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A characterization of the space of Sato-hyperfunctions on the unit circle

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§0. Introduction.

L. Waelbroeck [10] proved the following facts concerning the space $\mathscr{E}'(V)$ of Schwartz-distributions with compact support on a C^∞-manifold V. Namely:

1) The delta function map $\delta\colon V \to \mathscr{E}'(V)$ is a C^∞-map where $\mathscr{E}'(V)$ is considered as a b-space in his sense.
2) Any C^∞-map f of V into a b-space E uniquely factors through $\delta\colon V \to \mathscr{E}'(V)$ i.e. there exists a unique b-linear map $f^\wedge\colon \mathscr{E}'(V) \to E$ such that $f = f^\wedge \cdot \delta$.

In other words, the map $\delta\colon V \to \mathscr{E}'(V)$ is universal among C^∞-maps from V into any b-space and the b-space $\mathscr{E}'(V)$ is characterized up to b-isomorphisms by these two facts.

It is natural to ask what would happen if we replace C^∞-maps by C^ω-maps i.e. analytic maps. One of candidates by which $\mathscr{E}'(V)$ is replaced would be the space $\mathscr{B}_c(V)$ of Sato-hyperfunctions with compact support on a real analytic manifold V. The space $\mathscr{B}_c(V)$ has a structure of b-space but the delta function map $\delta\colon V \to \mathscr{B}_c(V)$ is not analytic even in the case where V is the unit circle $\boldsymbol{T}$ (an important remark by Prof. H. Komatsu). In the case of $V = \boldsymbol{T}$, the space $\mathscr{B}(\boldsymbol{T})$ can be considered as the inverse limit of Banach spaces B_N, i.e., $\mathscr{B}(\boldsymbol{T}) = \operatorname{inv}\lim_N B_N$. If we define an analytic map $f\colon \boldsymbol{T} \to \mathscr{B}(\boldsymbol{T})$ by requiring that the map $f\colon \boldsymbol{T} \to B_N$ be analytic for all N, then the delta function map $\delta\colon \boldsymbol{T} \to \mathscr{B}(\boldsymbol{T})$ is analytic and we can prove 1) and 2) replacing $\mathscr{E}'(V)$, C^∞-maps and b-spaces by $\mathscr{B}(\boldsymbol{T})$, C^ω-maps and iB-spaces, which are the inverse limits of Banach spaces, respectively. By a work of Gel'fand-Shilov [2], these iB-spaces have enough functionals and are too restrictive compared to the class of b-spaces of Waelbroeck. A b-space in the sense of Waelbroeck is the direct limit of Banach spaces and coincides with a ultrabornologic space in the sense of Bourbaki [1] if the space considered is a locally convex topological vector space (a remark by Prof. H. Komatsu). Of course, there are many non locally convex b-spaces such as L^0 which were main concern of Waelbroeck to introduce his notion of b-spaces. Therefore, we consider the inverse limit of b-spaces, which we call ib-space, namely, E is expressed with Banach spaces $E_{\alpha\beta}$ in the form

$$E = \text{inv}\ \lim_\alpha \text{dir}\ \lim_\beta E_{\alpha\beta}.$$

The class of *ib*-spaces includes *b*-spaces and *iB*-spaces. The notion of *ib*-maps is naturally defined. Especially, we are concerned with *ib*-space valued analytic maps $f: \boldsymbol{T} \to E = \text{inv}\ \lim_\alpha \text{dir}\ \lim_\beta E_{\alpha\beta}$ which, for every α, there is some β such that the map $f: E \to E_{\alpha\beta}$ is analytic. Remark that we are not considering topology on an *ib*-space itself but on each component $E_{\alpha\beta}$. Now we can state our theorem:

THEOREM *Let $\boldsymbol{T} = \{z \in \boldsymbol{C};\ |z| = 1\}$ be the unit circle in the complex plane $\boldsymbol{C}$. Then the space $\mathscr{B}(\boldsymbol{T})$ of Sato-hyperfunctions is an ib-space (actually an iB-space) and the delta function map $\delta: \boldsymbol{T} \to \mathscr{B}(\boldsymbol{T})$ is an ib-space valued analytic map such that for any ib-space E and for any ib-space valued analytic map $f: \boldsymbol{T} \to E$ there exists a unique ib-linear map $f^\wedge: \mathscr{B}(\boldsymbol{T}) \to E$ such that $f = f^\wedge \cdot \delta$.*

This theorem shows that the map $\delta: \boldsymbol{T} \to \mathscr{B}(\boldsymbol{T})$ is universal among analytic maps from $\boldsymbol{T}$ to *ib*-spaces and characterizes the space $\mathscr{B}(\boldsymbol{T})$ of Sato-hyperfunctions up to *ib*-isomorphisms among *ib*-spaces. Moreover, the map $f^\wedge: \mathscr{B}(\boldsymbol{T}) \to E$ is given by an integral with f as a kernel. In fact we show in §5 that in the case of $E = \mathscr{B}(\boldsymbol{T})$, these kernels are analytic functions of two variables:

$$f: \boldsymbol{T} \times \boldsymbol{C} \setminus \boldsymbol{T} \longrightarrow \boldsymbol{C}$$

vanishing at infinity, which is a special case of the kernel theorem of Köthe [6].

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§1. $\mathscr{B}(\boldsymbol{T})$ as an *ib*-space.

By a theorem due to Silva-Köthe-Grothendieck [3], [5], [7], we can represent the space $\mathscr{B}(\boldsymbol{T})$ as

$$\mathscr{B}(\boldsymbol{T}) = \mathscr{B}^+(\boldsymbol{T}) \oplus \mathscr{B}^-(\boldsymbol{T})$$

where

$$\mathscr{B}^+(\boldsymbol{T}) = \mathscr{O}(D) = \{\varphi: D \to \boldsymbol{C};\ \text{holomorphic}\},$$

$$\mathscr{B}^-(\boldsymbol{T}) = \mathscr{O}_0(\boldsymbol{C} \setminus \bar{D}) = \{\varphi: \boldsymbol{C} \setminus \bar{D} \longrightarrow \boldsymbol{C};\ \text{holomorphic and } \varphi(\infty) = 0\}$$

with $D = \{z \in \boldsymbol{C};\ |z| < 1\}$. We define, for each natural number N, a norm on $\mathscr{B}^-(\boldsymbol{T})$ by

$$\|\varphi\|_N^- = \sup_{|z| = 1 + 1/N} |\varphi(z)|, \quad \varphi \in \mathscr{B}^-(\boldsymbol{T}).$$

Let B_N^- be the completion of $\mathscr{B}^-(\boldsymbol{T})$ with respect to this norm. Then B_N^- is the space of continuous functions on

$$C \setminus \{z \in C;\ z < 1 + 1/N\}$$

which are holomorphic in $C \setminus \{z \in C;\ z \leqq 1 + 1/N\}$ and vanish at ∞. We see that

$$\mathscr{B}^-(T) = \text{inv lim}_N\, B_N^-$$

is an ib-space. Similarly

$$\mathscr{B}^+(T) = \text{inv lim}_N\, B_N^+.$$

§2. Analyticity of the delta function map δ: $T \to \mathscr{B}(T)$.

By definition, it is enough to show that for any natural number N, the map δ: $T \to B_N = B_N^+ \oplus B_N^-$ is analytic. It is known that the map δ is given by

$$\delta(t) = (\delta^+(t),\ \delta^-(t)) = (1/(t-z),\ 1/(z-t))$$
$$\in \mathcal{O}(D) \oplus \mathcal{O}_0(C \setminus \bar{D}) \subset B_N.$$

For instance, for $|z| > 1 + 1/N$, $1 - 1/2N < |\tau_\circ|$, $|\tau_\circ + h| < 1 + 1/2N$,

$$\frac{1}{z-(\tau_\circ + h)} = \frac{1}{z-\tau_\circ}\left(\frac{1}{1-/(z-\tau_\circ)}\right) = \frac{1}{z-\tau_\circ}\sum_{n=0}^{\infty}\frac{h^n}{(z-\tau_\circ)^n}$$

and

$$\|1/(z-\tau_\circ)^n\|_N^- = \sup_{|z|=1+1/N} |1/(z-\tau_\circ)^n| \leqq (2N)^n$$

so that for $|h| < 1/2N$, the series converges in B_N^-, i.e. δ^-: $T \to B_N^-$ is analytic. Similarly we see that δ^+: $T \to B_N^+$ is analytic.

§3. Existence of ib-linear map $f^\wedge$: $\mathscr{B}(T) \to E$.

Let f: $T \to E$ be an ib-space valued analytic map. By definition, if $E = \text{inv lim}_\alpha \text{dir lim}_\beta E_{\alpha\beta}$, then for each α, the map f: $T \to E_\alpha = \text{dir lim}_\beta E_{\alpha\beta}$ is analytic i.e. for some β, the map f: $T \to E_{\alpha\beta}$ into a banach space $E_{\alpha\beta}$ is analytic. Take a natural number N so that f can be extended holomorphically to

$$f^\sim:\ T_N^\circ = \{z \in C;\ 1 - 1/N < |z| < 1 + 1/N\} \longrightarrow E_{\alpha\beta}.$$

Let us define $f^\wedge$: $B_N \to E_{\alpha\beta}$ by the integral

$$f^\wedge(u) = \frac{1}{2\pi i}\oint_{\gamma_+} u^+(z) f^\sim(z)\,dz + \frac{1}{2\pi i}\oint_{\gamma_-} u^-(z)\, f^\sim(z)\,dz$$

where γ_+ is a negatively oriented circle of radius $1 - 1/2N$ and γ_- is a positively oriented circle of radius $1+1/2N$. $f^\wedge\colon B_N \to E_{\alpha\beta}$ is well defined, i.e. independent of the choices of contours γ_+, γ_- and extension $f^\sim$ of f, and defines an ib-linear map $f^\wedge\colon \mathscr{B}(\boldsymbol{T}) \to E$. Moreover, by definition, for any $t \in \boldsymbol{T}$

$$f^\wedge(\delta(t)) = \frac{1}{2\pi i}\oint_{\gamma_+} \frac{f^\sim(z)}{t-z}\,dz + \frac{1}{2\pi i}\oint_{\gamma_-} \frac{f^\sim(z)}{z-t}\,dz = f(t)$$

i.e. $f^\wedge \cdot \delta = f$.

§4. Unicity of the map $f^\wedge\colon \mathscr{B}(\boldsymbol{T}) \to E$.

It is enough to show that if $f^\wedge \cdot \delta = 0$ identically on $\boldsymbol{T}$, then $f^\wedge = 0$ identically on $\mathscr{B}(\boldsymbol{T})$. By the definition of ib-linear map

$$f^\wedge\colon \mathscr{B}(\boldsymbol{T}) \longrightarrow E = \text{inv}\lim_\alpha \text{dir}\lim_\beta E_{\alpha\beta},$$

for each α there are N and β such that $f^\wedge\colon B_N \to E_{\alpha\beta}$ is a bounded linear map between Banach spaces. Take a continuous linear functional $\varphi \in E'_{\alpha\beta}$. Then we have an equality for composed maps:

$$\varphi \cdot f^\wedge \cdot \delta = \varphi \cdot f.$$

By the duality theorem of Silva-Köthe-Grothendieck,

$$\varphi \cdot f^\wedge = 0.$$

Since $\varphi \in E'_{\alpha\beta}$ separate $E_{\alpha\beta}$, it follows that $f^\wedge = 0$ identically on $\mathscr{B}(\boldsymbol{T})$. This proves the unicity.

§5. Applications.

Our theorem states also that there exists a bijection $f \leftrightarrow f^\wedge$. For the case of $E = \boldsymbol{C}$, this shows that the dual space $\mathscr{B}'(\boldsymbol{T})$ of Sato-hyperfunctions is the space $\mathscr{A}(\boldsymbol{T})$ of analytic functions on $\boldsymbol{T}$, which is a part of Silva-Köthe-Grothendieck theorem. Novelty of our formulation is that the map $\delta\colon \boldsymbol{T} \to \mathscr{B}(\boldsymbol{T})$ is *analytic* and *universal* among analytic maps $f\colon \boldsymbol{T} \to E$.

Consider next the case $E = \mathscr{B}(\boldsymbol{T})$ itself. Then the space of ib-linear maps $f^\wedge\colon \mathscr{B}(\boldsymbol{T}) \to \mathscr{B}(\boldsymbol{T})$ is the space of linear operators on $\mathscr{B}(\boldsymbol{T})$. Köthe [6] has determined the space of inear operators as the space of two variable functions

$$f\colon \boldsymbol{T} \times \boldsymbol{C} \setminus \boldsymbol{T} \longrightarrow \boldsymbol{C}$$

which are analytic in two variables variables vanishing at $\boldsymbol{T} \times \infty$. To deduce this *kernel theorem* of Köthe from our theorem, it is enough to prove the following fact:

PROPOSITION. *$f: \boldsymbol{T}\to\mathscr{B}(\boldsymbol{T})$ is an ib-space valued analytic map if and only if the two variable function*

$$f: \boldsymbol{T}\times \boldsymbol{C}\setminus\boldsymbol{T}\longrightarrow \boldsymbol{C}$$

is analytic in each variable and vanishes at $\boldsymbol{T}\times\infty$.

PROOF. Suppose $f: \boldsymbol{T}\to\mathscr{B}(\boldsymbol{T})$ is an ib-space valued analytic map and take $(t_\circ, z_\circ)\in \boldsymbol{T}\times\boldsymbol{C}\setminus\boldsymbol{T}$. Assume $z_\circ\in\boldsymbol{C}\setminus\bar{D}$ for instance. Take N such that $|z_\circ|> 1+1/N$. For $(t, z)=(t_\circ+h, z)$ close to $(t_\circ, z)\in\boldsymbol{T}\times\boldsymbol{C}\setminus\boldsymbol{T}$,

$$f(t, z) = \textstyle\sum_{n=0}^{\infty} h^n f_n(t_\circ, z) \quad \text{in } B_N^-$$

This series converges uniformly in a small neighborhood of $(t_\circ, z_\circ)$ and each $f_n(t_\circ, z)$ is holomorphic in z. Hence $f(t, z)$ is analytic in two variables. By the definition of the space B_N^-, $f(t, z)$ vanishes at $z=\infty$.

Suppose, conversely, the map $f: \boldsymbol{T}\times\boldsymbol{C}\setminus\boldsymbol{T}\to\boldsymbol{C}$ is analytic in two variables and vanishes at $\boldsymbol{T}\times\infty$. Let us show that the ib-space valued map $f: \boldsymbol{T}\to\mathscr{B}(\boldsymbol{T})$ is analytic. For this, take a natural number N and let us prove first that f maps $\boldsymbol{T}$ into $B_N=B_N^+\oplus B_N^-$. Consider, for instance, $f(t, z)$ with $|z|>1+1/N$. For each $t\in\boldsymbol{T}$, $f(t, z)$ is holomorphic around z as long as $|z|>1$. Hence

$$\sup_{|z|=1+1/N}|f(t, z)| < +\infty.$$

By the compactness of $\boldsymbol{T}$ and $f(t, \infty)=0$, we conclude that

$$f(t, z)\in B_N^-.$$

Let us take $t_\circ\in\boldsymbol{T}$ and $t=t_\circ+h$; $|h|$ small. Then $f(t, z)=\sum_{n=0}^{\infty} h^n f_n(t_\circ, z)$ with

$$f_n(t_\circ, z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\tau, z)}{(\tau-t_\circ)^n}\,d\tau$$

where γ is a small contour negatively oriented around $t_\circ$, say $|\tau-t_\circ|=\varepsilon>0$. From

$$\|f_n(t_\circ, z)\|_N^- \leqq C\cdot\sup_{|\tau-t_\circ|=\varepsilon}\|f(\tau, z)\|_N^-/\varepsilon^n,$$

it follows that each coefficient $f_n(t_\circ, z)\in B_N^-$ and the series $f(t, z)=\sum_{n=0}^{\infty} h^n f_n(t_\circ, z)$ converges for small $|h|$, i.e. $f: \boldsymbol{T}\to B_N$ is analytic.

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