HIROSHIMA MATH. J.
16 (1986), 443–448

Serially finite Lie algebras

Shigeaki TÔGÔ

(Received September 19, 1985)

The purpose of this paper is to present several characterizations of serially finite Lie algebras.

For the class $\mathrm{L(ser)}\mathfrak{F}$ of serially finite Lie algebras, we shall show that over a field of characteristic 0

$$\mathrm{L(ser)}\mathfrak{F} = \mathrm{L}\mathfrak{F} \cap \mathrm{J(ser)}\mathfrak{F} = \mathrm{L(lsi)}\mathfrak{F} = \mathrm{J(lsi)}\mathfrak{F} = \mathrm{L(lasc)}\mathfrak{F} = \mathrm{J(lasc)}\mathfrak{F},$$

where $\mathrm{L}\mathfrak{F}$ is the class of locally finite Lie algebras, $\mathrm{L}(\Delta)\mathfrak{F}$ is the class of Lie algebras L such that any finite subset of L is contained in a finite-dimensional Δ-subalgebra, $\mathrm{J}(\Delta)\mathfrak{F}$ is the class of Lie algebras generated by finite-dimensional Δ-subalgebras ($\Delta=$ser, lsi, lasc), and $\mathrm{J(lsi)}\mathfrak{F}$ is the class of neoclassical Lie algebras introduced in [1, §13.2]. We shall give similar characterizations of subclasses $\mathrm{L(ser)}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ and $\mathrm{L(ser)}(\mathfrak{N} \cap \mathfrak{F})$ of $\mathrm{L(ser)}\mathfrak{F}$. Furthermore for the class $\mathrm{L}\mathfrak{N}$ of locally nilpotent Lie algebras, we shall show that $\mathrm{L}\mathfrak{N}=\mathrm{L(ser)}(\mathfrak{N} \cap \mathfrak{F})$ and $\mathrm{L}\mathfrak{N}$ coincides with the class of locally finite Lie algebras each of whose 1-dimensional subalgebras is weakly serial (resp. ω-step weakly ascendant).

1.

Throughout this paper, $\mathfrak{k}$ is a field of arbitrary characteristic unless otherwise specified, and L is a not necessarily finite-dimensional Lie algebra over $\mathfrak{k}$. When H is a subalgebra (resp. an ideal) of L, we denote $H \leq L$ (resp. $H \lhd L$).

Let $H \leq L$. For an ordinal ρ, H is a ρ-step weakly ascendant subalgebra (resp. a ρ-step ascendant subalgebra) of L, denoted by $H \leq^{\rho} L$ (resp. $H \lhd^{\rho} L$), if there exists an ascending chain $\{H_\sigma | \sigma \leq \rho\}$ of subspaces (resp. subalgebras) of L such that

(1) $H_0 = H$ and $H_\rho = L$,
(2) $[H_{\sigma+1}, H] \subseteq H_\sigma$ (resp. $H_\sigma \lhd H_{\sigma+1}$) for any ordinal $\sigma < \rho$,
(3) $H_\lambda = \cup_{\sigma<\lambda} H_\sigma$ for any limit ordinal $\lambda \leq \rho$.

H is a weakly ascendant subalgebra (resp. an ascendant subalgebra) of L, denoted by H wasc L (resp. H asc L), if $H \leq^{\rho} L$ (resp. $H \lhd^{\rho} L$) for some ordinal ρ. When ρ is finite, H is a weak subideal (resp. a subideal) of L and denoted by H wsi L (resp. H si L).

For a totally ordered set Σ, H is a weakly serial subalgebra (resp. a serial subalgebra) of type Σ of L, denoted by H wser L (resp. H ser L), if there exists a collection $\{\Lambda_\sigma, V_\sigma | \sigma \in \Sigma\}$ of subspaces (resp. subalgebras) of L such that

(1) $H \subseteq \Lambda_\sigma$ and $H \subseteq V_\sigma$ for all $\sigma \in \Sigma$,

(2) $\Lambda_\tau \subseteq V_\sigma \subseteq \Lambda_\sigma$ if $\tau < \sigma$,

(3) $L \backslash H = \cup_{\sigma \in \Sigma} (\Lambda_\sigma \backslash V_\sigma)$,

(4) $[\Lambda_\sigma, H] \subseteq V_\sigma$ (resp. $V_\sigma \lhd \Lambda_\sigma$) for all $\sigma \in \Sigma$.

Then any weakly ascendant (resp. ascendant) subalgebra of L is weakly serial (resp. serial).

H is a local subideal of L, denoted by H lsi L, if H si $\langle H, X \rangle$ for any finite subset X of L. We here introduce a similar concept. We call H a local ascendant subalgebra of L if H asc $\langle H, X \rangle$ for any finite subset X of L. We then write H lasc L.

A class of Lie algebras is a collection of Lie algebras over $\mathfrak{k}$ together with their isomorphic copies and the 0-dimensional Lie algebra. We denote by $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{N}$, $\mathrm{E}\mathfrak{A}$, $\mathfrak{E}$, $\mathrm{L}\mathfrak{F}$ and $\mathrm{L}\mathfrak{N}$ the classes of finite-dimensional, finitely generated, nilpotent, solvable, Engel, locally finite and locally nilpotent Lie algebras respectively.

Let $\mathfrak{X}$ be a class of Lie algebras and let Δ be any one of the relations ser, lsi, lasc, etc. We write $L \in \mathrm{L}(\Delta)\mathfrak{X}$ if for any finite subset X of L there exists a subalgebra H belonging to $\mathfrak{X}$ such that $X \subseteq H \,\Delta\, L$. Furthermore we write $L \in \mathrm{J}(\Delta)\mathfrak{X}$ if L is generated by a set of subalgebras H belonging to $\mathfrak{X}$ such that $H \,\Delta\, L$. Then over a field $\mathfrak{k}$ of characteristic 0 $\mathrm{J}(\mathrm{lsi})\mathfrak{F}$ is the class of neoclassical Lie algebras introduced in [1, §13.2].

LEMMA 1. *Let $L \in \mathfrak{F}$ and $H \leq L$. If H* wser *L (resp. H* ser *L), then H* wsi *L (resp. H* si *L).*

LEMMA 2 ([1, Proposition 13.2.4] and [2, Corollary 2.4]). *Let $L \in \mathrm{L}\mathfrak{F}$ and $H \leq L$. Then H* wser *L (resp. H* ser *L) if and only if $H \cap F$* wsi *F (resp. $H \cap F$* si *F) for any finite-dimensional subalgebra F of L.*

2.

Let $\mathfrak{X}$ be a class of Lie algebras. $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent) if in any Lie algebra the join of any pair of subideals (resp. ascendant subalgebras) belonging to $\mathfrak{X}$ is a subideal (resp. an ascendant subalgebra) belonging to $\mathfrak{X}$. We now call $\mathfrak{X}$ lsi-coalescent (resp. lasc-coalescent) if the condition is satisfied with local subideals (resp. local ascendant subalgebras) instead of subideals (resp. ascendant subalgebras).

Then we have

LEMMA 3. *If $\mathfrak{X}$ is coalescent (resp. ascendantly coalescent) and is a subcalss of $\mathfrak{G}$, then $\mathfrak{X}$ is lsi-coalescent (resp. lasc-coalescent). Especially the classes*

$\mathfrak{F}$, $\mathrm{E}\mathfrak{A} \cap \mathfrak{F}$ *and* $\mathfrak{N} \cap \mathfrak{F}$ *over a field of characteristic* 0 *are lsi-coalescent and lasc-coalescent.*

PROOF. The coalescence case is [1, Lemma 13.2.1]. The other case can be shown quite similarly. So we omit the proof.

We now show the following

THEOREM 1. *Let* $\mathfrak{X}$ *be a subclass of* $\mathfrak{F}$ *over a field* $\mathfrak{k}$.
a) *If* $\mathfrak{X}$ *is lsi-coalescent, then*

$$\mathrm{L}(\mathrm{ser})\mathfrak{X} = \mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})\mathfrak{X} = \mathrm{L}(\mathrm{lsi})\mathfrak{X} = \mathrm{J}(\mathrm{lsi})\mathfrak{X}.$$

b) *If* $\mathfrak{X}$ *is lasc-coalescent, then*

$$\mathrm{L}(\mathrm{ser})\mathfrak{X} = \mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})\mathfrak{X} = \mathrm{L}(\mathrm{lsi})\mathfrak{X} = \mathrm{J}(\mathrm{lsi})\mathfrak{X} = \mathrm{L}(\mathrm{lasc})\mathfrak{X} = \mathrm{J}(\mathrm{lasc})\mathfrak{X}.$$

PROOF. We shall show only b), since a) is similarly shown. Let $L \in \mathrm{L}\mathfrak{F}$ and let H be any serial subalgebra of L belonging to $\mathfrak{X}$. Then for any finite subset X of L, H ser $\langle H, X\rangle$. Since $L \in \mathrm{L}\mathfrak{F}$, $\langle H, X\rangle \in \mathfrak{F}$. Therefore by Lemma 1, H si $\langle H, X\rangle$. Hence H lsi L. Thus we have $\mathrm{L}(\mathrm{ser})\mathfrak{X} \leq \mathrm{L}(\mathrm{lsi})\mathfrak{X}$ and $\mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})\mathfrak{X} \leq \mathrm{J}(\mathrm{lsi})\mathfrak{X}$.

Next let $L \in \mathrm{J}(\mathrm{lasc})\mathfrak{X}$. For any finite subset X of L,

$$X \subseteq \langle H_1, \dots, H_n\rangle \qquad \text{with } H_i \text{ lasc } L \text{ and } H_i \in \mathfrak{X}\ (1 \leq i \leq n).$$

Put $H = \langle H_1, \dots, H_n\rangle$. Since $\mathfrak{X}$ is lasc-coalescent,

$$H \text{ lasc } L, \quad H \in \mathfrak{X}.$$

Hence $L \in \mathrm{L}\mathfrak{F}$. Furthermore for any finite-dimensional subalgebra F of L, H asc $\langle H, F\rangle$. Since $\langle H, F\rangle \in \mathfrak{F}$, H si $\langle H, F\rangle$ and therefore $H \cap F$ si F. Hence by Lemma 2, H ser L. Therefore $L \in \mathrm{L}(\mathrm{ser})\mathfrak{X}$. Thus $\mathrm{J}(\mathrm{lasc})\mathfrak{X} \leq \mathrm{L}(\mathrm{ser})\mathfrak{X}$.

Thus we have

$$\begin{array}{ccccccc} & & \mathrm{L}(\mathrm{ser})\mathfrak{X} & \leq & \mathrm{L}(\mathrm{lsi})\mathfrak{X} & \leq & \mathrm{L}(\mathrm{lasc})\mathfrak{X} & \\ & & \wedge\!| & & \wedge\!| & & \wedge\!| & \\ \mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})\mathfrak{X} & \leq & \mathrm{J}(\mathrm{lsi})\mathfrak{X} & \leq & \mathrm{J}(\mathrm{lasc})\mathfrak{X} & \leq & \mathrm{L}(\mathrm{ser})\mathfrak{X} \end{array}$$

and therefore the assertion holds.

As a consequence of Theorem 1 and Lemma 3, we have

THEOREM 2. *Over a field of characteristic* 0,
a) $\mathrm{L}(\mathrm{ser})\mathfrak{F} = \mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})\mathfrak{F} = \mathrm{L}(\mathrm{lsi})\mathfrak{F} = \mathrm{J}(\mathrm{lsi})\mathfrak{F} = \mathrm{L}(\mathrm{lasc})\mathfrak{F} = \mathrm{J}(\mathrm{lasc})\mathfrak{F}$,
b) $\mathrm{L}(\mathrm{ser})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) = \mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) = \mathrm{L}(\mathrm{lsi})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$
$= \mathrm{J}(\mathrm{lsi})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) = \mathrm{L}(\mathrm{lasc})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) = \mathrm{J}(\mathrm{lasc})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$,

(c) $\mathrm{L}(\mathrm{ser})(\mathfrak{N} \cap \mathfrak{F}) = \mathrm{L}\mathfrak{F} \cap \mathrm{J}(\mathrm{ser})(\mathfrak{N} \cap \mathfrak{F}) = \mathrm{L}(\mathrm{lsi})(\mathfrak{N} \cap \mathfrak{F})$
$= \mathrm{J}(\mathrm{lsi})(\mathfrak{N} \cap \mathfrak{F}) = \mathrm{L}(\mathrm{lasc})(\mathfrak{N} \cap \mathfrak{F}) = \mathrm{J}(\mathrm{lasc})(\mathfrak{N} \cap \mathfrak{F})$.

3.

To characterize locally nilpotent Lie algebras, we need some lemmas.

LEMMA 4 ([3, Lemma 2.1]). *Let H* wasc *L. Then for a finite subset X of L and finite subsets $Y_1, Y_2,\dots$ of H, there eixsts an integer $n=n(X, Y_1, Y_2,\dots)>0$ such that $[X, Y_1,\dots, Y_n]\subseteq H$.*

Let $\mathfrak{e}(L)$ denote the set of left Engel elements of L. Then

LEMMA 5 ([6, Lemma 2]). *For any $x\in L$, $x\in\mathfrak{e}(L)$ if and only if $\langle x\rangle\leq^{\omega}L$.*

Generalizing [5, Corollary to Theorem 5], we first characterize Engel algebras in the following

LEMMA 6. *For a Lie algebra L the following conditions are equivalent:*
a) $L\in\mathfrak{E}$
b) *For any $x\in L$, $\langle x\rangle$* wasc *L.*
c) *For any $x\in L$, $\langle x\rangle\leq^{\omega}L$.*
d) *For any $x\in L$, $\mathrm{ad}_L x$ is locally nilpotent.*

PROOF. b)$\Rightarrow$d) Let V be a finite-dimensional subspace of L and let $y_1, y_2,\dots, y_m$ be a basis of V. By Lemma 4 there exists an integer $n_i>0$ such that $[y_i, {}_{n_i}\langle x\rangle]\subseteq\langle x\rangle$. It follows that $[y_i, {}_{n_i+1}x]=0$. Putting $n=\max\{n_1+1,\dots, n_m+1\}$, we have $[V, {}_n x]=0$. Hence $\mathrm{ad}_L x$ is locally nilpotent.

d)$\Rightarrow$c) Let $x\in L$. For any $y\in L$ there exists an integer $n>0$ such that $[y, {}_n x]=0$. Hence $x\in\mathfrak{e}(L)$. By Lemma 5 $\langle x\rangle\leq^{\omega}L$.

Since c)$\Rightarrow$b) and a)$\Leftrightarrow$d) are evident, we have the equivalence of a),..., d).

By using Lemma 6, we now show the following theorem which is partly known (e.g., [4, Lemma 3.2] and [5, Corollary to Theorem 5]).

THEOREM 3. *Let $L\in\mathrm{L}\mathfrak{F}$. Then the following conditions are equivalent:*
a) $L\in\mathrm{L}\mathfrak{N}$.
b) $L\in\mathfrak{E}$.
c) *For any $H\leq L$, H* ser *L.*
d) *For any $H\leq L$, H* wser *L.*
e) *For any $x\in L$, $\langle x\rangle$* ser *L.*
f) *For any $x\in L$, $\langle x\rangle$* wser *L.*
g) *For any $x\in L$, $\langle x\rangle$* wasc *L.*
h) *For any $x\in L$, $\langle x\rangle\leq^{\omega}L$.*

i) *For any* $x \in L$, $\mathrm{ad}_L x$ *is locally nilpotent.*

PROOF. Taking account of Lemma 6, we have the following diagram of implications:

$$\begin{array}{ccccccccccccc} \mathrm{c}) & \Longrightarrow & \mathrm{e}) & & & & & & & & & & \\ \Downarrow & & \Downarrow & & & & & & & & & & \\ \mathrm{d}) & \Longrightarrow & \mathrm{f}) & \Longleftarrow & \mathrm{g}) & \Longleftrightarrow & \mathrm{h}) & \Longleftrightarrow & \mathrm{i}) & \Longleftrightarrow & \mathrm{b}). \end{array}$$

Since a)$\Rightarrow$b) is evident, it suffices to show that a)$\Rightarrow$c) and f)$\Rightarrow$a).

a)$\Rightarrow$c) Let $H \le L$. For any finite-dimensional subalgebra F of L, $F \in \mathrm{L}\mathfrak{N} \cap \mathfrak{F} \le \mathfrak{N}$. Then it is easy to see that $H \cap F \text{ si } F$. Hence by Lemma 2, H ser L.

f)$\Rightarrow$a) Let X be a finite subset of L. Take a finite-dimensional subalgebra F of L containing X. For any $x \in F$, $\langle x \rangle$ wser L and therefore by Lemma 1, $\langle x \rangle$ wsi F. It follows that $\mathrm{ad}_F x$ is nilpotent. By Engel's theorem $F \in \mathfrak{N}$. Therefore $L \in \mathrm{L}\mathfrak{N}$.

THEOREM 4. *Over any field* $\mathfrak{k}$

$$\mathrm{L}\mathfrak{N} = \mathrm{L}(\mathrm{ser})(\mathfrak{N} \cap \mathfrak{F}) = \mathrm{L}(\mathrm{wser})(\mathfrak{N} \cap \mathfrak{F}).$$

PROOF. Let $L \in \mathrm{L}\mathfrak{N}$ and let X be any finite subset of L. Since $\mathrm{L}\mathfrak{N} = \mathrm{L}(\mathfrak{N} \cap \mathfrak{F})$, there exists a subalgebra H of L belonging to $\mathfrak{N} \cap \mathfrak{F}$ such that $X \subseteq H$. By Theorem 3, H ser L. Hence $L \in \mathrm{L}(\mathrm{ser})(\mathfrak{N} \cap \mathfrak{F})$. Therefore $\mathrm{L}\mathfrak{N} \le \mathrm{L}(\mathrm{ser})(\mathfrak{N} \cap \mathfrak{F})$. Now we can easily conclude that the equalities hold.

Finally we examine further relations of the subclasses of $\mathrm{L}(\mathrm{ser})\mathfrak{F}$ stated above. Evidently

$$\mathrm{L}(\mathrm{ser})\mathfrak{F} \ge \mathrm{L}(\mathrm{ser})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) \ge \mathrm{L}(\mathrm{ser})(\mathfrak{N} \cap \mathfrak{F}) = \mathrm{L}\mathfrak{N}.$$

We remark that

$$\mathrm{L}(\mathrm{ser})\mathfrak{F} \ne \mathrm{L}(\mathrm{ser})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) \text{ and } \mathrm{L}(\mathrm{ser})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) \ne \mathrm{L}(\mathrm{ser})(\mathfrak{N} \cap \mathfrak{F}).$$

In fact, let L be the direct sum of a non-empty set of finite-dimensional non-abelian simple Lie algebras. Then $L \in \mathrm{L}(\lhd)\mathfrak{F} \le \mathrm{L}(\mathrm{ser})\mathfrak{F}$ and $L \notin \mathrm{L}(\mathrm{ser})(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$. Hence the first inequality holds. The other inequality is clear by considering a direct sum of finite-dimensional non-nilpotent solvable Lie algebras.

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Department of Mathematics,
Hiroshima Institute of Technology