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Maintenance of oscillations under the effect of a strongly bounded forcing term

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1. Introduction

Consider the forced differential equation

(1) $$L_n x + f(t, x) = h(t)$$

and the corresponding unforced equation

(2) $$L_n x + f(t, x) = 0,$$

where $n \geqq 2$ and L_n is the general disconjugate differential operator defined recursively by $L_0 x(t) = a_0(t)x(t)$ and

$$L_k x(t) = a_k(t)(L_{k-1}x(t))', \quad k = 1, 2,..., n.$$

We shall assume without further mention that the functions $a_i(t)$, $i=0, 1,..., n$, are positive and continuous on $[t_0, \infty)$ and the operator L_n is in the first canonical form in the sense that

(3) $$\int_{t_0}^{\infty} a_i^{-1}(t)dt = \infty, \quad i = 1, 2,..., n-1.$$

In what follows, the set of all real-valued functions $y(t)$ defined on $[t_y, \infty)$ and such that $L_i y(t)$, $i=0, 1,..., n$, exist and are continuous on $[t_y, \infty)$ will be denoted by $D(L_n)$.

The purpose of this paper is to examine the oscillatory behaviour of solutions of Eq. (1) by comparing with that of the associated unforced Eq. (2). More precisely, we shall show that the oscillation of solutions of Eq. (1) follows from the oscillation of solutions of Eq. (2) provided that the forcing term $h(t)$ is the n-th "quasi-derivative" of the function $p(t)$ for which $L_0 p(t)$ is strongly bounded in the sense that it assumes its maximum and minimum on every interval of the form $[T, \infty)$, $T \geqq t_0$ (cf. [17]). This means that we can derive oscillation criteria for Eq. (1) from other similar ones which are known for Eq. (2).

Comparison results of this type in the case $a_0(t) = \cdots = a_n(t) = 1$ were first given by Kartsatos [9–12] for the forcings $h(t)$ with the following properties: there exists a continuous function $p(t)$ such that $p^{(n)}(t) = h(t)$ on $[t_0, \infty)$ and either

(I) $\lim_{t\to\infty} p(t) = 0$ and $p(t)$ is oscillatory in $[t_0, \infty)$; or

(II) there exist sequences $\{t'_n\}_{n=1}^{\infty}$, $\{t''_n\}_{n=1}^{\infty}$ and constants q_1, q_2 such that $\lim_{n\to\infty} t'_n = \lim_{n\to\infty} t''_n = \infty$, $p(t'_n) = q_1$, $p(t''_n) = q_2$ and $q_1 \leqq p(t) \leqq q_2$ for $t \geqq t_0$.

Obviously, the class of strongly bounded functions contains the above types of forcings but it is not restricted to them. For example, the functions $p(t) = (1 + 1/t) \sin t$ and $p(t) = \exp(\sin t/t)$ are strongly bounded but they satisfy neither (I) nor (II). In this spirit our main result unifies some earlier Kartsatos' results on the maintenance of oscillations under the effect of a "small" or "periodic-like" forcings and at the same time extends them to more general forcing functions.

For other related results concerning Eq. (1) and corresponding functional differential equations and inequalities we refer the reader to the papers of Chen and Yeh [2, 3], Foster [4], Grace and Lalli [5, 6], Jaroš [7, 8], Kawano, Kusano and Naito [13], Kusano et al. [14–16], McCann [17], Onose [18, 19] and True [21].

2. Preliminaries

In proving our results we employ a technique developed in [9–12] to change Eq. (1) into an equation of the form (2). In order to obtain more general results and for technical reasons, it is more convenient to work with the differential inequality

$$x(L_n x + f_1(t, x) - h_1(t)) \leqq 0 \tag{4}$$

and the equation

$$L_n y + f_2(t, y) = h_2(t), \tag{5}$$

where $n \geqq 2$, L_n is as above and for the functions f_i and h_i, $i = 1, 2$, the following conditions are assumed to hold:

(a) $f_i \in C([t_0, \infty) \times R, R)$ and $x f_i(t, x) > 0$, $i = 1, 2$, for $x \neq 0$ and every fixed $t \geqq t_0$,

(b) $h_i \in C([t_0, \infty), R)$ and there exist functions $p_i \in D(L_n)$, $i = 1, 2$, such that $L_n p_i(t) = h_i(t)$ and $L_0 p_i(t)$ are bounded on $[t_0, \infty)$.

The results for Eqs. (1) and (2) follow then as immediate corollaries of corresponding comparison theorems for the inequality (4) and Eq. (5).

As usual, we restrict our considerations only to those solutions $x(t)$ of (4) (or (5)) which exist on some ray $[t_x, \infty)$, $t_x \geqq t_0$, and satisfy

$$\sup\{|x(s)| : s \geqq t\} > 0$$

for every $t \in [t_x, \infty)$. The oscillatory character of such solutions is considered in the usual sense, i.e. $x(t)$ is said to be oscillatory if it has arbitrarily large zeros in $[t_x, \infty)$ and it is said to be nonoscillatory otherwise.

We begin by analysing the asymptotic behaviour of the possible nonoscillatory solutions of (4). We consider only the possible positive solutions since the negative solutions have the analogous properties and the corresponding results for such solutions can be proved similarly.

So, let $x(t)$ be a positive solution of (4) defined on $[t_0, \infty)$. Put $u(t) = x(t) - p_1(t)$. Then we can rewrite the inequality (4) as

$$L_n u(t) + f_1(t, u(t) + p_1(t)) \leqq 0, \quad t \geqq t_0. \tag{6}$$

In view of (b) and the positivity of $x(t)$, we obtain that $L_n u(t) < 0$ for $t \geqq t_0$ which implies that $L_k u(t)$, $k = 0, 1, \ldots, n-1$, have to be eventually of constant sign, say for $t \geqq t_1 \geqq t_0$. In particular, $u(t)$ is either positive or negative for $t \geqq t_1$.

It is well-known that if

$$y(t)L_n y(t) < 0 \quad (\text{resp. } y(t)L_n y(t) > 0)$$

for all sufficiently large t, then according to a generalization of a familiar Kiguradze's Lemma (see [20, Lemma 2]) there exist an integer l, $0 \leqq l \leqq n$, $n+l$ is odd (resp. $n+l$ is even), and a $t_1 \geqq t_0$ such that

$$y(t)L_i y(t) > 0 \quad \text{on} \quad [t_1, \infty) \quad \text{for} \quad i = 0, 1, \ldots, l, \tag{7}$$

$$(-1)^{i-l} y(t)L_i y(t) > 0 \quad \text{on} \quad [t_1, \infty) \quad \text{for} \quad i = l, l+1, \ldots, n. \tag{8}$$

Since in our case $x(t)$ is positive and $L_0 p_1(t)$ is bounded, that is, $L_0 u(t)$ cannot be unbounded from below, from (7) and (8) it follows that $L_1 u(t)$ is always positive on $[t_1, \infty)$ regardless to the positivity or negativity of $u(t)$ or possibly $L_1 u(t) < 0$ for n odd and $L_0 x(t)$ bounded on $[t_1, \infty)$. Consequently, we have the following modifications of Kiguradze's lemmas for the positive solutions of the forced inequality (4).

LEMMA 1. *Let n be even. If $x(t)$ is a positive solution of (4) for $t \geqq t_1 \geqq t_0$, then there exist an odd integer l, $1 \leqq l \leqq n-1$, and a $t_2 \geqq t_1$ such that for $t \geqq t_2$ the function $u(t) = x(t) - p_1(t)$ satisfies*

$$L_i u(t) > 0 \quad \textit{for} \quad i = 1, 2, \ldots, l \tag{9}$$

and

$$(-1)^{i+1} L_i u(t) > 0 \quad \textit{for} \quad i = l, l+1, \ldots, n. \tag{10}$$

LEMMA 2. *Let n be odd. If $x(t)$ is a positive solution of (4) for $t \geqq t_1 \geqq t_0$, then either $L_0 x(t)$ is unbounded on $[t_1, \infty)$ and there exist an even integer l, $2 \leqq l \leqq n-1$, and a $t_2 \geqq t_1$ such that for all $t \geqq t_2$ the function $u(t) = x(t) - p_1(t)$ satisfies (9) for $i = 1, 2, \ldots, l$ and*

(11) $$(-1)^i L_i u(t) > 0 \quad \text{for} \quad i = l, l+1, \dots, n,$$

or $L_0 x(t)$ *is bounded on* $[t_1, \infty)$ *and there exists a* $t_3 \geqq t_1$ *such that* (11) *holds on* $[t_3, \infty)$ *for* $i = 1, 2, \dots, n$.

3. Comparison theorems

In this section we shall be concerned with the relationship between the nonoscillatory solutions of the inequality (4) and the equation (5). As an application of these results, the oscillation of the differential inequality (4) is compared to that of the equation (5). We again consider only the eventually positive solutions of (4) and (5) since the corresponding part of our results concerning eventually negative solutions can be formulated and proved in an analogous way.

THEOREM 1. *Let n be even. In addition to the conditions* (a) *and* (b) *suppose that*

(c) $f_1(t, x) \geqq f_2(t, x)$ *for* $x > 0$

and $f_2(t, x)$ *is nondecreasing in x for every fixed* $t \geqq t_0$,

(d) $L_0 p_1(t)$ *is strongly bounded from below in the sense that for every* $T \geqq t_0$ *there is a* $T_* \geqq T$ *such that*

$$L_0 p_1(T_*) = \min_{t \in [T, \infty)} L_0 p_1(t),$$

(e) $\lim_{t \to \infty} L_0 p_2(t) = 0$.

If the inequality (4) *has an eventually positive solution* $x(t)$, *then Eq.* (5) *has an eventually positive solution* $y(t)$ *such that* $y(t) \leqq x(t)$ *for all large t.*

PROOF. Assume that there exists a solution $x(t)$ of (4) which is defined and positive on $[t_1, \infty)$, $t_1 \geqq t_0$. From Lemma 1 it follows that there are an odd integer l, $1 \leqq l \leqq n-1$, and a $t_2 \geqq t_1$ such that the function $u(t) = x(t) - p_1(t)$ satisfies (9) and (10) for $t \geqq t_2$.

Now, integrating (4) n-times and using (9) and (10), we get

$$\begin{aligned} L_0 u(t) \geqq L_0 u(t_2) + \int_{t_2}^{t} \frac{1}{a_1(s_1)} \int_{t_2}^{s_1} \frac{1}{a_2(s_2)} \cdots \int_{t_2}^{s_{l-1}} \frac{1}{a_l(s_l)} \int_{s_l}^{\infty} \frac{1}{a_{l+1}(s_{l+1})} \cdots \\ \cdots \int_{s_{n-1}}^{\infty} \frac{f_1(s, x(s))}{a_n(s)} ds ds_{n-1} \cdots ds_1 \\ \equiv L_0 u(t_2) + \Phi_l(t, t_2; f_1(t, x(t))) \end{aligned}$$

for $t \geqq t_2$.

Choose $t_* \geqq t_2$ such that

$$L_0 p_1(t_*) = \min_{t \in [t_2, \infty)} L_0 p_1(t).$$

Then we have

$$L_0x(t) \geqq c - L_0p_1(t) + L_0p_1(t_*) + \Phi_l(t, t_*; f_1(t, x(t))) \tag{12}$$
$$\geqq c + \Phi_l(t, t_*; f_1(t, x(t)))$$

for $t \geqq t_*$, where $c = L_0x(t_*) > 0$. Since $\lim_{t\to\infty} L_0p_2(t) = 0$, there is a $t_3 \geqq t_*$ such that

$$c \geqq \frac{c}{2} + L_0p_2(t) > 0 \tag{13}$$

for $t \geqq t_3$. From (12), (13) and (c) we obtain

$$L_0x(t) \geqq \frac{c}{2} + L_0p_2(t) + \Phi_l(t, t_3; f_2(t, x(t)))$$

for $t \geqq t_3$. Using a result of Čanturija [1], we conclude that there exists a continuous solution $y(t)$ of the integral equation

$$L_0y(t) = \frac{c}{2} + L_0p_2(t) + \Phi_l(t, t_3; f_2(t, y(t))) \tag{14}$$

such that

$$L_0x(t) \geqq L_0y(t) \geqq \frac{c}{2} + L_0p_2(t) > 0$$

for $t \geqq t_3$. Differentiating (14) n-times, we see that $y(t)$ is the solution of (5) with desired properties.

THEOREM 2. *Let n be odd. In addition to* (a) *and* (b) *suppose that the conditions* (c) *and* (e) *of Theorem* 1 *are satisfied. If the inequality* (4) *has an eventually positive solution $x(t)$ such that*

$$\lim_{t\to\infty} (L_0x(t) - L_0p_1(t)) = const > -q_* \tag{15}$$

where $q_* = \liminf_{t\to\infty} L_0p_1(t)$, then Eq. (5) *has an eventually positive solution $y(t)$ such that $\lim_{t\to\infty} L_0y(t) = const > 0$.*

PROOF. Assume that there exists a solution $x(t)$ of (4) which is positive on $[t_1, \infty)$ and such that (15) holds. By Lemma 2, there is a $t_2 \geqq t_1$ such that the inequalities (11) hold for $t \geqq t_2$ and $i = 1, 2, \dots, n$.

Integrating (4) n-times and using (11), we obtain

$$L_0u(t) \geqq c + \int_t^\infty \frac{1}{a_1(s_1)} \int_{s_1}^\infty \frac{1}{a_2(s_2)} \cdots \int_{s_{n-1}}^\infty \frac{f_1(s, x(s)))}{a_n(s)} ds ds_{n-1} \cdots ds_1$$
$$\equiv c + \Psi(t; f_1(t, x(t))),$$

where $c = \lim_{t\to\infty} L_0u(t)$ and $t \geqq t_2$.

Now, in view of (15), there is a $t_3 \geqq t_2$ such that

$$c + L_0 p_1(t) \geqq \frac{c+q_*}{2} > 0$$

for $t \geqq t_3$ and taking (e) into account we further have

$$\frac{c+q_*}{2} \geqq \frac{c+q_*}{4} + L_0 p_2(t) > 0$$

for all sufficiently large t, say $t \geqq t_4 \geqq t_3$.

Therefore

$$L_0 x(t) \geqq \frac{c+q_*}{4} + L_0 p_2(t) + \Psi(t; f_2(t, x(t)))$$

for $t \geqq t_4$ and using the results of Čanturija [1] again, we conclude that there exists a continuous solution $y(t)$ of the integral equation

$$(16) \qquad L_0 y(t) = \frac{c+q_*}{4} + L_0 p_2(t) + \Psi(t; f_2(t, y(t)))$$

with property

$$L_0 x(t) \geqq L_0 y(t) \geqq \frac{c+q_*}{4} + L_0 p_2(t) > 0$$

for $t \geqq t_4$. It is easy to see that $y(t)$ is also a solution of Eq. (5) and that $\lim_{t\to\infty} L_0 y(t) = \text{const} > 0$. This completes the proof.

From Theorems 1 and 2 and the analogous results for eventually negative solutions, we obtain the following comparison theorem concerning oscillation.

THEOREM 3. *Consider the differential inequality* (4) *and the equation* (5) *subject to the conditions* (a), (b), (e) *and*:

(c′) $|f_1(t, x)| \geqq |f_2(t, x)|$ *for* $x \neq 0$ *and* $f_2(t, x)$ *is nondecreasing in* x *for every fixed* t,

(d′) $L_0 p_1(t)$ *is strongly bounded on* $[t_0, \infty)$ *in the sense that for every* $T \geqq t_0$ *there are* T^*, $T_* \geqq T$ *such that*

$$L_0 p_1(T_*) = \min_{t\in[T,\infty)} L_0 p_1(t), \quad L_0 p_1(T^*) = \max_{t\in[T,\infty)} L_0 p_1(t).$$

Suppose, moreover, that for n even, every solution $y(t)$ of (5) *is oscillatory and for n odd, every solution is either oscillatory or satisfies* $\lim_{t\to\infty} L_0 y(t) = 0$. *Then, if n is even, every solution $x(t)$ of* (4) *is oscillatory, while if n is odd, every solution is either oscillatory or such that*

$$(17) \qquad \lim_{t\to\infty} (L_0 x(t) - L_0 p_1(t)) = -q_* \quad \text{or} \quad -q^*,$$

where $q_* = \lim_{t\to\infty} (\min_{\tau\in[t,\infty)} L_0 p_1(\tau))$ *and* $q^* = \lim_{t\to\infty} (\max_{\tau\in[t,\infty)} L_0 p_1(\tau))$.

PROOF. Let n be even. Assume to the contrary that there exists a nonoscillatory solution $x(t)$ of (4). Without loss of generality, we may assume that this solution is positive on $[t_1, \infty)$, $t_1 \geqq t_0$. From Theorem 1 it follows that there exists an eventually positive solution of (5), a contradiction.

Let n be odd. We can exclude the existence of a nonoscillatory solution $x(t)$ of (4) such that $L_0x(t)$ is unbounded since this leads to the existence of a nonoscillatory solution $y(t)$ of (5) such that $\lim_{t\to\infty} L_0y(t)\neq 0$ (the proof is similar to that of Theorem 1 and we omit it). So, the only interesting case is the case of a possible nonoscillatory solution $x(t)$ of (4) for which $L_0x(t)$ is bounded. Let this solution $x(t)$ be positive on $[t_1, \infty)$, $t_1 \geqq t_0$. From Lemma 2 it follows, in particular, that $L_1u(t)=L_1(x(t)-p_1(t))<0$ on $[t_2, \infty)$ for some $t_2 \geqq t_1$. Consequently, $\lim_{t\to\infty} L_0u(t)=c$, where c is a constant.

Denote $q_1(t)=\min_{\tau\in[t,\infty)} L_0p_1(\tau)$ and put $z(t)=L_0u(t)+q_1(t)$. Then we have

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} (L_0u(t)+q_1(t)) = c + q_* = d.$$

If $d<0$, then $L_0u(t)+q_1(t)<0$ for sufficiently large t, say $t\geqq T\geqq t_2$. By (e′), there exists a $T_*\geqq T$ such that

$$\begin{aligned} L_0u(T_*) + q_1(T_*) &= L_0u(T_*) + L_0p_1(T_*) \\ &= L_0x(T_*) - L_0p_1(T_*) + L_0p_1(T_*) \\ &= L_0x(T_*) > 0, \end{aligned}$$

a contradiction.

If $d>0$, then we use Theorem 2 to conclude that there exists a positive solution $y(t)$ of (5) such that $\lim_{t\to\infty} L_0y(t)=\text{const} >0$, which is again a contradiction.

Thus, we conclude that $d=0$, which implies

$$\lim_{t\to\infty} (L_0x(t)-L_0p_1(t)) = \lim_{t\to\infty} (z(t)-q_1(t)) = -q_*.$$

A parallel argument holds if we assume that (4) has a negative solution $x(t)$ with $L_0x(t)$ bounded on $[t_1, \infty)$. In this case we prove that

$$\lim_{t\to\infty} (L_0x(t)-L_0p_1(t)) = -q^*.$$

This completes the proof.

When specialized to Eqs. (1) and (2), the above theorem yields the following result according to which the oscillatory character of Eq. (2) is maintained by adding a "strongly bounded" forcing term.

COROLLARY. *Consider Eqs.* (1) *and* (2) *subject to the following conditions*:

(f) *f: $[t_0, \infty)\times R\to R$ is continuous, $xf(t, x)>0$ for every $x\neq 0$ and $f(t, x)$ is nondecreasing in x for every fixed t,*

(g) *there is a function $p\in D(L_n)$ such that $L_np(t)=h(t)$ and $L_0p(t)$ is strongly bounded on $[t_0, \infty)$.*

Suppose, moreover, that for n even, every solution $x(t)$ of (2) *is oscillatory and* *for n odd, every solution is either oscillatory or satisfies* $\lim_{t\to\infty} L_0x(t)=0$. *Then, if n is even, every solution x(t) of* (1) *is oscillatory, while if n is odd, every solution is either oscillatory or such that*

$$\lim_{t\to\infty}(L_0x(t)-L_0p(t)) = -p_* \quad or \quad -p^*, \tag{18}$$

where $p_*=\lim_{t\to\infty}(\min_{\tau\in[t,\infty)} L_0p(\tau))$ *and* $p^*=\lim_{t\to\infty}(\max_{\tau\in[t,\infty)} L_0p(\tau))$,

REMARK 1. We remark here that Theorem 3 and Corollary actually hold for bounded solutions if the assumptions concern only the bounded solutions of Eq. (5) (or (2)).

REMARK 2. The above results can easily be extended to the functional differential equations

$$L_nx(t) + f(t, x(g_1(t)),\dots, x(g_m(t))) = h(t) \tag{19}$$

and

$$L_nx(t) + f(t, x(g_1(t)),\dots, x(g_m(t))) = 0, \tag{20}$$

or, more generaly, to the functional differential inequality

$$x(t)\{L_nx(t)+f_1(t, x(g_1(t)),\dots, x(g_m(t)))-h_1(t)\} \leqq 0 \tag{21}$$

and the equation

$$L_ny(t) + f_2(t, y(g_1(t)),\dots, y(g_m(t))) = h_2(t), \tag{22}$$

where L_n, h, h_1 and h_2 are as before and:
(i) g_i: $[t_0, \infty)\to R$, $1\leqq i\leqq m$, are continuous and $\lim_{t\to\infty} g_i(t)=\infty$, $1\leqq i\leqq m$;
(ii) f: $[t_0, \infty)\times R^m\to R$ is continuous, $x_1f(t, x_1,\dots, x_m)>0$ if $x_1x_i>0$, $1\leqq i\leqq m$, and $f(t, x_1,\dots, x_m)$ is nondecreasing in each x_i for every fixed $t\geqq t_0$;
(iii) f_1, f_2: $[t_0, \infty)\times R^m\to R$ are continuous, $x_1f_j(t, x_1,\dots, x_m)>0$, $j=1, 2$, if $x_1x_i>0$, $1\leqq i\leqq m$, $|f_1(t, x_1,\dots, x_m)|\geqq|f_2(t, x_1,\dots, x_m)|$ if $x_1x_i>0$, $1\leqq i\leqq m$, and $f_2(t, x_1,\dots, x_m)$ is nondecreasing in each x_i for every fixed $t\geqq t_0$.

The details of this extension are left to the reader.

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