HIROSHIMA MATH. J.
17 (1987), 47–66

# Genera and arithmetic genera of commutative rings

Akira OOISHI

(Received April 25, 1986)

## Introduction

Let $(R, \mathfrak{m})$ be a $d$-dimensional noetherian local ring and $I$ an $\mathfrak{m}$-primary ideal of $R$. Then it is well known that there exist (uniquely determined) integers $e_i$ $(0 \leqq i \leqq d)$ such that

$$\ell(R/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d$$

for all sufficiently large $n$. $e_0$ is the multiplicity of $I$. The significance of the other coefficients $e_i$ has been studied by some people (cf. [4], [10], [16], [17], [19], [23]). But it seems that much more remains to be done. The aim of this paper is to study the significance of the invariants $e_d$ and $e_0 - e_1 + \cdots + (-1)^d e_d$. We introduce the notion of *genus* and *arithmetic genus* of an ideal, and study the properties of local rings by these invariants. We also introduce the notion of *normal genus* and *normal arithmetic genus* by considering the function $\ell(R/\overline{I^{n+1}})$, where $\bar{J}$ denotes the integral closure of an ideal $J$. These invariants might be better to study singularities.

In §1, we investigate properties of general polynomial functions by introducing various invariants (e.g. the genus and the arithmetic genus) of a polynomial function.

In §2, we consider the Hilbert functions of graded modules. We express the genera by the local cohomology modules and study the relation of the genera and the vanishing of local cohomology modules.

In §3, the (normal) genus and the (normal) arithmetic genus of an ideal are defined, and we express them in terms of the sheaf cohomology of (normalized) blowing-up schemes.

In §4, we investigate the relation between the genera and the reduction exponents of ideals.

In §5 and §6, we examine the cases of dimension one and two (curve and surface singularities) in detail.

## Notation and terminology

All rings are commutative noetherian rings with unit.

Let $I$ be an ideal of a ring $R$. We denote by $\bar{I}$ the integral closure of $I$, i.e., the set of elements $x \in R$ such that $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ for some $a_i \in I^i$. We put $R(I) = \oplus_{n \geqq 0} I^n$, $\bar{R}(I) = \oplus_{n \geqq 0} \overline{I^n}$, $G(I) = \oplus_{n \geqq 0} I^n/I^{n+1}$ and $\bar{G}(I) = \oplus_{n \geqq 0} \overline{I^n}/\overline{I^{n+1}}$. If $(R, \mathfrak{m})$ is a local ring, then we put $G(R) = G(\mathfrak{m})$ and $\bar{G}(R) = \bar{G}(\mathfrak{m})$.

$\ell(M)$ denotes the length of an $R$-module $M$.

$Q(R)$ denotes the total quotient ring of $R$. For any $R$-submodules $I$ and $J$ of $Q = Q(R)$, put $(I: J) = \{x \in Q | xJ \subset I\}$ and $(I: J)_R = (I: J) \cap R$. $\bar{R}$ denotes the integral closure of $R$.

For $m, n \in \boldsymbol{Z}$, $n \geqq 0$, $\binom{m}{n}$ denotes the binomial coefficient with an agreement that $\binom{m}{n} = 0$ if $m < n$, and $\binom{0}{0} = 1$.

## § 1. Some invariants of polynomial functions

We assume that all functions $f$: $\boldsymbol{Z} \to \boldsymbol{Z}$ considered in this section satisfy the following condition: $f(n) = 0$ for all $n \ll 0$. We say that $f$ is a *polynomial function* if there is a (uniquely determined) polynomial $P_f \in \boldsymbol{Q}[t]$ such that $f(n) = P_f(n)$ for all $n \gg 0$. Denote by $\mathscr{P}$ the set of all polynomial functions and set $\mathscr{P}^+ = \{f \in \mathscr{P} | f(n) = 0$ for all $n < 0\}$. Put $(\Delta f)(n) = f(n) - f(n-1)$ and $(\nabla f)(n) = \sum_{i \leqq n} f(i)$. Then for any $f \in \mathscr{P}$, we have $\Delta f$, $\nabla f \in \mathscr{P}$, $\Delta(\nabla f) = f$, $\nabla(\Delta f) = f$ and $P_{\Delta f}(t) = P_f(t) - P_f(t-1)$. It is well known that if $f$ is a polynomial function, then there are (uniquely determined) integers $d \geqq 0$, $e_i$ $(0 \leqq i \leqq d)$, $e_0 \neq 0$, such that $(\nabla f)(n) = e_0 \binom{n+d}{d} - e_1 \binom{n+d-1}{d-1} + \cdots + (-1)^d e_d$ for all $n \gg 0$. Note that $d = \deg(P_f) + 1$ (resp. $d = 0$) if $P_f \neq 0$ (resp. $P_f = 0$). Put $d(f) = d$, $e_i(f) = e_i$, $e(f) = e_0$, $g(f) = e_d = (-1)^d P_{\nabla f}(-1)$, $\chi(f) = e_0 - e_1 + \cdots + (-1)^d e_d = P_{\nabla f}(0)$, $p_a(f) = (-1)^d(\chi(f) - \sum_{n \leqq 0} f(n))$, $n(f) = \min\{n | f(m) = P_f(m)$ for all $m > n\}$ and $F_f(t) = \sum_{n \in \boldsymbol{Z}} f(n) t^n \in \boldsymbol{Z}[[t]][t^{-1}]$. We call $d(f)$, $e_i(f)$, $e(f)$, $g(f)$, $\chi(f)$, $p_a(f)$, $n(f)$ and $F_f(t)$ the *dimension*, the *i*-th *Hilbert coefficient*, the *multiplicity*, the *genus*, the *Euler characteristic*, the *arithmetic genus*, the *postulation number* and the *Hilbert series* of $f$ respectively. Note that $\chi(\Delta f) = P_f(0)$, $\chi(f) = P_f(0) + (-1)^d g(f)$, and $g(f) - p_a(f) = (-1)^d(f(0) - P_f(0))$ if $f \in \mathscr{P}^+$. The following proposition may be well known (except for the assertion on the postulation number) and is easy to show. So we omit the proof.

PROPOSITION 1.1. *For any function $f$: $\boldsymbol{Z} \to \boldsymbol{Z}$, $f \neq 0$, the following conditions are equivalent*:

(1) *$f$ is a polynomial function.*

(2) *$F_f(t) = \varphi(t)/(1-t)^d$ for some integer $d \geqq 0$ and $\varphi(t) \in \boldsymbol{Z}[t, t^{-1}]$ such that $\varphi(1) \neq 0$.*

*Moreover, if these conditions are satisfied, then we have $d(f) = d$, $e(f) = \varphi(1)$*

*and* $n(f)=\deg(F_f)=\deg(\varphi)-d$. *We denote by* $\varphi_f$ *the (Laurent) polynomial* $\varphi$ *determined by* $f$.

REMARK 1.2. (1) If we define the product $f*g$ of $f$, $g\in\mathscr{P}$ by $(f*g)(n)=\sum_{i+j=n}f(i)g(j)$, then $\mathscr{P}$ becomes a commutative ring and $\mathscr{P}^+$ is a subring of $\mathscr{P}$ (the additions are usual ones). We have $F_{f*g}(t)=F_f(t)F_g(t)$, and $f$ is in $\mathscr{P}^+$ if and only if $\varphi_f\in \boldsymbol{Z}[t]$. Hence by corresponding $F_f$ to $f$, the ring $\mathscr{P}$ (resp. $\mathscr{P}^+$) is isomorphic to $\boldsymbol{Z}[t, t^{-1}, (1-t)^{-1}]$ (resp. $\boldsymbol{Z}[t, (1-t)^{-1}]$).

(2) For any $f\in\mathscr{P}$, we have $d(\nabla f)=d(f)+1$, $d(\Delta f)=d(f)-1$ if $d(f)\geqq 1$, $e_i(\nabla f)=e_i(f)$ $(0\leqq i\leqq d(f))$, $e_i(\Delta f)=e_i(f)$ $(0\leqq i<d(f))$, $F_f(t)=(1-t)F_{\nabla f}(t)=F_{\Delta f}(t)(1-t)^{-1}$ and $n(f)=n(\nabla f)+1=n(\Delta f)-1$.

THEOREM 1.3. *Let* $f$ *be a polynomial function and put* $d(f)=d$, $\varphi_f(t)=\sum_{n\in\boldsymbol{Z}}a_nt^n$, $C(f)=\sum_{n<0}f(n)$. ($C(f)=0$ *if* $f\in\mathscr{P}^+$.)

(1) *If* $d=0$, *then* $f(n)=a_n$ *for all* $n\in\boldsymbol{Z}$, $P_f=0$, $P_{\nabla f}=g(f)=\chi(f)=e(f)=\sum_{n\in\boldsymbol{Z}}a_n$ *and* $p_a(f)=\sum_{n\geqq 0}a_n$.

(2) *If* $d\geqq 1$, *then* $f(n)=\sum_{i\in\boldsymbol{Z}}a_i\binom{d+n-i+1}{d-1}$, $(\nabla f)(n)=\sum_{i\in\boldsymbol{Z}}a_i\binom{d+n-i}{d}$ *and* $f(n)-P_f(n)=(-1)^d\sum_{i\geqq d}a_{n+i}\binom{i-1}{d-1}$ *for all* $n\in\boldsymbol{Z}$.

(3) $a_n=\sum_{i+j=n,0\leqq i\leqq d}(-1)^i\binom{d}{i}f(j)$ *for all* $n\in\boldsymbol{Z}$.

(4) $f\in\mathscr{P}^+$ *if and only if* $a_n=0$ *for all* $n<0$, *i.e.,* $\varphi_f\in\boldsymbol{Z}[t]$. *In this case,* $a_0=f(0)$ *and* $a_1=f(1)-df(0)$.

(5) $e(f)=\sum_{n\in\boldsymbol{Z}}a_n$.

(6) $C(f)=\sum_{i<0}a_i\binom{d-i-1}{d}$ *and* $\sum_{n\leqq 0}f(n)=\sum_{i\leqq 0}a_i\binom{d-i}{d}$.

(7) *If* $n(f)\leqq 0$, *then* $g(f)=(-1)^d(f(0)-P_f(0)+C(f))=a_d+(-1)^dC(f)$, $\chi(f)=f(0)+C(f)$ *and* $p_a(f)=0$. *Hence* $n(f)<0$ *if and only if* $n(f)\leqq 0$ *and* $g(f)=(-1)^dC(f)$.

(8) *If* $n(f)>0$, *then* $g(f)=(-1)^d(\sum_{i=0}^{n(f)}(f(i)-P_f(i)+C(f))=\sum_{i=0}^{n(f)}a_{d+i}\binom{d+i}{d}+(-1)^dC(f)$, $\chi(f)=\sum_{i=1}^{n(f)}(f(i)-P_f(i))+f(0)+C(f)=(-1)^d\sum_{i=0}^{n(f)-1}a_{d+i+1}\binom{d+i}{d}+f(0)+C(f)$ *and* $p_a(f)=(-1)^d\sum_{i=1}^{n(f)}(f(i)-P_f(i))=\sum_{i=0}^{n(f)-1}a_{d+i+1}\binom{d+i}{d}$.

(9) *If* $f\in\mathscr{P}^+$, *then* $g(f)-p_a(f)=\sum_{i=0}^{n(f)}a_{d+i}\binom{d+i-1}{i}$.
*Hence* $g(f)\geqq p_a(f)\geqq 0$ *if* $a_n\geqq 0$ *for all* $n\geqq d$.

PROOF. (1) is trivial. (2) Since $\sum_{n\in\boldsymbol{Z}}f(n)t^n=F_f(t)=(\sum_{n\in\boldsymbol{Z}}a_nt^n)(1-t)^{-d}=(\sum_{n\in\boldsymbol{Z}}a_nt^n)(\sum_{n\in\boldsymbol{Z}}\binom{d+n-1}{d-1}t^n)$, we have $f(n)=\sum_{i+j=n}a_i\binom{d+j-1}{d-1}$ for all $n\in\boldsymbol{Z}$. The formula for $\nabla f$ follows from this. (Note that $\sum_{i\leqq m}\binom{n+i}{n}=\binom{m+n+1}{n+1}$ for all $m\in\boldsymbol{Z}$ and $n\geqq 0$.) For each $r\in\boldsymbol{Z}$, $r\geqq 0$, define $k\in\mathscr{P}$ by

$k(n)=\binom{n+r}{r}$. Then $P_k(n)=k(n)$ $(n\geqq 0)$, $P_k(n)=k(n)=0$ $(-r\leqq n<0)$ and $P_k(n)=(-1)^r\binom{-n-1}{r}$ $(n<-r)$. Hence $f(n)-P_f(n)=-\sum_{n-i<-(d-1)}(-1)^{d-1}a_i\binom{i-n-1}{d-1}=(-1)^d\sum_{n-i\leqq -d}a_i\binom{i-n-1}{d-1}=(-1)^d\sum_{j\geqq d}a_{n+j}\binom{j-1}{d-1}$. (3) Since $\sum_{n\in\mathbf{Z}}a_nt^n=\varphi_f(t)=(1-t)^dF_f(t)=(\sum_{i=0}^d(-1)^i\binom{d}{i}t^n)(\sum_{n\in\mathbf{Z}}f(n)t^n)$, we have $a_n=\sum_{i+j=n,0\leqq i\leqq d}(-1)^i\binom{d}{i}f(j)$ for all $n\in\mathbf{Z}$. (4) follows from (3). (5) Since $\binom{d+n-i-1}{d-1}$ is a polynomial function of dimension $d$, we have $e(f)=\sum_{n\in\mathbf{Z}}a_n$ by (2). (6) We may assume that $d\geqq 1$. Then $C(f)=\sum_{n<0}f(n)=\sum_{n<0}\sum_{i\in\mathbf{Z}}a_i\binom{d+n-i-1}{d-1}=\sum_{i<0}a_i\left(\sum_{n<0}\binom{d+n-i-1}{d-1}\right)=\sum_{i<0}a_i\binom{d-i-1}{d}$. The proof of the formula for $\sum_{n\leqq 0}f(n)$ is similar. Since (7) can be proved as (8), we have only to show (8). Put $n(f)=m$. Then for all $n\gg 0$, we have $(\nabla f)(n)=\sum_{i<0}f(i)+\sum_{i=0}^m f(i)+\sum_{i=m+1}^n f(i)=C(f)+\sum_{i=0}^m(f(i)-P_f(i))+\sum_{i=0}^n P_f(i)$. It is easy to see that the polynomial $P(t)$ determined by $P(n)=\sum_{i=0}^n P_f(i)$ for all $n\in\mathbf{Z}$, $n\geqq 0$ is a multiple of $t+1$. Hence $(-1)^d g(f)=P_{\nabla f}(-1)=\sum_{i=0}^m(f(i)-P_f(i))+C(f)$ and $\chi(f)=(-1)^d g(f)+P_f(0)=\sum_{i=1}^m(f(i)-P_f(i))+f(0)+C(f)$. Other assertions follow from (2). Q.E.D.

COROLLARY 1.4. *Suppose that* $f\in\mathscr{P}^+$, $f(0)=1$, *and put* $d(f)=d$, $e(f)=e$, $f(1)=v$.

(1) *In general, we have* $n(f)+d(f)=\deg(\varphi_f)\geqq 0$.

(2) $n(f)+d(f)=0$ *if and only if* $f(n)=\binom{n+d-1}{n}$ *for all* $n\geqq 1$.

(3) $n(f)+d(f)\leqq 1$ *if and only if* $f(n)=\binom{n+d-1}{n}+(v-d)\binom{n+d-2}{n-1}$ *for all* $n\geqq 1$. *In this case, we have* $v=e+d-1$.

(4) $n(f)+d(f)\leqq 2$ *if and only if* $f(n)=\binom{n+d-1}{n}+(v-d)\binom{n+d-2}{n-1}+(e-v+d-1)\binom{n+d-3}{n-2}$ *for all* $n\geqq 2$.

EXAMPLE 1.5 (Graded modules of polynomial growth). Let $A=\oplus_{n\geqq 0}A_n$ be a graded ring with $A_0=R$. Then a graded $A$-module $M=\oplus_{n\in\mathbf{Z}}M_n$ is said to be a graded $A$-module *of polynomial growth* if each $M_n$ is an $R$-module of finite length and $f(n)=\ell(M_n)$ is a polynomial function. In this case, put $H(M,n)=\ell(M_n)$ and we write $h(M,t)$, $d(M)$, $e_i(M)$, $e(M)$, $n(M)$, $F(M,t)$, $\varphi_M(t)$, $g(M)$, $\chi(M)$, $p_a(M)$ instead of $P_f(t)$, $d(f)$, $e_i(f)$, $e(f)$, $n(f)$, $F_f(t)$, $\varphi_f(t)$, $g(f)$, $\chi(f)$, $p_a(f)$ respectively. Note that $(\nabla H)(M,n)=H(M[X],n)$ with $\deg(X)=1$, and if $a\in A$ is a homogeneous $M$-regular element of degree $r$, then $F(M/aM,t)=(1-t^r)F(M,t)$ and $n(M/aM)=n(M)+r$. If $A$ is a homogeneous algebra over an artinian ring, then a finitely generated graded $A$-module $M$ is a graded $A$-module of polynomial growth with $d(M)=\dim(M)$ (see §2 for this case). If $X$ is a projective variety over an algebraically closed field $k$ and $D$ is an ample Cartier divisor on $X$, then $A=\oplus_{n\geqq 0} H^0(X, \mathcal{O}_X(nD))$ is a noetherian graded $k$-algebra of polynomial growth.

(1) Let $X$ be a $d$-dimensional abelian variety and $D$ an ample Cartier divisor on $X$. Put $A=\oplus_{n\geqq 0} H^0(X, \mathcal{O}_X(nD))$. Then $e:=e(A)=(D^d)$ and $H(A, n)=en^d/d!$ for all $n\geqq 1$. Hence we have $\varphi_A(t)=(1-t)^{d+1}+e/d! \sum_{0\leqq i<n<d} (-1)^i\binom{d+1}{i}(n-i)^d t^n$, $n(A)=0$, $p_a(A)=0$ and $g(A)=(-1)^{d+1}$. If $D$ is very ample, then $A$ is a Buchsbaum ring with $\dim(A)=d+1$, $\operatorname{depth}(A)=2$, $I(A)=\sum_{i=1}^d\binom{d}{i}\binom{d}{i-1}$, and $A$ is Cohen-Macaulay$\Leftrightarrow A$ is Gorenstein$\Leftrightarrow X$ is an elliptic curve (cf. [25]).

(2) If $X$ is a smooth projective surface of general type and $K$ is its canonical divisor, then the canonical ring $A=\oplus_{n\geqq 0} H^0(X, \mathcal{O}_X(nK))$ of $X$ is a 3-dimensional noetherian ring with $H(A, n)=(K^2)n(n-1)/2+\chi(\mathcal{O}_X)$ for all $n\geqq 2$ (cf. [2]). Hence $\varphi_A(t)=1+(v-3)t+(e-2v-q+4)t^2+(v+2q-3)t^3+(1-q)t^4$, where $e=(K^2)$, $v=p_g(X)$, $q=q(X)$. We have $g(A)=p_g(X)-2q(X)+1$, $p_a(A)=1-q(X)$, $|n(A)|\leqq 1$, and $n(A)\leqq 0$ (resp. $n(A)=-1$) if and only if $q(X)=1$, i.e., $p_a(A)=0$ (resp. $p_g(X)=q(X)=1$, i.e., $p_a(A)=g(A)=0$). If $K$ is ample, then $\operatorname{reg}(A)=4$, and $A$ is Cohen-Macaulay$\Leftrightarrow A$ is Gorenstein$\Leftrightarrow q(X)=0$. (Concerning $\operatorname{reg}(A)$, see [20].)

(3) Let $X$ be a Del Pezzo surface and $A=\oplus_{n\geqq 0} H^0(X, \mathcal{O}_X(-nK))$ its anti-canonical ring. Then $A$ is a 3-dimensional Gorenstein algebra and $H(A, n)=n(n+1)(9-r)/2+1$ for all $n\geqq 0$, where $r=9-(K^2)$. Hence $\varphi_A(t)=1+(7-r)t+t^2$, $n(A)=-1$ and $g(A)=p_a(A)=0$. (cf. Demazure, Lecture Notes in Math. vol. 777.)

EXAMPLE 1.6. Let $P$ be a $d$-dimensional lattice complex, i.e., a simplicial complex in $\boldsymbol{R}^m$ whose vertices are in $\boldsymbol{Z}^m$, then the function $f(n)=\operatorname{Card}(nP\cap \boldsymbol{Z}^m)$ ($n\geqq 0$) is a polynomial function such that $d(f)=d+1$ and $n(f)\leqq 0$. Hence $p_a(f)=0$. If $P$ is a $d$-dimensional lattice convex polytope in $\boldsymbol{R}^d$, then we have $n(f)<0$ and $e(f)=d!$ (the volume of $P$). Hence $p_a(f)=g(f)=0$ (cf. [1], [26]).

## § 2. The case of graded rings

In this section, $A=\oplus_{n\geqq 0} A_n=R[A_1]$ denotes a noetherian homogeneous algebra over an artinian local ring $R=A_0$ and $M=\oplus_{n\in \boldsymbol{Z}} M_n$ is a finitely generated graded $A$-module with $\dim(M)=d$. Put $P=A_+$.

LEMMA 2.1. *$H(M, n)-h(M, n)=\sum_{i=0}^d(-1)^i\ell([H_P^i(M)]_n)$ for all $n\in \boldsymbol{Z}$. In particular, if $M$ is Cohen-Macaulay, then $H(M, n)-h(M, n)=(-1)^d \ell([H_P^d(M)]_n)$ for all $n\in \boldsymbol{Z}$.*

Proof. Put $X=\mathrm{Proj}(A)$. Then $h(M, n)=\chi(X, \tilde{M}(n))=\sum_{i=0}^{\infty}(-1)^i \ell(H^i(X, \tilde{M}(n)))$ for all $n\in \boldsymbol{Z}$, $H_P^{i+1}(M)=\oplus_{n\in\boldsymbol{Z}} H^i(X, \tilde{M}(n))$ $(i\geqq 1)$, and we have the following exact sequence (cf. [6], (2.1.3), (2.5.3)):

$$0 \longrightarrow H_P^0(M) \longrightarrow M \longrightarrow \oplus_{n\in\boldsymbol{Z}} H^0(X, \tilde{M}(n)) \longrightarrow H_P^1(M) \longrightarrow 0.$$

Hence $h(M, n) = \sum_{i=1}^{\infty}(-1)^i\ell(H^i(X, \tilde{M}(n))) + \ell(H^0(X, \tilde{M}(n))) = \sum_{i=1}^{\infty}(-1)^i \ell([H_P^{i+1}(M)]_n) + \ell([H_P^1(M)]_n) + \ell(M_n) - \ell([H_P^0(M)]_n) = H(M, n) - \sum_{i=0}^{d}(-1)^i \ell([H_P^i(M)]_n)$. Q.E.D.

Corollary 2.2. *We have* $n(M)\leqq \mathrm{reg}(M)-\mathrm{depth}(M)$. *Moreover, if* $M$ *is Cohen-Macaulay, then* $n(M)=a(M):=\mathrm{reg}(M)-\dim(M)$.

Proof. If $n>\mathrm{reg}(M)-\mathrm{depth}(M)$, then for any $i\geqq \mathrm{depth}(M)$, we have $n+i\geqq n+\mathrm{depth}(M)>\mathrm{reg}(M)$, which implies that $[H_P^i(M)]_n=0$. Hence $H(M, n)=h(M, n)$ for all $n>\mathrm{reg}(M)-\mathrm{depth}(M)$, i.e., $n(M)\leqq \mathrm{reg}(M)-\mathrm{depth}(M)$. The second assertion follows from Lemma 2.1 immediately. Q.E.D.

Example 2.3. Assume that $R$ is a field $k$ and $A$ has a pure resolution:

$$0 \longrightarrow F_r \longrightarrow \cdots \longrightarrow F_1 \longrightarrow S \longrightarrow A \longrightarrow 0,\ F_i \cong S^{b_i}(-a_i) \quad (1\leqq i\leqq r),$$

where $S=k[X_1,\dots, X_v]$, $v=\mathrm{emb}(A)$, $0<a_1<\cdots<a_r$ and $r=\mathrm{hd}_S(A)$. Then we have $\mathrm{reg}(A)=a_r-r$ and $\varphi_A(t)=1+b_1t^{a_1}+\cdots+b_rt^{a_r}$. Hence $n(A)=\mathrm{reg}(A)-\mathrm{depth}(A)$.

Theorem 2.4. *Suppose that* $M=\oplus_{n\geqq 0} M_n$.

(1) *If* $n(M)\leqq 0$, *then* $p_a(M)=0$ *and* $g(M)=\sum_{i=0}^{d}(-1)^{d+i}\ell([H_P^i(M)]_0)$.

(2) $n(M)<0$ *if and only if* $n(M)\leqq 0$ *and* $g(M)=0$.

(3) *If* $n(M)>0$, *then* $g(M)=\sum_{n=0}^{n(M)}\sum_{i=0}^{d}(-1)^{d+i}\ell([H_P^i(M)]_n)$ *and* $p_a(M)=\sum_{n=1}^{n(M)}\sum_{i=0}^{d}(-1)^{d+i}\ell([H_P^i(M)]_n)$.

Proof. This follows from Theorem 1.3 and Lemma 2.1. Q.E.D.

Corollary 2.5. *Suppose that* $M=\oplus_{n\geqq 0} M_n$ *is Cohen-Macaulay.*

(1) $g(M)\geqq p_a(M)\geqq 0$ *and* $g(M)-p_a(M)=\ell([H_P^d(M)]_0)$.

(2) $p_a(M)=0$ *if and only if* $a(M)\leqq 0$. $g(M)=0$ *if and only if* $a(M)<0$ (or equivalently $g(M)=p_a(M)$).

(3) *If* $a(M)>0$, *then* $g(M)=\sum_{n=0}^{a(M)}\ell([H_P^d(M)]_n)$ *and* $p_a(M)=\sum_{n=1}^{a(M)}\ell([H_P^d(M)]_n)$.

Theorem 2.6. *Let* $A$ *be a Gorenstein homogeneous algebra over a field, and put* $\dim(A)=d$, $\mathrm{emb}(A)=v$, $e(A)=e$, $a(A)=a$.

(1) *If* $a(M)>0$, *then* $g(A)=\sum_{n=0}^{a(A)} H(A, n)$ *and* $p_a(A)=\sum_{n=0}^{a(A)-1} H(A, n)$.

(2) $a(A)\leqq 0$ (*resp.* $a(A)<0$, $a(A)=0$, $a(A)=1$) *if and only if* $p_a(A)=0$

*(resp.* $g(A)=0$, $g(A)=1$, $p_a(A)=1$).

(3) $g(A)\geqq 2\Leftrightarrow a(A)\geqq 1\Leftrightarrow g(A)\geqq \operatorname{emb}(A)+1\geqq 2$. $p_a(A)\geqq 2\Leftrightarrow a(A)\geqq 2\Leftrightarrow p_a(A)\geqq \operatorname{emb}(A)+1\geqq 2$. $a(A)=1$ *(resp.* $a(A)=2$) *if and only if* $g(A)=\operatorname{emb}(A)+1\geqq 2$ *(resp.* $p_a(A)=\operatorname{emb}(A)+1\geqq 2$).

(4) $g(A)\leqq\binom{v+a}{v}\leqq\binom{e}{v}$ *and* $p_a(A)\leqq\binom{v+a-1}{v}\leqq\binom{e-1}{v}$. $g(A)=\binom{v+a}{v}$ *(resp.* $p_a(A)=\binom{v+a-1}{v}$) *if and only if* $a(A)\leqq i(A)-1$ *(resp.* $a(A)\leqq i(A)$), *where* $i(A)=\min\left\{n \mid H(A, n)\neq\binom{v+n-1}{n}\right\}$, *the initial degree of* $A$. $g(A)=\binom{e}{v}$ *(resp.* $p_a(A)=\binom{e-1}{v}$) *if and only if A is a hypersurface or* $v=e+d-2$, $d\geqq 1$ *(resp. A is a hypersurface or* $v=e+d-2$).

(5) *If* $\operatorname{reg}(A)\geqq 3$ *and A is not a hypersurface, then* $g(A)\leqq\binom{[e/2]+2}{v}$ *and* $p_a(A)\leqq\binom{[e/2]+1}{v}$.

PROOF. Since $A(a)\cong K_A\cong \underline{\operatorname{Hom}}_k(H_P^d(A), k)$, we have $\ell([H_P^d(A)]_n)=\ell(A_{a-n})$ for all $n\in \boldsymbol{Z}$. This implies (1) by Theorem 2.4. (2) and (3) follow from (1). (4) $g(A)=\sum_{n=0}^{a} H(A, n)\leqq\sum_{n=0}^{a}\binom{v+n-1}{n}=\binom{v+a}{v}$, and the equality holds if and only if $i(A)\geqq a(A)+1$. Since $d+a=\operatorname{reg}(A)\leqq e+d-v$, we have $\binom{v+a}{v}\leqq\binom{e}{v}$, and the equality holds if and only if $A$ is a hypersurface or $v=e+d-2$ (i.e., $a+d=2$) (cf. [20], Proposition 13 and Theorem 15). The assertions for $p_a(A)$ are similarly proved. (5) Under the assumption, we have $\operatorname{reg}(A)\leqq e/2+d-v+{}+2$, i.e., $v+a\leqq e/2+2$ (cf. [20], Proposition 13). This implies the assertion.

Q.E.D.

EXAMPLE 2.7. (1) $g(A)$ and $p_a(A)$ are not necessarily non-negative. For example, put $A=k[X, Y]/(X^2, XY)$. Then we have $\varphi_A(t)=1+t-t^2$, $n(A)=1$, $a(A)=0$, $g(A)=p_a(A)=-1$.

(2) If $A$ is a complete intersection of type $(e_1, e_2)$ and $\operatorname{emb}(A)=v$, then $g(A)=\binom{e_1+e_2}{v}-\binom{e_1}{v}-\binom{e_2}{v}$ and $p_a(A)=\binom{e_1+e_2-1}{v}-\binom{e_1-1}{v}-\binom{e_2-1}{v}$. The formula for any complete intersection is similar.

(3) Let $A$ be a Cohen-Macaulay homogeneous algebra over a field. If $p_a(A)=1$, then $a(A)=1$. The canonical module $K_A$ of $A$ is Cohen-Macaulay, and we have $a(K_A)=0$, $g(K_A)=1$, $p_a(K_A)=0$.

(4) Let $A$ be the Stanley-Reisner ring $k[\Delta]$ (or more generally an ASL) on a simplicial complex $\Delta$ over a field $k$. Then $n(A)\leqq 0$ (cf. [27]) and we have $p_a(A)=0$ and $g(A)=(-1)^d(\chi(\Delta)-1)$, where $\chi(\Delta)$ is the Euler characteristic of $\Delta$, $d=\dim(\Delta)$.

## §3. Genera and arithmetic genera of local rings

Let $(R, \mathfrak{m}, k)$ be a noetherian local ring with $\dim(R)=d\geqq 1$, and let $I$ be an $\mathfrak{m}$-primary ideal of $R$. Then the Hilbert function $f(n)=H_I^0(n)=\ell(I^n/I^{n+1})$ of $I$ is a polynomial function. Also, if $R$ is analytically unramified, then the function $\bar{f}(n)=\bar{H}_I^0(n)=\ell(\overline{I^n}/\overline{I^{n+1}})$ is a polynomial function because $\bar{G}(I)$ is a finitely generated $G(I)$-module. We have $(\nabla f)(n)=\ell(R/I^{n+1})$ and $(\nabla\bar{f})(n)=\ell(R/\overline{I^{n+1}})$. We use the notation $e_i(I)$, $g(I)$, $\chi(I)$, $p_a(I)$ and $n(I)$ (resp. $\bar{e}_i(I)$, $\bar{g}(I)$, $\bar{\chi}(I)$, $\bar{p}_a(I)$ and $\bar{n}(I)$) instead of $e_i(f)$, $g(f)$, $\chi(f)$, $p_a(f)$ and $n(f)$ (resp. $e_i(\bar{f})$, $g(\bar{f})$, $\chi(\bar{f})$, $p_a(\bar{f})$ and $n(\bar{f})$). Note that $e_0(I)=\bar{e}_0(I)=e(I)$, the multiplicity of $I$, and we have $g(I)=g(G(I))$, $\bar{g}(I)=g(\bar{G}(I))$, etc. If $I=\mathfrak{m}$, we also put $g(\mathfrak{m})=g(R)$, $\bar{g}(\mathfrak{m})=\bar{g}(R)$, $p_a(\mathfrak{m})=p_a(R)$, $\bar{p}_a(\mathfrak{m})=\bar{p}_a(R)$, etc. We call $g(I)$, $p_a(I)$, $\bar{g}(I)$, $\bar{p}_a(I)$ the *genus*, the *arithmetic genus*, the *normal genus*, the *normal arithmetic genus* of $I$ respectively (cf. [22]). In what follows, when we consider $\bar{g}(I)$, $\bar{p}_a(I)$, etc, we always assume that $R$ is analytically unramified. Our aim of this section is to prove the following

THEOREM 3.1. *Assume that $R$ is Cohen-Macaulay, and put $X=\mathrm{Proj}\,(R(I))$, $D=\mathrm{Proj}\,(G(I))$, $\bar{X}=\mathrm{Proj}\,(\bar{R}(I))$ and $\bar{D}=\mathrm{Proj}\,(\bar{G}(I))$.*

(1) *If $d=1$, then $g(I)=\ell(H^0(X, \mathcal{O}_X)/R)$, $p_a(I)=g(I)-e(I)+\ell(R/I)$, $\bar{g}(I)=\ell(H^0(\bar{X}, \mathcal{O}_{\bar{X}})/R)=\ell(\bar{R}/R)$ and $\bar{p}_a(I)=\bar{g}(I)-e(I)+\ell(R/\bar{I})=\ell(I\bar{R}/\bar{I})$.*

(2) *If $d\geqq 2$, then $g(I)=\sum_{i=1}^{d-1}(-1)^{d+i+1}\ell(H^i(X, \mathcal{O}_X))$, $p_a(I)=g(I)+(-1)^d(\chi(D)-\ell(R/I))$, $\bar{g}(I)=\sum_{i=1}^{d-1}(-1)^{d+i+1}\ell(H^i(\bar{X}, \mathcal{O}_{\bar{X}}))$ and $\bar{p}_a(I)=\bar{g}(I)+(-1)^d(\chi(\bar{D})-\ell(R/\bar{I}))$, where $\chi(D)=\sum_{i=0}^{d-1}(-1)^i\ell(H^i(D, \mathcal{O}_D))$ and $\chi(\bar{D})=\sum_{i=0}^{d-1}(-1)^i\ell(H^i(\bar{D}, \mathcal{O}_{\bar{D}}))$. In particular, if $d=2$, then $g(I)=\ell(H^1(X, \mathcal{O}_X))$ and $\bar{g}(I)=\ell(H^1(\bar{X}, \mathcal{O}_{\bar{X}}))$.*

LEMMA 3.2. *Let $I$ be an ideal of a noetherian local ring $(R, \mathfrak{m})$, and let $F$ be a coherent module on $X=\mathrm{Proj}\,(R(I))$. Then $H^i(X, F)=0$ for all $i\geqq \ell(I):=\dim(G(I)/\mathfrak{m}G(I))$, and $\mathrm{Supp}_R(H^i(X, F))\subset V(I)$ for all $i\geqq 1$. In particular, if $I$ is $\mathfrak{m}$-primary, then $H^i(X, F)=0$ for all $i\geqq\dim(R)$ and $H^i(X, F)$ is an $R$-module of finite length for all $i\geqq 1$.*

PROOF. This follows from [6], (4.2.2). Q.E.D.

PROPOSITION 3.3. *Let $R$ be a noetherian ring, $M$ a finitely generated $R$-module and $I$ an ideal of $R$ which contains an $M$-regular element.*

(1) *Put $X=\mathrm{Proj}\,(R(I))$ and $F=R(I, M)^{\sim}$ the coherent module associated to $R(I, M)=\oplus_{n\geqq 0}I^nM$. Then*

$$H^0(X, F)\cong \lim_{n\to\infty}\mathrm{Hom}_R(I^n, I^nM)\cong \cup_{n=0}^{\infty}(I^nM: I^n)_{Q(M)},$$

*where $Q(M)=M_S$, $S=R-Z_R(M)$ the set of $M$-regular elements. More generally,* *we have* $H^0(X, F(r)) \cong \lim_{n\to\infty} \mathrm{Hom}_R(I^n, I^{n+r}M)$ *for all* $r \geqq 0$. *If* $\mathrm{depth}_I(M) \geqq 2$, *then* $H^0(X, F) \cong M$.

(2) *Assume that* $\bar{R}(I)$ *is noetherian, and put* $\bar{X} = \mathrm{Proj}(\bar{R}(I))$, $\bar{F} = \bar{R}(I, M)^{\sim}$, *where* $\bar{R}(I, M) = \oplus_{n\geqq 0} \overline{I^n M}$. *Then* $H^0(\bar{X}, \bar{F}(r)) = \lim_{n\to\infty} \mathrm{Hom}_R(\overline{I^n}, \overline{I^{n+r}M})$ *for all* $r \geqq 0$.

(3) *If* $R$ *is reduced and* $\bar{R}(I)$ *is noetherian, then* $H^0(\bar{X}, \mathcal{O}_{\bar{X}}(r)) \cong \overline{I^n}$ *for all* $r \geqq 0$.

PROOF. (1) Put $L = R(I, M)$, $A = H^0(X, F)$ and $B = \cup_{n=0}^{\infty} (I^n M : I^n)_{Q(M)}$. By the assumption , $I = (a_1, \ldots, a_v)$ for some $M$-regular elements $a_i$. Put $U_i = D_+(a_i)$, $a_i \in R(I)_1$. Then $H^0(U_i, F) = L_{(a_i)} = \{x/a_i^n \in Q(M) | x \in I^n M,\ n \geqq 0\}$, $H^0(U_i \cap U_j, F) = L_{(a_i a_j)}$ and $A = \mathrm{Ker}(\prod_i L_{(a_i)} \rightrightarrows \prod'_{i,j} L_{(a_i a_j)})$. Let $x$ be an element of $B$. Then $xI^n \subset I^n M$ for some $n$. Hence $x \in L_{(a_i)}$ for all $i$. Put $f(x) = (x, \ldots, x)$. Then it is easy to show that $f$ defines an isomorphism from $B$ to $A$. The fact $F(r) = R(I, I^r M)^{\sim}$ implies the second assertion. The proof of (2) is similar. (Note that $\mathcal{O}_{\bar{X}}(r) = \overline{I^r}\mathcal{O}_{\bar{X}} = I^r \mathcal{O}_{\bar{X}}$ for any $r \geqq 0$.) (3) follows from (2) and the following lemma. Q. E. D.

LEMMA 3.4. *Let* $I$ *and* $J$ *be ideals of a reduced noetherian ring* $R$. *Assume that* $I$ *contains an* $R$*-regular element. Then* $(\overline{IJ} : \bar{I})_R = \bar{J}$.

PROOF. We have only to show the inclusion $(\overline{IJ} : \bar{I})_R \subset \bar{J}$. Take an element $x$ of $(\overline{IJ} : \bar{I})_R$. If $R$ is an integral domain, then $\bar{I} = \cap IV \cap R$, where $V$ runs over all valuation overrings of $R$ (cf. [28]). For any $V$, put $IV = aV$. Then $x\bar{I}V \subset \overline{IJ}V$, $x\bar{I}V = xIV = xaV$ and $\overline{IJ}V = IJV = aJV$. Therefore $xa \in aJV$, i.e., $x \in JV$. Hence $x \in \cap JV \cap R = \bar{J}$. In general, we have $\bar{x} \in (\overline{IJR/\mathfrak{p}} : \overline{IR/\mathfrak{p}})_{R/\mathfrak{p}} = \overline{JR/\mathfrak{p}}$ for each minimal prime ideal $\mathfrak{p}$ of $R$ ($\bar{x}$ is the image of $x$ in $R/\mathfrak{p}$). Put $S = \prod_{\mathfrak{p}\in \mathrm{Min}(R)} R/\mathfrak{p}$. Then we have $x \in \overline{JS} \cap R = \bar{J}$. Q. E. D.

PROOF OF THEOREM 3.1. We assume that $d \geqq 2$. The case $d = 1$ is similar (or will be treated in §5). For each $r \geqq 1$, put $rD = \mathrm{Proj}(\oplus_{n\geqq 0} I^n/I^{n+r}) = \underline{\mathrm{Spec}}_X(\mathcal{O}_X/I^r\mathcal{O}_X)$, the closed subscheme of $X$ defined by $I^r\mathcal{O}_X$. From the exact sequence $0 \to I^r\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_{rD} \to 0$, we have the following exact sequence:

$$\begin{aligned} 0 &\longrightarrow H^0(I^r\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}_{rD}) \\ &\longrightarrow H^1(I^r\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_{rD}) \longrightarrow H^2(I^r\mathcal{O}_X) \longrightarrow \cdots. \end{aligned}$$

We have $H^0(\mathcal{O}_X) = R$ and $H^0(I^r\mathcal{O}_X) = I^r$, $H^i(I^r\mathcal{O}_X) = 0$ for all $r \gg 0$ and all $i \geqq 1$. Hence $\ell(R/I^r) = \chi(rD) - \sum_{i=1}^{d-1} (-1)^i \ell(H^i(\mathcal{O}_X))$ for all $r \gg 0$. Let $P$ be the polynomial associated to the function $\ell(R/I^{n+1})$. Then for each $r \geqq 1$, we have $\chi(\mathcal{O}_{rD}(n)) = \ell(I^n/I^{n+r}) = \ell(R/I^{n+r}) - \ell(R/I^n) = P(n+r-1) - P(n-1)$ for all $n \gg 0$. Hence $\chi(\mathcal{O}_{rD}) = P(r-1) - (-1)^d g(I)$. Therefore $(-1)^d g(I) = -\chi(rD) + P(r-1) =$

$-\chi(rD)+\ell(R/I^r)$ (for all $r\gg 0)=-\sum_{i=1}^{d-1}(-1)^i\ell(H^i(\mathcal{O}_X))$, $\chi(I)=P(0)=\chi(D)+(-1)^d g(I)$ and $p_a(I)=(-1)^d(\chi(I)-\ell(R/I))=g(I)+(-1)^d(\chi(D)-\ell(R/I))$. The proof for the normal genus case is similar. Q. E. D.

EXAMPLE 3.5. If $G(R)$ is reduced, then $\overline{\mathfrak{m}^n}=\mathfrak{m}^n$ for all $n\geqq 1$. Hence $\bar{g}(R)=g(R)$ and $\bar{p}_a(R)=p_a(R)$. For example, if $R=k[[X_1,\dots,X_v]]/(f)$, $e(R)=e$ and the initial form of $f$ is square-free, then $\bar{g}(R)=g(R)=\binom{e}{v}$ and $\bar{p}_a(R)=p_a(R)=\binom{e-1}{v}$.

## §4. Reduction exponents of ideals of local rings

In this section, $(R, \mathfrak{m}, k)$ denotes a Cohen-Macaulay local ring with infinite residue field and $\dim(R)=d\geqq 1$. In [21], we introduced the notion of *reduction exponent* $\delta_A(M)$ of a finitely generated graded module $M$ over a noetherian homogeneous $R$-algebra $A$: $\delta_A(M)=\min\{n \mid$ there is a minimal $M$-reduction $B$ of $A$ such that $B_1M_m=M_{m+1}$ for all $m\geqq n\}$. If $A=\oplus_{n\geqq 0}A_n$ is a (not necessarily homogeneous) noetherian graded ring with $A_0=R$ such that $A$ is a finitely generated $R[A_1]$-module, then we put $\delta(A)=\delta_{R[A_1]}(A)$. For an ideal $I$ of $R$, put $\delta(I)=\delta(R(I))$ and $\bar{\delta}(I)=\delta(\bar{R}(I))$. We call $\delta(I)$ and $\bar{\delta}(I)$ the *reduction exponent* and the *normal reduction exponent* of $I$ respectively. Hence $\delta(I)=\min\{n \mid$ there is a minimal reduction $J$ of $I$ such that $JI^n=I^{n+1}\}$ and $\bar{\delta}(I)=\min\{n \mid$ there is a minimal reduction $J$ of $\bar{I}$ such that $J\overline{I^m}=\overline{I^{m+1}}$ for all $m\geqq n\}$. Note that $\bar{\delta}(I)$ is well-defined if $R$ is analytically unramified. We also put $\delta(R)=\delta(\mathfrak{m})$ and $\bar{\delta}(R)=\bar{\delta}(\mathfrak{m})$.

THEOREM 4.1. *Let $I$ be an $\mathfrak{m}$-primary ideal of $R$.*

(1) *The following conditions are equivalent*:

(a) $\delta(I)=0$.

(b) *$I$ is a parameter ideal.*

(c) $\ell(I^n/I^{n+1})=\ell(R/I)\binom{n+d-1}{n}$ *for all* $n\geqq 1$.

(d) $e_1(I)=0$.

*If these conditions are satisfied, then $e_i(I)=0$ $(1\leqq i\leqq d)$, $n(I)=-d$ and $g(I)=p_a(I)=0$.*

(2) *Assume that $R$ is analytically unramified. Then the following conditions are equivalent*:

(a) $\bar{\delta}(I)=0$.

(b) *$\bar{I}$ is a parameter ideal.*

(c) $\ell(\overline{I^n}/\overline{I^{n+1}})=\ell(R/\bar{I})\binom{n+d-1}{n}$ *for all* $n\geqq 1$.

(d) $\bar{e}_1(I)=0$.

*If these conditions are satisfied, then $R$ is a regular local ring and $\overline{I^n}=\bar{I}^n$ $(n\geqq 1)$, $\bar{e}_i(I)=0$ $(1\leqq i\leqq d)$, $\bar{n}(I)=-d$, $\bar{g}(I)=\bar{p}_a(I)=0$.*

PROOF. We only prove (2). The proof of (1) is similar. (a)$\Rightarrow$(c)$\Rightarrow$(d) is clear. (d)$\Rightarrow$(b) follows from Lemma 4.2 below. (b)$\Rightarrow$(a): By [5], in this case, $R$ is a regular local ring and $\overline{I^n}=\bar{I}^n$ for all $n\geqq 1$. Q.E.D.

LEMMA 4.2 (cf. [19]). *$e_1(I)\geqq e(I)-\ell(R/I)\geqq 0$ and $\bar{e}_1(I)\geqq e(I)-\ell(R/\bar{I})\geqq 0$.*

PROOF. We prove only the second assertion. By Theorem 4.4, below, we have $\ell(R/\overline{I^{n+1}})\leqq e(I)\binom{n+d-1}{d}+\ell(R/\bar{I})\binom{n+d-1}{d-1}=e(I)\binom{n+d}{d}+(-e(I)+\ell(R/\bar{I}))\binom{n+d-1}{d-1}$ for all $n\geqq 0$. Hence $0\leqq(-e(I)+\ell(R/\bar{I})+\bar{e}_1(I))\binom{n+d-1}{d-1}+$ terms of degree $\leqq d-2$ for all $n\gg 0$. Therefore $-e(I)+\ell(R/\bar{I})+\bar{e}_1(I)\geqq 0$. Q.E.D.

THEOREM 4.3. *Let $I$ be an $\mathfrak{m}$-primary ideal of $R$. Then*

$$\ell(R/I^{n+1})\leqq e(I)\binom{n+d-1}{d}+\ell(R/I)\binom{n+d-1}{d-1}\qquad \text{for all}\quad n\geqq 0.$$

*In particular, $\ell(I/I^2)\leqq e(I)+(d-1)\ell(R/I)$. Moreover, the following conditions are equivalent:*

(1) $\ell(R/I^{n+1})=e(I)\binom{n+d-1}{d}+\ell(R/I)\binom{n+d-1}{d-1}$ *for all* $n\geqq 0$.

(2) $e_i(I)=0$ $(2\leqq i\leqq d)$ *and* $n(I)\leqq 0$.

(3) $\delta(I)\leqq 1$.

(4) $\ell(I/I^2)=e(I)+(d-1)\ell(R/I)$.

*If these conditions are satisfied, then $p_a(I)=0$, $e_1(I)=e(I)-\ell(R/I)$, $n(I)=-d$ or $-d+1$, $G(I)$ is Cohen-Macaulay, and $g(I)=0$ if $d\geqq 2$.*

PROOF. Let $J$ be a minimal reduction of $I$. Then $J$ is a parameter ideal and we have

$$\begin{aligned}\ell(R/I^{n+1}) &\leqq \ell(R/IJ^n)=\ell(R/J^n)+\ell(J^n/IJ^n)\\ &=\ell(R/J^n)+\ell(J^n/J^{n+1}\otimes_{R/J}R/I)\\ &=e(I)\binom{n+d-1}{d}+\ell(R/I)\binom{n+d-1}{d-1}\quad \text{for all}\quad n\geqq 0,\end{aligned}$$

because $J^n/J^{n+1}$ is a free $R/J$-module of rank $\binom{n+d-1}{d-1}$. In particular, $\ell(I/I^2)=e(I)+(d-1)\ell(R/I)-\ell(I^2/IJ)$. Hence (1)$\Leftrightarrow I^{n+1}=IJ^n$ for all $n\geqq 0\Leftrightarrow I^2=IJ$. This shows the equivalence of (1), (3) and (4). (1)$\Rightarrow$(2) is clear. (2)$\Rightarrow$(1): Since $\ell(R/I)=e_0(I)-e_1(I)$, we have $\ell(I/I^2)=e_0(I)d-e_1(I)(d-1)=e(I)+(d-1)\ell(R/I)$. Cohen-Macaulayness of $G(I)$ follows from [28]. Q.E.D.

The following theorem is analogous to Theorem 4.3. We omit the proof.

THEOREM 4.4. *Assume that $R$ is analytically unramified and $\bar{I}=I$. Then*

$$\ell(R/\overline{I^{n+1}}) \leqq e(I)\binom{n+d-1}{d} + \ell(R/I)\binom{n+d-1}{d-1} \qquad \textit{for all} \quad n \geqq 0$$

*and the following conditions are equivalent*:

(1) $\ell(R/\overline{I^{n+1}}) = e(I)\binom{n+d-1}{d} + \ell(R/I)\binom{n+d-1}{d-1}$ *for all* $n \geqq 0$.

(2) $\bar{e}_i(I) = 0$ $(2 \leqq i \leqq d)$, $\bar{n}(I) \leqq 0$ *and* $\overline{I^2} = I^2$.

(3) $\bar{\delta}(I) \leqq 1$.

(4) $\ell(I/I^2) = e(I) + (d-1)\ell(R/I)$ *and* $\overline{I^n} = I^n$ *for all* $n \geqq 1$.

Next, we recall some results of Sally about the reduction exponent of Cohen-Macaulay local rings (cf. [24]).

PROPOSITION 4.5. (1) $\delta(R) = 0 \Leftrightarrow \operatorname{reg}(G(R)) = 0 \Leftrightarrow R$ *is a regular local ring*.

(2) $\delta(R) \leqq 1 \Leftrightarrow \operatorname{reg}(G(R)) \leqq 1 \Leftrightarrow \operatorname{emb}(R) = e(R) + \dim(R) - 1$. *In this case, $G(R)$ is Cohen-Macaulay.*

(3) $\delta(R) = 2$ *if and only if* $\operatorname{reg}(G(R)) = 2$. *In this case, $G(R)$ is Cohen-Macaulay and* $\operatorname{emb}(R) \geqq e(R) + \dim(R) - 1 - r(R)$.

(4) *If $R$ is Gorenstein, then $\delta(R) = 2$ if and only if* $\operatorname{emb}(R) = e(R) + \dim(R) - 2$. *In this case, $G(R)$ is Gorenstein.*

(5) *If* $\operatorname{reg}(G(R)) = 3$, *then* $\delta(R) = 3$.

PROPOSITION 4.6 (cf. [14]). *For any ideal $I$ of $R$, we have $\delta(I^r) \leqq \dim(R)$ for all $r \gg 0$. Moreover, if $R$ is analytically unramified, then $\bar{\delta}(I^r) \leqq \dim(R)$ for all $r \gg 0$. (Here $R$ is not necessarily Cohen-Macaulay.)*

The proof of Proposition 4.6 requires some lemmas. For any real number $x$, put $\{x\} = \min\{n \in \boldsymbol{Z} \mid n \geqq x\}$.

LEMMA 4.7. *Let $A$ be a homogeneous algebra over a noetherian ring $R$ and let $M$ be a finitely generated graded $A$-module with* $\dim(M) = d$. *Then for any $r \geqq 1$, we have*

$$\operatorname{reg}(M^{(r)}) \leqq \{(\operatorname{reg}(M) - d + 1)/r + d - 1\}.$$

*In particular,* $\operatorname{reg}(M^{(r)}) \leqq d$ *for all* $r \gg 0$.

PROOF. If $n \geqq (\operatorname{reg}(M) - d + 1)/r + d - 1$, then $\operatorname{reg}(M) + 1 \leqq r\{(n+1) - d\} + d = r(n+1) + d(1-r) \leqq r(n+1) + i(1-r) = r(n+1-i) + i$ for all $0 \leqq i \leqq d$. Hence $[H_P^i(M)]_{r(n+1-i)} = 0$ for all $0 \leqq i \leqq d$. This implies that $\operatorname{reg}(M^{(r)}) \leqq n$. Q.E.D.

LEMMA 4.8. *Let $I$ be an ideal of a noetherian ring $R$, and put $A = R[It]$, $B = R[It, t^{-1}]$, $G = G(I)$. Then we have* $\operatorname{reg}(A) = \operatorname{reg}_A(B) = \operatorname{reg}(G)$.

PROOF. We may assume that $R$ is local. For a graded module $M$, put $T^n(M) = \oplus_{i+j=n}[H_P^i(M)]_j$. From the exactness of the sequence $0 \to B(1) \xrightarrow{t^{-1}} B \to G \to 0$, the sequence $T^{n+1}(B)=T^n(B(1))\to T^n(B)\to T^n(G)=0$ is exact for all $n>\operatorname{reg}(G)$. Since $T^{n+i}(B)=0$ for all $i\gg 0$, we have $T^n(B)=0$ for all $n>\operatorname{reg}(G)$, i.e., $\operatorname{reg}(B)\leqq \operatorname{reg}(G)$. If $n>\operatorname{reg}(B)$, then from the exact sequence $0=T^n(B)\to T^n(G)\to T^{n+1}(B(1))=T^{n+2}(B)=0$, we have $T^n(G)=0$, i.e., $\operatorname{reg}(G)\leqq\operatorname{reg}(B)$. Since $[B/A]_n=0$ for all $n\geqq 0$, we have $H^i_P(B/A)=0$ for all $i>0$ and $H^0_P(B/A)=B/A$. Hence the sequence $0\to H^0_P(A)\to H^0_P(B)\to B/A\to H^1_P(A)\to H^1_P(B)\to 0$ is exact, and $H^i_P(A)=H^i_P(B)$ for all $i\geqq 2$. From these facts, we get $\operatorname{reg}(A)=\operatorname{reg}_A(B)$. Q. E. D.

PROOF OF PROPOSITION 4.6. Since $G(I)^{(r)}=R(I^r)\otimes_R R/I$, we have $\delta(I^r)=\delta(R(I^r))=\delta(R(I^r)\otimes_R R/I)=\delta(G(I)^{(r)})\leqq\operatorname{reg}(G(I)^{(r)})\leqq\dim(G(I))=\dim(R)$ for all $r\gg 0$. The second assertion follows from the fact $\bar{R}(I^r)=R(\overline{I^r})$ for all $r\gg 0$.

Q. E. D.

REMARK 4.9 (cf. [14]). Assume that $R$ is analytically unramified, and put $\dim(R)=d$, $\bar{X}=\operatorname{Proj}(\bar{R}(I))$. Then $H^{d-1}(\bar{X}, \mathcal{O}_{\bar{X}})=0$ if and only if $\bar{\delta}(I^r)<\dim(R)$ for all $r\gg 0$.

PROPOSITION 4.10. *Assume that $d=1$ and let $I$ be an $\mathfrak{m}$-primary ideal of $R$. Then $\delta(I)=\operatorname{reg}(G(I))=n(I)+1$. If $R$ is analytically unramified, then we also have $\bar{\delta}(I)=\operatorname{reg}(\bar{G}(I))=\bar{n}(I)+1$.*

PROOF. Put $\delta(I)=n$ and let $xR$ be a minimal reduction of $I$ such that $xI^n=I^{n+1}$. Then, since $I^m/I^{m+1}\cong xI^m/xI^{m+1}=I^{m+1}/I^{m+2}$ for all $m\geqq n$, we have $\ell(I^m/I^{m+1})=e(I)$ for all $m\geqq n$, i.e., $n(I)\leqq n-1$. Also, since $xI^{n-1}\neq I^n$, we have $I^{n-1}/I^n\cong xI^{n-1}/xI^n\subsetneqq I^n/I^{n+1}$ and $\ell(I^{n-1}/I^n)<\ell(I^n/I^{n+1})=e(I)$. Therefore $n(I)=n-1$. By [21], $\delta(I)\leqq\operatorname{reg}(G(I))$. Put $A=G(I)$, $P=A_+$ and let $x$ be the image of $x$ in $A_1$. Consider the exact sequence $0\to K\to A\xrightarrow{x}A(1)\to C\to 0$. Then $C_m=K_{m-1}=0$ for all $m\geqq n$, and the following sequences are exact: $0\to K\to H^0_P(A)\to H^0_P(A(1))\to C$, $C\to H^1_P(A)\to H^1_P(A(1))\to 0$. Hence $[H^i_P(A)]_m\xrightarrow{x}[H^i_P(A)]_{m+1}$ is an isomorphism for all $m\geqq n$ and $i=0, 1$, which implies that $[H^i_P(A)]_m=0$. Therefore $\operatorname{reg}(G(I))\leqq n$. The second assertion is similarly proved. Q. E. D.

REMARK 4.11. (1) When $d=1$, $\delta(I)$ does not depend on the choice of a minimal reduction of $I$. Hence we define $\delta(I)$ by $\delta(I)=\delta(IR(X))$ even if $k$ is not an infinite field.

(2) After writing the manuscript, we noticed that recently K. Kubota obtained Theorem 4.3 independently (cf. Tokyo J. Math. **8** (1985), 439–448).

## §5. The case of dimension one (curve singularities)

In this section, let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring with the total quotient ring $Q$, and let $I$ be an $\mathfrak{m}$-primary ideal of $R$. Put $S=\bigcup_{n=0}^{\infty}(I^n: I^n)=\lim_{n\to\infty}\operatorname{End}_R(I^n)$, the *first nieghbourhood ring* of $R$ with respect to $I$ (cf. [12], [18]). Then $X=\mathrm{Proj}(R(I))$ is an affine scheme with $H^0(X, \mathcal{O}_X)=S$ (cf. Proposition 3.3). Some assertions in the following theorem may be well known (cf. [10], [12], [16], [19]). We give the proof for convenience.

THEOREM 5.1. (1) *$S$ is the smallest ring between $R$ and $Q$ such that $IS$ is a principal ideal. Moreover, $S$ is a finitely generated $R$-module.*

(2) $\ell(R/I^n)=e(I)n-\ell(S/R)$ *for all* $n\gg 0$.

(3) $e(I)=\ell(S/IS)$, $g(I)=\ell(S/R)$, $p_a(I)=g(I)-e(I)+\ell(R/I)=\ell(IS/I)=e(I)-\ell(I/I^2)+\ell(I^2S/I^2)$, *and* $g(I)\geqq p_a(I)\geqq 0$. *If $R$ is Gorenstein, then* $g(I)=\ell(R/\mathfrak{c})$, *where* $\mathfrak{c}=(R: S)$ *is the conductor of $S$ in $R$.*

(4) $\delta(I)=\min\{n \mid S=(I^n: I^n)\}$.

(5) $g(I)=0 \Leftrightarrow I$ *is a principal ideal* $\Leftrightarrow I^r$ *is a principal ideal for some* $r \Leftrightarrow \delta(I)=0 \Leftrightarrow p_a(I)=g(I) \Leftrightarrow \ell(I^n/I^{n+1})=e(I)$ *for all* $n\geqq 0$.

(6) $p_a(I)=0 \Leftrightarrow S=(I: I) \Leftrightarrow \delta(I)\leqq 1 \Leftrightarrow e(I)=\ell(I/I^2)$.

(7) $p_a(I)=e(I)-\ell(I/I^2)$ *if and only if* $\delta(I)\leqq 2$.

PROOF. We may assume that $k$ is an infinite field. Let $xR$ be a minimal reduction of $I$. Then $x$ is a non zero-divisor. (1) is well known. We have $IS=xS$. (2) For all $n\gg 0$, we have $I^n=I^nS=x^nS$ and $\ell(R/I^n)+\ell(S/R)=\ell(S/I^n)=\ell(S/x^nS)=\ell(S/xS)n=\ell(S/IS)n$. (3) follows from (2). If $R$ is Gorenstein, then $\ell(J/I)=\ell(I^{-1}/J^{-1})$ for any fractionary ideals $I\subset J$ of $R$. Hence $\ell(S/R)=\ell(R/\mathfrak{c})$. (4) If $\delta(I)\leqq n$, then we have $x^rI^n=I^{n+r}$ for all $r\geqq 0$. Hence $(I^{n+r}: I^{n+r})=(I^n: I^n)$ for all $r\geqq 0$, and we have $S=(I^n: I^n)$. Conversely, assume that $S=(I^n: I^n)$. Then for any $y\in I$, we have $yI^m\subset I^{m+1}=xI^m$ for some $m$. Hence $y/x\in(I^m: I^m)\subset S=(I^n: I^n)$, i.e., $yI^n\subset xI^n$. Therefore we get $xI^n=I^{n+1}$, i.e., $\delta(I)\leqq n$. (5) $g(I)=0\Leftrightarrow S=R\Leftrightarrow I$ is a principal ideal$\Leftrightarrow\delta(I)=0$. $g(I^r)=g(I)$ for all $r\geqq 1$ and $g(I)-p_a(I)=\ell(I/xI)$. This shows our assertion. (6) and (7) follow from (3) and (4).

Q. E. D.

COROLLARY 5.2. (1) $g(R)\geqq p_a(R)\geqq 0$ *and* $p_a(R)=g(R)-e(R)+1$.

(2) $g(R)=0\Leftrightarrow p_a(R)=g(R)\Leftrightarrow R$ *is a discrete valuation ring.*

(3) $p_a(R)=0\Leftrightarrow g(R)=e(R)-1\Leftrightarrow \mathrm{emb}(R)=e(R)$.

(4) $g(R)=1\Leftrightarrow e(R)=2\Leftrightarrow p_a(R)=0$, *$R$ is Gorenstein and is not a DVR.*

(5) *Assume that $R$ is Gorenstein. Then* $p_a(R)=1\Leftrightarrow g(R)=e(R)\Leftrightarrow\delta(R)=2\Leftrightarrow \mathrm{emb}(R)=e(R)-1$.

(6) $p_a(R)\geqq e(R)-\mathrm{emb}(R)$, *and* $p_a(R)=e(R)-\mathrm{emb}(R)$ *if and only if* $\delta(R)\leqq 2$.

PROOF. (1), (2), (3) and (6) follow from Theorem 5.1. (4) Assume that $g(R)=1$. Then $p_a(R)=0$ by (1) and (3). Hence $e(R)=g(R)-p_a(R)+1=2$. The other assertions are clear. (5) We have only to show the equivalence of $p_a(R)=1$ and $\mathrm{emb}(R)=e(R)-1$ (cf. Proposition 4.5). If $p_a(R)=1$, then since $\mathfrak{m}S/\mathfrak{m}=k$, we have $\mathfrak{c}=(R: S)\supset\mathfrak{m}^2$. Hence $\mathrm{emb}(R)\geqq\ell(\mathfrak{m}/\mathfrak{c})=\ell(R/\mathfrak{c})-\ell(R/\mathfrak{m})=g(R)-1=e(R)-1$. Therefore $\mathrm{emb}(R)=e(R)-1$ by (3). Conversely, assume that $\mathrm{emb}(R)$

$=e(R)-1$. Then $\mathfrak{m}^2 \subset \mathfrak{c}$ and $\ell(R/\mathfrak{c})=g(R) \geqq e(R)$ by (3). Hence $e(R)-1=\mathrm{emb}(R) \geqq \ell(\mathfrak{m}/\mathfrak{c})=\ell(R/\mathfrak{c})-\ell(R/\mathfrak{m}) \geqq e(R)-1$. Therefore $p_a(R)=\ell(R/\mathfrak{c})-e(R)+1=1$. Q.E.D.

From now on, we assume that $R$ is a one-dimensional analytically unramified local ring, i.e., $R$ is a one-dimensional reduced local ring whose integral closure $\bar{R}$ is a finitely generated $R$-module. Let $I$ be an $\mathfrak{m}$-primary ideal of $R$.

LEMMA 5.3. *$\bar{R}=\bigcup_{n=0}^{\infty}(\overline{I^n}:\overline{I^n})=H^0(\bar{X}, \mathcal{O}_{\bar{X}})$, where $\bar{X}=\mathrm{Proj}(\bar{R}(I))$.*

PROOF. Considering the base change $R \to R(X)$ and a minimal reduction, we may assume that $I=xR$ for a non zero-divisor $x$ of $R$. Then for all $n \gg 0$, we have $x^n\bar{R} \subset R$ and $\overline{x^nR}=x^n\bar{R} \cap R=x^n\bar{R}$. Hence $(\overline{I^n}:\overline{I^n})=(x^n\bar{R}: x^n\bar{R})=\bar{R}$ for all $n \gg 0$. This implies the first assertion. The second assertion follows from Proposition 3.3. Q.E.D.

The following Theorem 5.4 and Corollary 5.5 are analogous to Theorem 5.1 and Corollary 5.2. So we omit the proof.

THEOREM 5.4. (1) *$\ell(R/\overline{I^n})=e(I)n-\ell(\bar{R}/R)$ for all $n \gg 0$.*

(2) *$e(I)=\ell(\bar{R}/I\bar{R})$, $\bar{g}(I)=\ell(\bar{R}/R)$, $\bar{p}_a(I)=\bar{g}(I)-e(I)+\ell(R/\bar{I})=\ell(I\bar{R}/\bar{I})$, and $\bar{g}(I) \geqq \bar{p}_a(I) \geqq 0$. (Note that $\bar{g}(I)$ does not depend on $I$.) If $R$ is Gorenstein, then $\bar{g}(I)=\ell(R/\mathfrak{c})$, where $\mathfrak{c}=(R:\bar{R})$.*

(3) *$\bar{\delta}(I)=\min\{n \mid \bar{R}=(\overline{I^n}:\overline{I^n})\}=\min\{n \mid I^n \subset (R:\bar{R})\}$.*

(4) *$\bar{g}(I)=0 \Leftrightarrow \bar{p}_a(I)=\bar{g}(I) \Leftrightarrow e(I)=\ell(R/\bar{I}) \Leftrightarrow \bar{I}$ is a principal ideal $\Leftrightarrow R$ is a discrete valuation ring.*

(5) *$\bar{p}_a(I)=0 \Leftrightarrow \bar{R}=(\bar{I}:\bar{I}) \Leftrightarrow \bar{\delta}(I) \leqq 1 \Leftrightarrow \ell(\bar{I}/\bar{I}^2)=e(I)$ and $\overline{I^n}=\bar{I}^n$ for all $n \geqq 1$.*

(6) *Assume that $\bar{I}=I$. Then $\bar{p}_a(I)=e(I)-\ell(I/I^2)+\ell(I^2\bar{R}/I^2)$ and $\bar{p}_a(I)=e(I)-\ell(I/I^2)$ if and only if $\delta(I) \leqq 2$ and $\overline{I^n}=I^n$ for any $n \geqq 1$.*

(7) *$\bar{g}(I) \geqq g(I)$, and $\bar{g}(I)=g(I) \Leftrightarrow \overline{I^n}=I^n$ for all $n \gg 0 \Leftrightarrow \mathrm{Proj}(R(I))$ is normal.*

(8) *If $\bar{R}$ is local, then $\bar{\delta}(I)=\{\ell(R/\mathfrak{c})/e(I)\}$.*

COROLLARY 5.5. (1) *$\bar{g}(R) \geqq \bar{p}_a(R) \geqq p_a(R) \geqq 0$, $\bar{g}(R) \geqq g(R) \geqq p_a(R) \geqq 0$ and $\bar{p}_a(R)=\bar{g}(R)-e(R)+1$.*

(2) *$\bar{g}(R)=0 \Leftrightarrow \bar{g}(R)=\bar{p}_a(R) \Leftrightarrow R$ is a DVR.*

(3) *$\bar{p}_a(R)=0 \Leftrightarrow \bar{R}=(\mathfrak{m}:\mathfrak{m}) \Leftrightarrow R$ is a DVR or $(R:\bar{R})=\mathfrak{m} \Leftrightarrow \bar{\delta}(R) \leqq 1 \Leftrightarrow \mathrm{emb}(R)=e(R)$ and $\overline{\mathfrak{m}^n}=\mathfrak{m}^n$ for all $n \geqq 1$.*

(4) *$\bar{g}(R)=1 \Leftrightarrow \bar{p}_a(R)=0$ and $e(R)=2 \Leftrightarrow e(R)=2$ and $\overline{\mathfrak{m}^n}=\mathfrak{m}^n$ for all $n \geqq 1$.*

(5) *$\bar{g}(R)=2$ and $R$ is Gorenstein $\Leftrightarrow \bar{p}_a(R)=1$ and $e(R)=2$.*

(6) *$\bar{p}_a(R) \geqq e(R)-\mathrm{emb}(R)$ and $\bar{p}_a(R)=e(R)-\mathrm{emb}(R)$ if and only if $\delta(R) \leqq 2$ and $\overline{\mathfrak{m}^n}=\mathfrak{m}^n$ for all $n \geqq 1$.*

We call a maximal ideal of $\bar{R}/\mathfrak{m}\bar{R}\otimes_k \bar{k}$ a *geometric branch* of $R$, where $\bar{k}$ is the algebraic closure of $k$.

PROPOSITION 5.6. *Assume that $k$ is perfect. Then $\bar{p}_a(R)=0$ and the number of geometric branches of $R$ is equal to $e(R)$ if and only if $R$ is seminormal. (The only if part is valid without the assumption on $k$.)*

PROOF. We may assume that $R$ is not a $DVR$. Then $R$ is seminormal$\Leftrightarrow$ $(R: \bar{R})$ is a radical ideal of $\bar{R}\Leftrightarrow(R: \bar{R})=\mathfrak{m}$ and $\mathfrak{m}\bar{R}$ is a radical ideal of $\bar{R}\Leftrightarrow\bar{p}_a(R)=0$ and $\bar{R}/\mathfrak{m}\bar{R}$ is reduced. Hence our assertion follows from the following lemma.

Q. E. D.

LEMMA 5.7. *$e(R)\geqq$ the number of geometric branches of $R$, and the equality holds if and only if $\bar{R}/\mathfrak{m}\bar{R}$ is a geometrically reduced $k$-algebra.*

PROOF. Put $S=\bar{R}/\mathfrak{m}\bar{R}\otimes_k \bar{k}$. Then $S$ is a finite $\bar{k}$-algebra with $\dim_{\bar{k}}(S)=e(R)$. Put $\mathrm{Card}(\mathrm{Max}(S))=n$. Since the canonical homomorphism $f: S\to\prod_{\mathfrak{n}\in\mathrm{Max}(S)} S/\mathfrak{n}=\bar{k}^n$ is surjective, we have $\dim_{\bar{k}}(S)\geqq n$, and $f$ is an isomorphism (i.e., $S$ is reduced) if and only if $\dim_{\bar{k}}(S)=n$. Q. E. D.

EXAMPLE 5.8. Suppose that $k$ is algebraically closed and is contained in $R$.

(1) $\bar{g}(R)=1$ if and only if $\hat{R}\cong k[\![X, Y]\!]/(XY)$ or $k[\![X, Y]\!]/(Y^2-X^3)$. The "if" part is clear. Assume that $\bar{g}(R)=1$ and $R$ is complete. If $\mathrm{Card}(\mathrm{Max}(\bar{R}))=2$, then $R$ is a seminormal double point by Proposition 5.6. Hence $R\cong k[\![X, Y]\!]/(XY)$. If $\bar{R}$ is local, then $\bar{R}=k[\![t]\!]$, $\mathfrak{m}\bar{R}=t^2k[\![t]\!]$ and $R/k[\![t^2, t^3]\!]\subsetneqq\bar{R}/k[\![t^2, t^3]\!]\cong k$. Therefore $R=k[\![t^2, t^3]\!]\cong k[\![X, Y]\!]/(Y^2-X^3)$.

(2) (Double points) Assume that $\mathrm{char}(k)\neq 2$. If $e(R)=2$, then $\hat{R}\cong k[\![X, Y]\!]/(Y^2-X^r)$ for some $r$ (cf. [7], Ex. 5.14). It is easy to show that $g(R)=1$, $p_a(R)=0$, $\bar{g}(R)=\bar{p}_a(R)+1=r/2$ (if $r$ is even) or $(r-1)/2$ (if $r$ is odd). Hence $e(R)=2$ and $\bar{g}(R)=g$ if and only if $\hat{R}\cong k[\![X, Y]\!]/(Y^2-X^{2g})$ or $k[\![X, Y]\!]/(Y^2-X^{2g+1})$.

EXAMPLE 5.9. (Monomial curve singularities). Let $H$ be a numerical semigroup, i.e., an additive submonoid of $\boldsymbol{N}=\{0, 1, 2,\dots\}$ such that $\boldsymbol{N}-H$ is a finite set. Put $e(H)=\min\{n\in H|n>0\}$, $c(H)=\min\{n\in H|n+\boldsymbol{N}\subset H\}$ and $g(H)=\mathrm{Card}(\boldsymbol{N}-H)$. Fix a field $k$ and put $R=k[\![t^n|n\in H]\!]$. Then $\bar{R}=k[\![t]\!]$, $e(R)=e(H)$, $\bar{g}(R)=g(H)$, $\bar{p}_a(R)=g(H)-e(H)+1$ and $R$ is Gorenstein if and only if $c(H)=2g(H)$ (cf. [8]). We have $\bar{\delta}(R)=\{c(H)/e(H)\}$ by Theorem 5.4, (8).

(1) $\bar{p}_a(R)=0\Leftrightarrow g(H)=e(H)-1\Leftrightarrow e(H)=c(H)\Leftrightarrow H=H_e:=\langle e, e+1,\dots, 2e-1\rangle=\{0, e, e+1,\dots\}$, i.e., $H$ is an *ordinary semigroup*. In this case $r(R)=e(R)-1$. Hence $\bar{p}_a(R)=0$ and $R$ is Gorenstein if and only if $H=\langle 2, 3\rangle$.

(2) $\bar{p}_a(R)=1\Leftrightarrow g(H)=e(H)\Leftrightarrow H=H_{e,r}:=\{0, e, e+1,\dots, e+r, e+r+2,\dots\}=$

$\langle e, e+1,..., e+r, e+r+2,..., 2e-1\rangle$, $0\leqq r<e-3$ or $\langle e, e+1,..., 2e-2\rangle$, $e\geqq 3$. In this case, $\mathrm{emb}(R)=e(R)-1$, $c(H)=e(R)+r+2$. Hence $\bar{p}_a(R)=1$ and $R$ is Gorenstein if and only if $H=H_{e,e-2}=\langle e, e+1,..., 2e-2\rangle$, $e\geqq 3$.

(3) $g(H)=1\Leftrightarrow H=\langle 2, 3\rangle$. $g(H)=2\Leftrightarrow H=\langle 2, 5\rangle$ or $\langle 3, 4, 5\rangle$. $g(H)=3\Leftrightarrow H=\langle 2, 7\rangle$, $\langle 3, 4\rangle$, $\langle 3, 5, 7, 9\rangle$ or $\langle 4, 5, 6, 7\rangle$.

(4) Let $H=\{0, g, g+2, g+3,...\}$, $g\geqq 2$ (a *normal semigroup*). Then $\bar{g}(R)=g$, $g(R)=g-1$, $\bar{p}_a(R)=1$, $p_a(R)=0$ and $r(R)=g-1$.

(5) $e(H)=2$ if and only if $H=\langle 2, 2g+1\rangle$, $g\geqq 1$ (a *hyperelliptic semigroup*). In this case, $\bar{g}(R)=g$, $g(R)=1$, $\bar{p}_a(R)=g-1$ and $p_a(R)=0$.

(6) Let $H=\langle e, e+1\rangle$, $e\geqq 2$. Then $\bar{g}(R)=g(R)=e(e-1)/2$, $\bar{p}_a(R)=p_a(R)=(e-1)(e-2)/2$ and $\delta(R)=e-1$.

EXAMPLE 5.10 (Milnor number of curve singularities). Buchweitz and Greuel [3] defined the *Milnor number* $\mu(R)$ of a complex curve singularity $R$ by $\mu(R)=\ell(\mathrm{Coker}(c))$, where $c\colon R\to\Omega^1_{R/C}\to K_R$ is the canonical homomorphism into the canonical module of $R$, and they showed that $\mu(R)=2\bar{g}(R)-r(R)+1$, where $r(R)$ is the number of branches of $R$. Hence $\mu(R)\geqq\bar{g}(R)+\bar{p}_a(R)\geqq\bar{g}(R)$, and $\mu(R)=\bar{g}(R)+\bar{p}_a(R)$ (resp. $\mu(R)=\bar{g}(R)$) if and only if $e(R)=r(R)$ (resp. $\bar{p}_a(R)=0$ and $e(R)=r(R)$, i.e., $R$ is seminormal). $\mu(R)=1$ (resp. $\mu(R)=2$) if and only if $R$ is an ordinary double point (resp. $R$ is a seminormal triple point or $\hat{R}=\boldsymbol{C}[\![X, Y]\!]/(Y^2-X^3)$).

EXAMPLE 5.11. Let $X$ be a reduced noetherian scheme such that $\dim(\mathcal{O}_{X,x})=1$ for any closed point $x$ of $X$, and assume that $X$ has the finite and rational normalization $p\colon\bar{X}\to X$, i.e., $p$ is finite and $k(y)=k(p(y))$ for any closed point $y$ of $\bar{X}$. Let $I\subset\mathcal{O}_X$ be the largest ideal which defines the singular locus of $X$, and let $\mathrm{B}l(X)$ be the blowing-up of $X$ along $I$. Put $X_1=\mathrm{B}l(X)$ and $X_{n+1}=\mathrm{B}l(X_n)$ for all $n\geqq 1$. Define $g(X)=\sum_x g(\mathcal{O}_{X,x})$, $\bar{g}(X)=\sum_x \bar{g}(\mathcal{O}_{X,x})$, where $x$ runs over the set of closed points of $X$. Then $X_n=\bar{X}$ for all $n\gg 0$, and $\bar{g}(X)=\sum_{i=0}^{\infty} g(X_i)=\sum_{i=0}^{n-1} g(X_i)+\bar{g}(X_n)$ for any $n\geqq 1$. Therefore $\bar{g}(X)=\sum_{i=0}^{n} g(X_i)$ if and only if $X_n=\bar{X}$. In particular, after blowing-up $\bar{g}(X)-g(X)+1$ times, $X$ becomes normal.

## §6. The case of dimension two (surface singularities)

In this section, let $(R, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring with infinite residue field, and let $I$ be an $\mathfrak{m}$-primary ideal of $R$.

THEOREM 6.1. (1) (cf. [17]). *$g(I)=0$ if and only if $\delta(I^r)\leqq 1$ for some (or sufficiently large) $r\geqq 1$.*

(2) (cf. [22]). *Assume that $R$ is analytically unramified. Then the following conditions are equivalent:*

(a) $\bar{g}(I)=0$.

(b) $\bar{\delta}(I) \leqq 1$.

(c) $\bar{\delta}(I^r) \leqq 1$ *for some (or all)* $r \geqq 1$.

Proof. (1) If $\delta(I^r) \leqq 1$ for some $r$, then $g(I)=g(I^r)=0$ by Theorem 4.3. Conversely, if $g(I)=0$, then $g(I^r)=g(I)=0$ and $n(I^r) \leqq 0$ for all $r \gg 0$. Hence by Theorem 4.3, $\delta(I^r) \leqq 1$ for all $r \gg 0$.

(2) If $\bar{\delta}(I^r) \leqq 1$ for some $r$, then $\bar{g}(I)=\bar{g}(I^r)=0$ by Theorem 4.4. Conversely, assume that $\bar{g}(I)=0$ and put $\bar{X}=\operatorname{Proj}(\bar{R}(I))$ and $A=\bar{R}(I)$. Then $H_P^{i+1}(A)=\oplus_{n \in \boldsymbol{Z}} H^i(\bar{X}, \mathcal{O}_{\bar{X}}(n))$, $i \geqq 1$, and the following sequence is exact:

$$0 \longrightarrow H_P^0(A) \longrightarrow A \longrightarrow \oplus_{n \in \boldsymbol{Z}} H^0(\bar{X}, \mathcal{O}_{\bar{X}}(n)) \longrightarrow H_P^1(A) \longrightarrow 0.$$

By Lemma 3.2 and Proposition 3.3, (3), we have $H_P^0(A)=0$, $[H_P^1(A)]_n=0$ for all $n \geqq 0$, $[H_P^3(A)]_n=H^2(\bar{X}, \mathcal{O}_{\bar{X}}(n))=0$ for all $n \in \boldsymbol{Z}$ and $[H_P^2(A)]_0=H^1(\bar{X}, \mathcal{O}_{\bar{X}})=0$ by the assumption. Therefore we get $\bar{\delta}(I) \leqq \operatorname{reg}(\bar{R}(I)) \leqq 1$ (cf. [20]). Q. E. D.

Corollary 6.2. *Assume that $R$ is analytically unramified.*

(1) $\bar{g}(R)=0$ *if and only if* $\operatorname{emb}(R)=e(R)+1$ *and* $\overline{\mathfrak{m}^n}=\mathfrak{m}^n$ *for all* $n \geqq 1$.

(2) $\bar{g}(R)=0$ *and $R$ is Gorenstein if and only if $R$ is a regular local ring or* $e(R)=2$ *and* $\overline{\mathfrak{m}^n}=\mathfrak{m}^n$ *for all* $n \geqq 1$.

Proof. This follows from Theorem 6.1 and Theorem 4.4. Q. E. D.

Assume that $R$ is analytically unramified and normal. Then $R$ is said to be *pseudo-rational* if for any proper birational morphism $X \to \operatorname{Spec}(R)$ with $X$ a normal noetherian integral scheme, we have $H^1(X, \mathcal{O}_X)=0$. $R$ is said to be *rational* if $\operatorname{Spec}(R)$ has a desingularization $X \to \operatorname{Spec}(R)$ such that $H^1(X, \mathcal{O}_X)=0$. $R$ is pseudo-rational if and only if $\bar{g}(I)=0$ for any parameter (or $\mathfrak{m}$-primary) ideal $I$ of $R$, and $R$ is rational if and only if $R$ is pseudo-rational and is analytically normal. For these matters, see [11], [13], [14], [15].

Corollary 6.3. *Assume that $R$ is analytically unramified and normal. Then $R$ is pseudo-rational if and only if for any integrally closed $\mathfrak{m}$-primary ideal $I$ of $R$, we have $\ell(I/I^2)=e(I)+\ell(R/I)$ and $\overline{I^n}=I^n$ for all $n \geqq 1$ (in particular,* $\operatorname{emb}(R)=e(R)+1$ *and* $\overline{\mathfrak{m}^n}=\mathfrak{m}^n$ *for all* $n \geqq 1$). *In this case, $G(I)$ is Cohen-Macaulay for any integrally closed $\mathfrak{m}$-primary ideal $I$ of $R$.*

Example 6.4. (1) Put $R=k[\![X, Y]\!]$ and $I=(X^3, X^2Y, Y^3)$. Then $\bar{I}=\mathfrak{m}^3$, $(X^3, Y^3)I^2=I^3$, $\dim_k(\mathfrak{m}^{3n}/I^n)=1$ and $\ell(R/I^n)=9n(n-1)/2+6n+1$ for all $n \geqq 1$. Therefore we have $e(I)=9$, $n(I)=1$, $\delta(I)=2$, $p_a(I)=0$ and $g(I)=1$. Hence $g(\mathfrak{m})=\bar{g}(I)=0<g(I)$.

(2) Put $R=k[\![X, Y]\!]$ and $I=(X^4, X^3Y, XY^3, Y^4)$. Then $\bar{I}=\mathfrak{m}^4$, $I\mathfrak{m}^4=\mathfrak{m}^8$, $\overline{I^r}=I^r=\mathfrak{m}^{4r}$ for all $r \geqq 2$ and $(X^4, Y^4)I^2=I^3$. Therefore $\delta(I)=2$ and $\delta(I^r)=$

1 for all $r \geqq 2$. Hence the condition $g(I)=0$ does not imply $\delta(I) \leqq 1$ (cf. Theorem 6.1).

(3) (cf. [17]). Put $R=k[\![X, Y, Z]\!]/(Z^3)=k[\![x, y, z]\!]$ and $I=(x^2, y^2, xz, yz)$. Then $I\mathfrak{m}=\mathfrak{m}^3$, $\dim_k(\mathfrak{m}^2/I)=2$, $I^{n+1}=I^n(x^2, y^2)=I^2(x^2, y^2)^{n-1}$ and $\dim_k(\mathfrak{m}^{2n}/I^n)=n$ for all $n \geqq 2$. Therefore $\ell(R/I^{n+1})=6n^2+10n+5$ for all $n \geqq 1$, $n(I)=1$, $p_a(I)=-1$ and $g(I)=1$. Put $S=R[\![t]\!]$ and $J=(I, t)$. Then $\ell(S/J^{n+1})=2n^3+8n^2+11n+6$ for all $n \geqq 0$. Hence we have $p_a(J)=0$ and $g(J)=e_3(J)=-1<0$.

Assume that $R$ is analytically normal, and let $\tilde{X} \to \mathrm{Spec}(R)$ be a desingularization of $\mathrm{Spec}(R)$. Then we call $p_g(R):=\ell(H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}))$ the *geometric genus* of $R$. $p_g(R)$ does not depend on a desingularization (cf. [9], [13]).

PROPOSITION 6.5. *If $R$ is analytically normal, then $\bar{g}(I) \leqq p_g(R)$ for any $\mathfrak{m}$-primary ideal $I$ of $R$.*

PROOF. Put $\bar{X}=\mathrm{Proj}(\bar{R}(I))$ and let $\tilde{X} \xrightarrow{f} \bar{X}$ be a desingularization of $\bar{X}$. Then since $f_*(\mathcal{O}_{\tilde{X}})=\mathcal{O}_{\bar{X}}$, $H^1(\bar{X}, \mathcal{O}_{\bar{X}})=H^1(\bar{X}, f_*(\mathcal{O}_{\tilde{X}})) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is injective by Leray's spectral sequence. Hence $\bar{g}(I)=\ell(H^1(\bar{X}, \mathcal{O}_{\bar{X}})) \leqq \ell(H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}))=p_g(R)$. Q. E. D.

COROLLARY 6.6. *Assume that $R$ is analytically normal. Then $p_g(R)=1$ (i.e., $R$ is elliptic) if and only if $\bar{g}(I) \leqq 1$ for any $\mathfrak{m}$-primary ideal $I$ of $R$ and $\bar{g}(I)=1$ for some $\mathfrak{m}$-primary ideal $I$ of $R$.*

EXAMPLE 6.7. (1) Put $R=\boldsymbol{C}[\![X, Y, Z]\!]/(X^n+Y^n+Z^n)$, $n \geqq 2$. Then $p_g(R)=\bar{g}(R)=g(R)=\binom{n}{3}$ and $\bar{p}_a(R)=p_a(R)=\binom{n-1}{3}$.

(2) Put $R=\boldsymbol{C}[\![X, Y, Z]\!]/(Z^2-f(X, Y))$ and assume that $R$ is normal. Then as was pointed out by S. Itoh, $R(\mathfrak{m})$ is normal (or equivalently $\bar{g}(R)=0$) if and only if $f \notin (X, Y)^4$. If $R=\boldsymbol{C}[\![X, Y, Z]\!]/(X^2+Y^3+Z^6)$ (which is a simple elliptic singularity), then $p_g(R)=1$ and $\bar{g}(R)=0$. (Hence $\bar{g}(R)=0$ does not imply that $R$ is rational.) If $R=\boldsymbol{C}[\![X, Y, Z]\!]/(X^2+Y^4+Z^4)$, then $p_g(R)=\bar{g}(R)=1$ (cf. [9]).

## References

[1] U. Betke and P. McMullen, Lattice points in lattice polytope, Mh. Math. **99** (1985), 253–265.
[2] E. Bombieri and D. Husemoller, Classification and embeddings of surfaces, Proc. Symp. Pure Math. **29** (1975), 329–420.
[3] R. O. Buchweitz and G. M. Greuel, The Milnor number and deformations of complex curve singularities, Invent. Math. **58** (1980), 241–281.
[4] J. P. Fillmore, On the coefficients of the Hilbert Samuel polynomial, Math. Z. **97** (1967), 212–228.
[5] S. Goto, Integral closedness of complete-intersection ideals, preprint.

[6] A. Grothendieck and J. Dieudonné, Éléments de Géométrie Algébrique, Publ. Math. IHES, no. 11, 1961.
[7] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York–Heidelberg-Berlin, 1977.
[8] J. Herzog und E. Kunz, Die Wertehalbgruppe eines lokalen Rings der Dimension 1, S. B. Heiderberger Akad. Wiss. **2** (1971).
[9] T. Higuchi, E. Yoshinaga and K. Watanabe, Introduction to Complex Analysis of Several Variables, Morikita Shuppan, 1980 (in Japanese).
[10] D. Kirby, The reduction number of a one-dimensional local ring, J. London Math. Soc. **10** (1975), 471–481.
[11] J. Lipman, Rational singularities, with applications to algebraic surfaces and unique factorization, Publ. Math. IHES, no. 36, 1969, 195–279.
[12] J. Lipman, Stable ideals and Arf rings, Amer. J. Math. **93** (1971), 649–685.
[13] J. Lipman, Desingularization of two-dimensional schemes, Ann. of Math. **107** (1978), 151–207.
[14] J. Lipman and B. Teissier, Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals, Michigan Math. J. **28** (1981), 97–116.
[15] J. Lipman and A. Sathaye, Jacobian ideals and a theorem of Briançon-Skoda, Michigan Math. J. **28** (1981), 199–222.
[16] E. Matlis, The multiplicity and reduction number of a one-dimensional local ring, Proc. London Math. Soc. **26** (1973), 273–288.
[17] M. Narita, A note on the coefficients of Hilbert characteristic functions in semi-regular local rings, Proc. Camb. Phil. Soc. **59** (1963), 267–279.
[18] D. G. Northcott, On the notion of a first neighborhood ring, Proc. Camb. Phil. Soc. **53** (1959), 43–56.
[19] D. G. Northcott, A note on the coefficients of the abstract Hilbert function, J. London Math. Soc. **35** (1960), 209–214.
[20] A. Ooishi, Castelnuovo's regularity of graded rings and modules, Hiroshima Math. J. **12** (1982), 627–644.
[21] A. Ooishi, Reductions of graded rings and pseudo-flat graded modules, preprint.
[22] D. Rees, Hilbert functions and pseudo-rational local rings of dimension two, J. London Math. Soc. **24** (1981), 467–479.
[23] C. P. L. Rhodes, The Hilbert-Samuel polynomial in a filtered module, J. London Math. Soc. **3** (1971), 73–85.
[24] J. Sally, Tangent cones at Gorenstein singularities, Compositio Math. **40** (1980), 167–175.
[25] P. Schenzel, Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe, Lecture Notes in Math., vol. 907, Springer-Verlag, 1982.
[26] R. P. Stanley, Decompositions of rational convex polytopes, Ann. Discrete Math. **6** (1980), 333–342.
[27] R. P. Stanley, Combinatorics and Commutative Algebra, Birkhäuser, Boston-Basel-Stuttgart, 1983.
[28] G. Valla, On form rings which are Cohen-Macaulay, J. Algebra **58** (1979), 247–250.
[29] O. Zariski and P. Samuel, Commutative Algebra, vol. 2, Springer-Verlag, New York-Heidelberg-Berlin, 1960.

*Department of Mathematics,*
*Faculty of Science,*
*Hiroshima University*