_1>0$, then $C_0=C_0(u_0)>0$.

(vi) If $p_1=1+2/N$ and $u_0\in X_1$, then Corollary 3.2 implies that

$$\lim_{t\to\infty} t^\nu u(x, t) = 0 \quad \text{uniformly on} \quad E_C \quad \text{for any} \quad C>0,$$

$$\int_0^\infty\int_{\mathbf{R}^N} f(u(x, t))dxdt = \int_{\mathbf{R}^N} u_0(x)dx, \quad \text{and}$$

$$\lim_{t\to\infty} |u(\cdot, t)|_1 = 0.$$

Moreover if $|u_0|_1 > 0$, then for any compact subset G of $\mathbf{R}^N$ and any $t_0 > 1$, there exists a constant $M > 0$ such that

$$t^{\nu}(\log t)^{\nu} u(x, t) \geqq M \qquad \text{on} \quad G \times [t_0, \infty).$$

We next consider a function f such that $f(s)/s^p$ is nondecreasing for some $p > 0$. We can write

(2) $f(s) = s^p g(s)$,

where g is continuous, nondecreasing on $[0, \infty)$ and satisfies $0 < g(0) < g(s)$ for any $s > 0$. Then we have the same results as those for (1) with a_1, p_1 replaced respectively by $g(0)$, p.

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