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Removable sets for solutions of the Euler equations

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Introduction

Ahlfors and Beurling [1] gave a characterization of the removable singularities for the class of analytic functions with finite Dirichlet integral, in terms of extremal distances on the complex plane. This result was generalized and extended to the d-dimensional euclidean space R^d $(d \geqq 3)$ by many authors (see [3], [6], etc.).

Hedberg [3] gave some characterizations of removable sets for the class HD^p of all harmonic functions u with $\int |\nabla u|^p dx < \infty$ and for the subclass FD^p of HD^p consisting of functions with no flux. In [6] the author considered the notion of null sets for extremal distances of order p, namely, NED_p-sets, and characterized such null sets by the removability for a class of solutions of the Euler equation for the variational integral $\int |\nabla u|^p dx$.

In this paper we shall consider some classes consisting of solutions of the Euler equation for the variational integral $\int \psi(x, \nabla u)dx$, where $\psi(x, \tau)$: $R^d \times R^d \to R$ is strictly convex and continuously differentiable in τ and $\psi(x, \tau) \approx |\tau|^p$, and define the removable sets for these classes. More precisely, for any bounded domain G containing a compact set E, we shall consider the class $\mathscr{H}\mathscr{D}_\psi^p(G-E)$ (resp. $\mathscr{K}\mathscr{D}_\psi^p(G-E)$; $\widetilde{\mathscr{H}\mathscr{D}}_\psi^p(G-E)$) of all p-precise functions u (for p-precise functions, see [4, Chapter IV], [8]) such that

$$\int_{G-E} \langle \nabla_\tau \psi(x, \nabla u), \nabla \phi \rangle dx = 0$$

for every ϕ in $C_0^\infty(G-E)$ (resp. in $\{\phi \in C_0^\infty(G);\ \nabla\phi = 0$ on some neighborhood of $E\}$; in $C_0^\infty(G)$). A compact set E is said to be removable for $\mathscr{H}\mathscr{D}_\psi^p$ (resp. $\mathscr{K}\mathscr{D}_\psi^p$; $\widetilde{\mathscr{H}\mathscr{D}}_\psi^p$) if for some bounded domain G containing E every function in $\mathscr{H}\mathscr{D}_\psi^p(G-E)$ (resp. $\mathscr{K}\mathscr{D}_\psi^p(G-E)$; $\widetilde{\mathscr{H}\mathscr{D}}_\psi^p(G-E)$) can be extended to a function in $\mathscr{H}\mathscr{D}_\psi^p(G)$.

We shall see that E is removable for $\widetilde{\mathscr{H}\mathscr{D}}_\psi^p$ if and only if E is an NED_p-set (Theorem 1). This result is an improvement of [6, Theorem 2]. We shall show that E is removable for $\mathscr{H}\mathscr{D}_\psi^p$ if and only if E is removable for $HD^{p/(p-1)}$ (Theorem 2) and that E is removable for $\mathscr{K}\mathscr{D}_\psi^p$ if and only if E is removable for $FD^{p/(p-1)}$ in case $p \geqq 2$ (Theorem 3). The proofs of these theorems are based on the results obtained by Hedberg [3]. In the case that $\psi(x, \tau) = |\tau|^p$ for all

$(x, \tau)\in R^d\times R^d$, Theorem 3 is shown in [5].

§1. Preliminaries

Let p be a finite number such that $p>1$ and let G be a domain in R^d. For $f\in L^p(G)$, let $\|f\|_{p,G}$ be its L^p-norm, and for a vector field $v=(v_1, v_2,\dots, v_d)$ on G we define $\|v\|_{p,G}$ by $\||v|\|_{p,G}$. We denote by $C_0^\infty(G)$ the family of infinitely differentiable functions with compact support in G.

Let Γ be a family of curves in R^d. A non-negative Borel measurable function f is called admissible in association with Γ if $\int_\gamma f ds\geqq 1$ for each $\gamma\in\Gamma$, where ds is the line element. The p-module $M_p(\Gamma)$ is defined by $\inf_f\int f^p dx$, where the infimum is taken over all functions f admissible in association with Γ and dx is the volume element. A property will be said to hold p-almost everywhere ($=p$-a.e.) on Γ if the p-module of the subfamily of exceptional curves is zero. For a domain G and a compact subset $K\subset\partial G$, we denote by $\Gamma_G(K)$ the family of all curves in G each of which starts from a point of G and tends to K.

A real valued function u defined in a domain G is called a p-precise function, if it is absolutely continuous along p-a.e. curve in G and $|\nabla u|$ belongs to $L^p(G)$. We denote by $\mathscr{P}_p(G)$ the class of all p-precise functions on G. Every p-precise function u on G has a finite curvilinear limit $u(\gamma)$ along p-a.e. curve γ in G (see [4, Theorem 5.4]). The following results are known:

(1.1) Let u be a p-precise function on G such that $u(\gamma)=0$ for p-a.e. $\gamma\in\Gamma_G(\partial G)$. Then there is a sequence $\{\phi_n\}$ in $C_0^\infty(G)$ such that $\|\nabla(u-\phi_n)\|_{p,G}\to 0$ as $n\to\infty$ (see [4, Theorem 6.16]).

(1.2) Let Γ be a family of curves in G. Let $u_0, u_1, u_2,\dots$ be p-precise functions on G such that $u_n(\gamma)=$const. for p-a.e. $\gamma\in\Gamma$ for each $n\geqq 1$ and $\|\nabla(u_0-u_n)\|_{p,G}\to 0$ as $n\to\infty$. Then $u_0(\gamma)=$const. for p-a.e. $\gamma\in\Gamma$ (see [5, Lemma 1]).

Let ψ: $R^d\times R^d\to R$ satisfy the following (a)–(c):

(a) ψ is continuous.

(b) For each $x\in R^d$ the function $\tau\mapsto\psi(x, \tau)$ is strictly convex and continuously differentiable.

(c) There are constants $0<\alpha\leqq\beta<\infty$ such that

$$\alpha|\tau|^p\leqq\psi(x, \tau)\leqq\beta|\tau|^p$$

for all $(x, \tau)\in R^d\times R^d$.

By (b) the gradient $\nabla_\tau\psi(x, \tau)$ of ψ with respect to τ exists. The following inequalities are known:

$$(1.3)\qquad \langle\nabla_\tau\psi(x, \tau), \tau_1-\tau\rangle\leqq\psi(x, \tau_1)-\psi(x, \tau) \text{ for all } \tau, \tau_1\in R^d,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in R^d. In particular

(1.4) $\alpha|\tau|^p \leqq \langle \nabla_\tau \psi(x, \tau), \tau \rangle$,

(1.5) $|\nabla_\tau \psi(x, \tau)| \leqq c|\tau|^{p-1}$ for all $(x, \tau) \in R^d \times R^d$, where c is a constant depending only on d, p and β (see [2, Lemma 3.5]).

§2. Removable sets

Let E be a compact set in R^d and G be a bounded domain containing E. Throughout this paper we shall always assume that E is a compact set such that $m_d(E)=0$ (m_d denotes the d-dimensional Lebesgue measure) and E^c is a domain. Let

$$C_1^\infty(G; E) = \{\phi \in C_0^\infty(G);\ \nabla\phi = 0 \text{ on some neighborhood of } E\}.$$

We denote by $\mathscr{H}\mathscr{D}_\psi^p(G-E)$ (resp. $\mathscr{H}\mathscr{D}_\psi^p(G-E)$; $\widetilde{\mathscr{H}\mathscr{D}_\psi^p}(G-E)$) the class of all $u \in \mathscr{P}_p(G-E)$ satisfying the condition that

$$\int_{G-E} \langle \nabla_\tau \psi(x, \nabla u), \nabla\phi \rangle dx = 0$$

for every ϕ in $C_0^\infty(G-E)$ (resp. $C_1^\infty(G; E)$; $C_0^\infty(G)$). From the relations $C_0^\infty(G-E) \subset C_1^\infty(G; E) \subset C_0^\infty(G)$, it follows that every removable set for $\mathscr{H}\mathscr{D}_\psi^p$ is removable for $\mathscr{H}\mathscr{D}_\psi^p$ and every removable set for $\mathscr{H}\mathscr{D}_\psi^p$ is removable for $\widetilde{\mathscr{H}\mathscr{D}_\psi^p}$.

Let $\Omega = G-E$, and let $X(\Omega)$ be a linear subspace of $\{\phi|_\Omega;\ \phi \in C_0^\infty(G)\}$, where $\phi|_\Omega$ is the restriction of ϕ to Ω. Let

$$\bar{X}_p(\Omega) = \left\{ u \in \mathscr{P}_p(\Omega);\ \begin{array}{c} u(\gamma)=0 \text{ for } p\text{-a.e. } \gamma \in \Gamma_\Omega(\partial G), \\ \text{there is a sequence } \{\phi_n\} \text{ in } X(\Omega) \text{ such that} \\ \|\nabla(u-\phi_n)\|_{p,\Omega} \to 0\ (n\to\infty). \end{array} \right\}$$

LEMMA 1. *Let $X(\Omega)$ be as above. Then any $f \in \mathscr{P}_p(\Omega)$ can be decomposed into the form $f = u_0 + v_0$, where $u_0 \in \bar{X}_p(\Omega)$ and v_0 satisfies the condition that*

$$\int_\Omega \langle \nabla_\tau \psi(x, \nabla v_0), \nabla\phi \rangle dx = 0$$

for every ϕ in $X(\Omega)$.

PROOF. Let $I_\psi(g) = \int_\Omega \psi(x, \nabla g) dx$ for $g \in \mathscr{P}_p(\Omega)$ and choose $u_n \in \bar{X}_p(\Omega)$ such that

$$I_\psi(f-u_n) \longrightarrow a \equiv \inf\{I_\psi(f-u);\ u \in \bar{X}_p(\Omega)\} \quad (n\to\infty).$$

By (c), $\{\|\nabla(f-u_n)\|_{p,\Omega}\}$ is bounded. Hence by standard arguments in L^p-theory (Banach Saks' theorem, etc.) and by [4, Theorem 4.21] (also cf. [8, Theorem 4.3]), we find $v_0 \in \mathscr{P}_p(\Omega)$ and a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\nabla(f-u_{n_j}) \to \nabla v_0$ weakly in $L^p(\Omega)$ and

$$\|\nabla(f-v_0)-(\nabla u_{n_1}+\cdots+\nabla u_{n_k})/k\|_{p,\Omega} \longrightarrow 0 \quad (k\to\infty).$$

Since $(u_{n_1}+\cdots+u_{n_k})/k \in \overline{X}_p(\Omega)$, we may assume that $v_0(\gamma)=f(\gamma)$ for p-a.e. $\gamma \in \Gamma_\Omega(\partial G)$ (see (1.2)), so that $u_0 \equiv f-v_0 \in \overline{X}_p(\Omega)$. By (1.3) and (1.5), we see that

$$I_\psi(f-u_0)=I_\psi(v_0) \leqq \liminf_{j\to\infty} I_\psi(f-u_{n_j}),$$

which means that $I_\psi(f-u_0)=a$. Since $\overline{X}_p(\Omega)$ is a linear space, it follows that $v_0 = f-u_0$ is a solution of the Euler equation for I_ψ (cf. [2, Theorem 3.18]). The proof is completed.

We say that $C_0^\infty(G-E)$ (resp. $C_1^\infty(G; E)$) is dense in $\mathring{W}_1^p(G)$ if for each $\phi \in C_0^\infty(G)$ there is a sequence $\{\phi_n\}$ in $C_0^\infty(G-E)$ (resp. $C_1^\infty(G; E)$) such that $\|\nabla(\phi-\phi_n)\|_{p,G} \to 0$ as $n\to\infty$.

LEMMA 2. *If E is removable for $\mathscr{HD}_\psi^p$ (resp. $\mathscr{KD}_\psi^p$), then $C_0^\infty(G-E)$ (resp. $C_1^\infty(G; E)$) is dense in $\mathring{W}_1^p(G)$ for some bounded domain G containing E.*

PROOF. Suppose that E is removable for $\mathscr{HD}_\psi^p$. Then there is a bounded domain G containing E such that every function in $\mathscr{HD}_\psi^p(G-E)$ can be extended to a function in $\mathscr{HD}_\psi^p(G)$. Take $f \in C_0^\infty(G)$. Let $f_0=f|_{G-E}$ and $X(G-E)=C_0^\infty(G-E)$. By Lemma 1, f_0 can be decomposed into the form $f_0=u_0+v_0$, where $u_0 \in \overline{X}_p(G-E)$ and $v_0 \in \mathscr{HD}_\psi^p(G-E)$. Then $v_0(\gamma)=0$ for p-a.e. $\gamma \in \Gamma_{G-E}(\partial G)$. By assumption there is a p-precise function $\tilde{v}_0$ in $\mathscr{HD}_\psi^p(G)$ such that $v_0=\tilde{v}_0$ on $G-E$. Therefore

$$\int_G \langle \nabla_\tau\psi(x, \nabla\tilde{v}_0), \nabla\phi\rangle dx = 0$$

for every ϕ in $C_0^\infty(G)$. Since $\tilde{v}_0 \in \mathscr{P}_p(G)$ and $\tilde{v}_0(\gamma)=0$ for p-a.e. $\gamma \in \Gamma_G(\partial G)$, by using (1.1) and Hölder's inequality we have

$$\int_G \langle \nabla_\tau\psi(x, \nabla\tilde{v}_0), \nabla\tilde{v}_0\rangle dx = 0.$$

Hence $\tilde{v}_0=$const. a.e. in G by (1.4), and the constant must be 0. Hence $f_0 \in \overline{X}_p(G-E)$, i.e., there is a sequence $\{\phi_n\}$ in $C_0^\infty(G-E)$ such that $\|\nabla(f_0-\phi_n)\|_{p,G-E} \to 0$ $(n\to\infty)$. Since $m_d(E)=0$ we see that $\|\nabla(f-\phi_n)\|_{p,G}=\|\nabla(f_0-\phi_n)\|_{p,G-E} \to 0$ $(n\to\infty)$. Thus we conclude that $C_0^\infty(G-E)$ is dense in $\mathring{W}_1^p(G)$.

In the case that E is removable for $\mathscr{KD}_\psi^p$, we let $X(G-E)=C_1^\infty(G; E)$. Then the result for $C_1^\infty(G; E)$ is established in the same manner.

§ 3. Relations between removable sets

In [6] the author considered the notion of null sets for extremal distances of order p. A compact set E is called an NED_p-set if $M_p(\Gamma)=M_p(\Gamma_E)$ for all pairs of disjoint continua F_0 and F_1 in E^c, where Γ (resp. Γ_E) is the family of all curves connecting F_0 and F_1 in R^d (resp. E^c). The following lemma is known:

LEMMA 3 ([6, Theorems 1 and 2]). *The following statements are equivalent to each other:*

(1) *E is an NED_p-set.*

(2) *For some bounded domain (or any bounded domain) G containing E, every function in $\mathscr{P}_p(G-E)$ can be extended to a function in $\mathscr{P}_p(G)$.*

(3) $\int_{E^c} \frac{\partial u}{\partial x_i} dx=0$ $(i=1, 2,\dots, d)$ *for every u in $\mathscr{P}_p(E^c)$ which vanishes identically on a neighborhood of $\{\infty\}$.*

THEOREM 1. *E is removable for $\widetilde{\mathscr{H}\mathscr{D}}{}^p_\psi$ if and only if E is an NED_p-set.*

PROOF. Suppose that E is removable for $\widetilde{\mathscr{H}\mathscr{D}}{}^p_\psi$. Then there is a bounded domain G containing E such that every function in $\widetilde{\mathscr{H}\mathscr{D}}{}^p_\psi(G-E)$ can be extended to a function in $\mathscr{H}\mathscr{D}^p_\psi(G)$. Let $f\in\mathscr{P}_p(G-E)$ vanish identically on a neighborhood of ∂G and let $X(G-E)=\{\phi|_{G-E};\ \phi\in C_0^\infty(G)\}$. By Lemma 1, f can be decomposed into the form $f=u_0+v_0$, where $u_0\in\overline{X}_p(G-E)$ and $v_0\in\widetilde{\mathscr{H}\mathscr{D}}{}^p_\psi(G-E)$. Then $v_0(\gamma)=0$ for p-a.e. $\gamma\in\Gamma_{G-E}(\partial G)$. By assumption there is a p-precise function $\tilde{v}_0$ in $\mathscr{H}\mathscr{D}^p_\psi(G)$ such that $v_0=\tilde{v}_0$ on $G-E$. Since $\tilde{v}_0$ is an ACL function (cf. [4, Theorem 4.6], [8, Theorem 4.4]) and $m_d(E)=0$, by Fubini's theorem we have

$$\int_{G-E}\frac{\partial v_0}{\partial x_i}dx=\int_G\frac{\partial \tilde{v}_0}{\partial x_i}dx=0.$$

On the other hand there is a sequence $\{\phi_n\}$ in $C_0^\infty(G)$ such that $\|\nabla(u_0-\phi_n)\|_{p,G-E}\to 0$ as $n\to\infty$. Since $m_d(E)=0$ it follows that

$$\int_{G-E}\frac{\partial u_0}{\partial x_i}dx=\int_{G-E}\frac{\partial \phi_n}{\partial x_i}dx+\int_{G-E}\frac{\partial (u_0-\phi_n)}{\partial x_i}dx$$

$$=\int_{G}\frac{\partial \phi_n}{\partial x_i}dx+\int_{G-E}\frac{\partial (u_0-\phi_n)}{\partial x_i}dx\longrightarrow 0\quad (n\to\infty).$$

Thus we conclude that

$$\int_{G-E}\frac{\partial f}{\partial x_i}dx=0\quad (i=1, 2,\dots, d).$$

Next, let $u\in\mathscr{P}_p(E^c)$ vanish identically on a neighborhood of $\{\infty\}$. Set

$u_n = \max(-n, \min(u, n))$ for $n=1, 2,\dots$. Take a function $\phi \in C_0^\infty(R^d)$ such that $\mathrm{supp}\,\phi \subset G$ and $\phi = 1$ on a neighborhood of E. Since the restriction of ϕu_n to $G-E$ is a p-precise function on $G-E$ which vanishes identically on a neighborhood of ∂G, by the above result we obtain

$$\int_{G-E} \frac{\partial(\phi u_n)}{\partial x_i} dx = 0.$$

On the other hand, by Fubini's theorem we have

$$\int_{E^c} \frac{\partial(u_n(1-\phi))}{\partial x_i} dx = 0.$$

Hence

$$\int_{E^c} \frac{\partial u_n}{\partial x_i} dx = \int_{G-E} \frac{\partial(\phi u_n)}{\partial x_i} dx + \int_{E^c} \frac{\partial(u_n(1-\phi))}{\partial x_i} dx = 0.$$

Set $F_n = \{x \in E^c; |u(x)| \geqq n\}$ for $n=1, 2,\dots$. Since $m_d(F_n) \to 0$ as $n \to \infty$, we have

$$\int_{E^c} \frac{\partial u}{\partial x_i} dx = \int_{E^c} \frac{\partial(u-u_n)}{\partial x_i} dx = \int_{F_n} \frac{\partial u}{\partial x_i} dx \longrightarrow 0 \qquad (n \to \infty).$$

Thus we conclude that

$$\int_{E^c} \frac{\partial u}{\partial x_i} dx = 0 \quad (i=1, 2,\dots, d).$$

From Lemma 3 ((3)$\Rightarrow$(1)), it follows that E is an NED_p-set.

Conversely we suppose that E is an NED_p-set. By Lemma 3 ((1)$\Rightarrow$(2)) we can take a bounded domain G containing E such that every function in $\mathscr{P}_p(G-E)$ can be extended to a function in $\mathscr{P}_p(G)$. Let $u \in \widetilde{\mathscr{H}\mathscr{D}}{}_\psi^p(G-E)$. Then there is a p-precise function $\tilde{u}$ on G such that $u = \tilde{u}$ on $G-E$. Since $m_d(E)=0$ we have

$$\int_G \langle \nabla_\tau \psi(x, \nabla \tilde{u}), \nabla \phi \rangle dx = \int_{G-E} \langle \nabla_\tau \psi(x, \nabla u), \nabla \phi \rangle dx = 0$$

for every ϕ in $C_0^\infty(G)$. This implies that E is removable for $\widetilde{\mathscr{H}\mathscr{D}}{}_\psi^p$.

In [3], Hedberg considered the following classes of harmonic functions. For a domain G in R^d, denote by $HD^p(G)$ the class of all harmonic functions u on G with $\|\nabla u\|_{p,G} < \infty$, by $FD^p(G)$ the class of all $u \in HD^p(G)$ with no flux, i.e., $\int_c {*du} = \int_c \partial u/\partial\nu dS = 0$ for all $(d-1)$-cycles c in G. For a compact set E its p-capacity is defined by

$$C_p(E) = \inf\{\|\nabla \omega\|_{p,R^d}^p;\ \omega \in C_0^\infty(R^d),\ \omega \geqq 1 \text{ on } E\}.$$

In the case $p \geqq d$ ω's are restricted to $C_0^\infty(B)$ for some fixed large ball B containing

E in its interior. Throughout the rest of this paper we let $q=p/(p-1)$. Hedberg proved

THEOREM A ([3, Theorems 1.a and 2]). *The following statements are equivalent to each other*:

(1) $C_p(E)=0$.

(2) $C_0^\infty(G-E)$ *is dense in* $\mathring{W}_1^p(G)$ *for some bounded domain* G *containing* E.

(3) E *is removable for* HD^q.

THEOREM B ([3, Theorem 1.b]). $C_1^\infty(G; E)$ *is dense in* $\mathring{W}_1^p(G)$ *for some bounded domain* G *containing* E *if and only if* E *is removable for* FD^q.

LEMMA 4. *If* E *is an* NED_p*-set and is removable for* HD^q *(resp.* FD^q*), then* E *is removable for* $\mathscr{H}\mathscr{D}_\psi^p$ *(resp.* $\mathscr{K}\mathscr{D}_\psi^p$*).*

PROOF. Suppose that E is removable for HD^q. By Theorem A, $C_0^\infty(G-E)$ is dense in $\mathring{W}_1^p(G)$ for some bounded domain G containing E. Therefore, for each $\phi\in C_0^\infty(G)$ there is a sequence $\{\phi_n\}$ in $C_0^\infty(G-E)$ such that $\|\nabla(\phi-\phi_n)\|_{p,G}\to 0$ as $n\to\infty$. Since E is an NED_p-set, by Lemma 3 ((1)⇒(2)) every function u in $\mathscr{H}\mathscr{D}_\psi^p(G-E)$ can be extended to a function $\tilde{u}$ in $\mathscr{P}_p(G)$. By using Hölder's inequality we have

$$\int_G \langle \nabla_\tau\psi(x, \nabla\tilde{u}), \nabla\phi\rangle dx = \lim_{n\to\infty}\int_G \langle \nabla_\tau\psi(x, \nabla\tilde{u}), \nabla\phi_n\rangle dx$$

$$= \lim_{n\to\infty}\int_{G-E} \langle \nabla_\tau\psi(x, \nabla u), \nabla\phi_n\rangle dx = 0.$$

Hence $\tilde{u}\in\mathscr{H}\mathscr{D}_\psi^p(G)$. This implies that E is removable for $\mathscr{H}\mathscr{D}_\psi^p$.

In the case that E is removable for FD^q, $C_1^\infty(G; E)$ is dense in $\mathring{W}_1^p(G)$ for some bounded domain G containing E by Theorem B. The result for $\mathscr{K}\mathscr{D}_\psi^p$ is established in the same manner.

THEOREM 2. *E is removable for $\mathscr{H}\mathscr{D}_\psi^p$ if and only if E is removable for HD^q.*

PROOF. The only-if part follows from Lemma 2 and Theorem A ((2)⇒(3)).

Conversely, assume that E is removable for HD^q. By Lemma 4 it is enough to show that E is an NED_p-set. Let F_0, F_1 be disjoint continua in E^c and let Γ (resp. Γ_E) be the family of curves connecting F_0 and F_1 in R^d (resp. E^c). Take a bounded domain Ω disjoint from F_0 and F_1 such that $\Omega\supset E$. Since $C_p(E)=0$ by Theorem A, we can take a sequence $\{\omega_n\}$ in $C_0^\infty(\Omega)$ such that $\omega_n\geq 1$ on E for each n and $\|\nabla\omega_n\|_{p,\Omega}\to 0$ as $n\to\infty$ (see, e.g., [7, Lemma 4.2]). Obviously $|\nabla\omega_n|$ is admissible in association with $\Gamma-\Gamma_E$. Hence $M_p(\Gamma-\Gamma_E)=0$. From the inequalities $M_p(\Gamma_E)\leqq M_p(\Gamma)\leqq M_p(\Gamma_E)+M_p(\Gamma-\Gamma_E)$, it follows that $M_p(\Gamma_E)=M_p(\Gamma)$. This implies that E is an NED_p-set.

In the case that $\psi(x, \tau)=|\tau|^p$ for all $(x, \tau) \in R^d \times R^d$, we omit the subscript ψ in $\mathscr{K}\mathscr{D}^p_\psi$ and $\widetilde{\mathscr{H}\mathscr{D}}^p_\psi$. The following lemma is a relation between the removability for $\mathscr{K}\mathscr{D}^p$ and that for FD^q.

LEMMA 5 ([5, Theorem 11]). *If $p \geqq 2$, then E is removable for $\mathscr{K}\mathscr{D}^p$ if and only if E is removable for FD^q.*

THEOREM 3. *If $p \geqq 2$, then E is removable for $\mathscr{K}\mathscr{D}^p_\psi$ if and only if E is removable for FD^q.*

PROOF. The only-if part follows from Lemma 2 and Theorem B for $1 < p < \infty$.

Conversely, assume that E is removable for FD^q. By Lemma 5, E is removable for $\mathscr{K}\mathscr{D}^p$. Hence E is removable for $\widetilde{\mathscr{H}\mathscr{D}}^p$. From Theorem 1 it follows that E is an NED_p-set. By Lemma 4, we see that E is removable for $\mathscr{K}\mathscr{D}^p_\psi$.

REMARK. Theorems 1, 2 and 3 show that the removability for each of the classes $\mathscr{H}\mathscr{D}^p_\psi$, $\mathscr{K}\mathscr{D}^p_\psi$ and $\widetilde{\mathscr{H}\mathscr{D}}^p_\psi$ does not depend on the choice ψ as long as it satisfies (a)–(c) in §1.

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