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On wild knots which are weakly tame

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1. Introduction

In this paper, we are concerned mainly with knots, by which we mean topologically embedded circles in the 3-sphere S^3.

Let X be a subset of S^3. Then, X is *PL* if it is a subpolyhedron of S^3, *tame* if $h(X)$ is PL for some homeomorphism $h: S^3 \approx S^3$, and *wild* if it is not tame. Furthermore, X is *locally tame* at $x \in X$ if there are an open set $V \ni x$ in S^3 and a homeomorphism ϕ: $V \approx E^3$ such that $\phi(V \cap X)$ is a subpolyhedron of E^3 (E^n denotes the Euclidean n-space), and when X is a knot, X is *locally flat* at $x \in X$ if $\phi(V \cap X) = E^1$ in addition. For a knot $J \subset S^3$, we note that these local properties are equivalent to each other, and consider the closed subset

$$E(J) = \{x \in J \mid J \text{ is not locally tame at } x\} \subset J.$$

Then, Bing's theorem [2] says that J is tame if and only if $E(J)$ is empty.

We shall say that a knot $J \subset S^3$ is *wealky tame* if there is a PL knot $K \subset S^3$ such that the complement $S^3 - K$ is homeomorphic to $S^3 - J$, and *weakly flat* according to Duvall [7] if K is unknotted in addition; and we shall study several properties of such a knot J by taking notice of the set $E(J)$.

The main results are stated as follows.

Theorem I. *Assume that a knot $J \subset S^3$ is weakly tame, and let U be an open set in J. Then, J is locally tame at every point $x \in U$ if so is at every point $x \in U - C^*$, where C^* is a Cantor set in U.*

Corollary. *If a knot $J \subset S^3$ is weakly tame, then $E(J)$ has no isolated points. If J is locally tame at every point $x \in J - C^*$ for a Cantor set $C^* \subset J$ in addition, then it turns out that $E(J)$ is empty and J is tame.*

Theorem I means that $E(J)$ for a weakly tame knot J can not be 0-dimensional. In contrast with this we can find a weakly tame knot J with 1-dimensional $E(J)$: most significant one is given by the following

Theorem II. *For each PL knot $K \subset S^3$, there is a wild knot $J \subset S^3$ such that $S^3 - J$ is homeomorphic to $S^3 - K$ and J is everywhere wild, i.e., $E(J) = J$.*

A proof of Theorem I using Cannon's characterization of tame arcs in S^3

will be given in §2. We can give also an elementary proof by comparing a system of neighborhoods of a Cantor set C^* with the standard one, as described in the original version of the paper.

Theorem II is proved in §3. Bing [3] developed the "*hooked rug*" method, by which Alford constructed a "nice" wild 2-sphere in S^3 ([1]); it contains a wild knot J^* whose $E(J^*)$ is an arc (Rushing [14]). We show that this knot J^* is weakly flat (Theorem 3.1), and then prove Theorem II by taking J as a connected sum of K and infinitely many copies of this J^*.

The following notation and the terminologies are used in this paper:

$\approx$: homeomorphic, id: the identity map, $\emptyset$: empty set, $\cong$: isomorphic, $\boldsymbol{E}^n$: Euclidean n-space, $\boldsymbol{E}^n_+ = \boldsymbol{E}^{n-1} \times [0, \infty)$, $B^n = [-1, 1]^n$, $rB^n = [-r, r]^n$ $(r>0)$, $S^n = \partial B^{n+1}$: the n-sphere, d: a metric on S^n, diam X: the diameter of X, Cl X: the closure of X, Fr X: the frontier of X, $\mathrm{N}(X, r) = \{x \in S^3 | d(x, X) < r\}$ $(X \subset S^3)$.

For $X \subset S^3$, X is *locally polyhedral* at $x \in X$ if $X \cap V$ is polyhedral for some closed neighborhood V of x in S^3. When X is a compact n-manifold $(1 \leqslant n \leqslant 3)$, X is *locally flat* at $x \in X$ if it is locally tame at x by an open set $V \ni x$ and ϕ: $V \approx \boldsymbol{E}^3$ with $\phi(V \cap X) = \boldsymbol{E}^n_+$ or $\boldsymbol{E}^n$ according to $x \in \partial X$ or not in addition (these local properties are equivalent), and X is *locally flat* if so it at every point $x \in X$.

2. Proof of Theorem I

We first recall a characterization of tame arcs in S^3.

DEFINITION. An arc A in S^3 is said to *have* 1-ALG *complement* in S^3 if for each $\varepsilon>0$ there is a $\delta>0$ such that each loop in $S^3 - A$ which is null-homologous ($\boldsymbol{Z}$-coefficients) in a δ-subset of $S^3 - A$ bounds a singular ε-disk in $S^3 - A$.

THEOREM 2.1 (J. W. Cannon [5, Th. 3.16]). *An arc A in S^3 is tame if it has* 1-ALG *complement in S^3.*

We prove Theorem I by this theorem together with the following

PROPOSITION 2.2. *Let J be a knot in Theorem* I *and p be an arbitrary point of U. Then, for each open neighborhood W of p in S^3 there is an open neighborhood $V \subset W$ of p such that every loop in $V - J$ which is null-homologous in $V - J$ is null-homotopic in $W - J$.*

PROOF OF THEOREM I. Let A be an arc in U with Int $A \subset C^*$. For each $\varepsilon>0$, we define an open covering $\{V_x | x \in S^3\}$ of S^3 as follows:

$$V_x = \mathrm{N}(x, \min(\varepsilon/2, d(x, A)), \qquad \text{for} \quad x \in S^3 - A;$$

$V_x = V$ given by Proposition 2.2 for $p = x$ and $W = N(x, \varepsilon/2)$, for $x \in \operatorname{Int} A$; and

$V_x \ni x$ is an open ε-subset with $(V_x, A \cap V_x) \approx (E^3, E^1_+)$, for $x \in \partial A$.

Then, there is a Lebesgue number $\delta > 0$ for $\{V_x\}$, i.e., each δ-subset of S^3 is contained in some V_x. Thus, A has 1-ALG complement in S^3, and A is tame by Theorem 2.1.□

To prove Proposition 2.2, we prepare the following

LEMMA 2.3. *Suppose that a knot $J \subset S^3$ is weakly tame. Then, there is a sequence $\{P_n\}$ of locally flat solid tori in S^3 such that*

(1) $\operatorname{Int} P_n \supset P_{n+1}$, $\cap P_n = J$ *and* $P_n - \operatorname{Int} P_{n+1} \approx \partial P_n \times [0, 1]$, *and*
(2) *J is a deformation retract of P_n.*

PROOF. Let K be a PL knot with $S^3 - J \approx S^3 - K$ by assumption.

Case 1: *K is a trivial knot.* Let $h: S^3 - J \approx S^1 \times E^2$ be a homeomorphism, and put

$$Q_n = h^{-1}(S^1 \times nB^2), \quad P_n = S^3 - \operatorname{Int} Q_n.$$

Since Q_n is a locally flat solid torus in S^3, we note that P_n is a knot space. Since $J \subset \operatorname{Int} P_n$ is compact and J has codimension 2 in P_n, $P_n - J$ is connected and $\pi_1(P_n - J) \to \pi_1(P_n)$ is an epimorphism (see p. 329 of [11]). Note that $P_n - J \approx S^1 \times S^1 \times [0, \infty)$. Then, $\pi_1(P_n - J) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$, and so $\pi_1(P_n)$ is abelian. Hence, $\pi_1(P_n) \cong H_1(P_n) \cong \boldsymbol{Z}$ and P_n is a solid torus.

Case 2: *K is not trivial.* Take $h: S^3 - J \approx S^3 - K$ and a tubular neighborhood $K \times E^2$ of K; $S^3 \supset K \times E^2 \subset K \times \{0\} = K$.

$$Q_n = h^{-1}(S^3 - K \times \operatorname{Int}(1/n)B^2), \quad P_n = S^3 - \operatorname{Int} Q_n.$$

Then, the knot space Q_n is not a solid torus. It follows that P_n is a solid torus (cf. Rolfsen [13, Th. (4.C.1)]).

Clearly, $\{P_n\}$ satisfies the other conditions in (1). Since $J \approx S^1$ is an ANR, there are an open set $R \supset J$ in S^3 and a retraction $r: R \to J$. Then, there is an m such that $P_n \subset R$ for all $n \geqslant m$. Let $n \geqslant m$. Then, $r|P_n: P_n \to J$ is a retraction, and so

$$\boldsymbol{Z} \cong \pi_1(P_n) \xrightarrow{(r|P_n)_*} \pi_1(J) \cong \boldsymbol{Z}$$

is an isomorphism. Thus, $r|P_n$ is a deformation retraction. Let $n < m$. Then, by the last condition in (1), P_n is a deformation retract of P_n; and we see (2).□

PROOF OF PROPOSITION 2.2. By Bing [2, Th. 9], we may assume that J is locally polyhedral at every point of $U-C^*$. Also we may assume that $W \cap J \subset U$. Take a subarc I of $W \cap J$ such that $p \in \operatorname{Int} I$, and both end points a_0 and a_1 of I are contained in $U-C^*$. Then, there are disjoint PL disks D_0 and D_1 in W such that $D_i \cap J=\{a_i\}$ and J intersects D_i transversely at a_i $(i=0, 1)$. By Lemma 2.3, there is a locally flat solid torus $P \subset S^3$ such that $P \cap (\partial D_0 \cup \partial D_1)=\emptyset$, $J \subset \operatorname{Int} P$ and J is a deformation retract of P. Let X and X' be the components of $\operatorname{Int} P-(D_0 \cup D_1)$ containing $\operatorname{Int} I$ and $J-I$, respectively.

Claim 1. $X \neq X'$.

Suppose that $X=X'$. Take a point $q \in J-I$. Then, there is an arc $H \subset X$ joining p and q. Let $H' \ni a_0$ be the subarc of J which joins p and q. Then, the loop $H \cup H'$ in $\operatorname{Int} P$ intersects D_0 transversely at a_0 and $(H \cup H') \cap D_0=\{a_0\}$. Thus, $H \cup H'$ is homotopic to J in $\operatorname{Int} P$, because J is a deformation retract of P, $J \cap D_0=\{a_0\}$ and J intersect transversely at a_0. But, $(H \cup H') \cap D_1=\emptyset$; this is a contradiction. Claim 1 follows.

Thus, $Y=X \cap W$ is an open neighborhood of $\operatorname{Int} I$ in S^3 and $Y \cap X'=\emptyset$. Take subdisks E_0 and E_1 of D_0 and D_1, respectively, such that $a_i \in \operatorname{Int} E_i$ and $E_i \subset \operatorname{Int} P$ $(i=0, 1)$. By Lemma 2.3, there is a locally flat solid torus $P' \subset \operatorname{Int} P$ such that $J \subset \operatorname{Int} P'$, J is a deformation retract of P', $P'-J \approx S^1 \times S^1 \times [0, \infty)$ and $P' \cap ((D_0 \cup D_1 \cup \operatorname{Fr} Y)-\operatorname{Int}(E_0 \cup E_1))=\emptyset$. Let V be the component of $\operatorname{Int} P'-(D_0 \cup D_1)$ containing $\operatorname{Int} I$. Then, $V \subset Y$ by the same reason as in Claim 1.

Let $f: S^1=\partial B^2 \to V$ be a loop which is null-homologous in $V-J$. Then, f is also null-homologous in $\operatorname{Int} P'-J$; hence f extends to a map $f: B^2 \to \operatorname{Int} P'-J$. We may assume that f is in general position with respect to $E_0 \cup E_1$; hence $f^{-1}(E_0 \cup E_1)$ is a finite union of disjoint circles in $\operatorname{Int} B^2$. Let B' be the closure of the component of $B^2-f^{-1}(E_0 \cup E_1)$ which contains ∂B^2.

Claim 2. *For each component* $L \subset f^{-1}(E_i)$ $(i=0, 1)$, *the loop* $f|L: L \to E_i-\{a_i\}$ *is null-homotopic.*

If not, then some non-zero multiple of ∂E_i is homotopic to $f|L$ in S^3-J and $f|L$ is null-homotopic in S^3-J. Therefore the linking number of J and ∂E_i is zero. This is a contradiction, and Claim 2 follows.

Thus, $f|B': B' \to (V \cup E_0 \cup E_1)-J$ extends to a map $f': B^2 \to (V \cup E_0 \cup E_1)-J$; hence $f: S^1 \to V$ bounds a singular disk in $W-J$. □

Thus, the proof of Theorem I is completed.

3. Proof of Theorem II

THEOREM 3.1. *There is a wild knot* $J^* \subset S^3$ *which satisfies the following conditions.*

(a) *J^* is weakly flat.*

(b) *$E(J^*)$ is a non-empty subarc A^* of J^*.*

(c) *There are an open subset $U \subset S^3$ and a homeomorphism h: $(U, U \cap J^*) \approx (E^3, E^1)$ such that $D^* - J^* \approx (\partial D^* - J^*) \times [0, \infty)$ where $D^* = S^3 - \operatorname{Int} h^{-1}(B^3)$.*

PROOF. Alford [1] constructed a wild 3-cell B^* in S^3 such that $S^3 - B^* \approx E^3$ and $A^* = \{x \in \partial B^* | \partial B^*$ is not locally tame at $x\}$ is a non-empty arc on ∂B^*. B^* and A^* are the limits of PL 3-cells $\{B_n \subset S^3\}$ and PL arcs $\{A_n \subset \partial B_n\}$ respectively: B_0 is a PL 3-cell and A_0 is a PL arc on ∂B_0. B_n is obtained from B_{n-1} by adding "cubes-with-eyebolts" to ∂B_{n-1} along A_{n-1} and removing a thin slice from the loop of each cube-with-eyebolt. There is a homeomorphism f_n: $B_{n-1} \approx B_n$ such that $f_n | \operatorname{Cl}(B_{n-1} - C_{n-1}) = id$ where C_{n-1} is a regular neighborhood of A_{n-1} in B_{n-1}, and $A_n = f_n(A_{n-1})$. $\{f_n f_{n-1} \cdots f_1 : B_0 \to B_n\}$ is a Cauchy sequence converging to the embedding f^*: $B_0 \to S^3$ such that $B^* = f^*(B_0)$ and $A^* = f^*(A_0)$. By the construction (cf. Bing [3, §4] and Gillman [9, §3]), we have PL cubes-with-handles $\{M_n\}$ such that

(M1) $\operatorname{Int} M_n \supset M_{n+1}$ and $\cap_n M_n = A^*$,

(M2) $M_n \cap B_{n-1} = C_{m-1}$ and $M_n \cap B_n = f_n(C_{n-1})$.

Now we take a PL circle J_0 on ∂B with $J_0 \supset A_0$, and put $J^* = f^*(J_0)$. Then, J^* is a wild knot in S^3 with $E(J^*) = A^*$ by [14]; hence J^* satisfies (b).

Next, we prove (a). By applying the "stretching argument" to a regular neighborhood of B_n ($n \geqslant 1$) as used in [9, §§4–5], we can construct PL 3-cells $\{N_n\}$ satisfying the following

(N1) $\operatorname{Int} N_n \supset N_{n+1}$ and $\cap N_n = B^*$,

(N2) $M_n \cap N_i$ is a PL 3-cell and $M_n \cap N_i \cap \partial W_n$ is a PL disk for $i > n$, where $W_n = M_n \cap N_n$.

Let p: $S^3 \to S^3/A^*$ be the projection. Since B^* is locally tame at every $x \in B^* - A^*$ and B^* is cellular in S^3 by (N1), it follows from Meyer [12, Th. 2] that

$$S^3 \approx S^3/B^* \approx S^3/A^* \supset \partial B^*/A^* \approx S^2$$

and $\partial B^*/A^*$ is locally tame at every point of $\partial B^*/A^* - p(A^*)$.

Now we show that $\partial B^*/A^*$ is flat in S^3/A^*. Since B^*/A^* is a 3-cell, it is sufficient to show that $\operatorname{Cl}(S^3/A^* - B^*/A^*)$ is a 3-cell, and this is equivalent to that $S^3/A^* - B^*/A^*$ is $1-$LC at $p(A^*)$ by Bing [4, Th. 2]. Here, for closed subset $A \subset X$ in S^3, $S^3 - X$ is 1-LC at A if each open set $U \subset A$ in S^3 contains an open set $V \subset A$ such that each loop in $V - X$ is null-homotopic in $U - X$. Thus, it is sufficient to show that $S^3 - B^*$ is 1-LC at A^*. By (M1) and (N1–2),

$$\operatorname{Int} W_n \supset W_{n+1}, \quad \cap_n W_n = A^* \quad \text{and} \quad W_n - B^* = \cup_{i \geqslant n+1} (W_n - N_i).$$

Moreover, $\mathrm{Cl}(W_n - N_i) = \mathrm{Cl}(W_n - (M_n \cap N_i))$ is a PL 3-cell for $i \geqslant n+1$ by (N2). Thus, $S^3 - B^*$ is 1-LC at A^*, and $\partial B^*/A^*$ is flat in S^3/A^*. From this, J^*/A^* is flat in S^3/A^*. Then, we see (a), because

$$S^3 - J^* \approx (S^3/A^*) - (J^*/A^*) \approx S^3 - S^1 \approx S^1 \times E^2.$$

Finally we verify (c). Take $\phi: (S^3/A^*, J^*/A^*) \approx (S^3, S^1)$, and choose an open set $U' \subset S^3 - \phi p(A^*)$ and $h': (U', U' \cap S^1) \approx (E^3, E^1)$. Furthermore, put $U = p^{-1}\phi^{-1}(U') \subset S^3 - A^*$ and $h = h'\phi p: (U, U \cap J^*) \approx (E^3, E^1)$. Then, we have

$$D^* - J^* \approx B^3 - B^1 \approx (\partial D^* - J^*) \times [0, \infty) \qquad \text{for} \quad D^* = S^3 - \operatorname{Int} h^{-1}(B^3).$$

This completes the proof of Theorem 3.1 □

LEMMA 3.2. *Suppose that J^* and D^* are as in Theorem 3.1. Let $g: S^1 \to S^3$ be an embedding such that there is an open set $U' \subset S^3$ with $h': (U', U' \cap g(S^1)) \approx (E^3, E^1)$. Take a locally flat 3-cell $D' = h'^{-1}(B^3)$ in S^3 and a subarc $C' = g^{-1}(D')$ of S^1. Then, there is an embedding $f: S^1 \to S^3$ with the following (1)–(4):*

(1) $f | S^1 - \operatorname{Int} C' = g | S^1 - \operatorname{Int} C'$.

(2) $f(\operatorname{Int} C') \subset \operatorname{Int} D'$.

(3) $(D', D' \cap f(S^1)) \approx (D^*, D^* \cap J^*)$.

(4) *There is* $\phi: S^3 - g(S^1) \approx S^3 - f(S^1)$ *such that* $\phi = \mathrm{id}$ *on* $S^3 - (g(S^1) \cup \operatorname{Int} D')$.

PROOF. Suppose that J^*, U, h and D^* are as in Theorem 3.1. Then, there is an embedding $e: S^1 \to S^3$ such that $J^* = e(S^1)$. Take a subarc $C = e^{-1}(D) \subset S^1$ where $D = h^{-1}(B^3)$, and put

$$S^1 \# S^1 = (S^1 - \operatorname{Int} C') \cup_{\bar{g}} (S^1 - \operatorname{Int} C), \quad \bar{g} = e^{-1}h^{-1}h': \partial C' \longrightarrow \partial C,$$

$$\text{and} \quad S^3 \# S^3 = (S^3 - \operatorname{Int} D') \cup_{\bar{h}} (S^3 - \operatorname{Int} D), \quad \bar{h} = h^{-1}h': \partial D' \longrightarrow \partial D.$$

Then, there are $p: S^1 \approx S^1 \# S^1$ and $q: S^3 \approx S^3 \# S^3$ such that $p | S^1 - \operatorname{Int} C' = \mathrm{id}$ and $q | S^3 - \operatorname{Int} D' = \mathrm{id}$. We can define an embedding $g': S^1 \# S^1 \to S^3 \# S^3$ by

$$g' | S^1 - \operatorname{Int} C' = g | S^1 - \operatorname{Int} C' \quad \text{and} \quad g' | S^1 - \operatorname{Int} C = e | S^1 - \operatorname{Int} C.$$

Therefore, we get an embedding $f = q^{-1}g'p: S^1 \to S^3$. Clearly, f satisfies (1)–(3). From (c) of Theorem 3.1, we can easily verify (4). □

LEMMA 3.3. *Let J^* and D^* be the ones in Theorem 3.1. Let $f_0: S^1 \to S^3$ be a PL embedding, and V_n $(n \geqslant 1)$ be connected open sets in S^1, which forms a basis of open sets. Then, there are $B \subset A \subset \{1, 2, \dots\}$, $D_n \subset U_n \subset S^3$, $h_n: U_n \approx E^3$ and $C_n \subset V_n$ for $n \in A$, embeddings $f_n: S^1 \to S^3$ and $\phi_n: S^3 - f_n(S^1) \approx S^3 - f_n(S^1)$ for*

$n\geqslant 1$, *which satisfy the following conditions* (F1)–(F6):

(F1) *If* $n\notin A$, *then* $f_{n-1}(V_n)$ *is everywhere wild*, $f_n=f_{n-1}$ *and* $\phi_n=id$.

(F2) *For each* $n\in A$, U_n *is open in* S^3, $h_n\colon (U_n, U_n\cap f_{n-1}(S^1))\approx(E^3, E^1)$, $D_n=h_n^{-1}(B^3)\subset U_n$ *is a locally flat* 3-*cell with* $\operatorname{diam} D_n<1/2^n$, *and* $C_n=f_{n-1}^{-1}(D_n)\subset V_n$ *is a subarc of* S^1 *with* $\operatorname{diam} C_n<1/n$.

(F3) *If* $n<m$, *then either* $D_n\cap D_m=\emptyset=C_n\cap C_m$, *or* $D_m\subset\operatorname{Int} D_n$ *and* $C_m\subset\operatorname{Int} C_n$.

(F4) $f_n|S^1-\operatorname{Int} C_n=f_{n-1}|S^1-\operatorname{Int} C_n$, $f_n(\operatorname{Int} C_n)\subset\operatorname{Int} D_n$, $(D_n, D_n\cap f_n(S^1))\approx(D^*, D^*\cap J^*)$ and $\phi_n|S^3-(f_{n-1}(S^1)\cup\operatorname{Int} D_n)=id$.

(F5) *If* $n\in B$, *then* $D_n\cap\bar{D}_n=\emptyset$ *where* $\bar{D}_n=\bigcup_{i<n}D_i$.

(F6) *If* $n\in A-B$, *then* $D_n\subset\operatorname{Int} D_i$ *for some* $i<n$. *If* k *is the smallest integer of such* i *in addition, then*

$$D_n\subset\operatorname{Int} D_k-\phi_{n-1}\cdots\phi_k h_k^{-1}(K_n)\quad\textit{where}\quad K_n=B^1\times(B^2-(1/n)B^2).$$

PROOF. The requirements in the lemma with (F1)–(F6) are defined by induction on n as follows:

Case 1: $V_n-\bar{C}_n\neq\emptyset$ *where* $\bar{C}_n=\bigcup_{i<n}C_i$. Let $n\in B$ and $n\in A$. By (F4) in the inductive assumptions, we have

$$f_{n-1}\,|\,V_n-\bar{C}_n=f_0\,|\,V_n-\bar{C}_n\quad\text{and}\quad f_{n-1}(V_n-\bar{C}_n)\subset S^3-\bar{D}_n.$$

Then, there are an open set $U_n\subset S^3-\bar{D}_n$ with $U_n\cap f_{n-1}(S^1)\subset f_{n-1}(V_n-\bar{C}_n)$ and $h_n\colon (U_n, U_n\cap f_{n-1}(S^1))\approx(E^3, E^1)$. Put $D_n=h_n^{-1}(B^3)$ and $C_n=f_{n-1}^{-1}(D_n)$. We may assume that $\operatorname{diam} D_n<1/2^n$ and $\operatorname{diam} C_n<1/n$. Then, (F2), (F3) and (F5) hold. By Lemma 3.2, we get an embedding $f_n\colon S^1\to S^3$ with (F4).

Case 2: $V_n\subset E(f_{n-1})$, *where* $E(f_{n-1})=f_{n-1}^{-1}(E(f_{n-1}(S^1)))$. Set $n\notin A$, $f_n=f_{n-1}$ and $\phi_n=id$.

Case 3: $V_n\subset\bar{C}_n$ *and* $V_n-E(f_{n-1})\neq\emptyset$. Set $n\in A$ and $m\notin B$. Since $\bar{C}_n$ is a finite union of pairwise disjoint arcs, (F3) implies that $V_n\subset\bigcup_{i<n}\operatorname{Int} C_i$. Take a point $p\in V_n-E(f_{n-1})$ and put

$$j(p)=\max\{i\,|\,p\in\operatorname{Int} C_i\},\quad k(p)=\min\{i\,|\,p\in\operatorname{Int} C_i\}\quad\text{and}$$
$$C(p)=\bigcup\{C_i\,|\,C_i\subset\operatorname{Int} C_{j(p)}\}.$$

Then, $V_n\cap(\operatorname{Int} C_{j(p)}-C(p))\ni p$ is open in S^1. (F4) shows that

$$f_{n-1}=f_{j(p)}\quad\text{on}\quad\operatorname{Int} C_{j(p)}-C(p)\quad\text{and}$$
$$\phi_{n-1}=\phi_{j(p)}\quad\text{on}\quad\operatorname{Int} D_{j(p)}-D(p),$$

where $D(p)=\bigcup\{D_i\,|\,D_i\subset\operatorname{Int} D_{j(p)}\}$. Moreover

$$N=(\operatorname{Int} D_{j(p)}-D(p))-\phi_{n-1}\cdots\phi_{k(p)}h_{k(p)}^{-1}(K_n)$$

is a neighborhood of $f_{n-1}(p)$ in S^3. Since $p \notin E(f_{n-1})$, i.e., $f_{n-1}(S^1)$ is locally flat at $f_{n-1}(p)$, there is an open set $U_n \ni f_{n-1}(p)$ in Int N such that

$$U_n \cap f_{n-1}(S^1) \subset f_{n-1}(V_n \cap (\mathrm{Int}\, C_{j(p)} - C(p))).$$

Then, we can define h_n, D_n and C_n with (F2), (F3) and (F6), and an embedding f_n with (F4) by Lemma 3.2.□

PROOF OF THEOREM II. Let $g: S^1 \to S^3$ be a PL embedding with $g(S^1)=K$. By using Lemma 3.3 for $f_0=g$, we define J as follows.

Since $d(f_n, f_{n-1})<1/2^n$ by (F2) and (F4), $\{f_n\}$ is a Cauchy sequence converging to a continuous map f: $S^1 \to S^3$. We show that f is an embedding, and put $J=f(S^1)$.

By induction, it is easy to check that, for all $i \geqslant n$,

$$f_i(\mathrm{Int}\, C_n) \subset \mathrm{Int}\, D_n \quad \text{and} \quad f_i(S^1 - \mathrm{Int}\, C_n) \subset S^3 - \mathrm{Int}\, D_n.$$

From this, we have

(i) $f(\mathrm{Int}\, C_n) \subset \mathrm{Int}\, D_n$ $(n \geqslant 1)$ and

(ii) $f(S^1 - \mathrm{Int}\, C_n) \subset S^3 - \mathrm{Int}\, D_n$ $(n \geqslant 1)$.

To see (i), we take $x \in \mathrm{Int}\, C_n$. Suppose that $x \in \mathrm{Int}\, C_k \subset C_k \subset \mathrm{Int}\, C_n$ for some $k>n$. Then, $f_i(x) \in \mathrm{Int}\, D_k$ for each $i \geqslant k$, and so $f(x)=\lim f_i(x) \in D_k \subset \mathrm{Int}\, D_n$. Suppose that $x \notin \mathrm{Int}\, C_i$ $(i>n)$. Then, $f_i(x)=f_n(x)$ $(i \geqslant n)$, and so $f(x)=f_n(x) \in \mathrm{Int}\, D_n$. Thus, (i) holds. (ii) is also easy to varify.

Now let x, $y \in S^1$ be distinct points.

Case 1. If $x \in \mathrm{Int}\, C_n$ and $y \in S^1 - \mathrm{Int}\, C_n$, then $f(x) \in \mathrm{Int}\, D_n$ and $f(y) \in S^3 - \mathrm{Int}\, D_n$ by (i) and (ii), and so $f(x) \neq f(y)$.

Case 2. If x, $y \in S^1 - \bigcup_{n \in A} \mathrm{Int}\, C_n$, then $f(x)=g(x) \neq g(y)=f(y)$.

Case 3. If x, $y \in \mathrm{Int}\, C_n$ and x, $y \in S^3 - \mathrm{Int}\, C_i$ for every $i>n$, then we have $f(x)=f_n(x) \neq f_n(y)=f(y)$. Thus, f is an embedding.

We shall see that J is everywhere wild by proving the following claims A1–3:

Claim A.1. *$E(f_n) \subset E(f_i)$ and $E(f_n) \cap C_i = \emptyset$ $(n<i)$.*

This claim is shown by induction.

Claim A.2. *For each $n \notin A$, $f(V_n)=f_n(V_n)$ is everywhere wild.*

In fact, let $n \notin A$. Then, $f_n(V_n)=f_{n-1}(V_n)$ is everywhere wild by (F1). Thus, $V_n \subset E(f_n)$ and so $V_n \subset S^1 - C_i$ $(i>n)$ by claim A.1. Then, $f_i | V_n = f_n | V_n$ $(i>n)$, and hence we have Claim A.2.

Claim A.3. *$\bigcup_{n \notin A} V_n$ is dense in S^1.*

If $n \in A$, then $V_n \supset C_n \supset \mathrm{Int}\, C_n$ $\mathrm{Int}\, E(f_n) \neq \emptyset$ by (F4). Hence, $\mathrm{Int}\, C_n \cap \mathrm{Int}\, E(f_n) \supset V_i$ for some $i > n$. Thus, we have the claim.

Since $E(f)$ is a closed subset of S^1, Claims A.2 and A.3 show that $E(f) = S^1$, i.e., J is everywhere wild.

Finally, we shall prove that $S^3 - J \approx S^3 - K$ by showing the following Claims B1–6: For each n, we define closed sets $A(n, i)$ $(i \in B)$ of $S^3 - K = S^3 - g(S^1)$ by

$$A(n, i) = \begin{cases} h_i^{-1}(B^1 \times ((1/n)B^2 - \{0\})) = h_i^{-1}(B^1 \times (1/n)B^2) - K & (i < n) \\ h_i^{-1}(B^1 \times (B^2 - \{0\})) = D_i - K & (i \geqslant n). \end{cases}$$

Claim B.1. *The collection* $\{A(n, i)\}_{i \in B}$ *is locally finite in* $S^3 - K$.

Suppose that this claim is false. Then, there are a point $y \in S^3 - K$ and a sequence $\{y_k\}$ in $S^3 - K$ converging to y such that $y_k \in A(n, i(k))$ for a sequence $i(1) < i(2) < \cdots$ in B. Since $A(n, i(k)) \subset D_{i(k)}$, $\lim \mathrm{diam}\, D_{i(k)} = 0$ and $D_{i(k)} \cap K = f_{i(k)-1} \cdot (C_{i(k)}) \neq \emptyset$, we see that $y \in K$. This is a contradiction; and the claim follows.

By virtue of this claim, we can define open sets X_n of $S^3 - K$ by

$$X_n = (S^3 - K) - \cup_{i \in B} A(n, i) \quad (n \geqslant 1).$$

Claim B.2. $X_1 \neq X_2 \subset \cdots$ *and* $\cup_n X_n = S^3 - K$.

This follows easily from the definition of $\{X_n\}$.

Claim B.3. $\phi_{n-1} \cdots \phi_1(X_n) \subset S^3 - (f_{n-1}(S^1) \cup \tilde{D}_n)$, *where* $\tilde{D}_n = \cup_{i > n} D_i$.

We prove this by induction on n. This holds for $n = 1$ since $X_1 = S^3 - (K \cup \cup_{i \in B} D_i) = S^3 - (g(S^1) \cup \tilde{D}_1)$ by (F5).

If $n - 1 \notin B$, then $X_n = X_{n-1}$ and

$$\begin{aligned} \phi_{n-1} \cdots \phi_1(X_n) &\subset \phi_{n-1}(S^3 - (f_{n-2}(S^1) \cup \tilde{D}_{n-1})) \quad \text{(by induction hypothesis)} \\ &= \phi_{n-1}(S^3 - (f_{n-1}(S^1) \cup \tilde{D}_{n-1})) \subset S^3 - (f_{n-1}(S^1) \cup \tilde{D}_n) \quad \text{(by (F4))}. \end{aligned}$$

If $n - 1 \in B$, then $X_n = X_{n-1} \cup h_{n-1}^{-1}(K_n)$ and

$$\phi_{n-1} \cdots \phi_1 h_{n-1}^{-1}(K_n) = \phi_{n-1} h_{n-1}^{-1}(K_n), \quad \phi_{n-1} h_{n-1}^{-1}(K_n) \cap f_{n-1}(S^1) = \emptyset.$$

Thus, it suffices to show that

$$\phi_{n-1} h_{u-1}^{-1}(K_n) \cap D_i = \emptyset \qquad \text{for each} \quad i \geqslant n.$$

This is trivial in case of $D_i \cap D_{n-1} = \emptyset$. If $D_i \subset \mathrm{Int}\, D_{n-1}$, then (F6) shows that

$$\begin{aligned} D_i &\subset \mathrm{Int}\, D_{n-1} - \phi_{i-1} \cdots \phi_{n-1} h_{n-1}^{-1}(K_i) \\ &\subset \mathrm{Int}\, D_{n-1} - \phi_{i-1} \cdots \phi_{n-1} h_{n-1}^{-1}(K_n) = \mathrm{Int}\, D_{n-1} - \phi_{n-1} h_{n-1}^{-1}(K_n). \end{aligned}$$

Therefore, we see Claim B.3.

By (F4), we see that $S^3-(f_{n-1}(S^1)\cup\tilde{D}_n)=S^3-(f(S^1)\cup\tilde{D}_n)$. Therefore, by Claim B.3, an embedding $\psi_n: X_n\to S^3-J=S^3-f(S^1)$ can be defined by

$$\psi_n = \phi_{n-1}\cdots\phi_1 | X_n \qquad \text{for each} \quad n.$$

Claim B.4. $\psi_{n+1}|X_n=\psi_n$.

Since $\phi_n|\phi_{n-1}\cdots\phi_1(X_n)=\mathrm{id}$ by Claim B.3 and (F4), we see Claim B.4.

Claim B.5. *For each* $y\in S^3-J$, $\{n|y\in D_n\}$ *is a finite set.*

Suppose that there is a sequence $n(1)<n(2)<\cdots$ such that $y\in D_{n(k)}$. Then, $D_{n(1)}\supset D_{n(2)}\supset\cdots$, $\lim\operatorname{diam} D_{n(k)}=0$, $C_{n(1)}\supset C_{n(2)}\supset\cdots$, $\lim\operatorname{diam} C_{n(k)}=0$, by (F3). Thus, $\{y\}=\bigcap_k D_{n(k)}=f(\bigcap_k C_{n(k)})\subset J$; and the claim follows.

Claim B.6. $S^3-J=\bigcup_n \psi_n(X_n)$.

For $y\in S^3-J$, put $k=\min\{n|y\in D_n\}$, $j=\max\{n|y\in D_n\}$ and $z=\phi_k^{-1}\cdots\phi_j^{-1}(y)\in D_k-K$. Then, $z\in X_n$ for some $n>j$. Thus,

$$\psi_n(z) = \phi_{n-1}\cdots\phi_1(z) = \phi_j\cdots\phi_k(z) = y,$$

and the claim holds.

Now, by Claims B.2, B.4 and B.6, we have a hoeomorphism $\psi: S^3-K\approx S^3-J$ given by $\psi|X_n=\psi_n$.

This completes the proof of Theorem II.□

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