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Super Grassmann hierarchies

—A multicomponent theory—

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0. Introduction

This paper is concerned with a supersymmetric extension of certain completely integrable nonlinear systems.

Supersymmetry is a concept originated from the unification theory in particle physics. It is a formalism for describing Bose fields and Fermi fields simultaneously. It has been imported to mathematics recently. In the first place V. G. Kac [6] established the theory of Lie superalgebras. Representation theory of some infinite dimensional Lie superalgebras is studied prosperously ([3, 6, 12] and references cited there). B. Kostant [7], D. A. Leites [10] and A. Rogers [14] have developed the theory of supermanifolds. Some of nonlinear integrable differential equations have proved to have supersymmetric extensions. Among them are the two-dimensional Toda lattice [13], the Korteweg-de Vries (KdV) equation [9], the Liouville and the sine-Gordon equation [1] and so on [4, 8]. Yu. I. Manin and A. O. Radul [11] gave a supersymmetric extension of the one-component Kadomtsev-Petviashvili (KP) hierarchy as the Lax equations. They also gave the variational formalism.

The KP hierarchy was introduced by M. Sato and Y. Sato (cf. [15, 16, 17]). It can be seen that the multicomponent KP hierarchy includes, through the reduction procedure, the KdV equation, the Boussinesq equation, the nonlinear Schrödinger equation and the Toda lattice. Sato's fundamental theorem says that the KP hierarchy is a dynamical system on the infinite dimensional Grassmann manifold UGM.

The KP hierarchy can be treated in various aspects (cf. [2]). Among others the linearization equations or the Sato equations are most important. They are the equations

$$\partial W/\partial t_n^{(\alpha)} = B_n^{(\alpha)} W - W E_{\alpha\alpha} \partial_x^n,$$
$$B_n^{(\alpha)} = (W E_{\alpha\alpha} \partial_x^n W^{-1})_+ \qquad (n=1, 2,\dots; \alpha=0, 1,\dots, r-1)$$

for $W=\sum_{j\geq 0} w_j \partial_x^{-j} \in \mathscr{E}(0)^{mon}$ (see section 1).

The finite dimensional version of the KP hierarchy is named by K. Ueno the Grassmann hierarchy. In the theory of Grassmann hierarchies the fundamental role is played by a linear algebraic equation, which is called the Grassmann equation. Ueno and the author gave in [19, 20] a supersymmetric extension of one-component Grassmann hierarchies from the viewpoint of the Grassmann equation. Our approach is slightly different from [11].

In the present paper we treat a supersymmetric extension of multicomponent Grassmann hierarchies.

Let θ be a Grassmann number and put $\Theta=\partial_\theta+\theta\partial_x$. Super Grassmann hierarchies are described by using the super microdifferential operator $W=\sum_{j=0}^{m} w_j\Theta^{-j}$. Naturally the odd time evolution is introduced as well as the even time evolution. In the theory of multicomponent super Grassmann hierarchies the matrix valued super Grassmann equation is the central object.

The plan of this paper is as follows. We review one-component and multicomponent Grassmann hierarchies in Section 1. Section 2 is devoted to the preliminaries for superanalysis. One-component super Grassmann hierarchies are reviewed briefly in Section 3. In Section 4 we formulate multicomponent super Grassmann hierarchies and induce the Sato equations for them (Theorem 4.1). In Section 5 we prove our main theorem (Theorem 5.1) which gives an expression of solutions using superdeterminants.

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1. Grassmann hierarchies

To clarify our motivation, we summarize in this section the theory of *Grassmann hierarchies* according to Sato's lecture [16].

Let $\mathscr{K}$ be a differential field of one variable x. Namely there is an additive map $\partial_x\colon \mathscr{K}\to\mathscr{K}$ with the property $\partial_x(fg)=\partial_x(f)g+f\partial_x(g)$ for $f,\ g\in\mathscr{K}$. Put $\mathscr{C}=\{f\in\mathscr{K};\ \partial_x(f)=0\}$, the constant field of $\mathscr{K}$. Let $\mathscr{D}=\mathscr{K}[\partial_x]$, $\mathscr{E}=\mathscr{K}((\partial_x^{-1}))$ be the ring of differential operators, the ring of microdifferential operators respectively. The ring structure is defined through the Leibniz rule

$$\partial_x^n\cdot f=\sum_{\nu=0}^{\infty}\binom{n}{\nu}f^{(\nu)}\cdot\partial_x^{n-\nu}\quad (n\in\boldsymbol{Z}),$$

where $f\in\mathscr{K}$, $f^{(\nu)}=\partial_x^\nu(f)$, $\binom{n}{\nu}=n(n-1)\cdots(n-\nu+1)/\nu!$. Let $\mathscr{D}(m)$, $\mathscr{E}(m)$ be the subspaces of $\mathscr{D}$, $\mathscr{E}$ consisting of operators of order less than or equal to m, so that $\mathscr{D}=\bigcup_{m\geq 0}\mathscr{D}(m)$, $\mathscr{E}=\bigcup_{m\in\boldsymbol{Z}}\mathscr{E}(m)$. We denote by $\mathscr{D}(m)^{mon}$, $\mathscr{E}(m)^{mon}$ the subspaces of monic operators of the form $\partial_x^m+(\textit{lower order terms})$. For an operator $P=\sum_{n\in\boldsymbol{Z}}p_n\partial_x^n\in\mathscr{E}$, we put $(P)_+=\sum_{n\geq 0}p_n\partial_x^n\in\mathscr{D}$.

Fix two positive integers $N>m$. Let us consider the following linear equation, which is called the *Grassmann equation*:

$$\tilde{w}\Phi\Xi = 0, \tag{1.1}$$

where $\tilde{w}=(w, 1, 0,\dots, 0)$, $w=(w_m, w_{m-1},\dots, w_1)$, $w_j\in\mathscr{K}$, $\Phi=\exp(x\Lambda_N)$, $\Lambda_N=(\delta_{i+1,j})_{0\leq i,j<N}$ and $\Xi\in Mat(N, m; \mathscr{C})$ with rank $\Xi=m$. We call such a matrix Ξ an *N-dimensional m-frame*, and the totality of N-dimensional m-frames is denoted by $FR(N, m; \mathscr{C})$. The equation (1.1) for the unknown vector w is solved uniquely for any $\Xi\in FR(N, m; \mathscr{C})$. We denote by $A_{l_0l_1\cdots l_{m-1}}$ the determinant of the $m\times m$-matrix consisting of l_i-th rows of $A\in Mat(N, m; \mathscr{K})$. Then the solution is given by

$$w_j = (-)^j(\Phi\Xi)_{01\cdots m-j-1,m-j+1\cdots m}/(\Phi\Xi)_{01\cdots m-1} \quad (j=1,\dots, m).$$

On the space $FR(N, m; \mathscr{C})$ there is an action of $GL(m; \mathscr{C})$ from the right by multiplication. If Ξ is replaced by Ξg ($g\in GL(m; \mathscr{C})$), then $(\Phi\Xi g)_{l_0l_1\cdots l_{m-1}}=(\Phi\Xi)_{l_0l_1\cdots l_{m-1}}(\det g)$. Thus the solution w is invariant under $GL(m; \mathscr{C})$. Therefore we have the one-to-one correspondence:

$$\{w;\ \textit{solution of}\ (1.1)\} \simeq FR(N, m; \mathscr{C})/GL(m; \mathscr{C}).$$

The right hand side is nothing but the Grassmann manifold $GM(m, N-m)$.

Next we introduce the *time evolution* or *deformation* of w in the Grassmann manifold, and consider the Grassmann hierarchy. The time variables are denoted by $t_1, t_2, t_3,\dots$. We give now the time evolution of the frame Ξ by $\Xi(t)=\exp(\eta(t, \Lambda_N))\Xi$, where $\eta(t, \lambda)=\sum_{n=1}^{\infty} t_n\lambda^n$. The time evolution of w is described as follows. There exist functions $b_i^{(n)}$ ($n=1, 2,\dots; i=0, 1,\dots, n$) for which the relations

$$(\partial\tilde{w}/\partial t_n+\tilde{w}\Lambda_N^n) - \sum_{i=0}^{n} b_i^{(n)}\Big(\sum_{\nu=0}^{n-i}\binom{n-i}{\nu}(\partial^\nu\tilde{w}/\partial x^\nu)\Lambda_N^{n-i-\nu}\Big) = 0$$

hold. These equations are induced from

$$(\partial/\partial t_n-\sum_{i=0}^{n} b_i^{(n)}\partial_x^{n-i})(\tilde{w}\Phi\Xi(t)) = 0,$$

and from the unique solvability of (1.1). Put $W=\sum_{j=0}^{m} w_j\partial_x^{m-j}$ ($w_0=1$) and $B_n=\sum_{i=0}^{n} b_i^{(n)}\partial_x^{n-i}$. These operators are monic of order m and n respectively. By an easy claculation we see that

$$\partial W/\partial t_n = B_nW - W\partial_x^n. \tag{1.2}$$

More elegant way to obtain (1.2) is as follows. Let $\psi=(\psi_0,\dots, \psi_{m-1})$ be the 0-th row of the matrix $\Phi\Xi(t)$. The Grassmann equation (1.1) is equivalent to the equation

(1.3) $$W\psi = 0.$$

By differentiating (1.3) with respect to t_n, and by using the relation $\partial\psi/\partial t_n = \partial^n\psi/\partial x^n$, we get $(\partial W/\partial t_n + W\partial_x^n)\psi = 0$. By using the division theorem of differential operators we have

$$\partial W/\partial t_n + W\partial_x^n = B_n W + R_n$$

for some $B_n \in \mathscr{D}(n)^{mon}$ and $R_n \in \mathscr{D}(m-1)$. Then we have $R_n\psi_k = 0$ for linearly independent functions $\psi_0, \ldots, \psi_{m-1}$. This says that $R_n = 0$, and the equations (1.2) are obtained. From (1.2), the differential operator B_n is rewritten as $B_n = (W\partial_x^n W^{-1})_+$. The equations (1.2) are called the *Sato equations* for the Grassmann hierarchy. Solutions are expressed by

$$w_j = p_j(-\tilde{\partial}_t)((\Phi\Xi(t))_{01\cdots m-1})/(\Phi\Xi(t))_{01\cdots m-1},$$

where $p_j(t)$ are the Schur polynomials defined by $\exp(\eta(t, \lambda)) = \sum_{j\geq 0} p_j(t)\lambda^j$ and $\tilde{\partial}_t = (\partial/\partial t_1, 2^{-1}\partial/\partial t_2, 3^{-1}\partial/\partial t_3, \ldots)$ ([17; p. 269]). Put $\Xi_0 = {}^t(I_m, 0) \in FR(N, m; \mathscr{C})$. The determinant $\tau(t, \Xi) = (\Phi\Xi(t))_{01\cdots m-1} = \det({}^t\Xi_0\Phi\Xi(t))$ is an important quantity and is called the *τ-function*. The nonlinear evolution equations (1.2) for w are transformed to Hirota's bilinear equations for the τ-function ([17; Theorem 2]).

Next we consider the multicomponent Grassmann hierarchy. Let r be the number of components. The Grassmann equation is

(1.4) $$\tilde{\mathscr{W}}\Phi_r\Xi = 0,$$

where $\tilde{\mathscr{W}} = (\tilde{w}^{(\alpha\beta)})_{0\leq\alpha,\beta<r}$, $\tilde{w}^{(\alpha\beta)} = (w^{(\alpha\beta)}, \delta_{\alpha\beta}, 0, \ldots, 0)$, $w^{(\alpha\beta)} = (w_m^{(\alpha\beta)}, \ldots, w_1^{(\alpha\beta)})$, $w_j^{(\alpha\beta)} \in \mathscr{K}$, $\Phi_r = \operatorname{diag}(\exp(x\Lambda_N), \ldots, \exp(x\Lambda_N))$ and $\Xi \in FR(rN, rm; \mathscr{C})$. For a frame Ξ, which satisfies the condition

(1.5) $$\det(\operatorname{diag}({}^t\Xi_0, \ldots, {}^t\Xi_0)\Phi_r\Xi) \neq 0,$$

the equation (1.4) is uniquely solved by using Cramer's formula. Note that if Ξ satisfies (1.5), then so does Ξg for $g \in GL(rm; \mathscr{C})$. Like the one-component case we have the following correspondence:

(1.6) $$\{w^{(\alpha\beta)}, 0 \leq \alpha, \beta < r; \textit{solution of } (1.4)\} \simeq \{\Xi \in FR(rN, rm; \mathscr{C}); \Xi \textit{ satisfies } (1.5)\}/GL(rm; \mathscr{C}).$$

Recall that the Grassmann manifold $GM(m, n)$ is decomposed into cells:

$$GM(m, n) = \amalg_Y GM(m, n)^Y,$$

where Y runs over all Young diagrams included in the $m \times n$-rectangular Young diagram (For the cell decomposition of Grassmann manifolds by means of

Young diagrams, one can find a detailed exposition in [16]). The right hand side of (1.6) is a subset of the Grassmann manifold $GM(rm, r(N-m))$. For example, if $r=2$ and $m=1$, $N=2$, then the right hand side of (1.6) is the union of cells $GM(2, 2)^Y$, where $Y=\square, \square\square, \begin{smallmatrix}\square\\ \square\end{smallmatrix}, \begin{smallmatrix}\square\square\\ \square\end{smallmatrix}$, and a subset of the generic cell $GM(2, 2)^{\phi}$.

We prepare the time variables $t_n^{(\alpha)}$ $(n=1, 2,\dots; \alpha=0, 1,\dots, r-1)$ and define the time evolution of Ξ by $\Xi(t)=\operatorname{diag}(\exp(\eta(t^{(0)}, \Lambda_N)),\dots, \exp(\eta(t^{(r-1)}, \Lambda_N)))\Xi$. The τ-function $\tau(t, \Xi)$ is defined by the left hand side of (1.5). The diagonal components $w_j^{(\alpha\alpha)}$ are expressed as

$$w_j^{(\alpha\alpha)} = p(-\tilde{\partial}_{t^{(\alpha)}})(\tau(t, \Xi))/\tau(t, \Xi).$$

One can see that there exist $r\times r$-matrices $b_{ni}^{(\alpha)}$ $(n=1, 2,\dots; i=0, 1,\dots, n; \alpha=0, 1,\dots, r-1)$ with $b_{n0}^{(\alpha)}=E_{\alpha\alpha}$ for which the relations

$$(\partial\tilde{\mathscr{W}}/\partial t_n^{(\alpha)} + \tilde{\mathscr{W}}(\Lambda_N^{(\alpha)})^n) - \textstyle\sum_{i=0}^n b_{ni}^{(\alpha)}\Big(\sum_{\nu=0}^{n-i}\binom{n-i}{\nu}\tilde{\mathscr{W}}^{(n-i-\nu)}(\Lambda_{N,r})^{\nu}\Big) = 0$$

hold, where we have put $\Lambda_N^{(\alpha)}=E_{\alpha\alpha}\otimes\Lambda_N$, $\Lambda_{N,r}=\operatorname{diag}(\Lambda_N,\dots, \Lambda_N)$. Putting $W=\sum_{j=0}^m w_j\partial_x^{-j}$ and $B_n^{(\alpha)}=\sum_{i=0}^n b_{ni}^{(\alpha)}\partial_x^{n-i}$, we have

$$\begin{aligned}
B_n^{(\alpha)} W &= \textstyle\sum_{i=0}^n \sum_{j=0}^m b_{ni}^{(\alpha)}\Big(\sum_{\nu=0}^{n-i}\binom{n-i}{\nu} w_j^{(n-i-\nu)}\partial_x^{\nu}\Big)\partial_x^{-j} \\
&= \textstyle\sum_{j=0}^m E_{\alpha\alpha}w_j\partial_x^{n-j} + \sum_{j=0}^m \sum_{\nu=0}^{n-1}\binom{n}{\nu}E_{\alpha\alpha}w_j^{(n-\nu)}\partial_x^{\nu-j} \\
&\quad + \textstyle\sum_{j=0}^m \sum_{i=1}^n b_{ni}^{(\alpha)}\Big(\sum_{\nu=0}^{n-i}\binom{n-i}{\nu} w_j^{(n-i-\nu)}\partial_x^{\nu-j}\Big) \\
&= \textstyle\sum_{j=0}^m w_jE_{\alpha\alpha}\partial_x^{n-j} - \sum_{j=0}^m [w_j, E_{\alpha\alpha}]\partial_x^{n-j} \\
&\quad + \textstyle\sum_{j=0}^m \sum_{\nu=0}^{n-1} E_{\alpha\alpha}w_j^{(n-\nu)}\partial_x^{\nu-j} \\
&\quad + \textstyle\sum_{j=0}^m \sum_{i=1}^n \sum_{\nu=0}^{n-i}\binom{n-i}{\nu} b_{ni}^{(\alpha)}w_j^{(n-i-\nu)}\partial_x^{\nu-j} \\
&= WE_{\alpha\alpha}\partial_x^n + \partial W/\partial t_n^{(\alpha)}.
\end{aligned}$$

Thus we obtain the Sato equations for the r-component Grassmann hierarchy:

$$\partial W/\partial t_n^{(\alpha)} = B_n^{(\alpha)}W - WE_{\alpha\alpha}\partial_x^n. \tag{1.7}$$

Here we have $B_n^{(\alpha)}=(WE_{\alpha\alpha}\partial_x^nW^{-1})_+$. Clearly $B_1^{(\alpha)}=E_{\alpha\alpha}\partial_x$. If $r=2$, then one sees

$$B_2^{(0)} = \begin{bmatrix} \partial_x^2+(w_1^{(01)}w_1^{(10)}-2(w_1^{(00)})_x) & -w_1^{(01)}\partial_x+(w_1^{(01)}w_1^{(11)}-2(w_1^{(01)})_x) \\ w_1^{(10)}\partial_x-w_1^{(00)}w_1^{(10)} & -w_1^{(01)}w_1^{(10)} \end{bmatrix},$$

$$B_2^{(1)} = \begin{bmatrix} -w_1^{(01)}w_1^{(10)} & w_1^{(01)}\partial_x-w_1^{(01)}w_1^{(11)} \\ -w_1^{(10)}\partial_x+(w_1^{(00)}w_1^{(10)}-2(w_1^{(10)})_x) & \partial_x^2+(w_1^{(01)}w_1^{(10)}-2(w_1^{(11)})_x) \end{bmatrix}.$$

The (00)-component of $B_n^{(0)}$ is, provided that $w_j^{(\alpha\beta)}=0$ for $(\alpha, \beta)\neq(0, 0)$, nothing but B_n of one-component theory.

For a solution W of (1.7) we define the Lax operator by $L=W\partial_x W^{-1}$, and define $C^{(\alpha)}=WE_{\alpha\alpha}W^{-1}$. Then they satisfy the following *Lax equations*:

$$\partial L/\partial t_n^{(\alpha)} = [B_n^{(\alpha)}, L], \quad \partial C^{(\beta)}/\partial t_n^{(\alpha)} = [B_n^{(\alpha)}, C^{(\beta)}]$$

with $B_n^{(\alpha)}=(C^{(\alpha)}L^n)_+$.

Taking the formal limit N, $m\to\infty$, one obtains the KP hierarchy.

2. Superdeterminats, superfields, super microdifferential operators

Let $\mathscr{C}^{\boldsymbol{N}}$ be the infinite dimensional vector space over $\mathscr{C}$ spanned by $\{e_n\}_{n\in\boldsymbol{N}}$. Put $\mathscr{A}=\Lambda(\mathscr{C}^{\boldsymbol{N}})$, the exterior algebra of $\mathscr{C}^{\boldsymbol{N}}$. This is a typical example of super-commutative superalgebras. Namely $\mathscr{A}$ has the decomposition $\mathscr{A}=\mathscr{A}_{[0]}\oplus\mathscr{A}_{[1]}$, where $\mathscr{A}_{[0]}$ (resp. $\mathscr{A}_{[1]}$) is the vector space spanned by elements of the form $e_{i_0}\wedge e_{i_1}\wedge\cdots\wedge e_{i_{k-1}}$ with even (resp. odd) k, $\mathscr{A}_{[i]}\cdot\mathscr{A}_{[j]}\subset\mathscr{A}_{[i+j]}$ ($[i]$ denotes $i(\mathrm{mod}\,2)$), and $ab=(-)^{ij}ba$ for $a\in\mathscr{A}_{[i]}$, $b\in\mathscr{A}_{[j]}$. The subspace $\mathscr{A}_{[0]}$ (resp. $\mathscr{A}_{[1]}$) is called the even (resp. odd) part of $\mathscr{A}$. There is a canonical projection ε: $\mathscr{A}\to\mathscr{C}$, which is called the *body map*. An element $a\in\mathscr{A}$ is invertible if and only if $\varepsilon(a)\neq 0$. For if $\varepsilon(a)\neq 0$, then

$$\varepsilon(a)^{-1}\textstyle\sum_{n\geqslant 0}(-)^n((a-\varepsilon(a))/\varepsilon(a))^n$$

converges and gives a^{-1}.

For positive integers m and n, we define

$$\begin{aligned} Mat(m\,|\,n;\mathscr{A})_{[\nu]} = \{X=(A_{\alpha\beta})_{0\leqslant\alpha,\beta<2};\ & A_{00}\in Mat(m, m;\mathscr{A}_{[\nu]}),\\ & A_{01}\in Mat(m, n;\mathscr{A}_{[1+\nu]}),\ A_{10}\in Mat(n, m;\mathscr{A}_{[1+\nu]}),\\ & A_{11}\in Mat(n, n;\mathscr{A}_{[\nu]})\}\end{aligned}$$

for $\nu=0$, 1, and put $Mat(m\,|\,n;\mathscr{A})=Mat(m\,|\,n;\mathscr{A})_{[0]}\oplus Mat(m\,|\,n;\mathscr{A})_{[1]}$. Define

$$GL(m\,|\,n;\mathscr{A}) = \{X\in Mat(m\,|\,n;\mathscr{A})_{[0]};\ X\ \textit{is invertible}\}.$$

The invertibility of $(A_{\alpha\beta})\in Mat(m\,|\,n;\mathscr{A})_{[0]}$ is equivalent to that of $\varepsilon(A_{00})$ and $\varepsilon(A_{11})$. We define the *superdeterminants* of $X=(A_{\alpha\beta})\in GL(m\,|\,n;\mathscr{A})$ by

$$\begin{aligned}\mathrm{sdet}\,X &= \det(A_{00}-A_{01}A_{11}^{-1}A_{10})/\det A_{11},\\ \mathrm{s}^{-1}\mathrm{det}\,X &= \det(A_{11}-A_{10}A_{00}^{-1}A_{01})/\det A_{00}.\end{aligned}$$

The following remarkable property of the superdeterminants is of importance in our argument.

PROPOSITION 2.1 ([10]). *Let* $X, Y \in GL(m\,|\,n; \mathscr{A})$. *Then* $XY \in GL(m\,|\,n; \mathscr{A})$ *and* 1) $\operatorname{sdet} XY = (\operatorname{sdet} X)(\operatorname{sdet} Y)$, $s^{-1}\det XY = (s^{-1}\det X)(s^{-1}\det Y)$, 2) $(\operatorname{sdet} X)\cdot(s^{-1}\det X) = 1$.

For positive integers $M > m$, $N > n$, define

$$\begin{aligned} FR(M\,|\,N, m\,|\,n; \mathscr{A}) = \{ & \Xi^{\vee} = (\Xi_{\alpha\beta})_{0<\alpha,\beta<2};\ \Xi_{00} \in Mat(M, m; \mathscr{A}_{[0]}), \\ & \Xi_{01} \in Mat(M, n; \mathscr{A}_{[1]}),\ \Xi_{10} \in Mat(N, m; \mathscr{A}_{[1]}), \\ & \Xi_{11} \in Mat(N, n; \mathscr{A}_{[0]}),\ \operatorname{rank} \varepsilon(\Xi_{00}) = m, \\ & \operatorname{rank} \varepsilon(\Xi_{11}) = n\}. \end{aligned}$$

An element $FR(M\,|\,N, m\,|\,n; \mathscr{A})$ is called an *$M\,|\,N$-dimensional $m\,|\,n$-superframe.* On the space $FR(M\,|\,N, m\,|\,n; \mathscr{A})$ there is an action of $GL(m\,|\,n; \mathscr{A})$ by the right multiplication. Therefore we can define the *super Grassmann manifold* by

$$GM(m\,|\,n, (M-m)\,|\,(N-n); \mathscr{A}) = FR(M\,|\,N, m\,|\,n; \mathscr{A})/GL(m\,|\,n; \mathscr{A}).$$

Let θ be an abstract Grassmann number, i.e., $\theta^2 = 0$. We assume that $x\theta = \theta x$. Denote $\mathscr{K}[\theta] = \mathscr{K} \oplus \mathscr{K}\theta$. We consider the superalgebra $\mathscr{S} = \mathscr{K}[\theta] \otimes \mathscr{A} = \mathscr{S}_{[0]} \oplus \mathscr{S}_{[1]}$ of superfields. A superfield f is of the form

$$f = f_{00} + \theta f_{01} + f_{10} + \theta f_{11},$$

where $f_{00}, f_{11} \in \mathscr{K} \otimes \mathscr{A}_{[0]}$, $f_{01}, f_{10} \in \mathscr{K} \otimes \mathscr{A}_{[1]}$. Put

$$f^* = f_{00} + \theta f_{01} - f_{10} - \theta f_{11}.$$

The derivation ∂_θ acts on $\mathscr{S}$ and on $Mat(r, r; \mathscr{S})$ by $\partial_\theta(f) = f_{01} + f_{11}$. We define a square root of ∂_x by $\Theta = \partial_\theta + \theta\partial_x : \mathscr{S}_{[\nu]} \to \mathscr{S}_{[1+\nu]}$. Namely

$$\Theta(f) = f_{11} + \theta(f_{10})_x + f_{01} + \theta(f_{00})_x.$$

We often use the notation $\mathring{f}$ instead of $\Theta(f)$. One easily checks that $\Theta^2 = \partial_x$. The inverse element of Θ is given by $\Theta^{-1} = \theta + \partial_\theta \partial_x^{-1}$. For an integer n, the super Leibniz rule reads

$$\Theta^{2n} \cdot f = \textstyle\sum_{j>0} \binom{n}{j} f^{(j)} \Theta^{2n-2j},$$

$$\Theta^{2n+1} \cdot f = \textstyle\sum_{j>0} \binom{n}{j} \mathring{f}^{(j)} \Theta^{2n-2j} + (-)^{\nu} \sum_{j>0} \binom{n}{j} f^{(j)} \Theta^{2n+1-2j}$$

for $f \in \mathscr{S}_{[\nu]}$. Adding Θ^{-1} to the superalgebra $\mathscr{S}$, we get the space of *super microdifferential operators* $\mathscr{E}^{1|1} = \mathscr{S}((\Theta^{-1}))$, which plays a role of the ring of microdifferential operators $\mathscr{E}$. We consider $\mathscr{E}^{1|1}$ itself and $Mat(r, r; \mathscr{E}^{1|1})$.

The product of two elements is defined through the super Leibniz rule. The $\boldsymbol{Z}_2$-gradation $\mathscr{E}^{1|1}=\mathscr{E}^{1|1}{}_{[0]}\oplus\mathscr{E}^{1|1}{}_{[1]}$ is given by

$$\mathscr{E}^{1|1}{}_{[\nu]} = \{\textstyle\sum_{n\in\boldsymbol{Z}} p_n\Theta^n \in \mathscr{E}^{1|1};\ p_n \in \mathscr{S}_{[n+\nu]}\}\,.$$

We put $\mathscr{E}^{1|1}(m)=\mathscr{S}[[\Theta^{-1}]]\Theta^m$ for $m\in\boldsymbol{Z}$, so that $\mathscr{E}^{1|1}=\bigcup_{m\in\boldsymbol{Z}}\mathscr{E}^{1|1}(m)$. There is a subsuperalgebra $\mathscr{D}^{1|1}=\mathscr{S}[\Theta]$, whose element is called a *super differential operator*. Through the direct sum decomposition $\mathscr{E}^{1|1}=\mathscr{D}^{1|1}\oplus\mathscr{E}^{1|1}(-1)$, any element $P\in\mathscr{E}^{1|1}$ is uniquely represented as $P=(P)_+ + (P)_-$, where $(P)_+\in\mathscr{D}^{1|1}$ and $(P)_-\in\mathscr{E}^{1|1}(-1)$. An operator $P=\sum_{n\in\boldsymbol{Z}} p_n\Theta^n\in\mathscr{E}^{1|1}(m)$ (resp. $Mat(r, r; \mathscr{E}^{1|1}(m))$) $(p_m\neq 0)$ is invertible if and only if p_m is invertible in $\mathscr{S}$ (resp. $Mat(r, r; \mathscr{S})$), and in that case $P^{-1}\in\mathscr{E}^{1|1}(-m)$ (resp. $Mat(r, r; \mathscr{E}^{1|1}(-m))$. We remark that the operator $W=\sum_{j\geq 0} w_j\Theta^{-j}$ with $w_0=1$ (resp. $w_0=I$) is always invertible.

3. One-component super Grassmann hierarchies

In this section we review briefly the theory of one-component *super Grassmann hierarchies* according to [19, 20]. Equations in this section will be generalized in section 4 to the multicomponent case.

We start with the following *super Grassmann equation*:

$$\tilde{\boldsymbol{w}}\Phi\Xi = 0, \tag{3.1}$$

where $\tilde{\boldsymbol{w}}=(\boldsymbol{w}, 1, 0,\dots, 0)$, $\boldsymbol{w}=(w_m,\dots, w_1)$, $w_j\in\mathscr{S}$, $\Phi=\exp(\theta\Lambda_N+x\Lambda_N^2)$ and $\Xi=(\xi_{ij})_{0<i<N,0<j<m}\in Mat(N, m; \mathscr{A})$ with $\xi_{ij}\in\mathscr{A}_{[i+j]}$ and $\operatorname{rank}\varepsilon(\Xi)=m$. For the sake of simplicity we assume that N and m are even numbers. For a matrix $X=(x_{ij})_{0<i<K,0<j<L}\in Mat(K, L; \mathscr{S})$ with $x_{ij}\in\mathscr{S}_{[i+j]}$, denote $X^\vee=(X_{\alpha\beta})_{0<\alpha,\beta<2}$, where $X_{\alpha\beta}=(x_{2i+\alpha, 2j+\beta})$. Then $\Xi^\vee\in FR(N/2\,|\,N/2, m/2\,|\,m/2; \mathscr{A})$ and equation (3.1) is rewritten as

$$(w_m, w_{m-2},\dots, w_2, 1, 0,\dots, 0;\ w_{m-1}, w_{m-3},\dots, w_1, 0,\dots, 0)\Phi^\vee\Xi^\vee = 0. \tag{3.2}$$

This equation is uniquely solved so that $w_j\in\mathscr{S}_{[j]}$ for any $N/2\,|\,N/2$-dimensional $m/2\,|\,m/2$-superframe $\Xi^\vee$. Solutions are expressed by means of the superdeterminants. From Proposition 2.1 (1), one obtains the correspondence:

$$\{\boldsymbol{w};\ \textit{solution of}\ (3.1)\} \simeq GM(m/2\,|\,m/2, (N-m)/2\,|\,(N-m)/2; \mathscr{A})\,.$$

We introduce the time evolution and consider the super Grassmann hierarchy. The even time variables are denoted by $t_2, t_4, t_6,\dots$ and the odd ones are denoted by $t_1, t_3, t_5,\dots$. We assume that they satisfy the commutation and anticommutation relations $[t_{2n}, t_m]=0$, $[t_{2n-1}, t_{2m-1}]_+=0$. Put $\Gamma_N=((-)^{i+1}\delta_{i+1,j})_{0<i,j<N}$, which has the properties $\Gamma_N^2=-\Lambda_N^2$ and $[\Gamma_N, \Lambda_N]_+=0$. The time evolution $\Xi(t)$ is

defined by $\Xi(t)=\exp(\eta(t, \Gamma_N))\Xi$. Define the differential operators ([11])

$$\Theta_{2n} = \partial/\partial t_{2n}, \quad \Theta_{2n-1} = \partial/\partial t_{2n-1} + \sum_{m>1} t_{2m-1}\partial/\partial t_{2n+2m-2}$$

for $n \geqslant 1$. These operators satisfy the bracket relations:

$$\begin{aligned}(3.3) \qquad & [\Theta, \Theta_{2n}] = [\Theta, \Theta_{2n-1}]_+ = 0, \\ & [\Theta_{2n}, \Theta_m] = 0, \quad [\Theta_{2n-1}, \Theta_{2m-1}]_+ = 2\Theta_{2m+2m-2}.\end{aligned}$$

Define an even operator $W=\sum_{j=0}^{m} w_j\Theta^{-j}$ ($w_0=1$) for the solution w of the super Grassmann equation $\tilde{w}\Phi\Xi(t)=0$. The Sato equations for the super Grassmann hierarchy have the following form ([19, 20]):

$$\begin{aligned}(3.4) \qquad & \Theta_{2n}(W) = (-)^n(B_{2n}W - W\Theta^{2n}), \\ & \Theta_{2n-1}(W) = (-)^{n+m}(B_{2n-1}W - W\Theta^{2n-1}),\end{aligned}$$

where $B_n=(W\Theta^n W^{-1})_+$. Here we give some B_n's.

$$\begin{aligned}B_1 &= \Theta + 2w_1, \quad B_2 = \Theta^2 \quad (=\partial_x), \\ B_3 &= \Theta^3 + 2w_1\Theta^2 - \dot{w}_1\Theta + (2w_3 - \dot{w}w_1 - 2w_1w_2 + (w_1)_x - \dot{w}_2), \\ B_4 &= \Theta^4 - 2(w_1)_x\Theta - 2((w_1)_x w_1 + (w_2)_x).\end{aligned}$$

We define the Lax operator by $L=W\Theta W^{-1}$. Note that this is an odd operator and that $\dot{u}_1+2u_2=0$ if $L=\sum_{i>0} u_i\Theta^{1-i}$. The time evolution of L can be calculated as follows.

$$\begin{aligned}(3.5) \qquad \Theta_{2n-1}(L) &= \Theta_{2n-1}(W\Theta W^{-1}) \\ &= \Theta_{2n-1}(W)\Theta W^{-1} - W\Theta\Theta_{2n-1}(W^{-1}) \\ &= (-)^{n+m}\{B_{2n-1}W - W\Theta^{2n-1}\}\Theta W^{-1} \\ &\quad + (-)^{n+m}W\Theta W^{-1}\{B_{2n-1}W - W\Theta^{2n-1}\}W^{-1} \\ &= (-)^{n+m}\{B_{2n-1}L - 2W\Theta^{2n}W^{-1} + LB_{2n-1}\} \\ &= (-)^{n+m}\{[B_{2n-1}, L]_+ - 2L^{2n}\}.\end{aligned}$$

Similarly we have

$$(3.6) \qquad \Theta_{2n}(L) = (-)^n[B_{2n}, L].$$

Here we remark that $B_n=(L^n)_+$. Equations (3.5) and (3.6) are the Lax equations for the super Grassmann hierarchy.

Solutions of (3.4) are expressed by means of the superdeterminants. Define the j-th reference point in $FR(N/2\,|\,N/2,\ m/2\,|\,m/2;\ \mathscr{A})$ by

$$(3.8) \qquad {}^t\Xi^\vee_{2j} = \begin{bmatrix} {}^t\Xi_j & 0 \\ 0 & {}^t\Xi_0 \end{bmatrix},$$

where $({}^t\Xi_j)_{ik}=\delta_{ik}$ for $0\leqslant i<m/2-j$, $=\delta_{i+1,k}$ for $m/2-j\leqslant i<m/2$ $(0\leqslant k<N/2)$. Put $\tau(t,\Xi^\vee)=\mathrm{sdet}({}^t\Xi^\vee_0\Phi^\vee\Xi^\vee)$ and $\tau_j(t,\Xi^\vee)=\mathrm{sdet}({}^t\Xi^\vee_{2j}\Phi^\vee\Xi^\vee)$ for $\Xi=\Xi(t)$. Here we state the fundamental result of [19, 20].

THEOREM 3.1. 1) $\Theta(\tau(t,\Xi^\vee))=\Theta_1(\tau(t,\Xi^\vee))$.
2) $w_{2j}=(-)^j\tau_j(t,\Xi^\vee)/\tau(t,\Xi^\vee)$,
$w_{2j+1}=(-)^j(\Theta+\Theta_1)(\tau_j(t,\Xi^\vee))/2\tau(t,\Xi^\vee)$.
Especially,
3) $w_1=\Theta(\log\tau(t,\Xi^\vee))=\Theta_1(\log\tau(t,\Xi^\vee))$.

Let us consider the body part $\varepsilon(W)=\sum_{j=0}^{m/2}\varepsilon(w_{2j})\partial_x^{-j}$, $\varepsilon(B_{2n})=(\varepsilon(W)\partial_x^n\cdot\varepsilon(W)^{-1})_+$, and kill the odd time variables. Then, from (3.4), after changing the signature and indices $t_{4n}\to t_{2n}$, $t_{4n+2}\to -t_{2n+1}$, $w_{2j}\to w_j$, $B_{2n}\to B_n$, the Sato equations (1.2) for the ordinary Grassmann hierarchy are recovered.

4. Multicompoennt super Grassmann hierarchies

In this section we generalize the equations in the preceeding section to the multicomponent case. We should consider the matrix $(w_j^{(\alpha\beta)})_{0<\alpha,\beta<r}$ for the r-component case. Recall that $w_j\in\mathscr{S}_{[j]}$ in the super Grassmann equation (3.1). There are two choices in the generalization to determine the parity of $w_j^{(\alpha\beta)}$. The first one is $w_j^{(\alpha\beta)}\in\mathscr{S}_{[j]}$ for all α and β, the second is $w_j^{(\alpha\beta)}\in\mathscr{S}_{[j+\alpha+\beta]}$. We adopt here the second one. The r-component super Grassmann equation is of the following form:

$$(4.1) \qquad \tilde{\mathscr{W}}\Phi_r\Xi=0,$$

where $\tilde{\mathscr{W}}=(\tilde{w}^{(\alpha\beta)})_{0<\alpha,\beta<r}$, $\tilde{w}^{(\alpha\beta)}=(w^{(\alpha\beta)},\delta_{\alpha\beta},0,\dots,0)$, $w^{(\alpha\beta)}=(w_m^{(\alpha\beta)},\dots,w_1^{(\alpha\beta)})$, $w_j^{(\alpha\beta)}\in\mathscr{S}$, $\Phi_r=\mathrm{diag}(\exp(\theta\Lambda_N+x\Lambda_N^2),\dots,\exp(\theta\Lambda_N+x\Lambda_N^2))\in Mat(rN,rN;\mathscr{S})$, $\Xi={}^t({}^t\Xi^{(0)},\dots,{}^t\Xi^{(r-1)})\in Mat(rN,rm;\mathscr{A})$, $\Xi^{(\alpha)}=(\xi_{ij}^{(\alpha)})_{0<i<N,0<j<rm}$ if α is even, $=(\xi_{i+1,j}^{(\alpha)})_{0<i<N,0<j<rm}$ if α is odd, with $\xi_{ij}^{(\alpha)}\in\mathscr{A}_{[i+j]}$ and $\mathrm{rank}\,\varepsilon(\Xi)=rm$. Set $\mathscr{W}=(w^{(\alpha\beta)})_{0<\alpha,\beta<r}$.

For a matrix $X=(x_{ij}^{(\alpha)})_{0<\alpha<r,0<i<K,0<j<L}$ with $x_{ij}^{(\alpha)}\in\mathscr{S}_{[i+j]}$, denote $X^\vee=(X_{\mu\nu})_{0<\mu,\nu<2}$, where $X_{\mu\nu}=(x_{2i+\mu,2j+\nu}^{(\alpha)})$, so that entries in X_{00}, X_{11} are even and entries in X_{01}, X_{10} are odd.

For the sake of simplicity we take even r, N, m for a while. We introduce the even time variables $t_2^{(\alpha)}$, $t_4^{(\alpha)},\dots$ and odd ones $t_1^{(\alpha)}$, $t_3^{(\alpha)},\dots$ for $0\leqslant\alpha<r$. They satisfy $[t_{2n}^{(\alpha)},t_m^{(\beta)}]=0$, $[t_{2n-1}^{(\alpha)},t_{2m-1}^{(\beta)}]_+=0$ for n, $m\geqslant 1$, $0\leqslant\alpha$, $\beta<r$. The time evolution of $\Xi=\Xi(0)$ is defined as usual by

$$\Xi(t) = \operatorname{diag}(\exp(\eta(t^{(0)}, \Gamma_N)), \dots, \exp(\eta(t^{(r-1)}, \Gamma_N)))\Xi. \tag{4.2}$$

We consider the r-component super Grassmann equation (4.1) for $\Xi = \Xi(t)$. For a superframe $\Xi^\vee \in FR(rN/2 \mid rN/2, rm/2 \mid rm/2; \mathscr{A})$ which satisfies the condition

$$\operatorname{diag}(\underbrace{{}^t\Xi_0, \dots, {}^t\Xi_0}_{2r})\Phi_r^\vee \Xi^\vee \in GL(rm/2 \mid rm/2; \mathscr{S}) \tag{4.3}$$

($\Xi_0 \in FR(N/2, m/2; \mathscr{C})$ is defined as in section 1), the equation (4.1) is uniquely solved. Hence we have the correspondence:

$\{w^{(\alpha\beta)}, 0 \leqslant \alpha, \beta < r$; *solution of* (4.1)$\} \simeq$

$\{\Xi^\vee \in FR(rN/2 \mid rN/2, rm/2 \mid rm/2; \mathscr{A})$; $\Xi^\vee$ *satisfies* (4.3)$\}/GL(rm/2 \mid rm/2; \mathscr{A})$.

Define the differential operators

$$\Theta_{2n}^{(\alpha)} = \partial/\partial t_{2n}^{(\alpha)}, \quad \Theta_{2n-1}^{(\alpha)} = \partial/\partial t_{2n-1}^{(\alpha)} + \textstyle\sum_{m>1} t_{2m-1}^{(\alpha)} \partial/\partial t_{2n+2m-2}^{(\alpha)}.$$

They satisfy the same bracket relations as (3.3), and if $\alpha \neq \beta$, then $[\Theta_n^{(\alpha)}, \Theta_m^{(\beta)}]_+ = 0$ for odd n, m, and $[\Theta_n^{(\alpha)}, \Theta_m^{(\beta)}] = 0$ otherwise.

Now let us calculate some fundamental quantities for the simplest case, i.e., $r=2$, $N=4$ and $m=2$. First we have

$$\exp(\eta(t^{(\alpha)}, \Gamma_4)) = \begin{bmatrix} 1 & p_1^{(\alpha)} & p_2^{(\alpha)} & p_3^{(\alpha)} \\ & 1 & \bar{p}_1^{(\alpha)} & \bar{p}_2^{(\alpha)} \\ & & 1 & p_1^{(\alpha)} \\ & 0 & & 1 \end{bmatrix},$$

where $p_1^{(\alpha)} = -\bar{p}_1^{(\alpha)} = -t_1^{(\alpha)}$, $p_2^{(\alpha)} = \bar{p}_2^{(\alpha)} = -t_2^{(\alpha)}$ and $p_3^{(\alpha)} = t_3^{(\alpha)} + t_1^{(\alpha)} t_2^{(\alpha)}$. We can write

$$\Phi_2 \Xi(t) = \Phi_2(t)\Xi = \operatorname{diag}\left(\begin{bmatrix} 1 & a^{(\alpha)} & b^{(\alpha)} & c^{(\alpha)} \\ & 1 & \bar{a}^{(\alpha)} & \bar{b}^{(\alpha)} \\ & & 1 & a^{(\alpha)} \\ & 0 & & 1 \end{bmatrix}, \alpha=0, 1\right)\Xi,$$

where $a^{(\alpha)} = \theta + p_1^{(\alpha)}$, $\bar{a}^{(\alpha)} = \theta + \bar{p}_1^{(\alpha)}$, $b^{(\alpha)} = x + \theta\bar{p}_1^{(\alpha)} + p_2^{(\alpha)}$, $\bar{b}^{(\alpha)} = x + \theta p_1^{(\alpha)} + \bar{p}_2^{(\alpha)}$ and $c^{(\alpha)} = \theta x + x p_1^{(\alpha)} + \theta\bar{p}_2^{(\alpha)} + p_3^{(\alpha)}$. We see that

$$\Theta(\Phi_2(t)) = \Lambda_{4,2}\Phi_2(t), \quad \Theta_1^{(\alpha)}(\Phi_2(t)) = \Gamma_4^{(\alpha)}\Phi_2(t),$$

where $\Lambda_{4,2} = \operatorname{diag}(\Lambda_4, \Lambda_4)$, $\Gamma_4^{(\alpha)} = \operatorname{diag}(\Gamma_4\delta_{\alpha 0}, \Gamma_4\delta_{\alpha 1})$. Differentiating (4.1) by

$\Theta_1^{(\alpha)}$ and by Θ, we get

$$(4.4)\qquad (\Theta_1^{(\alpha)}(\tilde{\mathscr{W}})+\tilde{\mathscr{W}}^*\Gamma_4^{(\alpha)})\Phi_2(t)\Xi = 0,$$

$$(4.5)\qquad (\Theta(\tilde{\mathscr{W}})+\tilde{\mathscr{W}}^*\Lambda_{4,2})\Phi_2(t)\Xi = 0,$$

where $\tilde{\mathscr{W}}^*$ denotes the matrix obtained form $\tilde{\mathscr{W}}$ by changing the signature of the odd elements. If we put

$$b_1^{(\alpha)} = \begin{bmatrix} 2w_1^{(00)}\delta_{\alpha 0} & -w_1^{(01)} \\ -w_1^{(10)} & 2w_1^{(11)}\delta_{\alpha 1} \end{bmatrix},$$

then we can see that in the equation

$$\{(\Theta_1^{(\alpha)}(\tilde{\mathscr{W}})+\tilde{\mathscr{W}}^*\Gamma_4^{(\alpha)})+E_{\alpha\alpha}(\Theta(\tilde{\mathscr{W}})+\tilde{\mathscr{W}}^*\Lambda_{4,2})+b_1^{(\alpha)}\tilde{\mathscr{W}}\}\Phi_2(t)\Xi = 0,$$

the matrix in the braces is of the form $\tilde{\mathscr{R}}=(\tilde{\boldsymbol{r}}^{(\alpha\beta)})_{0<\alpha,\beta<2}$, where $\tilde{\boldsymbol{r}}^{(\alpha\beta)}=(r_1^{(\alpha\beta)},\dots, r_1^{(\alpha\beta)}, 0,\dots, 0)$. By the unique solvability of (4.1), the matrix $\tilde{\mathscr{R}}$ must be zero. We define $W=I+w_1\Theta^{-1}+w_2\Theta^{-2}$ $(w_j=(w_j^{(\alpha\beta)})_{0<\alpha,\beta<2})$ and $B_1^{(\alpha)}=E_{\alpha\alpha}\Theta+b_1^{(\alpha)}$. Then, by the super Leibniz rule,

$$(4.6)\quad B_1^{(\alpha)}W-(-)^\alpha JWE_{\alpha\alpha}\Theta = \textstyle\sum_j (E_{\alpha\alpha}\dot{w}_j+b_1^{(\alpha)}w_j)\Theta^{-j}$$

$$+\sum_{j:even}(-)^\alpha \begin{bmatrix} 0 & -w_j^{(01)} \\ w_j^{(10)} & 0 \end{bmatrix}\Theta^{1-j} + \sum_{j:odd}\begin{bmatrix} -2w_j^{(00)}\delta_{\alpha 0} & w_j^{(01)} \\ w_j^{(10)} & -2w_j^{(11)}\delta_{\alpha 1} \end{bmatrix}\Theta^{1-j},$$

where we have put $J=E_{00}-E_{11}$. The above argument says that the right hand side of (4.6) is nothing but $-\Theta_1^{(\alpha)}(W)=-\sum_{j=0}^{2}\Theta_1^{(\alpha)}(w_j)\Theta^{-j}$. Thus we have obtained the time evolution $\Theta_1^{(\alpha)}(W)$ of the super microdifferential operator W. This argument is valid for the general case as follows.

THEOREM 4.1. *For a solution $\tilde{\mathscr{W}}$ of the super Grassmann equation* (4.1), *put $W=\sum_{j=0}^{m} w_j\Theta^{-j}$ $(w_j=(w_j^{(\alpha\beta)})_{0<\alpha,\beta<r}, w_0=I)$. Then the time evolution of W is given by the following Sato equations.*

$$(4.7;\text{a})\qquad \Theta_{2n}^{(\alpha)}(W) = (-)^n(B_{2n}^{(\alpha)}W-WE_{\alpha\alpha}\Theta^{2n}),$$

$$(4.7;\text{b})\qquad \Theta_{2n-1}^{(\alpha)}(W) = (-)^{n+m}(B_{2n-1}^{(\alpha)}W-(-)^\alpha JWE_{\alpha\alpha}\Theta^{2n-1}),$$

where $J=\sum_{\alpha=0}^{r-1}(-)^\alpha E_{\alpha\alpha}$. The operators $B_n^{(\alpha)}$ are given by

$$(4.8)\qquad B_{2n}^{(\alpha)} = (WE_{\alpha\alpha}\Theta^{2n}W^{-1})_+,\quad B_{2n-1}^{(\alpha)} = (-)^\alpha J(WE_{\alpha\alpha}\Theta^{2n-1}W^{-1})_+.$$

PROOF. Since $B_{2n-1}^{(\alpha)}=E_{\alpha\alpha}\Theta^{2n-1}+(l.o.t.)$, we have

$$B_{2n-1}^{(\alpha)}W = E_{\alpha\alpha}W^*\Theta^{2n-1} + (l.o.t.)$$
$$= (E_{\alpha\alpha}W^*\Theta^{2n-1} - (-)^\alpha JWE_{\alpha\alpha}\Theta^{2n-1}) + (-)^\alpha JWE_{\alpha\alpha}\Theta^{2n-1} + (l.o.t.)$$

(*l.o.t.* = *lower order terms*). Here

$$E_{\alpha\alpha}W^*\Theta^{2n-1} - (-)^\alpha JWE_{\alpha\alpha}\Theta^{2n-1} = (-)^\alpha \textstyle\sum_j u_j^{(\alpha)}\Theta^{2n-1-j},$$

where $u_j^{(\alpha)} = ((u_j^{(\alpha)})^{(\beta\gamma)})_{0<\beta,\gamma<r}$, with $(u_j^{(\alpha)})^{(\beta\gamma)} = (-)^{j+\gamma}w_j^{(\alpha\gamma)}\delta_{\alpha\beta} - (-)^\beta w_j^{(\beta\alpha)}\delta_{\alpha\gamma}$. On the other hand, from the equation

$$(\Theta_{2n-1}^{(\alpha)} + (-)^{n+m-1}B_{2n-1}^{(\alpha)})\tilde{\mathscr{W}}\Phi_r\Xi = 0,$$

we get

$$\Theta_{2n-1}^{(\alpha)}(\tilde{\mathscr{W}}) + \tilde{\mathscr{W}}^*(\Gamma_N^{(\alpha)})^{2n-1} + (-)^{n+m-1}E_{\alpha\alpha}\tilde{\mathscr{W}}^*(\Lambda_{N,r})^{2n-1} + (l.o.t.) = 0.$$

The super Leibniz rule implies (4.7; b). The equation (4.7; a) can be obtained similarly. Multiplying W^{-1} from the right, we have

$$\Theta_{2n}^{(\alpha)}(W)W^{-1} = (-)^n(B_{2n} - WE_{\alpha\alpha}\Theta^{2n}W^{-1}),$$
$$\Theta_{2n-1}^{(\alpha)}(W)W^{-1} = (-)^{n+m}(B_{2n-1} - (-)^\alpha JWE_{\alpha\alpha}\Theta^{2n-1}W^{-1}).$$

The left hand sides are operators of negative order. Thus the super differential operators $B_n^{(\alpha)}$ are obtained by taking the differential operator part of the right hand sides as desired. ∎

We see

$$B_2^{(\alpha)} = E_{\alpha\alpha}\Theta^2 + (-)^\alpha \begin{bmatrix} 0 & w_1^{(01)} \\ -w_1^{(10)} & 0 \end{bmatrix}\Theta + (-)^\alpha \begin{bmatrix} 0 & -w_2^{(01)} \\ w_2^{(10)} & 0 \end{bmatrix}$$

for $r=2$, and

$$B_1^{(\alpha)} = E_{\alpha\alpha}\Theta + \begin{bmatrix} 2w_1^{(00)}\delta_{\alpha 0} & -w_1^{(01)}(\delta_{\alpha 0}+\delta_{\alpha 1}) & w_1^{(02)}(\delta_{\alpha 0}+\delta_{\alpha 2}) \\ -w_1^{(10)}(\delta_{\alpha 0}+\delta_{\alpha 1}) & 2w_1^{(11)}\delta_{\alpha 1} & -w_1^{(12)}(\delta_{\alpha 1}+\delta_{\alpha 2}) \\ w_1^{(20)}(\delta_{\alpha 0}+\delta_{\alpha 2}) & -w_1^{(21)}(\delta_{\alpha 1}+\delta_{\alpha 2}) & 2w_1^{(22)}\delta_{\alpha 2} \end{bmatrix},$$

$$B_2^{(0)} = E_{00}\Theta^2 + \begin{bmatrix} 0 & -w_1^{(01)} & -w_1^{(02)} \\ w_1^{(10)} & 0 & 0 \\ w_1^{(20)} & 0 & 0 \end{bmatrix}\Theta$$
$$+ \left[\begin{matrix} w_1^{(01)}w_1^{(10)} - w_1^{(02)}w_1^{(20)} & -w_2^{(01)} - w_1^{(01)}w_1^{(10)} + w_1^{(02)}w_1^{(21)} \\ w_2^{(10)} + w_1^{(10)}w_1^{(00)} & -w_1^{(10)}w_1^{(01)} \\ w_2^{(20)} + w_1^{(20)}w_1^{(00)} & -w_2^{(20)}w_1^{(01)} \end{matrix}\right.$$

$$\left.\begin{matrix} -w_2^{(02)}+w_1^{(01)}w_1^{(12)}-w_1^{(02)}w_1^{(22)} \\ w_1^{(10)}w_1^{(02)} \\ w_1^{(20)}w_1^{(02)} \end{matrix}\right]$$

for $r=3$.

These examples suggest the following equality.

PROPOSITION 4.2. $\sum_{\alpha=0}^{r-1} B_2^{(\alpha)}=\partial_x$.

PROOF. If we put $W^{-1}=\sum_{j\geq 0} v_j\Theta^{-j}$, then we see that $v_0=I$, $v_1=-w_1$ and $v_2=-w_2+w_1w_1^*$. Hence

$$\begin{aligned}\textstyle\sum_{\alpha=0}^{r-1} B_2^{(\alpha)} &= (W\partial_x W^{-1})_+ \\ &= ((\Theta^2+w_1\Theta+w_2)(I+v_1\Theta^{-1}+v_2\Theta^{-2}))_+ \\ &= \Theta^2 + (w_1+v_1)\Theta + (v_2+w_1v_1^*+w_2) \\ &= \partial_x . \quad \blacksquare\end{aligned}$$

5. τ-functions

In this section we define the τ-function of the multicomponent super Grassmann hierarchy and prove a representation formula of a solution.

Put $X=\Phi_r\Xi(t)$. Using the check operator ($\vee$), we put

$$X_0 = \begin{cases} \mathrm{diag}\,(\underbrace{{}^t\Xi_0(m/2),\dots,{}^t\Xi_0(m/2)}_{2r})X^\vee & \text{if } m \text{ is even,} \\ \mathrm{diag}\,(\underbrace{{}^t\Xi_0((m+1)/2),\, {}^t\Xi_0((m-1)/2),\, {}^t\Xi_0((m+1)/2),\dots,{}^t\Xi_0((m-1)/2)}_{2r})X^\vee & \\ & \text{if } m \text{ is odd,}\end{cases}$$

where $\Xi_0(k)={}^t(I_k,\,0)\in FR(N/2,\,k;\,\mathscr{C})$. We consider the superframe $\Xi^\vee$ for which $X_0\in GL(rm/2\,|\,rm/2;\,\mathscr{S})$ if rm is even, $\in GL((rm+1)/2\,|\,(rm-1)/2;\,\mathscr{S})$ if rm is odd (cf. (4.3)). Then we can define

$$\tau(t,\,\Xi^\vee) = \mathrm{sdet}\, X_0, \tag{5.1}$$

which does not vanish.

We set, for $\Xi(t)={}^t({}^t\Xi^{(0)}(t),\dots,{}^t\Xi^{(r-1)}(t))$,

$$\exp(\theta\Lambda_N+x\Lambda_N^2)\Xi^{(\alpha)}(t) = \begin{cases} (a_{ij}^{(\alpha)})_{0<i<N,0<j<rm} & (\alpha \quad \text{is even}), \\ (a_{i+1,j}^{(\alpha)})_{0<i<N,0<j<rm} & (\alpha \quad \text{is odd}). \end{cases}$$

Clearly one has

$$\Theta(a_{ij}^{(\alpha)}) = a_{i+1,j}^{(\alpha)}, \quad \Theta_1^{(0)}(a_{ij}^{(\alpha)}) = (-)^{i+1}\delta_{\alpha 0}a_{i+1,j}^{(\alpha)}. \tag{5.2}$$

Put

$$\mathscr{W}^{\wedge} = \begin{bmatrix} (w_{[\beta]}^{(2\alpha,\beta)}) & (w_{[\beta+1]}^{(2\alpha,\beta)}) \\ (w_{[\beta]}^{(2\alpha+1,\beta)}) & (w_{[\beta+1]}^{(2\alpha+1,\beta)}) \end{bmatrix},$$

where

$$w_{[\gamma]}^{(\alpha\beta)} = \begin{cases} (w_m^{(\alpha\beta)}, w_{m-2}^{(\alpha\beta)}, \ldots, w_2^{(\alpha\beta)}) & \text{for } [\gamma] = 0, \\ (w_{m-1}^{(\alpha\beta)}, w_{m-3}^{(\alpha\beta)}, \ldots, w_1^{(\alpha\beta)}) & \text{for } [\gamma] = 1, \end{cases}$$

if m is even, and

$$w_{[\gamma]}^{(\alpha\beta)} = \begin{cases} (w_m^{(\alpha\beta)}, w_{m-2}^{(\alpha\beta)}, \ldots, w_1^{(\alpha\beta)}) & \text{for } [\gamma] = 0, \\ (w_{m-1}^{(\alpha\beta)}, w_{m-3}^{(\alpha\beta)}, \ldots, w_2^{(\alpha\beta)}) & \text{for } [\gamma] = 1, \end{cases}$$

if m is odd. Then the super Grassmann equation is equivalent to

$$\mathscr{W}^{\wedge} X_0 = -(A_{\mu\nu})_{0 \leqslant \mu,\nu < 2}, \tag{5.3}$$

where $A_{00}=(a_{m,2j}^{(2\alpha)})$, $A_{01}=(a_{m,2k+1}^{(2\alpha)})$, $A_{10}=(a_{m+1,2j}^{(2\beta+1)})$ and $A_{11}=(a_{m+1,2k+1}^{(2\beta+1)})$ with $0 \leqslant \alpha < r/2$ (resp. $0 \leqslant \alpha < (r+1)/2$), $0 \leqslant \beta < r/2$ (resp. $0 \leqslant \beta < (r-1)/2$)) if r is even (resp. odd), and $0 \leqslant j < rm/2$ (resp. $0 \leqslant j < (rm+1)/2$), $0 \leqslant k < rm/2$ (resp. $0 \leqslant k < (rm-1)/2$) if rm is even (resp. odd).

Now we can state our main theorem.

THEOREM 5.1. $w_1^{(\alpha\alpha)} = (-)^{\alpha}\Theta_1^{(\alpha)}(\log \tau(t, \Xi^{\vee}))$ *for* $0 \leqslant \alpha < r$.

For the proof of this theorem, we use the following lemma.

LEMMA 5.2. *Let* $Z=(z_{ab})_{0 \leqslant a,b < M}$ *be an even matrix and* $Y=(y_{ab})_{0 \leqslant a,b < M}$ *be an odd matrix. If* $\tilde{\Theta}(Z)=YZ$ *for an odd vector field* $\tilde{\Theta}$, *then* $\tilde{\Theta}(\det Z)= (\operatorname{tr} Y)(\det Z)$.

PROOF OF LEMMA 5.2. Denote Z_{ab} the (a, b)-cofactor of Z. By using the chain rule of the differentiation, one sees that

$$\begin{aligned} \tilde{\Theta}(\det Z) &= \textstyle\sum_{a,b=0}^{M-1} (\partial(\det Z)/\partial z_{ab})\tilde{\Theta}(z_{ab}) \\ &= \textstyle\sum_{a,b=0}^{M-1} Z_{ab} \sum_{c=0}^{M-1} y_{ac} z_{cb} \\ &= \textstyle\sum_{a,c=0}^{M-1} y_{ac} \sum_{b=0}^{M-1} Z_{ab} z_{cb} \\ &= \textstyle\sum_{a,c=0}^{M-1} y_{ac}\delta_{ac}(\det Z) \\ &= (\operatorname{tr} Y)(\det Z). \quad \blacksquare \end{aligned}$$

PROOF OF THEOREM 5.1. We give a proof for the case that r and $m=2s$ are even and that $\alpha=0$. Other cases are, mutatis mutandis, verified.

In the following we use the indices which run over integers with the conditions: $0 \leqslant \beta < r$; $1 \leqslant \eta < m$, odd; $1 \leqslant \nu < rm$, odd; $0 \leqslant j < rm-1$, even; $2[\beta] \leqslant k < m+2[\beta]$, even, where $[\beta] = \beta \bmod 2$.

Let $X_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Put $Q = \det A$ and $G = D - CA^{-1}B = {}^t({}^t(g_{\eta\nu}^{(0)}), \dots, {}^t(g_{\eta\nu}^{(r-1)}))$. We denote by $G[\boldsymbol{g}_\eta^{(\beta)} \to \boldsymbol{p}_\eta]$ the matrix obtained by substituting the vector $\boldsymbol{p}_\eta = (p_{\eta\nu})$ instead of the row vector $\boldsymbol{g}_\eta^{(\beta)} = (g_{\eta\nu}^{(\beta)})$ in G. Putting $A^{-1}B = Q^{-1}(f_{i\nu})$, we have

$$g_{\eta\nu}^{(\beta)} = a_{\eta\nu}^{(\beta)} - Q^{-1} \sum_j a_{\eta j}^{(\beta)} f_{j\nu}.$$

We put

$$g_{k\nu}^{(\beta)} = a_{k\nu}^{(\beta)} - Q^{-1} \sum_j a_{kj}^{(\beta)} f_{j\nu},$$

which is equal to zero, because of the trivial identity $B - A(A^{-1}B) = 0$.

By Cramer's formula the solution $w_1^{(00)}$ is expressed as

$$w_1^{(00)} = -Q^{-1} \det G[\boldsymbol{g}_{m-1}^{(0)} \to \boldsymbol{g}_m^{(0)}]/s^{-1}\det X_0.$$

Now, by differentiating by $\Theta_1^{(0)}$ the equation $g_{k\nu}^{(\beta)} = 0$, we have

$$\begin{aligned} -a_{k+1,\nu}^{(0)}\delta_{\beta 0} + \Theta_1^{(0)}(Q)Q^{-2} \sum_j a_{kj}^{(\beta)} f_{j\nu} + Q^{-1} \sum_j a_{k+1,j}^{(\beta)} f_{j\nu}\delta_{\beta 0} \\ - Q^{-1} \sum_j a_{kj}^{(\beta)} \Theta_1^{(0)}(f_{j\nu}) = 0. \end{aligned}$$

Notice that $Q^{-1} \sum_j a_{kj}^{(\beta)} f_{j\nu} = a_{k\nu}^{(\beta)}$. Therefore the following matrix equality holds:

$$-\delta_{\beta 0} Q(g_{\eta\nu}^{(\beta)}) + \Theta_1^{(0)}(Q)(a_{k\nu}^{(\beta)}) = (a_{kj}^{(\beta)})(\Theta_1^{(0)}(f_{j\nu})).$$

Thus we have the matrix equality

$$\Theta_1^{(0)}(QA^{-1}B) = \Theta_1^{(0)}(Q)A^{-1}B - QA^{-1}G^{(0)}, \tag{5.4}$$

where $G^{(0)} = {}^t({}^t(g_{\eta\nu}^{(0)}), {}^t0, \dots, {}^t0)$. Next we calculate $\Theta_1^{(0)}(g_{\eta\nu}^{(\beta)})$.

$$\begin{aligned} \Theta_1^{(0)}(g_{\eta\nu}^{(\beta)}) &= g_{\eta+1,\nu}^{(0)}\delta_{\beta 0} + \Theta_1^{(0)}(Q)Q^{-2} \sum_j a_{\eta j}^{(\beta)} f_{j\nu} \\ &\quad + Q^{-1} \sum_j a_{\eta j}^{(\beta)} \Theta_1^{(0)}(f_{j\nu}) \\ &= g_{\eta+1,\nu}^{(0)}\delta_{\beta 0} - \Theta_1^{(0)}(Q)Q^{-1} g_{\eta\nu}^{(\beta)} + e_{\eta\nu}^{(\beta)}, \end{aligned} \tag{5.5}$$

where we have put

$$e_{\eta\nu}^{(\beta)} = Q^{-1}\{\Theta_1^{(0)}(Q)a_{\eta\nu}^{(\beta)} + \sum_j a_{\eta j}^{(\beta)} \Theta_1^{(0)}(f_{j\nu})\}.$$

The $rs \times rs$-matrix E, which has $e_{\eta\nu}^{(\beta)}$ as entires, is expressed as

$$E = Q^{-1}\{\Theta_1^{(0)}(Q)D + C\Theta_1^{(0)}(QA^{-1}B)\}. \tag{5.6}$$

From (5.4) it follows that

$$C\Theta_1^{(0)}(QA^{-1}B) = CA^{-1}\{\Theta_1^{(0)}(Q)B - QG^{(0)}\}$$
$$= -\Theta_1^{(0)}(Q)(D-G) - QCA^{-1}G^{(0)}.$$

Substituting the above to (5.6), we get

$$E = Q^{-1}\{\Theta_1^{(0)}(Q)G - QCA^{-1}G^{(0)}\} \tag{5.7}$$
$$= Q^{-1}\{\Theta_1^{(0)}(Q)G - (h_{\eta j}^{(\beta)})G^{(0)}\},$$

where we have defined $h_{\eta j}^{(\beta)}$ by $CA^{-1} = Q^{-1}(h_{\eta j}^{(\beta)})$. If we put $\Theta_1^{(0)}(A)A^{-1} = Q^{-1}\cdot(h'^{(\beta)}_{\eta j})$, then $h_{\eta j}^{(0)} = -h'^{(0)}_{\eta j}$ and $h'^{(\beta)}_{\eta j} = 0$ for $\beta \neq 0$.

We show that

$$Q^{-1}\det G[\boldsymbol{g}_{m-1}^{(0)} \to \boldsymbol{g}_m^{(0)}] = \Theta_1^{(0)}(Q^{-1}\det G). \tag{5.8}$$

The right side is equal to

$$-\Theta_1^{(0)}(Q)Q^{-2}\det G + Q^{-1}\textstyle\sum_\beta \sum_\eta \det G[\boldsymbol{g}_\eta^{(\beta)} \to \Theta_1^{(0)}(\boldsymbol{g}_\eta^{(\beta)})]$$
$$= (\textit{the left hand side}) - (rs+1)\Theta_1^{(0)}(Q)Q^{-2}\det G$$
$$+ Q^{-1}\textstyle\sum_\beta \sum_\eta \det G[\boldsymbol{g}_\eta^{(\beta)} \to \boldsymbol{e}_\eta^{(\beta)}].$$

Using Lemma 5.2, putting $Z = A$ and $Y = (h'^{(\beta)}_{\eta j})$, we have

$$(\textit{the third term}) = Q^{-2}\{rs\Theta_1^{(0)}(Q) + \textstyle\sum_\eta h'^{(0)}_{\eta,\eta-1}\}\det G$$
$$= (rs+1)\Theta_1^{(0)}(Q)Q^{-2}\det G.$$

Finally we use (2) of Proposition 2.1 to obtain

$$\Theta_1^{(0)}(\log(\mathrm{s}^{-1}\det X_0)) = -\Theta_1^{(0)}(\log(\mathrm{sdet}\, X_0)). \quad \blacksquare$$

It is plausible that, taking the limit N, $m \to \infty$, one would obtain the super KP hierarchy. We hope that the link between the super KP hierarchy and the Lie superalgebra $\mathfrak{gl}(\infty \mid \infty)$, which is constructed in [18] by means of the free field operators, will be revealed in the near future.

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