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Integral closures of ideals of the principal class

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1. Introduction

Let A be a noetherian ring. For an ideal I of A. we denote by I^* the integal closure of I. In [1], we proved that if $\mathfrak{q}$ is an ideal of A generated by an A-regular sequence, then $\mathfrak{q}^n \cap (\mathfrak{q}^{n+1})^* = \mathfrak{q}^n\mathfrak{q}^*$ for every $n \geqq 1$. In this note, we shall prove the following

THEOREM. *Assume that A is locally quasi-unmixed, and let $\mathfrak{q}$ be an ideal of the principal class of A. Then $\mathfrak{q}^n \cap (\mathfrak{q}^{n+1})^* = \mathfrak{q}^n\mathfrak{q}^*$ for every $n \geqq 1$.*

We shall prove the Theorem by using a method similar to the one that we used in the proof of [1], Theorem 1. The assumption that A is locally quasi-unmixed is essential; in Section 4, we shall show this by an example.

C. Huneke communicated to the author that he and M. Hochster proved the Theorem in the case that A is a quasi-unmixed local ring containing a field and $\mathfrak{q}$ is generated by a part of a system of generators for A.

2. Some preliminary results

In this section, we assume that A is locally quasi-unmixed and $\mathfrak{q}$ is an ideal of the principal class with $\operatorname{ht}\mathfrak{q} \geqq 2$. Let $R = \sum_{n\in Z} \mathfrak{q}^n t^n$ and $R' = \sum_{n \in Z} (\mathfrak{q}^n)^* t^n$.

Let $x_1, \dots, x_d$ be elements of A which generate $\mathfrak{q}$, where $d = \operatorname{ht}\mathfrak{q}$. First, we need the following result.

LEMMA 1. *$x_1^n, \dots, x_d^n$ are $(\mathfrak{q}^n)^*$-independent for all $n \geqq 1$, i.e. every form $f(X_1, \dots, X_d) \in A[X_1, \dots, X_d]$ such that $f(x_1^n, \dots, x_d^n) = 0$ has all its coefficients in $(\mathfrak{q}^n)^*$.*

PROOF. See [3], (4.14.2).

LEMMA 2. (i) *$(\mathfrak{q}^{n+1})^* \colon x_i = (\mathfrak{q}^n)^*$ for every $n \geqq 0$ and i.*

(ii) *For a fixed s with $2 \leqq s \leqq d$, let J be the ideal of R generated by t^{-1}, $x_1 t, \dots, x_s t$. Then $H_J^2(R')_n = 0$ for $n \leqq 0$.*

PROOF. See [2], Proposition 3.13, and the proof of [1], Proposition 16.

LEMMA 3. *$(\mathfrak{q}^{n+1})^* \cap (x_1, x_2)^n = \mathfrak{q}^*(x_1, x_2)^n$ for every $n \geqq 0$.*

PROOF. Our proof is very similar to that of [1], Proposition 6. Let $x=x_1$, $y=x_2$ and $u=t^{-1}$. Let $N=(u, xt, yt)R$. By Lemma 2, $H_N^2(R')_0=0$. We then use the Čech complex $0\to R'_{xt}\times R'_{yt}\times R'_u\to R'_{xtyt}\times R'_{xtu}\times R'_{ytu}\to R'_{xtytu}\to 0$ to compute $H_N^2(R')_0=0$. Let $z\in(\mathfrak{q}^{n+1})^*\cap\mathfrak{q}^n$ and write $z=ax^n+by$ with $a\in A$ and $b\in(x, y)^{n-1}$. Using an argument similar to the proof of [1], Proposition 6, we have $zx^{m-n}y^{m-1}=a'x^m+b'y^m$ for some integer $m>n$ and a', $b'\in(\mathfrak{q}^m)^*$. Since $(a'-ay^{m-1})x^m+(b'+bx^{m-n})y^m=0$, it follows from Lemma 1 that $b'-bx^{m-n}\in(\mathfrak{q}^m)^*$ and $a'-ay^{m-1}\in(\mathfrak{q}^m)^*$, and hence, by Lemma 2, $a\in\mathfrak{q}^*$, because $ay^{m-1}\in(\mathfrak{q}^m)^*$. In the case $n=1$, since $by\in(\mathfrak{q}^2)^*$, we have $b\in\mathfrak{q}^*$. So assume that $n>1$. Since $bx^{m-n}\in(\mathfrak{q}^m)^*$, it follows from Lemma 2 that $b\in(\mathfrak{q}^n)^*$, and hence $b\in(\mathfrak{q}^n)^*\cap(x, y)^{n-1}$. Therefore, by the induction on n, $b\in\mathfrak{q}^*(x, y)^{n-1}$; hence $z=ax^n+by\in\mathfrak{q}^*(x, y)^n$. This completes the proof.

3. Proof of the Theorem

In this section, we shall prove the Theorem. Let A be a locally quasi-unmixed noetherian ring, and let $x_1,\dots, x_d$ be elements of A such that $\mathfrak{q}=(x_1,\dots, x_d)$ is an ideal of the principal class with $\operatorname{ht}\mathfrak{q}=d$.

THEOREM. $(\mathfrak{q}^{n+1})^*\cap\mathfrak{q}^n=\mathfrak{q}^n\mathfrak{q}^*$ for every $n\geqq 0$.

PROOF. We use induction on d. In the case $d=2$, the assertion follows from Lemma 3. So assume that $d\geqq 3$ and that the assertion has been established for ideals of the principal class of locally unmixed noetherian rings with height less than d. Then we may assume that A is local. Let $R=\sum_{n\in\boldsymbol{Z}}\mathfrak{q}^nt^n$ and $R'=\sum_{n\in\boldsymbol{Z}}(\mathfrak{q}^n)^*t^n$. Let $P_1,\dots, P_r$ be the prime divisors of $t^{-1}R'$, and let $V_i=(R'_{P_i})_{red}$ for each i. By [1], Lemma 4, each V_i is a DVR; and moreover, for $a\in A$ and $n\in\boldsymbol{N}$, we have $a\in(\mathfrak{q}^n)^*$ if and only if $a\in\mathfrak{q}^nV_i$ for each i. Therefore we may assume that $x_1V_i=\mathfrak{q}V_i$ for each i (If necessary, replace A by $A(X)$). Let B be the A-subalgebra of A_{x_1} generated by x_2/x_1, and let $I=(x_1, x_3,\dots, x_d)B$. Note here that I is an ideal of the principal class. It follows from our choice of x_1 that the canonical homomorphism $A\to V_i$ factors through the canonical homomorphism $A\to B$ for each i, and hence $(I^{n+1})^*\cap A=(\mathfrak{q}^{n+1})^*$ for every n. Let $\{M_j\}$ (resp. $\{N_i\}$) be the set of monomials in $x_1,\dots, x_d$ (resp. $x_1, x_3,\dots, x_d$) with degree n. Each M_j(resp. N_i) is an element of A. Let $z\in(\mathfrak{q}^{n+1})^*\cap\mathfrak{q}^n$, and write $z=\sum a_jM_j$ with $a_j\in A$. Since $z\in(I^{n+1})^*\cap I^n=I^*I^n$, we can write $z=\sum b_iN_i$ with $b_i\in I^*$. For each N_i, we put $T_i=\{M_j\mid M_j(x_1)+M_j(x_2)=N_i(x_1)$ and $M_j(x_k)=N_i(x_k)$ for $k\geqq 3\}$, and we define $w_i\in A$ by $w_i=\sum_{M_j\in T_i}a_jx_1^{M_j(x_1)}x_2^{M_j(x)}$, (For a monomial $M=x_1^{e(1)}\cdots x_d^{e(d)}$ in $x_1,\dots, x_d$, $M(x_i)$ denotes the integer $e(i)$ for every i.) We put $n(i)=N_i(x_1)$ for simplicity. Then $\sum_i(b_i-w_i/x_1^{n(i)})N_i=0$. By Lemma 1, we have $b_i-w_i/x_1^{n(i)}\in I^*$, and hence $w_i\in(I^{n(i)+1})^*$. Therefore $w_i\in(I^{n(i)+1})^*\cap$

$A=(\mathfrak{q}^{n(i)+1})^*$, and moreover, by Lemma 3, we have $w_i \in (\mathfrak{q}^{n(i)+1})^* \cap (x_1, x_2)^{n(i)} = (x_1, x_2)^{n(i)}\mathfrak{q}^*$. Thus we can write $w_i = \sum_k c_k x_1^{n(i)-k} x_2^k$ with $c_k \in \mathfrak{q}^*$. It then follows from Lemma 1 that $a_j - c_k \in (x_1, x_2)^*$ if $M_j(x_2)=k$; consequently $a_j \in \mathfrak{q}$ for every j. This proves the assertion for $d \geqq 2$.

Finally, we must prove the Theorem in the case $\operatorname{ht} \mathfrak{q}=1$. So assume that $\mathfrak{q}=xA$ with $\operatorname{ht}\mathfrak{q}=1$. Let I be the nilradical of A. It is clear that every $(x^n A)^*$ contains I. Since x is A/I-regular, we have $I \cap x^n A = Ix^n$ for every n. Therefore by [1], Theorem 1, $(x^{n+1}A)^* \cap x^n A \subseteq ((xA)^* x^n A + I) \cap x^n A = (xA)^* x^n A + I \cap x^n A = (xA)^* x^n A$, and hence $(x^{n+1}A)^* \cap x^n A = (xA)^* x^n A$. This completes the proof of the Theorem.

4. An example

Let k be a field, and let X, Y, Z and W be four variables. Let $B=k[\![X, Y, Z, W]\!]$, and consider two prime ideals P_1, P_2 of B defined as follows: $P_1=WB$ and $P_2=(X^3-Y^4, Z^3-W^4)B$. Let $A=B/P_1 \cap P_2$, and denote by x, y, z and w the images of X, Y, Z and W respectively. Note here that, since $\operatorname{ht} P_1=1$ and $\operatorname{ht} P_2=2$, A is not quasi-unmixed and $\dim A=3$. Let $\mathfrak{q}=(x, y, z)A$. $\mathfrak{q}$ is a parameter ideal of A.

We shall now prove that

$$(\mathfrak{q}^2)^* \cap \mathfrak{q} \neq \mathfrak{q}^*\mathfrak{q}.$$

We first show that $\mathfrak{q}^*=(x, y, z, w^2)$. Since $(w^2)^3=w^2z^3 \in \mathfrak{q}^3$, we have $w^2 \in \mathfrak{q}^*$. Let t and u be two variables, and let $f: A \to k[\![t, u]\!]$ be the homomorphism defined by $f(x)=t^4$, $f(y)=t^3$, $f(z)=u^4$ and $f(w)=u^3$. Since $f(w)=u^3 \notin (f(\mathfrak{q}^*)k[\![t, u]\!])^*$ $(=(t^3, u^4)^*=(t^3, t^2u^2, tu^3, u^4))$, we have $w \notin \mathfrak{q}^*$. Therefore $\mathfrak{q}^*=(x,y,z,w^2)$ and $\mathfrak{q}^*\mathfrak{q}=(x^2,xy,xz,y^2,yz,z^2,xw^2,yw^2,zw^2)$. Since $f(\mathfrak{q}^*\mathfrak{q})k[\![t,u]\!] = (t^6, t^3u^4, u^8) \not\ni f(xw)=t^4u^3$, we have $xw \notin \mathfrak{q}^*\mathfrak{q}$. But $xw \in (\mathfrak{q}^2)^*$, because $(xw)^{18}=(x^3w)^6w^{12}=(y^4w)^6w^{12}=y^{24}(w^5)^3w^3=y^{24}(z^3w)^3w^3=y^{24}z^9w(z^3w) \in (\mathfrak{q}^2)^{18}$. Consequently, $(\mathfrak{q}^2)^* \cap \mathfrak{q} \neq \mathfrak{q}^*\mathfrak{q}$.

References

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