HIROSHIMA MATH. J.
16 (1986), 367–386

Classes of generalized soluble Lie algebras

Masanobu HONDA

(Received August 23, 1985)

Introduction

A class $\mathfrak{X}$ of Lie algebras is said to be a class of generalized soluble Lie algebras if every soluble Lie algebra is an $\mathfrak{X}$-algebra and every finite-dimensional $\mathfrak{X}$-algebra is soluble. As relatively large classes of generalized soluble Lie algebras we know the classes $\hat{\mathrm{E}}(\lhd)\mathfrak{A}$ and $\hat{\mathrm{E}}\mathfrak{A}$, which are the Lie-theoretic analogues of the class of *SI*-groups and the class of *SN*-groups respectively. In group theory Mal'cev [6] has proved that the class of *SI*-groups, the class of *SN*-groups and the class of *Z*-groups are L-closed. The first purpose of this paper is to prove the Lie-theoretic analogue of this result.

Generalizing the class $\mathfrak{R}$ of residually central Lie algebras, Amayo [2] has introduced a relatively large class, denoted by $\mathfrak{R}^{(1)}$ in this paper, of generalized soluble Lie algebras. In the recent paper [5] we have introduced the class $\mathfrak{R}_{(\infty)}$ of residually (ω)-central Lie algebras. The second purpose of this paper is to introduce and investigate various classes of Lie algebras generalizing the class $\mathfrak{R}$. Most of them are classes of generalized soluble Lie algebras.

In Section 2, following [8, §8.2] it can be more generally proved that the classes $\hat{\mathrm{E}}\mathfrak{A}$, $\hat{\mathrm{E}}(\lhd)\mathfrak{A}$ and $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$ are L-closed, where $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$ is the class of Lie algebras having central series (Theorems 2.2 and 2.6). We shall also show that every finite-dimensional subalgebra of an $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$-algebra (resp. a hypocentral Lie algebra) is serial (resp. descendant) (Theorem 2.9).

In Section 3 we shall develop some results analogous to those of [5, §2] by using the class $\mathfrak{R}_{(*)}$, naturally including the class $\mathfrak{R}_{(\infty)}$, of generalized soluble Lie algebras. Especially, we shall show that $\mathfrak{R}_{(*)} \cap \mathfrak{M}^{(*)} = \grave{\mathrm{E}}\mathfrak{A}$ (Theorem 3.5), where $\mathfrak{M}^{(*)}$ is a class of Lie algebras generalizing quasi-artinian Lie algebras.

Section 4 is devoted to investigating the classes $\mathfrak{R}^*$, $\mathfrak{R}^{(*)}$, $\mathfrak{R}^{(1)}_{(*)}$ and $\mathfrak{R}^{(*)}_{(*)}$, naturally including the class $\mathfrak{R}^{(1)}$, of generalized soluble Lie algebras. We shall show that $\mathfrak{R}^{(1)} = \mathfrak{R}^* = \mathfrak{R}^{(*)} = (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{R}^{(1)} = (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{R}^* = (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{R}^{(*)}$ and $\mathfrak{R}^{(1)}_{(*)} = \mathfrak{R}^{(*)}_{(*)} = (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{R}^{(1)}_{(*)} = (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{R}^{(*)}_{(*)}$ (Theorem 4.3). We shall also show that $\mathfrak{R}^{(*)}_{(*)} \cap \text{Min-}\lhd = \acute{\mathrm{E}}(\lhd)\mathfrak{A} \cap \text{Min-}\lhd$ (Theorem 4.6).

In Section 5 we shall investigate the classes $\mathfrak{R}_{(1)}$ and $\mathfrak{R}_*$ which are between the classes $\mathfrak{R}$ and $\mathfrak{R}_{(*)}$. In particular, we shall present a sufficient condition for a Lie algebra to be contained in the class $\mathfrak{R}^{(1)}$ and consequently show that $\mathfrak{R}_{(1)}$ is a subclass of the class $\mathfrak{R}^{(1)}$ (Theorem 5.2).

1.

Throughout this paper we always consider not necessarily finite-dimensional Lie algebras over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified, and mostly follow [3] for the use of notations and terminology.

Let L be a Lie algebra over $\mathfrak{k}$ and $\mathfrak{X}$ a class of Lie algebras. $\mathfrak{X}$ is said to be a class of generalized soluble (resp. nilpotent) Lie algebras if $\mathfrak{X} \cap \mathfrak{F} \leq \mathrm{E}\mathfrak{A} \leq \mathfrak{X}$ (resp. $\mathfrak{X} \cap \mathfrak{F} \leq \mathfrak{N} \leq \mathfrak{X}$). As a relatively large class of generalized nilpotent Lie algebras, we know the class $\mathfrak{R}$ of residually central Lie algebras, where L is residually central if $x \in L \setminus \{0\}$ implies $x \notin [x, L]^L$. In fact, since [2, Theorem 3.5] (or [9, Corollary to Theorem 3.3]) states that

$$\mathfrak{R} \cap \text{Min-}\lhd \leq \mathfrak{Z} \cap \mathrm{E}\mathfrak{A},$$

$\mathfrak{R}$ is a class of generalized nilpotent Lie algebras. In this paper we introduce the classes $\mathfrak{R}^{(1)}$, $\mathfrak{R}_{(1)}$ and $\mathfrak{R}_*$, naturally including the class $\mathfrak{R}$, as follows:

$$\begin{aligned} &L \in \mathfrak{R}^{(1)} && \text{iff} \quad x \in L \setminus \{0\} \quad \text{implies} \quad x \notin ([x, L]^L)^{(1)}; \\ &L \in \mathfrak{R}_{(1)} && \text{iff} \quad x \in L \setminus \{0\} \quad \text{implies} \quad x \notin [x, L^{(1)}]^L; \\ &L \in \mathfrak{R}_* && \text{iff} \quad x \in L \setminus \{0\} \quad \text{implies} \quad x \notin [x, L^*]^L, \end{aligned}$$

where we denote by L^* the intersection of all the terms in the transfinite lower central series for L. Among them the class $\mathfrak{R}^{(1)}$ has been studied in [2, p. 16].

On the other hand, as relatively large classes of generalized soluble Lie algebras, we know the classes $\hat{\mathrm{E}}\mathfrak{A}$, $\hat{\mathrm{E}}(\lhd)\mathfrak{A}$, $\acute{\mathrm{E}}\mathfrak{A}$, $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$, $\grave{\mathrm{E}}\mathfrak{A} = \grave{\mathrm{E}}(\lhd)\mathfrak{A}$, $\mathfrak{R}_{(\infty)}$ and $\mathfrak{R}^{(1)}$. $\hat{\mathrm{E}}\mathfrak{X}$ (resp. $\hat{\mathrm{E}}(\lhd)\mathfrak{X}$) is the class of Lie algebras L having a family $\mathscr{S} = \{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ of subalgebras (resp. ideals) of L for some totally ordered set Σ such that

(a) $V_\sigma \lhd \Lambda_\sigma$ and $\Lambda_\sigma / V_\sigma \in \mathfrak{X}$ for all $\sigma \in \Sigma$;
(b) $\Lambda_\sigma \leq V_\tau$ if $\sigma < \tau$;
(c) $L \setminus \{0\} = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \setminus V_\sigma)$.

Then $\mathscr{S}$ is called a series (resp. an ideal series) of L (of type Σ) with $\mathfrak{X}$-factors. When Σ is well-ordered, $\mathscr{S}$ is called an ascending series (resp. ideal series) of L with $\mathfrak{X}$-factors. When Σ is reversely well-ordered, $\mathscr{S}$ is called a descending series (resp. ideal series) of L with $\mathfrak{X}$-factors. $L \in \acute{\mathrm{E}}\mathfrak{X}$ (resp. $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$) if L has an ascending series (resp. ideal series) with $\mathfrak{X}$-factors. $L \in \grave{\mathrm{E}}\mathfrak{X}$ (resp. $\grave{\mathrm{E}}(\lhd)\mathfrak{X}$) if L has a descending series (resp. ideal series) with $\mathfrak{X}$-factors. From the definitions it is clear that $\hat{\mathrm{E}}\mathfrak{A}$, $\hat{\mathrm{E}}(\lhd)\mathfrak{A}$, $\acute{\mathrm{E}}\mathfrak{A}$, $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ and $\grave{\mathrm{E}}\mathfrak{A} = \grave{\mathrm{E}}(\lhd)\mathfrak{A}$ are classes of generalized soluble Lie algebras. The class $\mathfrak{R}_{(\infty)}$, strictly including the class $\mathfrak{R}$, is defined in [5] by

$$L \in \mathfrak{R}_{(\infty)} \quad \text{iff} \quad x \in L \setminus \{0\} \quad \text{implies} \quad x \notin [x, L^{(\omega)}]^L.$$

Then by [5, Theorem 2.3] we have

$$\mathfrak{R}_{(\infty)} \cap \text{qmin-}\lhd = \text{E}\mathfrak{A},$$

where qmin-$\lhd$, strictly including the class Min-$\lhd$, is the class of quasi-artinian Lie algebras. In [1] L is said to be quasi-artinian if for any descending chain $I_1 \geq I_2 \geq \cdots$ of ideals of L there exists an integer $n>0$ such that $[I_n, L^{(n)}] \leq \bigcap_{i\geq 1} I_i$. On the other hand, Amayo has indicated in [2, p. 16] that

$$\mathfrak{R}^{(1)} \cap \text{Min-}\lhd \leq \acute{\text{E}}(\lhd)\mathfrak{A}.$$

Therefore $\mathfrak{R}_{(\infty)}$ and $\mathfrak{R}^{(1)}$ are indeed classes of generalized soluble Lie algebras.

In this paper we introduce the class $\mathfrak{R}_{(*)}$, naturally including the class $\mathfrak{R}_{(\infty)}$, and the classes $\mathfrak{R}^*$, $\mathfrak{R}^{(*)}$, $\mathfrak{R}^{(1)}_{(*)}$ and $\mathfrak{R}^{(*)}_{(*)}$, naturally including the class $\mathfrak{R}^{(1)}$, as follows:

$$\begin{array}{llll} L\in\mathfrak{R}_{(*)} & \text{iff} & x\in L\setminus\{0\} & \text{implies} \quad x\notin[x, L^{(*)}]^L; \\ L\in\mathfrak{R}^{*} & \text{iff} & x\in L\setminus\{0\} & \text{implies} \quad x\notin([x, L]^L)^*; \\ L\in\mathfrak{R}^{(*)} & \text{iff} & x\in L\setminus\{0\} & \text{implies} \quad x\notin([x, L]^L)^{(*)}; \\ L\in\mathfrak{R}^{(1)}_{(*)} & \text{iff} & x\in L\setminus\{0\} & \text{implies} \quad x\notin([x, L^{(*)}]^L)^{(1)}; \\ L\in\mathfrak{R}^{(*)}_{(*)} & \text{iff} & x\in L\setminus\{0\} & \text{implies} \quad x\notin([x, L^{(*)}]^L)^{(*)}, \end{array}$$

where we denote by $L^{(*)}$ the intersection of all the terms in the transfinite derived series for L.

Concerning L^* and $L^{(*)}$ the following lemma is elementary.

LEMMA 1.1. *Let $I\lhd L$ and $H\leq L$. Then:*

(1) *$H^* \leq L^*$ and $H^{(*)} \leq L^{(*)}$.*

(2) *$(H^*+I)/I \leq ((H+I)/I)^*$ and $(H^{(*)}+I)/I \leq ((H+I)/I)^{(*)}$.*

(3) *If $H \cap I = \{0\}$ then $(H^*+I)/I = ((H+I)/I)^*$ and $(H^{(*)}+I)/I = ((H+I)/I)^{(*)}$.*

(4) *$(L^{(*)})^{(*)} = L^{(*)} \leq L^*$.*

2.

In this section, following [8, §8.2] we shall first show that for any {Q, R}-closed class $\mathfrak{X}$ of Lie algebras the classes $\acute{\text{E}}\mathfrak{X}$ and $\acute{\text{E}}(\lhd)\mathfrak{X}$ are L-closed. We shall secondly show that in a Lie algebra having a central series (resp. a descending central series) every finite-dimensional subalgebra is serial (resp. descendant).

We begin by expressing the concepts of a series and an ideal series in functional forms. Let L be a Lie algebra over $\mathfrak{k}$. Assume that L has a series (resp. an ideal series) $\{\Lambda_\sigma, V_\sigma: \sigma\in\Sigma\}$ of some type Σ (with $\mathfrak{D}$-factors). To each $x\in L\setminus\{0\}$ there corresponds a unique $\sigma(x)\in\Sigma$ such that $x\in\Lambda_{\sigma(x)}\setminus V_{\sigma(x)}$. For any $x\in L\setminus\{0\}$ we clearly see that $x\in\Lambda_\sigma$ iff $\sigma\geq\sigma(x)$, and that $x\in V_\sigma$ iff $\sigma>\sigma(x)$. We define a binary function $f_L: L\times L\to\{0, 1\}$ as follows; for any x, $y\in L$

$$f_L(x, y) = \begin{cases} 0 & \text{if } x = 0 \text{ or if } x, y \neq 0 \text{ and } \sigma(x) \leq \sigma(y), \\ 1 & \text{otherwise.} \end{cases} \qquad (*)$$

Then we can easily verify that the function f_L satisfies the following conditions (i)–(iv) and (v) (resp. (v′)), where $x, y, z \in L$ and $\alpha, \beta \in \mathfrak{k}$:

(i) If $f_L(x, y) = f_L(y, z) = 0$ then $f_L(x, z) = 0$.
(ii) Either $f_L(x, y) = 0$ or $f_L(y, x) = 0$.
(iii) If $f_L(x, 0) = 0$ then $x = 0$.
(iv) If $f_L(x, z) = f_L(y, z) = 0$ then $f_L(\alpha x + \beta y, z) = f_L([x, y], z) = 0$.
(v) If $f_L(x, y) = 1$ then $f_L(x, [x, y]) = 1$.
(v′) $f_L([x, y], x) = 0$.

Conversely, assume that there exists a binary function f_L: $L \times L \to \{0, 1\}$ satisfying the conditions (i)–(iv) and (v) (resp. (v′)). Let $x \sim y$ mean that $f_L(x, y) = f_L(y, x) = 0$. By (i), (ii) and (iii) the relation $\sim$ is an equivalence relation on L and $\{x \in L: x \sim 0\} = \{0\}$. Let Σ denote the family of all $\sim$-equivalence classes except $\{0\}$. For $\sigma, \tau \in \Sigma$, we write $\sigma < \tau$ if $\sigma \neq \tau$ and $f_L(\sigma, \tau) = \{0\}$. Then by (i) and (ii) the relation $<$ is a total order on Σ. We now define a family $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ of subsets of L as follows; for each $\sigma \in \Sigma$

$$\Lambda_\sigma = \{x \in L: f_L(x, \sigma) = \{0\}\}, \ V_\sigma = \begin{cases} \cup_{\tau < \sigma} \Lambda_\tau & \text{if } \{\tau \in \Sigma: \tau < \sigma\} \neq \emptyset, \\ \{0\} & \text{otherwise.} \end{cases} \qquad (**)$$

By (i) and (iv) $\{\Lambda_\sigma: \sigma \in \Sigma\}$ is a totally ordered chain of subalgebras of L. It follows that $V_\sigma \leq \Lambda_\sigma$ for any $\sigma \in \Sigma$. If $\tau < \sigma$ then $\Lambda_\tau \leq V_\sigma$. It is not hard to show that $L \setminus \{0\} = \cup_{\sigma \in \Sigma} (\Lambda_\sigma \setminus V_\sigma)$. By using (i) and (v) (resp. (i) and (v′)) we can easily see that $V_\sigma \lhd \Lambda_\sigma$ (resp. V_σ, $\Lambda_\sigma \lhd L$) for all $\sigma \in \Sigma$. Therefore $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ is a series (resp. an ideal series) of L of type Σ (with $\mathfrak{O}$-factors).

Let $\mathscr{F}_\infty$ be the free Lie algebra over $\mathfrak{k}$ on a countably infinite set $\{t_1, t_2, \cdots\}$. An elements of $\mathscr{F}_\infty$ is called a word.

LEMMA 2.1. *Let L be a Lie algebra, Ω a set of words and $\mathfrak{V}_\Omega$ the variety determined by Ω. Then $L \in \acute{\mathrm{E}}\mathfrak{V}_\Omega$ (resp. $\acute{\mathrm{E}}(\lhd)\mathfrak{V}_\Omega$) if and only if there exists a binary function f_L: $L \times L \to \{0, 1\}$ satisfying the conditions* (i)–(iv), (v) (*resp.* (v′)) *and*

(vi) *If $y \neq 0$ and $f_L(x_i, y) = 0$ $(1 \leq i \leq n)$, then $f_L(y, w(x_1, \ldots, x_n)) = 1$,*

where $w = w(t_1, \ldots, t_n) \in \Omega$ and $x_i, y \in L$ $(1 \leq i \leq n)$.

PROOF. Assume that $L \in \acute{\mathrm{E}}\mathfrak{V}_\Omega$ (resp. $\acute{\mathrm{E}}(\lhd)\mathfrak{V}_\Omega$) and let $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ be a series (resp. an ideal series) of L of type Σ with $\mathfrak{V}_\Omega$-factors. Then the binary function f_L: $L \times L \to \{0, 1\}$ defined by (*) satisfies the conditions (i)–(iv) and (v) (resp. (v′)). Let $w = w(t_1, \ldots, t_n) \in \Omega$ and x_i, $y \in L$ $(1 \le i \le n)$. Suppose that $y \ne 0$ and $f_L(x_i, y) = 0$ $(1 \le i \le n)$. Then $x_i \in \Lambda_{\sigma(y)}$ $(1 \le i \le n)$. Since $\Lambda_{\sigma(y)}/V_{\sigma(y)} \in \mathfrak{B}_\Omega$, we have $w(x_1, \ldots, x_n) \in V_{\sigma(y)}$. Hence $w(x_1, \ldots, x_n) = 0$ or $\sigma(w(x_1, \ldots, x_n)) < \sigma(y)$. This implies $f_L(y, w(x_1, \ldots, x_n)) = 1$. Therefore f_L satisfies the conditions (i)–(iv), (v) (resp. (v′)) and (vi).

Conversely, assume that there exists a binary function f_L: $L \times L \to \{0, 1\}$ satisfying the conditions (i)–(iv), (v) (resp. (v′)) and (vi). Let $\{\Lambda_\sigma, V_\sigma: \sigma \in \Sigma\}$ be the series (resp. the ideal series) of L defined by (**). We show that $\Lambda_\sigma/V_\sigma \in \mathfrak{B}_\Omega$ for all $\sigma \in \Sigma$. Let $\sigma \in \Sigma$, $w = w(t_1, \ldots, t_n) \in \Omega$ and $x_i \in \Lambda_\sigma$ $(1 \le i \le n)$. Suppose that $w(x_1, \ldots, x_n) \notin V_\sigma$. Since $f_L(x_i, \sigma) = \{0\}$ $(1 \le i \le n)$, by (vi) we have $f_L(\sigma, w(x_1, \ldots, x_n)) = \{1\}$. We can find a $\tau \in \Sigma$ such that $w(x_1, \ldots, x_n) \in \tau$. Then we have $\tau < \sigma$. Hence $w(x_1, \ldots, x_n) \in \Lambda_\tau \le V_\sigma$, a contradiction. Therefore we have $w(x_1 + V_\sigma, \ldots, x_n + V_\sigma) = 0$. It follows that $\Lambda_\sigma/V_\sigma \in \mathfrak{B}_\Omega$. Thus we obtain $L \in \hat{\mathrm{E}}\mathfrak{B}_\Omega$ (resp. $\hat{\mathrm{E}}(\lhd)\mathfrak{B}_\Omega$).

Now we have the first main theorem of this section, which corresponds to [8, Theorem 8.23].

THEOREM 2.2. *For any variety $\mathfrak{B}$ of Lie algebras, the classes $\hat{\mathrm{E}}\mathfrak{B}$ and $\hat{\mathrm{E}}(\lhd)\mathfrak{B}$ are* L*-closed. In other words, for any* {Q, R}*-closed class $\mathfrak{X}$ of Lie algebras, the classes $\hat{\mathrm{E}}\mathfrak{X}$ and $\hat{\mathrm{E}}(\lhd)\mathfrak{X}$ are* L*-closed.*

PROOF. It is well known (cf. [3, p. 257]) that a class $\mathfrak{X}$ of Lie algebras is a variety if and only if $\mathfrak{X}$ is {Q, R}-closed. Hence it suffices to prove the first half of the theorem. Let $\mathfrak{B}$ be a variety of Lie algebras. Then there exists a set Ω of words determining $\mathfrak{B}$. Let $L \in \mathrm{L}\hat{\mathrm{E}}\mathfrak{B}$ (resp. $\mathrm{L}\hat{\mathrm{E}}(\lhd)\mathfrak{B}$). We denote by $\mathscr{L}$ the set of $\hat{\mathrm{E}}\mathfrak{B}$-subalgebras (resp. $\hat{\mathrm{E}}(\lhd)\mathfrak{B}$-subalgebras) of L. Then $\mathscr{L}$ is a local system on L in the sense of [8, p. 94]. It follows from Lemma 2.1 that for each $H \in \mathscr{L}$ there exists a binary function f_H: $H \times H \to \{0, 1\}$ satisfying the conditions (i)–(iv), (v) (resp. (v′)) and (vi) which are obtained by replacing L with H. Owing to [8, Lemma 8.22], there exists a binary function f_L: $L \times L \to \{0, 1\}$ such that, given any finite subset $\{(x_i, y_i): 1 \le i \le m\}$ of $L \times L$, there exists an $H \in \mathscr{L}$ for which $(x_i, y_i) \in H \times H$ and $f_L(x_i, y_i) = f_H(x_i, y_i)$ $(1 \le i \le m)$. Since each of the conditions (i)–(iv), (v) (resp. (v′)) and (vi) involves a finite number of elements of L, the function f_L also satisfies the conditions (i)–(iv), (v) (resp. (v′)) and (vi). Using Lemma 2.1 again, we have $L \in \hat{\mathrm{E}}\mathfrak{B}$ (resp. $\hat{\mathrm{E}}(\lhd)\mathfrak{B}$).

We regard the class $\mathfrak{A}$ as the variety determined by the set of the single word $[t_1, t_2]$. Then as an immediate consequence of Theorem 2.2 we have the following

COROLLARY 2.3. (1) $\mathrm{L}\hat{\mathrm{E}}\mathfrak{A} = \hat{\mathrm{E}}\mathfrak{A}$ *and* $\mathrm{L}\hat{\mathrm{E}}(\lhd)\mathfrak{A} = \hat{\mathrm{E}}(\lhd)\mathfrak{A}$.

(2) $\mathrm{L}\acute{\mathrm{E}}\mathfrak{A} \le \hat{\mathrm{E}}\mathfrak{A}$ *and* $\mathrm{L}\acute{\mathrm{E}}(\lhd)\mathfrak{A} \cup \mathrm{L}\grave{\mathrm{E}}\mathfrak{A} \le \hat{\mathrm{E}}(\lhd)\mathfrak{A}$.

REMARK. By making use of [3, Corollary 6.5.3] and [2, Theorem 4.6], we see that if $\mathfrak{k}$ has zero characteristic then $\mathrm{L}\acute{\mathrm{E}}\mathfrak{A} \neq \acute{\mathrm{E}}\mathfrak{A}$. In his recent paper [4] Ikeda has proved that $\mathrm{L}\acute{\mathrm{E}}(\lhd)\mathfrak{A} \neq \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ ([4, Corollary 3.4]) and that if every countable dimensional subalgebra of a Lie algebra L belongs to $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$ then $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ ([4, Corollary 2.10]). Moreover, we have $\mathrm{L}\grave{\mathrm{E}}\mathfrak{A} \neq \grave{\mathrm{E}}\mathfrak{A}$. In fact, we consider the McLain Lie algebra $\mathscr{L}_{\mathfrak{k}}(\boldsymbol{Q})$ over $\mathfrak{k}$, where $\boldsymbol{Q}$ is the set of rational numbers with natural ordering (cf. [3, p. 111]). Then it is well known ([10, p. 96]) that $\mathscr{L}_{\mathfrak{k}}(\boldsymbol{Q})$ is perfect and locally nilpotent. Therefore we have $\mathscr{L}_{\mathfrak{k}}(\boldsymbol{Q}) \in \mathrm{L}\grave{\mathrm{E}}\mathfrak{A} \setminus \grave{\mathrm{E}}\mathfrak{A}$.

Next we introduce the Lie-theoretic analogue of the concept of marginal subgroups of groups (cf. [7, p. 9]). Let I be an ideal of a Lie algebra L. For a word $w = w(t_1,\dots, t_n)$, I is said to be w-marginal in L if $w(x_1,\dots, x_n) = w(y_1,\dots, y_n)$ whenever x_i, $y_i \in L$ and $x_i \equiv y_i \bmod I$ $(1 \leq i \leq n)$. Let Ω be a set of words and $\mathfrak{B}_\Omega$ the variety determined by Ω. Then I is said to be $\mathfrak{B}_\Omega$-marginal in L if I is w-marginal in L for all $w \in \Omega$. Clearly if I is $\mathfrak{B}_\Omega$-marginal in L then $I \in \mathfrak{B}_\Omega$. Since the variety $\mathfrak{A}$ is determined by $\{[t_1, t_2]\}$, we can easily see that I is $\mathfrak{A}$-marginal in L if and only if I is central in L (i.e. $I \leq \zeta_1(L)$). Let J be an ideal of L contained in I. We say that I/J is a $\mathfrak{B}_\Omega$-marginal factor of L if I/J is a factor of some ideal series of L and is $\mathfrak{B}_\Omega$-marginal in L/J. Then we define the classes $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{B}}_\Omega$, $\acute{\mathrm{E}}(\lhd)\hat{\mathfrak{B}}_\Omega$ and $\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{B}}_\Omega$ of Lie algebras as follows:

$L \in \hat{\mathrm{E}}(\lhd)\hat{\mathfrak{B}}_\Omega$ iff L has an ideal series with $\mathfrak{B}_\Omega$-marginal factors;
$L \in \acute{\mathrm{E}}(\lhd)\hat{\mathfrak{B}}_\Omega$ iff L has an ascending ideal series with $\mathfrak{B}_\Omega$-marginal factors;
$L \in \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{B}}_\Omega$ iff L has a descending ideal series with $\mathfrak{B}_\Omega$-marginal factors.

In particular, we have

LEMMA 2.4. (1) $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} = \{L \in \mathfrak{D}$: *L has a central series*$\}$.
(2) $\acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} = \{L \in \mathfrak{D}$: *L has an ascending central series*$\} = \mathfrak{Z}$.
(3) $\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} = \{L \in \mathfrak{D}$: *L has a descending central series*$\} = \{L \in \mathfrak{D}: L^* = \{0\}\}$.

REMARK. It has been indicated in [9, p. 58] that every Lie algebra having a central series is residually central. It follows from Lemma 2.4 (1) that

$$\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} \leq \mathfrak{R}.$$

In particular, $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$ is a class of generalized nilpotent Lie algebras.

We are able to express the concept of ideal series with marginal factors of Lie algebras in functional form.

LEMMA 2.5. *Let L be a Lie algebra, Ω a set of words and $\mathfrak{B}_\Omega$ the variety determined by Ω. Then $L \in \hat{\mathrm{E}}(\lhd)\hat{\mathfrak{B}}_\Omega$ if and only if there exists a binary function*

f_L: $L\times L\to\{0, 1\}$ *satisfying the conditions* (i)–(iv), (v′) *and*

(vii) *If* $z\neq 0$ *and* $f_L(x_i-y_i, z)=0$ $(1\leq i\leq n)$, *then* $f_L(z, w(x_1,\dots, x_n)-w(y_1,\dots, y_n))=1$,

where $w=w(t_1,\dots, t_n)\in\Omega$ *and* x_i, y_i, $z\in L$ $(1\leq i\leq n)$.

PROOF. Assume that $L\in\acute{\mathrm{E}}(\lhd)\hat{\mathfrak{V}}_\Omega$ and let $\{\Lambda_\sigma, V_\sigma\colon \sigma\in\Sigma\}$ be an ideal series of L with $\mathfrak{V}_\Omega$-marginal factors. Then the binary function f_L: $L\times L\to\{0, 1\}$ defined by $(*)$ satisfies the conditions (i)–(iv) and (v′). Let $w=w(t_1,\dots, t_n)\in\Omega$ and x_i, y_i, $z\in L$ $(1\leq i\leq n)$. Suppose that $z\neq 0$ and $f_L(x_i-y_i, z)=0$ $(1\leq i\leq n)$. Then $x_i-y_i\in\Lambda_{\sigma(z)}$ $(1\leq i\leq n)$. Since $\Lambda_{\sigma(z)}/V_{\sigma(z)}$ is a $\mathfrak{V}_\Omega$-marginal factor, we have $w(x_1,\dots, x_n)-w(y_1,\dots, y_n)\in V_{\sigma(z)}$. It follows that $w(x_1,\dots, x_n)-w(y_1,\dots, y_n)=0$ or $\sigma(w(x_1,\dots, x_n)-w(y_1,\dots, y_n))<\sigma(z)$. Hence we have $f_L(z, w(x_1,\dots, x_n)-w(y_1,\dots, y_n))=1$.

Conversely, assume that there exists a binary function f_L: $L\times L\to\{0, 1\}$ satisfying the conditions (i)–(iv), (v′) and (vii). Let $\{\Lambda_\sigma, V_\sigma\colon \sigma\in\Sigma\}$ be the ideal series of L defined by $(**)$. We show that Λ_σ/V_σ is a $\mathfrak{V}_\Omega$-marginal factor of L for any $\sigma\in\Sigma$. Let $\sigma\in\Sigma$, $w=w(t_1,\dots, t_n)\in\Omega$ and x_i, $y_i\in L$ $(1\leq i\leq n)$. Suppose that $x_i\equiv y_i \bmod \Lambda_\sigma$ $(1\leq i\leq n)$. Then $f_L(x_i-y_i, \sigma)=\{0\}$ $(1\leq i\leq n)$. It follows from (vii) that $f_L(\sigma, w(x_1,\dots, x_n)-w(y_1,\dots, y_n))=\{1\}$. Suppose that $0\neq w(x_1,\dots, x_n)-w(y_1,\dots, y_n)\in\tau\in\Sigma$. Then we have $\tau<\sigma$. It follows that $w(x_1,\dots, x_n)-w(y_1,\dots, y_n)\in\Lambda_\tau\leq V_\sigma$. Therefore Λ_σ/V_σ is w-marginal in L/V_σ. Thus we have $L\in\acute{\mathrm{E}}(\lhd)\hat{\mathfrak{V}}_\Omega$.

Now we have the second main theorem of this section, which corresponds to [8, Theorem 8.24].

THEOREM 2.6. *Let* Ω *be a set of words and* $\mathfrak{V}_\Omega$ *the variety determined by* Ω. *Then the class* $\acute{\mathrm{E}}(\lhd)\hat{\mathfrak{V}}_\Omega$ *is* L-*closed.*

PROOF. By using Lemma 2.5, we can prove the theorem as in the proof of Theorem 2.2.

By making use of Lemma 2.4 and Theorem 2.6, we have

COROLLARY 2.7. (1) $\mathrm{L}\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}=\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$.
(2) $\mathrm{L}\mathfrak{N}=\mathrm{L}\mathfrak{Z}=\mathrm{L}\acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}\leq\mathrm{LR}\mathfrak{N}\leq\mathrm{L}\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}\leq\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$.

REMARK. Both of the classes $\acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}=\mathfrak{Z}$ and $\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$ are not L-closed. In fact, the McLain Lie algebra $\mathscr{L}_t(Q)$ is locally nilpotent and is neither hypercentral nor hypocentral.

It is well known that if $L\in\acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}=\mathfrak{Z}$ then every subalgebra of L is ascendant in L. On the other hand, it is not known whether every subalgebra of an $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$-

algebra (resp. an $\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$-algebra) L is serial (resp. descendant) in L or not. However, we can prove that every finite-dimensional subalgebra of an $\hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$-algebra (resp. an $\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$-algebra) L is serial (resp. descendant) in L. To do this we need the following lemma concerning vector spaces.

LEMMA 2.8. *Let V be a vector space over $\mathfrak{k}$, U a subspace of V and X a finite-dimensional subspace of V. Assume that there exist a totally ordered set Σ and a family $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ of subspaces of V such that*

(a) $U \subseteq V_\sigma \subseteq \Lambda_\sigma$ *for all* $\sigma \in \Sigma$;

(b) $\Lambda_\sigma \subseteq V_\tau$ *if* $\sigma < \tau$;

(c) $V \smallsetminus U = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \smallsetminus V_\sigma)$.

Then we have $V \smallsetminus (U+X) = \bigcup_{\sigma \in \Sigma} ((\Lambda_\sigma + X) \smallsetminus (V_\sigma + X))$.

PROOF. By using induction on $n = \dim(X)$, we show the result. It is clear for $n=0$. Let $n>0$ and assume that the result is true for $n-1$. There are an $(n-1)$-dimensional subspace X_0 of X and a non-zero element x of X such that $X = X_0 + \mathfrak{k}x$. For each $\sigma \in \Sigma$, set $\Lambda'_\sigma = \Lambda_\sigma + X_0$, $V'_\sigma = V_\sigma + X_0$, $\Lambda''_\sigma = \Lambda_\sigma + X$ and $V''_\sigma = V_\sigma + X$. Then by inductive hypothesis the family $\{\Lambda'_\sigma, V'_\sigma : \sigma \in \Sigma\}$ satisfies the following conditions:

(a′) $U + X_0 \subseteq V'_\sigma \subseteq \Lambda'_\sigma$ for all $\sigma \in \Sigma$;

(b′) $\Lambda'_\sigma \subseteq V'_\tau$ if $\sigma < \tau$;

(c′) $V \smallsetminus (U + X_0) = \bigcup_{\sigma \in \Sigma} (\Lambda'_\sigma \smallsetminus V'_\sigma)$.

It follows from (b′) and (c′) that for any $v \in V \smallsetminus (U+X_0)$ there exists a unique $\sigma(v) \in \Sigma$ such that $v \in \Lambda'_{\sigma(v)} \smallsetminus V'_{\sigma(v)}$. In the case that $x \in U + X_0$, by (a′) and (c′) we have

$$V \smallsetminus (U+X) = V \smallsetminus (U+X_0) = \bigcup_{\sigma \in \Sigma} (\Lambda'_\sigma \smallsetminus V'_\sigma) = \bigcup_{\sigma \in \Sigma} (\Lambda''_\sigma \smallsetminus V''_\sigma).$$

So we consider the case that $x \notin U + X_0$. Let $v \in V \smallsetminus (U+X)$. For each of the cases

$$1)\ \sigma(x) < \sigma(v), \qquad 2)\ \sigma(v) < \sigma(x), \qquad 3)\ \sigma(x) = \sigma(v),$$

we show that $v \in \Lambda''_\sigma \smallsetminus V''_\sigma$ for some $\sigma \in \Sigma$.

Case 1). By (a′) and (b′) $x \in \Lambda'_{\sigma(x)} \subseteq V'_{\sigma(v)} \subseteq \Lambda'_{\sigma(v)}$. It follows that $\Lambda''_{\sigma(v)} = \Lambda'_{\sigma(v)}$ and $V''_{\sigma(v)} = V'_{\sigma(v)}$. Hence we have $v \in \Lambda''_{\sigma(v)} \smallsetminus V''_{\sigma(v)}$.

Case 2). Suppose that $v \in V''_{\sigma(v)} = V'_{\sigma(v)} + \mathfrak{k}x$ and write $v = u + \alpha x$ ($u \in V'_{\sigma(v)}$, $0 \neq \alpha \in \mathfrak{k}$). Then by (a′) and (b′) we have $x = (v-u)/\alpha \in V'_{\sigma(x)}$, a contradiction. Therefore we have $v \in \Lambda''_{\sigma(v)} \smallsetminus V''_{\sigma(v)}$.

Case 3). We may suppose that $v \in V''_{\sigma(v)} = V'_{\sigma(v)} + \mathfrak{k}x$. Write $v = w + \beta x$ ($w \in V'_{\sigma(v)}$, $0 \neq \beta \in \mathfrak{k}$). Then $w \notin U + X_0$. Since $w \in V'_{\sigma(v)} \smallsetminus V'_{\sigma(w)}$, by (a′) and (b′) we have $V'_{\sigma(w)} \subseteq V'_{\sigma(v)}$. It is clear that $V'_{\sigma(v)} \cap \mathfrak{k}x = \{0\}$. If $v \in V''_{\sigma(w)} = V'_{\sigma(w)} + \mathfrak{k}x$, then by modular law

$$w = v - \beta x \in V'_{\sigma(v)} \cap (V'_{\sigma(w)} + \mathfrak{k}x) = V'_{\sigma(w)} + (V'_{\sigma(v)} \cap \mathfrak{k}x) = V'_{\sigma(w)},$$

a contradiction. Hence we have $v \notin V''_{\sigma(w)}$. Since $v = w + \beta x \in \Lambda'_{\sigma(w)} + \mathfrak{k}x = \Lambda''_{\sigma(w)}$, we obtain $v \in \Lambda''_{\sigma(w)} \setminus V''_{\sigma(w)}$.

In every case we have shown that $v \in \bigcup_{\sigma\in\Sigma}(\Lambda''_\sigma \setminus V''_\sigma)$. Thus we have $V \setminus (U+X) \subseteq \bigcup_{\sigma\in\Sigma}(\Lambda''_\sigma \setminus V''_\sigma)$. The converse inclusion is trivial from (a′). This completes the proof.

We can now prove the third main theorem of this section.

THEOREM 2.9. *Let L be a Lie algebra over* $\mathfrak{k}$.

(1) *If $L \in \hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$, then every finite-dimensional subalgebra of L is serial in L.*

(2) *If $L \in \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$, then every finite-dimensional subalgebra of L is descendant in L.*

PROOF. Let F be a finite-dimensional subalgebra of L. If $L \in \hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$, then by Lemma 2.4 (1) L has a central series $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ of some type Σ. For each $\sigma \in \Sigma$, set $\Lambda'_\sigma = \Lambda_\sigma + F$ and $V'_\sigma = V_\sigma + F$. Then by Lemma 2.8 we have $L \setminus F = \bigcup_{\sigma\in\Sigma}(\Lambda'_\sigma \setminus V'_\sigma)$. For any $\sigma \in \Sigma$, since $\Lambda_\sigma / V_\sigma$ is central, we have $V'_\sigma \lhd \Lambda'_\sigma$. Hence $\{\Lambda'_\sigma, V'_\sigma : \sigma \in \Sigma\}$ is a series from F to L and therefore F is serial in L. Especially, if $L \in \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$ then we may suppose that Σ is a reversely well-ordered set. Thus F is descendant in L.

It has been proved in [2, Theorem 4.6] that $\mathfrak{Gr} \leq \mathrm{L}\mathfrak{N}$, where $\mathfrak{Gr}$ is the class of Gruenberg Lie algebras, that is, $\mathfrak{Gr}$ is the class of Lie algebras in which every 1-dimensional subalgebra is ascendant. Here we analogously define the classes $\hat{\mathfrak{Gr}}$ and $\grave{\mathfrak{Gr}}$ of Lie algebras as follows:

$L \in \hat{\mathfrak{Gr}}$ iff every 1-dimensional subalgebra of L is serial in L;

$L \in \grave{\mathfrak{Gr}}$ iff every 1-dimensional subalgebra of L is descendant in L.

Then by Corollary 2.7 (2) and Theorem 2.9 we have

$$\mathrm{L}\mathfrak{N} \leq \hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} \leq \hat{\mathfrak{Gr}} \quad \text{and} \quad \mathrm{R}\mathfrak{N} \leq \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} \leq \grave{\mathfrak{Gr}} \leq \hat{\mathfrak{Gr}}.$$

It follows that $\grave{\mathfrak{Gr}}$ contains all free Lie algebras. Since every non-abelian free Lie algebra is not locally nilpotent, we have

$$\mathfrak{Gr} \leq \mathrm{L}\mathfrak{N} < \hat{\mathfrak{Gr}} \quad \text{and} \quad \grave{\mathfrak{Gr}} \not\leq \mathfrak{Gr}.$$

Considering the example described in [4, p. 119], we have $\mathfrak{Gr} \not\leq \grave{\mathfrak{Gr}}$. On the other hand, the following result shows that $\hat{\mathfrak{Gr}}$ is a class of generalized nilpotent Lie algebras.

PROPOSITION 2.10. $\mathrm{L}\mathfrak{N}=\mathrm{L}\mathfrak{F}\cap\hat{\mathfrak{G}}\mathfrak{r}=\mathrm{L}\,\mathrm{Min}\cap\hat{\mathfrak{G}}\mathfrak{r}$.

PROOF. By using [3, Proposition 13.2.4], we can easily see that every subalgebra of a locally nilpotent Lie algebra is serial. It follows that $\mathrm{L}\mathfrak{N}\leq\mathrm{L}\mathfrak{F}\cap\hat{\mathfrak{G}}\mathfrak{r}\leq\mathrm{L}\,\mathrm{Min}\cap\hat{\mathfrak{G}}\mathfrak{r}$. Let $L\in\mathrm{L}\,\mathrm{Min}\cap\hat{\mathfrak{G}}\mathfrak{r}$ and let H be a finitely generated subalgebra of L. Then we have $H\in\mathrm{Min}$. Let $x\in H$. Since $L\in\hat{\mathfrak{G}}\mathfrak{r}$, there exists a series $\{\Lambda_\sigma, V_\sigma: \sigma\in\Sigma\}$ from $\langle x\rangle$ to H of some type Σ. We may assume that $V_\sigma\neq\Lambda_\sigma$ for all $\sigma\in\Sigma$. Then $V_\sigma<V_\tau$ iff $\sigma<\tau$. Since every non-empty subset of $\{V_\sigma: \sigma\in\Sigma\}$ has a minimal element, Σ must be a well-ordered set. Thus we have $\langle x\rangle$ asc H, so that $H\in\mathfrak{G}\mathfrak{r}$. Owing to [2, Theorem 4.6], we have $H\in\mathfrak{G}\cap\mathrm{L}\mathfrak{N}=\mathfrak{F}\cap\mathfrak{N}$. Hence $L\in\mathrm{L}\mathfrak{N}$ and therefore $\mathrm{L}\,\mathrm{Min}\cap\mathfrak{G}\mathfrak{r}\leq\mathrm{L}\mathfrak{N}$.

3.

From the definitions clearly we have

$$\mathfrak{N}\leq\mathfrak{N}_{(\infty)}\leq\mathfrak{N}_{(*)}.$$

In this section we shall develop some results analogous to those of [5, §2] by using $\mathfrak{N}_{(*)}$ instead of $\mathfrak{N}_{(\infty)}$.

We begin with the following result corresponding to [5, Lemma 2.1].

PROPOSITION 3.1. (1) $\{\mathrm{s},\mathrm{R}\}\mathfrak{N}_{(*)}=\mathfrak{N}_{(*)}$.

(2) $\mathrm{\acute{E}}\mathfrak{A}<\mathfrak{N}_{(*)}$.

PROOF. (1) By Lemma 1.1 (1) clearly we have $\mathrm{s}\mathfrak{N}_{(*)}=\mathfrak{N}_{(*)}$. Using Lemma 1.1 (2), we can easily show as in the proof of [5, Lemma 2.1] that $\mathrm{R}\mathfrak{N}_{(*)}=\mathfrak{N}_{(*)}$.

(2) If $L\in\mathrm{\acute{E}}\mathfrak{A}$ then $L^{(*)}=\{0\}$. It follows that $\mathrm{\acute{E}}\mathfrak{A}\leq\mathfrak{N}_{(*)}$. We consider the McLain Lie algebra $L=\mathscr{L}_{\mathfrak{k}}(Q)$ over $\mathfrak{k}$. Then $L\in\mathrm{L}\mathfrak{N}\leq\mathfrak{N}\leq\mathfrak{N}_{(*)}$. Since $L^{(1)}=L$, we have $L\notin\mathrm{\acute{E}}\mathfrak{A}$.

We here introduce the class $\mathfrak{M}^{(*)}$ of Lie algebras, naturally generalizing that of quasi-artinian Lie algebras, as follows:

$L\in\mathfrak{M}^{(*)}$ iff for any descending chain $I_1\geq I_2\geq\cdots$ of ideals of L contained in $L^{(*)}$ there exists an integer $n=n(I_1, I_2,\dots)>0$ such that $I_n/\bigcap_{i\geq 1}I_i\leq\zeta_1(L^{(*)}/\bigcap_{i\geq 1}I_i)$.

We present some equivalent conditions for a Lie algebra to be an $\mathfrak{M}^{(*)}$-algebra in the following

LEMMA 3.2. *For a Lie algebra L, the following conditions are equivalent*:

(1) $L\in\mathfrak{M}^{(*)}$.

(2) *For any descending chain* $I_1 \geq I_2 \geq \cdots$ *of ideals of* L *contained in* $L^{(*)}$, *there are integers* $n, r>0$ *such that* $I_n/\bigcap_{i\geq 1} I_i \leq \zeta_r(L^{(*)}/\bigcap_{i\geq 1} I_i)$.

(3) *For any descending chain* $I_1 \geq I_2 \geq \cdots$ *of ideals of* L, *there is an integer* $n>0$ *such that* $[I_n, L^{(*)}] \leq \bigcap_{i\geq 1} I_i$.

(4) *For any descending chain* $I_1 \geq I_2 \geq \cdots$ *of ideals of* L, *there are integers* $n, r>0$ *such that* $[I_{n,r} L^{(*)}] \leq \bigcap_{i\geq 1} I_i$.

PROOF. It is sufficient to show that (2) implies (3). Let $I_1 \geq I_2 \geq \cdots$ be a descending chain of ideals of L. Then $[I_1, L^{(*)}] \geq [I_2, L^{(*)}] \geq \cdots$ is a descending chain of ideals of L contained in $L^{(*)}$. By (2) there are integers $n, r>0$ such that $[I_n, L^{(*)}]/\bigcap_{i\geq 1}[I_i, L^{(*)}] \leq \zeta_r(L^{(*)}/\bigcap_{i\geq 1}[I_i, L^{(*)}])$. Since $L^{(*)}$ is perfect, we have

$$[I_n, L^{(*)}] = [I_{n,r+1} L^{(*)}] \leq \bigcap_{i\geq 1} [I_i, L^{(*)}] \leq \bigcap_{i\geq 1} I_i .$$

Hence (2) implies (3) and therefore the conditions (1)–(4) are equivalent.

It is easy to see that if $L \in \text{qmin-}\lhd$ then $L^{(*)} = L^{(n)}$ for some $n<\omega$. We now denote by $\mathfrak{X}_0$ the class of Lie algebras L such that $L^{(*)} = L^{(n)}$ for some $n<\omega$. Then we have the following result characterizing the classes qmin-$\lhd$ and $\mathfrak{M}^{(*)}$.

PROPOSITION 3.3. (1) $\mathfrak{M}^{(*)} \cap \mathfrak{X}_0 = \text{qmin-}\lhd$.
(2) $\mathfrak{M}^{(*)}\grave{E}\mathfrak{A} = \mathfrak{M}^{(*)}$.

PROOF. (1) By using Lemma 3.2 we have $\mathfrak{M}^{(*)} \cap \mathfrak{X}_0 \leq \text{qmin-}\lhd$. The converse inclusion is evident.

(2) Let $L \in \mathfrak{M}^{(*)}\grave{E}\mathfrak{A}$. Then there exists an ideal I of L such that $I \in \mathfrak{M}^{(*)}$ and $L/I \in \grave{E}\mathfrak{A}$. By Lemma 1.1 we have $L^{(*)} = (L^{(*)})^{(*)} = I^{(*)}$. Let $I_1 \geq I_2 \geq \cdots$ be a descending chain of ideals of L contained in $L^{(*)}$. Since $L^{(*)} = I^{(*)}$ and $I \in \mathfrak{M}^{(*)}$, there exists an integer $n>0$ such that $[I_n, L^{(*)}] = [I_n, I^{(*)}] \leq \bigcap_{i\geq 1} I_i$. Hence we have $L \in \mathfrak{M}^{(*)}$.

It is clear that qmin-$\lhd \cup \grave{E}\mathfrak{A} \leq \mathfrak{M}^{(*)}$. Furthermore, we have

PROPOSITION 3.4. qmin-$\lhd \cup \grave{E}\mathfrak{A} < \mathfrak{M}^{(*)}$.

PROOF. Let S be a 3-dimensional simple Lie algebra over $\mathfrak{k}$ with basis $\{x, y, z\}$ such that

$$[x, y] = z, \quad [y, z] = x, \quad [z, x] = y,$$

and M the McLain Lie algebra $\mathscr{L}_{\mathfrak{k}}(\boldsymbol{Z})$ over $\mathfrak{k}$, where $\boldsymbol{Z}$ is the set of integers with natural ordering. Then M has basis $\{a_{ij}: i, j \in \boldsymbol{Z}, i<j\}$ such that

$$[a_{ij}, a_{kl}] = \delta_{jk}a_{il} - \delta_{li}a_{kj} .$$

Since $M^n = \langle a_{ij}: i, j \in \boldsymbol{Z}, j-i \geq n\rangle$ $(1 \leq n < \omega)$, we have $M^\omega = \{0\}$, so that $M \in$

$\mathrm{R}\mathfrak{N} \leq \mathrm{RE}\mathfrak{A} \leq \grave{\mathrm{E}}\mathfrak{A}$. Define $L = S \oplus M$. Then by Proposition 3.3 (2) we have $L \in \mathfrak{M}^{(*)}$. Since $L^{(*)} = S \neq S \oplus M^{(n)} = L^{(n)}$ $(n < \omega)$, we have $L \notin \mathfrak{X}_0 \cup \grave{\mathrm{E}}\mathfrak{A}$, so that $L \notin$ qmin-$\lhd \cup \grave{\mathrm{E}}\mathfrak{A}$. Therefore we obtain qmin-$\lhd \cup \grave{\mathrm{E}}\mathfrak{A} < \mathfrak{M}^{(*)}$.

The following result, corresponding to [5, Theorem 2.3], is the main theorem of this section.

THEOREM 3.5. $\mathfrak{X} \cap \mathfrak{Y} = \grave{\mathrm{E}}\mathfrak{A}$ *for any class* $\mathfrak{X}$ *of Lie algebras such that* $\grave{\mathrm{E}}\mathfrak{A} \leq \mathfrak{X} \leq \mathfrak{R}_{(*)}$ *and any class* $\mathfrak{Y}$ *of Lie algebras such that* $\grave{\mathrm{E}}\mathfrak{A} \leq \mathfrak{Y} \leq \mathfrak{M}^{(*)}$.

PROOF. It suffices to prove that $\mathfrak{R}_{(*)} \cap \mathfrak{M}^{(*)} \leq \grave{\mathrm{E}}\mathfrak{A}$. Let $L \in \mathfrak{R}_{(*)} \cap \mathfrak{M}^{(*)}$ and assume that $L \notin \grave{\mathrm{E}}\mathfrak{A}$. Set $I = L^{(*)}$. Then $I^{(\alpha)} = I \neq \{0\}$ for all ordinals α. It follows from [3, Lemma 8.1.1] that $\zeta_*(I) < I$. First we show that if $x \in I \setminus \zeta_\alpha(I)$ then $x \notin [x, I]^L + \zeta_\alpha(I)$, by using transfinite induction on α. It is clear for $\alpha = 0$. Let $\alpha > 0$ and suppose that the result is true for all $\beta < \alpha$. Then it is also true for α if α is a limit ordinal. So we consider the case that α is not a limit ordinal. Let $x \in [x, I]^L + \zeta_\alpha(I)$ and write $x = y + z$ $(y \in [x, I]^L,\ z \in \zeta_\alpha(I))$. Then we have $[x, I]^L \leq [y, I]^L + \zeta_{\alpha-1}(I)$. Hence by inductive hypothesis we have $y \in \zeta_{\alpha-1}(I)$, so that $x = y + z \in \zeta_\alpha(I)$. This completes the induction.

Next we construct a sequence $(x_i)_{i=1}^{\infty}$ of elements of $I \setminus \zeta_*(I)$ such that for any $i \geq 1$

$$x_i \notin [x_i, I]^L + \zeta_*(I) \quad \text{and} \quad x_{i+1} \in [x_i, I]^L + \zeta_*(I).$$

There is an $x_1 \in I \setminus \zeta_*(I)$. Then $x_1 \notin [x_1, I]^L + \zeta_*(I)$. Let $i \geq 1$ and suppose that it has been constructed up to x_i. Since $x_i \notin \zeta_*(I)$, there exists an $x_{i+1} \in [x_i, I]^L + \zeta_*(I)$ such that $x_{i+1} \notin \zeta_*(I)$. Then we have $x_{i+1} \notin [x_{i+1}, I]^L + \zeta_*(I)$. Therefore we can inductively show that such a sequence exists actually.

Set $I_i = [x_i, I]^L + \zeta_*(I)$ $(i \geq 1)$. Then $I_1 > I_2 > \cdots$ is a strictly descending chain of ideals of L contained in $L^{(*)}$. Since $L \in \mathfrak{M}^{(*)}$, there exists an integer $n > 0$ such that $[I_n, I] \leq \bigcap_{i \geq 1} I_i$. Since I is perfect, we have

$$[x_n, I]^L \leq [[x_n, I], I]^L \leq [[x_n, I]^L, I] \leq [I_n, I] \leq I_{n+1}.$$

It follows that $I_n \leq I_{n+1}$, a contradiction. Therefore we have $\mathfrak{R}_{(*)} \cap \mathfrak{M}^{(*)} \leq \grave{\mathrm{E}}\mathfrak{A}$.

By making use of Proposition 3.3 (1) and Theorem 3.5, we have

COROLLARY 3.6. $\mathfrak{R}_{(*)} \cap$ qmin-$\lhd = \mathrm{E}\mathfrak{A}$.

It is immediately deduced from Corollary 3.6 that $\mathfrak{R}_{(*)}$ is a class of generalized soluble Lie algebras.

4.

In this section we shall first characterize the classes $\mathfrak{N}^{(1)}$ and $\mathfrak{N}^{(*)}_{(*)}$, and secondly prove that Amayo's result ([2, p. 16]), described in §1, is also true for $\mathfrak{N}^{(*)}_{(*)}$ instead of $\mathfrak{N}^{(1)}$.

We begin with the following

PROPOSITION 4.1. (1) $\{\mathrm{S}, \mathrm{R}\}\mathfrak{N}^{(*)}_{(*)} = \mathfrak{N}^{(*)}_{(*)}$.

(2) $\acute{\mathrm{E}}(\lhd)\mathfrak{A} \le \mathfrak{N}^{(1)} \le \mathfrak{N}^* \le \mathfrak{N}^{(*)} \le \mathfrak{N}^{(*)}_{(*)}$ *and* $\mathfrak{N}_{(*)} \le \mathfrak{N}^{(1)}_{(*)} \le \mathfrak{N}^{(*)}_{(*)}$.

PROOF. (1) is easily proved from Lemma 1.1.

(2) Let $L \in \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ and $x \in L \setminus \{0\}$. L has an ideal series $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ with $\mathfrak{A}$-factors. Then $x \in \Lambda_\sigma \setminus V_\sigma$ for some $\sigma \in \Sigma$. Since $[x, L]^L \le \Lambda_\sigma$, we have $([x, L]^L)^{(1)} \le V_\sigma$, so that $x \notin ([x, L]^L)^{(1)}$. Hence $L \in \mathfrak{N}^{(1)}$ and therefore $\acute{\mathrm{E}}(\lhd)\mathfrak{A} \le \mathfrak{N}^{(1)}$. It is clear that $\mathfrak{N}^{(1)} \le \mathfrak{N}^*$ and $\mathfrak{N}_{(*)} \le \mathfrak{N}^{(1)}_{(*)} \le \mathfrak{N}^{(*)}_{(*)}$. Using Lemma 1.1, we have $\mathfrak{N}^* \le \mathfrak{N}^{(*)} \le \mathfrak{N}^{(*)}_{(*)}$.

REMARK. We shall prove in Theorem 4.3 below that $\mathfrak{N}^{(1)} = \mathfrak{N}^* = \mathfrak{N}^{(*)}$ and $\mathfrak{N}^{(1)}_{(*)} = \mathfrak{N}^{(*)}_{(*)}$. On the other hand, it has been indicated in [2, p. 16] that the class $\mathfrak{N}^{(1)}$ is $\{\mathrm{S}, \mathrm{R}, \mathrm{L}\}$-closed. It follows that the classes $\mathfrak{N}^*$ and $\mathfrak{N}^{(*)}$ are $\{\mathrm{S}, \mathrm{R}, \mathrm{L}\}$-closed.

Before showing the first main theorem of this section, we need

LEMMA 4.2. *Let* $x \in L$ *and* $X \subseteq L$. *Then the following conditions are equivalent*:

(1) $x \notin ([x, X]^L)^{(1)}$.

(2) $x \notin ([x, X]^L)^*$.

(3) $x \notin ([x, X]^L)^{(*)}$.

PROOF. (1)$\Rightarrow$(2)$\Rightarrow$(3) is clear from Lemma 1.1 (4). So we show that (3) implies (1). Set $I = [x, X]^L$ and assume that $x \in I^{(1)}$. Since $I^{(1)} \lhd L$, we have $\langle x \rangle^L \le I^{(1)}$. Obviously $I = [x, X]^L \le [\langle x \rangle^L, X]^L \le \langle x \rangle^L$. It follows that $\langle x \rangle^L = I = I^{(1)}$. Hence we have $x \in I^{(*)}$. Therefore (3) implies (1).

We now have the first main theorem of this section, which characterizes the classes $\mathfrak{N}^{(1)}$ and $\mathfrak{N}^{(*)}_{(*)}$.

THEOREM 4.3. (1) *The following classes coincide with each other*:

$$\mathfrak{N}^{(1)},\ \mathfrak{N}^*,\ \mathfrak{N}^{(*)},\ (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{N}^{(1)},\ (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{N}^*,\ (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{N}^{(*)}.$$

(2) *The following classes coincide with each other*:

$$\mathfrak{N}^{(1)}_{(*)},\ \mathfrak{N}^{(*)}_{(*)},\ (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{N}^{(1)}_{(*)},\ (\grave{\mathrm{E}}\mathfrak{A})\mathfrak{N}^{(*)}_{(*)}.$$

PROOF. We here only prove (1), since (2) is proved similarly. By using Lemma 4.2, we can easily see that $\mathfrak{R}^{(1)}=\mathfrak{R}^{*}=\mathfrak{R}^{(*)}$. Let $L\in(\acute{\mathrm{E}}\mathfrak{A})\mathfrak{R}^{(*)}$ and $x\in L\setminus\{0\}$. There exists an ideal I of L such that $I\in\acute{\mathrm{E}}\mathfrak{A}$ and $L/I\in\mathfrak{R}^{(*)}$. If $x\notin I$ then by Lemma 1.1 (2) $x+I\notin(([x, L]^{L})^{(*)}+I)/I$ and so $x\notin([x, L]^{L})^{(*)}$. If $x\in I$ then by Lemma 1.1 (1) $([x, L]^{L})^{(*)}\leq I^{(*)}=\{0\}$ and so $x\notin([x, L]^{L})^{(*)}$. Hence $L\in\mathfrak{R}^{(*)}$ and therefore $(\acute{\mathrm{E}}\mathfrak{A})\mathfrak{R}^{(*)}=\mathfrak{R}^{(*)}$. This completes the proof.

In [2, p. 16] Amayo has indicated without proof that if M is a minimal ideal of an $\mathfrak{R}^{(1)}$-algebra L then $M\in\mathfrak{A}$ and $L/M\in\mathfrak{R}^{(1)}$, and that $\mathfrak{R}^{(1)}\cap\mathrm{Min}\text{-}\lhd\leq\acute{\mathrm{E}}(\lhd)\mathfrak{A}$. We shall next show that these results also hold for $\mathfrak{R}^{(*)}_{(*)}$ instead of $\mathfrak{R}^{(1)}$. To do this we need

LEMMA 4.4. *If M is a minimal ideal of a Lie algebra L, then $(L/M)^{(*)}=(L^{(*)}+M)/M$.*

PROOF. We can find a sufficiently large ordinal σ such that $(L/M)^{(*)}=(L/M)^{(\sigma)}$ and $M^{(*)}=M^{(\sigma)}$. First we consider the case that $M\leq L^{(\alpha)}$ for all $\alpha\leq\sigma$. By transfinite induction on α we can easily see that $(L/M)^{(\alpha)}=L^{(\alpha)}/M$ for all $\alpha\leq\sigma$. It follows that $(L/M)^{(*)}=L^{(*)}/M$. Next we consider the case that $M\not\leq L^{(\alpha)}$ for some $\alpha\leq\sigma$. Then there exists the least ordinal $\mu\leq\sigma$ with respect to $M\not\leq L^{(\mu)}$. Clearly μ is non-zero and is not a limit ordinal. Since $M\leq L^{(\alpha)}$ for all $\alpha\leq\mu-1$, we have $(L/M)^{(\mu-1)}=L^{(\mu-1)}/M$, so that $(L/M)^{(\mu)}=(L^{(\mu)}+M)/M$. By the minimality of M we have $L^{(\mu)}\cap M=\{0\}$. Using Lemma 1.1, we have

$$(L/M)^{(*)} = ((L/M)^{(\mu)})^{(*)} = ((L^{(\mu)}+M)/M)^{(*)} = ((L^{(\mu)})^{(*)}+M)/M = (L^{(*)}+M)/M.$$

PROPOSITION 4.5. *Let $L\in\mathfrak{R}^{(*)}_{(*)}$. If M is a minimal ideal of L, then $M\in\mathfrak{A}$ and $L/M\in\mathfrak{R}^{(*)}_{(*)}$.*

PROOF. By Theorem 4.3 (2) we may prove the proposition for $\mathfrak{R}^{(1)}_{(*)}$ instead of $\mathfrak{R}^{(*)}_{(*)}$. Assume that $M\notin\mathfrak{A}$. Then there exists an $a\in M\setminus\zeta_1(M)$. Since $L\in\mathfrak{R}^{(1)}_{(*)}$, we have $a\notin([a, L^{(*)}]^{L})^{(1)}$. By the minimality of M we see that $[a, L^{(*)}]^{L}=\{0\}$ or $[a, L^{(*)}]^{L}=M$, and that M is perfect. By Lemma 1.1 (1) $M=M^{(*)}\leq L^{(*)}$. If $[a, L^{(*)}]^{L}=\{0\}$, then $[a, M]\subseteq[a, L^{(*)}]=\{0\}$ and so $a\in\zeta_1(M)$, a contradiction. If $[a, L^{(*)}]^{L}=M$, then $a\notin M^{(1)}=M$, a contradiction. Therefore we have $M\in\mathfrak{A}$.

Now we show that $L/M\in\mathfrak{R}^{(1)}_{(*)}$. Let $x\in L\setminus M$ and set $I=[x, L^{(*)}]^{L}$. By using Lemma 4.4 we have

$$([x+M, (L/M)^{(*)}]^{L/M})^{(1)} = (I^{(1)}+M)/M.$$

Assume that $x\in I^{(1)}+M$ and write $x=y+z$ ($y\in I^{(1)}$, $z\in M$). By the minimality of M we have $[I^{(1)}, M]=\{0\}$ or $[I^{(1)}, M]=M$. First we consider the case that $[I^{(1)}, M]=\{0\}$. Set $Y=[y, L^{(*)}]^{L}$ and $Z=[z, L^{(*)}]^{L}$. Then $I\leq Y+Z$, $Y\leq I^{(1)}$ and $Z\leq M$. Since $[I^{(1)}, M]=\{0\}$ and $M\in\mathfrak{A}$, we have $I^{(1)}\leq(Y+Z)^{(1)}=Y^{(1)}$, so that $y\in Y^{(1)}=([y, L^{(*)}]^L)^{(1)}$. Hence $y=0$ and therefore $x=z\in M$, a contradiction. Next we consider the case that $[I^{(1)}, M]=M$. Since $M\leq I^{(1)}$, we have $x\in I^{(1)}=([x, L^{(*)}]^L)^{(1)}$. Hence $x=0\in M$, a contradiction. Therefore we have $x\notin I^{(1)}+M$, so that $x+M\notin([x+M, (L/M)^{(*)}]^{L/M})^{(1)}$. Thus we obtain $L/M\in\mathfrak{R}_{(*)}^{(1)}$.

We now set about showing the second main theorem of this section.

THEOREM 4.6. $\mathfrak{R}_{(*)}^{(*)}\cap\text{Min-}\lhd=\acute{\text{E}}(\lhd)\mathfrak{A}\cap\text{Min-}\lhd$.

PROOF. By Proposition 4.1 (2) and Theorem 4.3 (2) it suffices to prove that $\mathfrak{R}_{(*)}^{(1)}\cap\text{Min-}\lhd\leq\acute{\text{E}}(\lhd)\mathfrak{A}$. Let $L\in\mathfrak{R}_{(*)}^{(1)}\cap\text{Min-}\lhd$. We shall construct a strictly ascending series $\{L_\alpha: \alpha\geq 0\}$ of ideals of L such that $L_{\alpha+1}/L_\alpha\in\mathfrak{A}$ and $L/L_\alpha\in\mathfrak{R}_{(*)}^{(1)}$ for all $\alpha\geq 0$. Define $L_0=\{0\}$. Let $\alpha>0$ and assume that $\{L_\beta: \beta<\alpha\}$ has been constructed. First we consider the case that α is not a limit ordinal. If $L_{\alpha-1}=L$ then $L\in\acute{\text{E}}(\lhd)\mathfrak{A}$. If $L_{\alpha-1}\neq L$, then $\{0\}\neq L/L_{\alpha-1}\in\mathfrak{R}_{(*)}^{(1)}\cap\text{Min-}\lhd$. Let $L_\alpha/L_{\alpha-1}$ be a minimal ideal of $L/L_{\alpha-1}$. Then by Theorem 4.3 (2) and Proposition 4.5 we have $L_\alpha/L_{\alpha-1}\in\mathfrak{A}$ and $L/L_\alpha\in\mathfrak{R}_{(*)}^{(1)}$. Next we consider the case that α is a limit ordinal. Define $L_\alpha=\bigcup_{\beta<\alpha}L_\beta$. Let $x\in L$ and suppose that $x+L_\alpha\in([x+L_\alpha, (L/L_\alpha)^{(*)}]^{L/L_\alpha})^{(1)}$. Since $L\in\text{Min-}\lhd$, it is not hard to see that $(L/I)^{(*)}=(L^{(*)}+I)/I$ for any $I\lhd L$. Hence we have $x\in([x, L^{(*)}]^L)^{(1)}+L_\alpha$. It follows that $x\in([x, L^{(*)}]^L)^{(1)}+L_\beta$ for some $\beta<\alpha$. Then we have $x+L_\beta\in([x+L_\beta, (L/L_\beta)^{(*)}]^{L/L_\beta})^{(1)}$. Since $L/L_\beta\in\mathfrak{R}_{(*)}^{(1)}$, we have $x\in L_\beta\leq L_\alpha$. Therefore $L/L_\alpha\in\mathfrak{R}_{(*)}^{(1)}$. Thus we can inductively construct such a series. By set-theoretic consideration we see that $L=L_\sigma$ for some ordinal σ. Therefore we have $L\in\acute{\text{E}}(\lhd)\mathfrak{A}$. This completes the proof.

COROLLARY 4.7. (1) $\mathfrak{R}_{(*)}^{(*)}\cap\text{Min-}\lhd\cap\text{Max-}\lhd\leq\text{E}\mathfrak{A}$. *In particular,* $\mathfrak{R}_{(*)}^{(*)}$ *is a class of generalized soluble Lie algebras.*

(2) *If* $\mathfrak{k}$ *has non-zero characteristic, then* $\mathfrak{R}_{(*)}^{(*)}\cap\text{Min-}\lhd\cap\text{Max-}\lhd=\text{E}\mathfrak{A}\cap\mathfrak{F}$.

PROOF. (1) is directly deduced from Theorem 4.6.

(2) Since $\mathfrak{k}$ has non-zero characteristic, owing to [3, Corollary 11.2.3] we have $\acute{\text{E}}(\lhd)\mathfrak{A}\cap\text{Min-}\lhd\cap\text{Max-}\lhd=\text{E}\mathfrak{A}\cap\mathfrak{F}$. Therefore the result follows from Theorem 4.6.

REMARK. If $\mathfrak{k}$ has zero characteristic, then $\mathfrak{R}_{(*)}^{(*)}\cap\text{Min-}\lhd\cap\text{Max-}\lhd>\text{E}\mathfrak{A}\cap\mathfrak{F}$. In fact, let L be the Hartley algebra (cf. [3, Example 6.3.6]) over $\mathfrak{k}$. Then it is well known that $L\in\text{E}\mathfrak{A}\cap\text{Min-}\lhd\cap\text{Max-}\lhd$ and $L\notin\mathfrak{F}$.

5.

In this section we shall investigate the classes $\mathfrak{R}_{(1)}$ and $\mathfrak{R}_*$. Concerning them the following proposition is elementary.

Proposition 5.1. (1) $\{\mathrm{s}, \mathrm{R}, \mathrm{L}\}\mathfrak{N}_{(1)}=\mathfrak{N}_{(1)}$ *and* $\{\mathrm{s}, \mathrm{R}\}\mathfrak{N}_{*}=\mathfrak{N}_{*}$.
(2) $\mathfrak{N}\leq\mathfrak{N}_{(1)}\leq\mathfrak{N}_{*}\leq\mathfrak{N}_{(*)}$.

Proof. (1) Obviously $\{\mathrm{s}, \mathrm{R}\}\mathfrak{N}_{(1)}=\mathfrak{N}_{(1)}$. Let $L\in \mathrm{L}\mathfrak{N}_{(1)}$ and assume that $L\notin \mathfrak{N}_{(1)}$. Then there exists an $x\in L\setminus\{0\}$ such that $x\in[x, L^{(1)}]^{L}$. We can find a finite subset X of L such that $x\in[x, \langle X\rangle^{(1)}]^{\langle X\rangle}$. Set $H=\langle x, X\rangle$. Then $H\in\mathfrak{N}_{(1)}$ and $x\in[x, H^{(1)}]^{H}$. Hence $x=0$, a contradiction. Therefore we have $\mathrm{L}\mathfrak{N}_{(1)}=\mathfrak{N}_{(1)}$. By using Lemma 1.1, easily we have $\{\mathrm{s}, \mathrm{R}\}\mathfrak{N}_{*}=\mathfrak{N}_{*}$.

(2) $\mathfrak{N}\leq\mathfrak{N}_{(1)}\leq\mathfrak{N}_{*}$ is trivial. It follows from Lemma 1.1 (4) that $\mathfrak{N}_{*}\leq\mathfrak{N}_{(*)}$.

Next we prove that $\mathfrak{N}_{(1)}$ is a subclass of the class $\mathfrak{N}^{(1)}$. To do this we present a sufficient condition for a Lie algebra to be contained in the class $\mathfrak{N}^{(1)}$ in the following

Theorem 5.2. *Let L be a Lie algebra over $\mathfrak{k}$. If $x\in L\setminus\{0\}$ implies $x\notin \bigcap_{n<\omega}[x, L^{n+1}]^{L}$, then $L\in\mathfrak{N}^{(1)}$. In particular, $\mathfrak{N}_{(1)}\leq\mathfrak{N}^{(1)}$.*

Proof. It suffices to prove the first half of the theorem, since the latter half is immediately deduced from the first half. Let $x\in L\setminus\{0\}$. By using induction on n we first show that for any $n<\omega$

$$([x, L]^{L})^{(n)} \subseteq [x, L^{n+1}]^{L}.$$

It is clear for $n=0$. Let $n>0$ and assume that the result is true for $n-1$. Then

$$([x, L]^{L})^{(n)} \subseteq ([x, L^{n}]^{L})^{(1)} \subseteq \textstyle\sum_{k<\omega}[[x, L^{n}, {}_{k} L], L^{n+1}].$$

Set $I_{k}=[[x, L^{n}, {}_{k} L], L^{n+1}]$ $(k<\omega)$. Clearly $I_{0}\subseteq[x, L^{n+1}]^{L}$. If $I_{k}\subseteq[x, L^{n+1}]^{L}$, then by the Jacobi identity

$$I_{k+1} \subseteq [I_{k}, L] + [[x, L^{n}, {}_{k} L], L^{n+1}] \subseteq [x, L^{n+1}]^{L}.$$

Hence by the second induction on k we have $I_{k}\subseteq[x, L^{n+1}]^{L}$ $(k<\omega)$. Thus

$$([x, L]^{L})^{(n)} \subseteq \textstyle\sum_{k<\omega} I_{k} \subseteq [x, L^{n+1}]^{L}.$$

This completes the first induction. Since $x\notin \bigcap_{n<\omega}[x, L^{n+1}]^{L}$, there exists an $n=n(x)<\omega$ such that $x\notin[x, L^{n+1}]^{L}$. Then we have $x\notin([x, L]^{L})^{(n)}$, so that $x\notin([x, L]^{L})^{(*)}$. It follows from Lemma 4.2 that $x\notin([x, L]^{L})^{(1)}$. Therefore we have $L\in\mathfrak{N}^{(1)}$.

In Proposition 5.1 (2) we have given relationships among four classes. Among them $\mathfrak{N}$ and $\mathfrak{N}_{(*)}$ are respectively a class of generalized nilpotent Lie algebras and a class of generalized soluble Lie algebras. Concerning $\mathfrak{N}_{(1)}$ and $\mathfrak{N}_{*}$

among them we next consider whether a similar fact will be shown or not. In [2, Theorem 3.5] (or [9, Corollary to Theorem 3.3]) it has been proved that

$$\mathfrak{N} \cap \text{Min-}\lhd \leq \mathfrak{Z} \cap \text{E}\mathfrak{A}.$$

If this holds for the class $\mathfrak{N}_{(1)}$ instead of the class $\mathfrak{N}$, then $\mathfrak{N}_{(1)}$ will be a class of generalized nilpotent Lie algebras. By Corollary 3.6 and Proposition 5.1 (2) we have

$$\mathfrak{N}_{(1)} \cap \text{Min-}\lhd \leq \mathfrak{N}_{*} \cap \text{Min-}\lhd \leq \text{E}\mathfrak{A}.$$

However, the following proposition shows that

$$\mathfrak{N}_{(1)} \cap \text{Min-}\lhd \not\leq \mathfrak{Z}.$$

PROPOSITION 5.3. $\mathfrak{N}_{(1)} \cap \mathfrak{F} \not\leq \mathfrak{N}$.

PROOF. Let L be a 2-dimensional non-abelian Lie algebra over $\mathfrak{k}$. Then it is well known that L has basis $\{x, y\}$ such that $[x, y]=x$. We claim that $L \in \mathfrak{N}_{(1)}$. Assume, to the contrary, that there exists a $z \in L \setminus \{0\}$ such that $z \in [z, L^{(1)}]^L$. Clearly $\{I : I \lhd L\} = \{\{0\}, L^{(1)} = \langle x \rangle, L\}$. Hence $z \in [z, L^{(1)}]^L = \langle x \rangle$ and therefore $z \in [\langle x \rangle, \langle x \rangle]^L = \{0\}$, a contradiction. Thus we obtain $L \in \mathfrak{N}_{(1)}$. Since $L \in \mathfrak{F} \setminus \mathfrak{N}$, we have $\mathfrak{N}_{(1)} \cap \mathfrak{F} \not\leq \mathfrak{N}$.

From this proposition both of the classes $\mathfrak{N}_{(1)}$ and $\mathfrak{N}_{*}$ are not classes of generalized nilpotent Lie algebras. Therefore we have

$$\mathfrak{N} < \mathfrak{N}_{(1)}.$$

On the other hand, by the following proposition we can see that both of the classes $\mathfrak{N}_{(1)}$ and $\mathfrak{N}_{*}$ are not necessarily classes of generalized soluble Lie algebras.

PROPOSITION 5.4. *If $\mathfrak{k}$ has non-zero characteristic, then* $\mathfrak{A}^3 \cap \mathfrak{F} \not\leq \mathfrak{N}_{*}$.

PROOF. Let $\mathfrak{k}$ have characteristic $p>0$ and let A be an abelian Lie algebra over $\mathfrak{k}$ with basis $\{a_0, a_1, \dots, a_{p-1}\}$. Define $x, y \in \text{Der}(A)$ as follows:

$$a_0 x = a_{p-1}, \quad a_i x = a_{i-1} \quad (1 \leq i \leq p-1);$$

$$a_i y = -i a_i \quad (0 \leq i \leq p-1).$$

Set $M = \langle x, y \rangle \leq \text{Der}(A)$. From the definitions we have $[x, y]=x$. Form the split extension $L = A \dotplus M$ of A by M. Then $L \in \mathfrak{A}^3 \cap \mathfrak{F}$. It is easy to see that $L^* = L^2 = A + \langle x \rangle$. Therefore we have $a_0 = [a_0, {}_p x] \in [a_0, L^*]^L$, so that $L \notin \mathfrak{N}_{*}$.

By this proposition we see that if $\mathfrak{k}$ has non-zero characteristic then

$$\mathfrak{N}_{(1)} < \mathfrak{N}^{(1)} \quad \text{and} \quad \mathfrak{N}_* < \mathfrak{N}_{(*)}.$$

Finally we shall present interesting subclasses of the classes $\mathfrak{N}_{(1)}$ and $\mathfrak{N}_*$. To do this we denote by $\hat{\mathrm{E}}(\mathrm{ch})\hat{\mathfrak{A}}$ the class of Lie algebras which have series, consisting of characteristic ideals, with $\mathfrak{A}$-marginal factors. It is clear that $L \in \hat{\mathrm{E}}(\mathrm{ch})\hat{\mathfrak{A}}$ iff L has a central series consisting of characteristic ideals. Then by Lemma 2.4 we have

$$\acute{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} \cup \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}} \leq \hat{\mathrm{E}}(\mathrm{ch})\hat{\mathfrak{A}} \leq \hat{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}.$$

Moreover, we have

PROPOSITION 5.5. (1) $(\hat{\mathrm{E}}(\mathrm{ch})\hat{\mathfrak{A}})\mathfrak{A} \leq \mathfrak{N}_{(1)}$.
(2) $(\hat{\mathrm{E}}(\mathrm{ch})\hat{\mathfrak{A}})(\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}) \leq \mathfrak{N}_*$.

PROOF. We here only prove (2), since (1) is proved similarly. Let $L \in (\hat{\mathrm{E}}(\mathrm{ch})\hat{\mathfrak{A}})(\grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}})$. Then there exists an ideal I of L such that $I \in \hat{\mathrm{E}}(\mathrm{ch})\hat{\mathfrak{A}}$ and $L/I \in \grave{\mathrm{E}}(\lhd)\hat{\mathfrak{A}}$. I has a central series $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ consisting of ideals of L. By Lemmas 1.1 (2) and 2.4 (3) we have $L^* \leq I$. Let $x \in L \setminus \{0\}$ and assume that $x \in [x, L^*]^L$. Since $x \in I \setminus \{0\}$, $x \in \Lambda_\sigma \setminus V_\sigma$ for some $\sigma \in \Sigma$. Then we have $x \in [x, L^*]^L \leq [\Lambda_\sigma, I]^L \leq V_\sigma$, a contradiction. Thus we have $L \in \mathfrak{N}_*$.

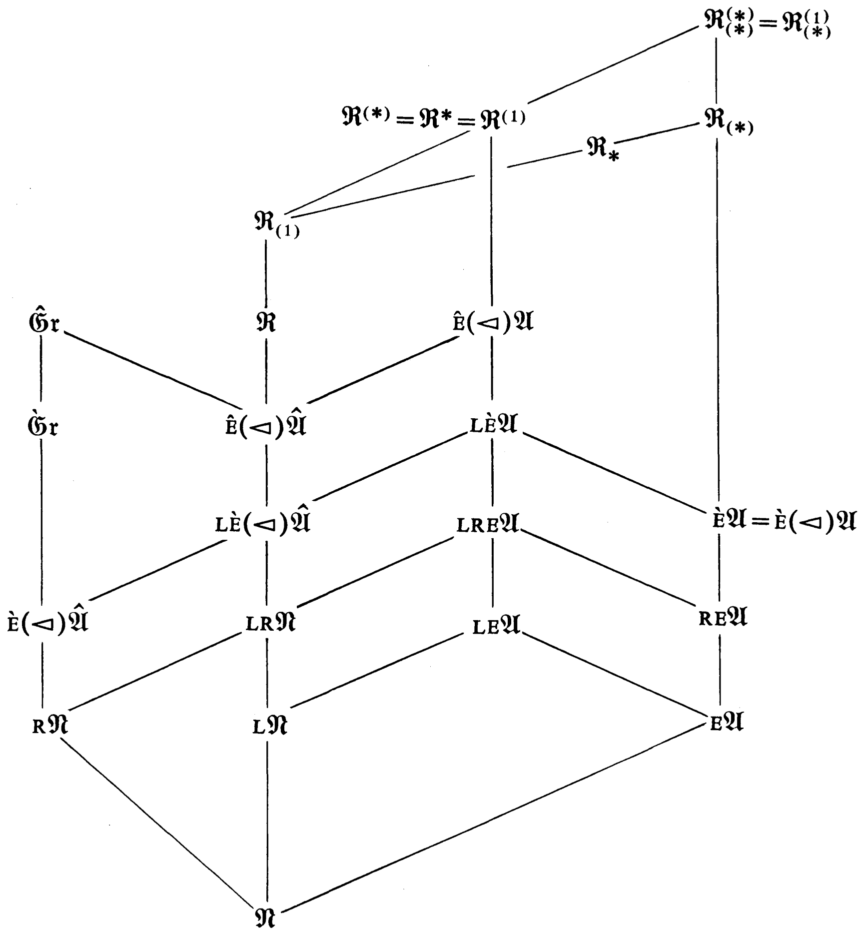
By Lemma 2.4 and Proposition 5.5 we see that the class $\mathfrak{N}_{(1)}$ contains all hypercentral-by-abelian and all hypocentral-by-abelian Lie algebras, and that the class $\mathfrak{N}_*$ contains all hypercentral-by-hypocentral and all hypocentral-by-hypocentral Lie algebras. It has been proved in [4, Corollary 3.7] that if $\mathfrak{k}$ has zero characteristic then $\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \leq \mathfrak{Z}\mathfrak{A}$. Therefore we obtain

COROLLARY 5.6. *If $\mathfrak{k}$ has zero characteristic, then* $\acute{\mathrm{E}}(\lhd)(\mathfrak{A} \cap \mathfrak{F}) \leq \mathfrak{N}_{(1)}$.

REMARK. In contrast with Proposition 5.4, it is directly deduced from Corollary 5.6 that if $\mathfrak{k}$ has zero characteristic then $\mathrm{E}\mathfrak{A} \cap \mathfrak{F} \leq \mathfrak{N}_{(1)}$.

6.

By the lattice diagram of the following figure, we illustrate the known inclusions between well-known classes and the various classes we have defined in this paper.



In this figure, every class including $\mathrm{E}\mathfrak{A}$ is a class of generalized soluble Lie algebras, and every class included in $\mathfrak{R}$ or $\hat{\mathfrak{G}}\mathfrak{r}$ is a class of generalized nilpotent Lie algebras.

References

[1] F. A. M. Aldosray: On Lie algebras with finiteness conditions, Hiroshima Math. J. **13** (1983), 665–674.

[2] R. K. Amayo: Ascendant subalgebras of Lie algebras, preprint, Universität Bonn, 1975.

[3] R. K. Amayo and I. Stewart: Infinite-dimensional Lie Algebras, Noordhoff, Leyden, 1974.

[4] T. Ikeda: Hyperabelian Lie algebras, Hiroshima Math. J. **15** (1985), 601–617.

[5] F. Kubo and M. Honda: Quasi-artinian Lie algebras, Hiroshima Math. J. **14** (1984), 563–570.

[6] A. I. Mal'cev: On a general method for obtaining local theorems in group theory, Ivanov. Gos. Ped. Inst. Učen. Zap. **1** (1941), 3–9.
[7] D. J. S. Robinson: Finiteness Conditions and Generalized Soluble Groups I, Springer, Berlin, 1972.
[8] D. J. S. Robinson: Finiteness Conditions and Generalized Soluble Groups II, Springer, Berlin, 1972.
[9] G. E. Stevens: Topics in the theory of infinite dimensional Lie algebras, Ph. D. thesis, Univ. of Michigan, 1974.
[10] I. Stewart: Structure theorems for a class of locally finite Lie algebras, Proc. London Math. Soc. (3) **24** (1972), 79–100.

Niigata College of Pharmacy