HIROSHIMA MATH. J.
17 (1987), 627–649

Existence for nonlinear functional differential equations

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(Received September 11, 1986)

1. Introduction

In this paper we prove two local existence results concerning strongly nonlinear functional differential equations of the form

$$(1)\qquad \begin{aligned} &\frac{du}{dt}(t) + Au(t) \ni F(t, u_t), \quad 0 \leqq t \leqq T \\ &u(s) = \varphi(s), \quad -r \leqq s \leqq 0. \end{aligned}$$

With few exceptions, our notation follows that of Hale [11] and Fitzgibbon [9]. Thus, X is a real Banach space, $A: D(A) \subset X \to 2^X$ is an m-accretive operator (possibly nonlinear and multi-valued), U is a nonempty subset in $C([-r, 0]; X)$, $F: [0, T] \times U \to X$ is a given mapping, and $\varphi \in U$ satisfies $\varphi(0) \in \overline{D(A)}$. We recall that for each $u \in C([-r, T]; X)$ and $t \in [0, T]$, $u_t: [-r, 0] \to X$ is defined by $u_t(s) := u(t+s)$, for $s \in [-r, 0]$.

We note that equations of the form (1) have been intensively studied over the past several years by many authors, using mainly the semigroup techniques initiated by Webb [26]. Roughly speaking, the semigroup approach to (1) consists in applying the Crandall and Liggett generation theorem [4, Theorem 1.3, p. 104] to prove the unique solvability of an equivalent equation obtained by an appropriate reformulation of the original problem in a suitably chosen state space. This method is no longer effective if we do not assume some Lipschitz-like condition on F with respect to its second argument. Therefore, to compensate the lack of regularity of F we have to assume some additional "appropriate compactness" properties on the operator A. In the case where A is linear and densely defined, it has been shown that a very natural compactness assumption on A, which along with the continuity of F ensures the local solvability of (1), is the compactness of the generated semigroup. This very useful remark which goes back to Pazy [18] has been widely used in the study of various classes of, either semilinear functional differential equations, or fully nonlinear differential and integro-differential equations. See for instance [7, 9, 10, 12, 20–23, 25].

The aim of this paper is to show that similar compactness arguments can be employed to conclude the local solvability of (1) when both A and F are nonlinear. Thus, our existence results which are mainly based on a compactness criterion due to the second author [24] apply to new classes of functional differential equations including all those considered in [9, 19, 20, 26] — in the quasi-autonomous case.

Here we have to point out that, although we are merely concerned with the quasi-autonomous case, i.e., the case in which A does not depend on t, our proofs can be adapted to handle the nonautonomous case as well. To this aim we ought to use the notion of integral solution for a nonautonomous equation as defined in [13], or that of DS-limit solution combined with (i) in Theorem 4.1 in [16]. However, we do not touch upon such generalizations in order to make the ideas of the proofs as transparent as possible by avoiding unpleasant technicalities.

This paper consists of seven sections. The next two sections are devoted to some background material and auxiliary results. In Sections 4 and 5 we prove the main local existence results, while in Section 6 we focus our attention on the continuation of the solutions. Finally, the last section contains some examples which illuminate the effectiveness of the abstract theory.

ACKNOWLEDGEMENTS. The authors would like to express their warmest thanks to Professor S. Oharu for suggesting several improvements in the presentation.

2. Preliminaries

In all that follows X is a real Banach space whose norm is denoted by $\|\cdot\|$. If $x, y \in X$, then $(x, y)_+([x, y]_+)$ is the right directional derivative of $2^{-1}\|\cdot\|^2$ (of $\|\cdot\|$) evaluated at x in the direction y, i.e.,

$$(x, y)_+ := \lim_{h \downarrow 0} (2h)^{-1}(\|x+hy\|^2 - \|x\|^2),$$

and

$$[x, y]_+ := \lim_{h \downarrow 0} h^{-1}(\|x+hy\| - \|x\|).$$

We recall that $A\colon D(A) \subset X \to 2^X$ is said to be *accretive* if

$$[x_1 - x_2, y_1 - y_2]_+ \geqq 0$$

for $x_i \in D(A)$, $y_i \in Ax_i$, $i = 1, 2$. An accretive operator A is said to be *m-accretive*, if for each $\lambda > 0$ the mapping $I_d + \lambda A$ is surjective, where I_d is the identity on X.

Let $A\colon D(A) \subset X \to 2^X$ be an m-accretive operator and consider the following quasi-autonomous nonlinear evolution equation

$$\text{(2)} \qquad \begin{cases} \dfrac{du}{dt}(t) + Au(t) \ni f(t), & a \leqq t \leqq b \\ u(a) = u_0, \end{cases}$$

where $f \in L^1([a, b]; X)$ and $u_0 \in \overline{D(A)}$.

DEFINITION 1. A function $u: [a, b] \to \overline{D(A)}$ is called a *strong solution* of (2) on $[a, b]$ if $u(a) = u_0$, $u(t) \in D(A)$ for a.a. $t \in [a, b]$, u belongs to $W^{1,1}([a, b]; X)$ and it satisfies the equation in (2) for a.a. $t \in [a, b]$. For the definition and properties of the spaces $W^{k,p}([a, b]; X)$, $(k, p) \in \boldsymbol{N} \times [1, +\infty]$, see [4, p. 18].

DEFINITION 2. A function $u: [a, b] \to \overline{D(A)}$ is called an *integral solution* of (2) on $[a, b]$ if $u(a) = u_0$, $u \in C([a, b]; X)$, and u satisfies

$$\|u(t) - x\| \leqq \|u(s) - x\| + \int_s^t [u(\tau) - x, f(\tau) - y]_+ d\tau$$

for $x \in D(A)$, $y \in Ax$, and $a \leqq s \leqq t \leqq b$. See [14, Definition 3.5.1, p. 104].

It turns out that each strong solution of (2) on $[a, b]$ is an integral solution of (2) on the same interval. We note that if X is not reflexive, then the problem (2) may have no strong solution even though $u_0 \in D(A)$ and f is smooth. However, if A is m-accretive, then for each $(u_0, f) \in \overline{D(A)} \times L^1([a, b]; X)$ the problem (2) has a unique integral solution u on $[a, b]$. See [5], [4, Theorem 2.1, p. 124] and [14, Theorem 3.5.1, p. 104].

In order to exhibit the dependence of u on (u_0, f) we shall denote this integral solution by $I(u_0, f)$. The inequality below describes the continuous (in fact Lipschitz-continuous) dependence of I on (u_0, f). In particular, it is regarded as a uniqueness result. Namely, we have

$$\|u(t) - v(t)\| \leqq \|u(s) - v(s)\| + \int_s^t \|f(\tau) - g(\tau)\| d\tau \tag{3}$$

for $a \leqq s \leqq t \leqq b$, where $u = I(u_0, f)$, and $v = I(v_0, g)$. See [5], [4, Theorem 2.1, p. 124] and [14, Theorem 3.5.1, p. 104]. With the same notation as in Definition 2, we also have

$$\|u(t) - v(t)\|^2 \leqq \|u(s) - v(s)\|^2 + 2\int_s^t (u(\tau) - v(\tau), f(\tau) - g(\tau))_+ d\tau \tag{4}$$

for $a \leqq s \leqq t \leqq b$. See [5], [4] and [14] loc cit.

In this paper we mostly assume that A is m-accretive. Hence a semigroup $\{S(t); t \geqq 0\}$ of (possibly nonlinear) contraction operators from $\overline{D(A)}$ into itself can be constructed via the Crandall-Liggett theorem. In this case we say for simplicity that A generates a semigroup on $\overline{D(A)}$.

3. Basic compactness results

The purpose of the present section is to collect some basic compactness results we shall use in the proofs of our local existence theorems.

We begin by recalling that a subset G in $L^1([a, b]; X)$ is called *uniformly integrable* if for each $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that

$$\int_E \|f(t)\| dt < \varepsilon$$

for each measurable subset E in $[a, b]$ whose Lebesgue measure is less than $\delta(\varepsilon)$, and this holds uniformly for $f \in G$.

REMARK 1. We may easily verify, by using Hölder's inequality, that each bounded subset of $L^p([a, b]; X)$ with $1<p\leqq+\infty$ is uniformly integrable. We also note that, since $[a, b]$ is compact, each uniformly integrable subset is bounded in $L^1([a, b]; X)$.

The next result, due to the second author [24], is the main compactness argument we shall use in what follows.

THEOREM 1. *Let A: $D(A)\subset X\to 2^X$ be an m-accretive operator, $u_0 \in \overline{D(A)}$ and G a uniformly integrable subset in $L^1([a, b]; X)$. Then, the following conditions are equivalent.*

(i) *The set $\{I(u_0, f); f\in G\}$ is relatively compact in $C([a, b]; X)$.*

(ii) *There exists a dense subset D in $[a, b]$ such that for each $t\in D$, the set $\{I(u_0, f)(t); f\in G\}$ is relatively compact in X.*

A useful instance of Theorem 1 which is a slight extension of some previous results due to Baras [3] and the second author [21] is given below.

THEOREM 2. *Let A: $D(A)\subset X\to 2^X$ be an m-accretive operator which generates a semigroup $\{S(t); S(t)\colon \overline{D(A)}\to\overline{D(A)}, t\geqq 0\}$ such that $S(t)$ is compact for each $t>0$. Then, for each $u_0\in\overline{D(A)}$ and each uniformly integrable subset G in $L^1([a, b]; X)$, the set $\{I(u_0, f); f\in G\}$ is relatively compact in $C([a, b]; X)$.*

For a simple proof of Theorem 2 based on Theorem 1 see [24] and [25].
Another consequence of Theorem 1 we need later is the following:

THEOREM 3. *Let A: $D(A)\subset X\to 2^X$ be an m-accretive operator such that $(I_d+\lambda A)^{-1}$ is compact for each $\lambda>0$, let $u_0\in\overline{D(A)}$, and let G be a uniformly integrable subset of $L^1([a, b]; X)$. Then, the following conditions are equivalent.*

(i) *The set $\{I(u_0, f); f\in G\}$ is relatively compact in $C([a, b]; X)$.*

(iii) *The set $\{I(u_0, f); f\in G\}$ is equicontinuous from the right at each $t\in[a, b[$.*

PROOF. Obviously (i) implies (iii). To prove the converse statement we shall use Theorem 1. First of all, let us recall that

$$\|J_\lambda x - x\| \leqq \frac{4}{\lambda}\int_0^\lambda \|S(\tau)x - x\| d\tau \tag{5}$$

for each $x \in \overline{D(A)}$ and $\lambda > 0$, where $J_\lambda := (I_d + \lambda A)^{-1}$ and $\{S(t);\ t \geqq 0\}$ is the semi-group generated by A. For the proof of (5) the reader is referred to [6], or [25].

Now, we shall show that (iii) together with the compactness of J_λ implies (ii) in Theorem 1 with $D = [a, b[$. Since u_0 is fixed in $\overline{D(A)}$, we write for simplicity in notation as

$$u^f := I(u_0, f)$$

for each $f \in G$. Using (5), we then have

$$\begin{aligned}\|J_\lambda u^f(t) - u^f(t)\| &\leqq \frac{4}{\lambda}\int_0^\lambda \|S(\tau)u^f(t) - u^f(t)\| d\tau \\ &\leqq \frac{4}{\lambda}\int_0 \|S(\tau)u^f(t) - u^f(t+\tau)\| d\tau + \frac{4}{\lambda}\int_0^\lambda \|u^f(t+\tau) - u^f(t)\| d\tau\end{aligned}$$

for $f \in G$, $t \in [a, b[$ and $\lambda > 0$ with $t + \lambda \in [a, b]$. At this point, we observe that the mapping $s \to S(s-t)u^f(t)$ is the unique integral solution of the problem

$$\frac{dv}{ds}(s) + Av(s) \ni 0, \quad t \leqq s \leqq t + \tau$$

$$v(t) = u^f(t),$$

and therefore, from (3), we easily deduce

$$\|S(\tau)u^f(t) - u^f(t+\tau)\| \leqq \int_t^{t+\tau} \|f(s)\| ds$$

for $f \in G$, $t \in [a, b[$, and $\tau > 0$ with $t + \tau \in [a, b]$. Consequently,

$$\|J_\lambda u^f(t) - u^f(t)\| \leqq \frac{4}{\lambda}\int_0^\lambda \int_t^{t+\lambda} \|f(s)\| ds \cdot d\tau + \frac{4}{\lambda}\int_0^\lambda \|u^f(t+\tau) - u^f(t)\| d\tau$$

for $f \in G$, $t \in [a, b[$, and $\lambda > 0$ with $t + \lambda \in [a, b]$. From this inequality, taking into account that G is uniformly integrable and $\{u^f; f \in G\}$ is equicontinuous from the right at each $t \in [a, b[$, we deduce that

$$\lim_{\lambda \downarrow 0} \|J_\lambda u^f(t) - u^f(t)\| = 0$$

for each $t \in [a, b[$, the limit existing *uniformly* for $f \in G$. On the other hand, $\{u^f(t); f \in G\}$ is bounded in X because G is bounded in $L^1([a, b]; X)$. See Remark 1 and (3). Hence, inasmuch as J_λ is compact for each $\lambda > 0$, the above formula shows that $\{u^f(t); f \in G\}$ is precompact in X. Thus for each $t \in [a, b[$

the set $\{u^f(t); f \in G\}$ is relatively compact in X, and therefore by Theorem 1 it follows that $\{u^f; f \in G\}$ is relatively compact in $C([a, b]; X)$, as claimed.

REMARK 2. The compactness of J_λ for each $\lambda > 0$ is a strictly weaker condition than the compactness of $S(t)$ for each $t > 0$. See [6] and [25]. Roughly speaking, the compactness of $S(t)$ for each $t > 0$ is a specific property of operators A arising in the study of nonlinear parabolic equations and systems, while the compactness of J_λ for each $\lambda > 0$ is a general property of not only operators as mentioned above but also of those arising in the study of nonlinear hyperbolic equations and systems.

Now, let us recall that a subset G in $L^1([a, b]; X)$ is called *equiintegrable* if it is uniformly integrable and, in addition

$$\lim_{h \downarrow 0} \int_a^{b-h} \|f(t+h) - f(t)\| dt = 0$$

holds uniformly for $f \in G$.

REMARK 3. Using Weil's Criterion of Compactness [8, Theorem 4.20.1, p. 269], we easily conclude that any relatively compact subset of $L^1([a, b]; X)$ is equiintegrable, but the converse statement is no longer true unless X is finite dimensional.

An important consequence of Theorem 3 is Theorem 4 below.

THEOREM 4. *Let A: $D(A) \subset X \to 2^X$ be an m-accretive operator with $(I_d + \lambda A)^{-1}$ compact for each $\lambda > 0$. Then, for each $u_0 \in \overline{D(A)}$ and each equiintegrable subset G in $L^1([a, b]; X)$, the set $\{I(u_0, f); f \in G\}$ is relatively compact in $C([a, b]; X)$.*

PROOF. With the same notations as in the proof of Theorem 3, it readily follows from (3) that

$$\|u^f(t+h) - u^f(t)\| \leqq \|u^f(a+h) - u_0\| + \int_a^t \|f(s+h) - f(s)\| ds$$

for $f \in G$, $t \in [a, b[$ and $h > 0$ with $t + h \in [a, b]$. Also by (3) we get

$$\begin{aligned} \|u^f(a+h) - u_0\| &\leqq \|u^f(a+h) - S(h)u_0\| + \|S(h)u_0 - u_0\| \\ &\leqq \int_a^{a+h} \|f(s)\| ds + \|S(h)u_0 - u_0\| \end{aligned}$$

for $f \in G$, and $h > 0$ with $a + h \leqq b$. Therefore

$$\|u^f(t+h) - u^f(t)\| \leqq \|S(h)u_0 - u_0\| + \int_a^{a+h} \|f(s)\| ds + \int_a^t \|f(s+h) - f(s)\| ds$$

for each $f \in G$, $t \in [a, b[$ and $h > 0$ with $t + h \in [a, b]$.

Now, taking into account the condition that $\{S(t); t \geqq 0\}$ is strongly continuous at the origin and G is equiintegrable, we conclude that $\{u^f; f \in G\}$ is equicontinuous from the right at each $t \in [a, b[$. Thus Theorem 3 applies, and this completes the proof.

The next lemma gives a simple sufficient condition in order that a subset G of $L^1([a, b]; X)$ be equiintegrable.

LEMMA 1. *If G is bounded in $W^{1,1}([a, b]; X)$, then G is equiintegrable.*

PROOF. Let G be a bounded subset of $W^{1,1}([a, b]; X)$ and define the set $H := \{g_f; f \in G\}$, where

$$g_f(t) := \int_a^t \left\| \frac{df}{ds}(s) \right\| ds$$

for $f \in G$ and $t \in [a, b]$. Clearly, H is bounded in $W^{1,1}([a, b]; R)$. Since this Sobolev space is compactly embedded in $L^1([a, b]; R)$, H is actually relatively compact in $L^1([a, b]; R)$. Therefore, by Remark 3, H is equiintegrable in $L^1([a, b]; R)$.

At this point we observe that

$$\lim_{h \downarrow 0} \int_a^{b-h} \|f(s+h) - f(s)\| ds = \lim_{h \downarrow 0} \int_a^{b-h} \left\| \int_t^{t+h} \frac{df}{ds}(s) ds \right\| dt$$

$$\leqq \lim_{h \downarrow 0} \int_a^{b-h} \int_t^{t+h} \left\| \frac{df}{ds}(s) \right\| ds \cdot dt = \lim_{h \downarrow 0} \int_a^{b-h} |g_f(t+h) - g_f(t)| dt = 0$$

uniformly for $f \in G$. Finally, $W^{1,1}([a, b]; X)$ is continuously embedded in $L^p([a, b]; X)$ for $p \in]1, +\infty[$, and so we conclude that G is bounded in the space $L^p([a, b]; X)$. Hence, by Remark 1, G is uniformly integrable. This remark in conjunction with the above relation shows that G is equiintegrable, and this completes the proof.

Now, Theorem 4 together with Lemma 1 yields:

THEOREM 5. *Let $A: D(A) \subset X \to 2^X$ be an m-accretive operator such that $(I_d + \lambda A)^{-1}$ is compact for each $\lambda > 0$. Then, for each $u_0 \in \overline{D(A)}$ and each bounded subset G of $W^{1,1}([a, b]; X)$, the set $\{I(u_0, f); f \in G\}$ is relatively compact in $C([a, b]; X)$.*

We conclude this section with a result concerning the weak-strong continuity of the mapping $f \to I(u_0, f)$ from $W^{1,1}([a, b]; X)$ into $C([a, b]; X)$ —here u_0 is fixed in $\overline{D(A)}$— in the specific case when the dual of X is uniformly convex.

First, we need the following lemma.

LEMMA 2. *Let X be a real Banach space whose dual X^* is uniformly convex, let (u_n), (v_k) be two sequences in $C([a, b]; X)$, and (f_n), (g_k) two sequences in $L^1([a, b]; X)$. If $\lim u_n = u$, $\lim v_k = v$ strongly in $C([a, b]; X)$, $\lim f_n = f$ and $\lim g_k = g$ weakly in $L^1([a, b]; X)$, then*

$$\lim_{n,k} \int_a^b (u_n(s) - v_k(s), f_n(s) - g_k(s))_+ ds$$
$$= \int_a^b (u(s) - v(s), f(s) - g(s))_+ ds.$$

PROOF. Since X^* is uniformly convex, the duality mapping $\mathscr{J}$ on X is single-valued and uniformly continuous on bounded subsets in X. See [4, Proposition 1.5, p. 14]. On the other hand it is well known that

$$(x, y)_+ = \mathscr{J}(x)(y)$$

for each $(x, y) \in X \times X$. See [14, p.10]. Therefore

$$\left| \int_a^b (u_n(s) - v_k(s), f_n(s) - g_k(s))_+ ds - \int_a^b (u(s) - v(s), f(s) - g(s))_+ ds \right|$$
$$\leqq \left| \int_a^b \mathscr{J}(u_n(s) - v_k(s))(f_n(s) - g_k(s)) ds - \int_a^b \mathscr{J}(u(s) - v(s))(f_n(s) - g_k(s)) ds \right|$$
$$+ \left| \int_a^b \mathscr{J}(u(s) - v(s))(f_n(s) - g_k(s)) ds - \int_a^b \mathscr{J}(u(s) - v(s))(f(s) - g(s)) ds \right|$$

for $n, k \in N$. Since $\lim_{n,k} (f_n - g_k) = f - g$ weakly in $L^1([a, b]; X)$ and $\mathscr{J}(u(\cdot) - v(\cdot))$ belongs to the dual of $L^1([a, b]; X)$, the second term on the right-hand side of the above inequality approaches 0 when $n, k \to +\infty$. Clearly $\{f_n - g_k;\, n, k \in \boldsymbol{N}\}$ is bounded in $L^1([a, b]; X)$, say by $M > 0$, and thus

$$\left| \int_a^b \mathscr{J}(u_n(s) - v_k(s))(f_n(s) - g_k(s)) ds - \int_a^b \mathscr{J}(u(s) - v(s))(f_n(s) - g_k(s)) ds \right|$$
$$\leqq M \cdot \sup \{ \| \mathscr{J}(u_n(s) - v_k(s)) - \mathscr{J}(u(s) - v(s)) \|;\, s \in [a, b] \}$$

for $n, k \in \boldsymbol{N}$. Now, recalling that $\lim_{n,k} (u_n - v_k) = u - v$ strongly in $C([a, b]; X)$, and using the fact that $\mathscr{J}$ is uniformly continuous on bounded subsets of X, we conclude that

$$\left| \int_a^b \mathscr{J}(u_n(s) - v_k(s))(f_n(s) - g_k(s)) ds - \int_a^b \mathscr{J}(u(s) - v(s))(f_n(s) - g_k(s)) ds \right|$$

approaches 0 when $n, k \to \infty$. This completes the proof.

THEOREM 6. *Let X be a real Banach space whose dual is uniformly convex, let $A: D(A) \subset X \to 2^X$ be an m-accretive operator such that $(I_d + \lambda A)^{-1}$ is compact* *for each* $\lambda>0$, *and let* u_0 *be a fixed element in* $\overline{D(A)}$. *Then, the mapping* $f\to I(u_0, f)$ *is sequentially continuous from each bounded subset of* $W^{1,1}([a, b]; X)$ *endowed with the induced weak topology of* $L^1([a, b]; X)$ *into* $C[(a, b]; X)$ *endowed with its strong topology. In particular, the mapping* $f\to I(u_0, f)$ *is sequentially continuous from* $W^{1,p}([a, b]; X)$ *endowed with its weak topology into* $C([a, b]; X)$ *endowed with its strong topology, for each* $p\in[1, +\infty]$.

PROOF. Let G be a bounded subset of $W^{1,1}([a, b]; X)$. From Theorem 5 it follows that $\{I(u_0, f); f\in G\}$ is relatively compact in $C([a, b]; X)$. Therefore, to conclude the proof, it suffices to show that for each sequence (f_n) in G which converges weakly in $L^1([a, b]; X)$ to $f\in G$, the sequence $(I(u_0, f_n))$ —denoted for simplicity by (u_n)— has at most one limit point. To this aim, let (u_m) and (u_k) be two subsequences of (u_n) such that $\lim_m u_m=u$ and $\lim_k u_k=v$ strongly in $C([a, b]; X)$. From (4) we obtain

$$\|u_m(t)-u_k(t)\|^2 \leqq 2\int_a^t (u_m(s)-u_k(s), f_m(s)-f_k(s))_+ ds$$

for m, k and $t\in[a, b]$. Now, using Lemma 2, we conclude that

$$\lim_{m,k}(u_m-u_k) = u - v = 0$$

strongly in $C([a, b]; X)$, and this completes the proof.

4. The case in which F is continuous

Let $r>0$, let U be a nonempty subset of $C([-r, 0]; X)$, and let $\varphi\in U$ with $\varphi(0)\in\overline{D(A)}$. Let F: $[0, T]\times U\to X$ be a given mapping.

DEFINITION 3. A function u: $[-r, T]\to\overline{D(A)}$ is called an *integral solution* of the problem (1) on $[0, T]$ if

- (I_1) $u(s)=\varphi(s)$ for each $s\in[-r, T]$,
- (I_2) $u_t\in U$ for each $t\in[0, T]$,
- (I_3) the mapping f: $[0, T]\to X$ defined by $f(t):=F(t, u_t)$ for a.a. $t\in]0, T[$ belongs to $L^1([0, T]; X)$,
- (I_4) u is an integral solution of (2) on $[0, T]$ in the sense of Definition 2, with f as above and $u_0=\varphi(0)$.

The main result in this section is Theorem 7 below.

THEOREM 7. *Let* A: $D(A)\subset X\to 2^X$ *be an m-accretive operator which generates a semigroup* $\{S(t);\, t\geqq 0\}$ *on* $\overline{D(A)}$ *such that* $S(t)$ *is compact for each* $t>0$, *let* U *be a nonempty open subset of* $C([-r, 0]; X)$, *and let* F: $[0, T]\times U\to X$ *be a continuous mapping. Then for each* $\varphi\in U$ *with* $\varphi(0)\in\overline{D(A)}$, *there exists* $T_0=$

$T_0(\varphi) \in]0, T]$ *such that the problem* (1) *has at least one integral solution on* $[0, T_0]$.

PROOF. The idea of the proof consists in showing that a suitably defined operator has at least one fixed point which is an integral solution of (1) defined on $[0, T_0]$ with $T_0 \in]0, T]$.

To this aim, we choose $h>0$, $M>0$ and $T_1 \in]0, T]$ such that

$$B(\varphi, h) := \{\psi \in C([-r, 0]; X); \|\psi - \varphi\|_C \leqq h\} \subset U \tag{6}$$

and

$$\|F(t, \psi)\| \leqq M \tag{7}$$

for $t \in [0, T_1]$ and $\psi \in B(\varphi, h)$, where $\|\cdot\|_C$ denotes the usual sup-norm on $C([-r, 0]; X)$. Next, we choose $T_0 \in]0, \min\{T_1, r\}]$ such that

$$\|\varphi(t+s) - \varphi(s)\| \leqq h/3 \tag{8}$$

for $-r \leqq s \leqq 0$ and $t \in [0, T_0]$ with $t+s \leqq 0$, and

$$\|S(t)\varphi(0) - \varphi(0)\| + T_0 \cdot M \leqq \frac{2}{3} \cdot h \tag{9}$$

for each $t \in [0, T_0]$. We note that this is always possible since U is open in $C([-r, 0]; X)$, F is continuous, $\varphi \in U$, $\varphi(0) \in \overline{D(A)}$, and the semigroup $\{S(t); t \geqq 0\}$ is strongly continuous at the origin.

We define

$$K_0 := \{y;\ y \in C([0, T_0]; X),\ y(0) = \varphi(0)\},$$

which is clearly closed in $C([0, T_0]; X)$. Furthermore, for each $y \in K_0$, we define $\hat{y}: [-r, T_0] \to X$ by

$$\hat{y}(t) := \begin{cases} \varphi(t) & \text{if } \ -r \leqq t \leqq 0, \\ y(t) & \text{if } \ 0 < t \leqq T_0, \end{cases} \tag{10}$$

and set

$$K := \{y;\ y \in K_0,\ \hat{y}_t \in B(\varphi, h) \text{ for each } t \in [0, T_0]\}.$$

Obviously K is nonempty (by (8) the constant function $y: [0, T_0] \to X$, defined by $y(t) := \varphi(0)$ for each $t \in [0, T_0]$, belongs to K), bounded, closed and convex in $C([0, T_0]; X)$.

We then define the operator $P: K \to K_0$ by $(Py)(t) := u(t)$ for each $y \in K$, $t \in [0, T_0]$, where u is the unique integral solution of the problem

$$\frac{du}{dt}(t) + Au(t) \ni F(t, \hat{y}_t), \quad 0 \leqq t \leqq T_0,$$

$$u(0) = \varphi(0).$$

We note that, in view of (6), (7) and the definition of K, the operator P is defined on all of K.

Next, we prove that $P(K) \subset K$. To this aim, observe that for $y \in K$ we have $(Py)_t \in B(\varphi, h)$ if and only if

$$\|u(t+s) - \varphi(s)\| \leqq h, \quad u = Py,$$

for $t \in [0, T_0]$ and $-r \leqq s \leqq 0$.

If $t+s > 0$, we have

$$\|u(t+s) - \varphi(s)\| \leqq \|u(t+s) - S(t+s)\varphi(0)\| + \|S(t+s)\varphi(0) - \varphi(0)\| + \|\varphi(0) - \varphi(s)\|,$$

and thus, by (3), (7), (8) and (9), we obtain

$$\begin{aligned}\|u(t+s) - \varphi(s)\| &\leqq \int_0^{T_0} \|F(\theta, \hat{y}_\theta)\| d\theta + \|S(t+s)\varphi(0) - \varphi(0)\| + \|\varphi(0) - \varphi(s)\| \\ &\leqq T_0 \cdot M + \|S(t+s)\varphi(0) - \varphi(0)\| + \|\varphi(0) - \varphi(s)\| \\ &\leqq \frac{2}{3} \cdot h + \frac{1}{3} \cdot h = h.\end{aligned}$$

If $t+s \leqq 0$, by (8), we have

$$\|u(t+s) - \varphi(s)\| = \|\varphi(t+s) - \varphi(s)\| \leqq h/3 < h,$$

and consequently $P(K) \subset K$.

Now, from (3), (7), the continuity of F and Lebesgue's Dominated Convergence Theorem, it readily follows that P is continuous from K into itself with respect to the induced norm topology on both domain and range.

Finally, by (7) and the definition of K, we conclude that the set $G := \{f_y;\ f_y \in L^1([0, T_0]; X),\ f_y(t) := F(t, \hat{y}_t)$ for a.a. $t \in]0, T_0[,\ y \in K\}$ is bounded in $L^\infty([0, T_0]; X)$. Hence, by Remark 1, the set G defined as above is uniformly integrable in $L^1([0, T_0]; X)$. Thus, Theorem 2 applies, and consequently $P(K)$ is relatively compact in $C([0, T_0]; X)$. From Schauder's Fixed Point Theorem it then follows that P has at least one fixed point which, obviously, gives an integral solution of (1) on $[0, T_0]$, thereby completing the proof.

5. The case in which F is an integral operator

Let $C_A([a, b]; X)$ be the set of all functions φ in $C([a, b]; X)$ such that $\varphi(s) \in$

$D(A)$ for a.a. $s \in]a, b[$ and consider the problem

$$\text{(11)} \qquad \begin{aligned} &\frac{du}{dt}(t) + Au(t) \ni \int_{t-r}^{t} k(t-s)g(u(s))ds, \quad 0 \leqq t \leqq T \\ &u(s) = \varphi(s), \quad -r \leqq s \leqq 0, \end{aligned}$$

where k is an operator kernel, $g\colon D(A) \to X$ is a given mapping and $\varphi \in C_A([-r, 0]; X)$ satisfies $\varphi(0) \in D(A)$.

We note that, by a simple change of variable, the problem (11) may be rewritten in the form (1) with $F\colon [0, T] \times C_A([-r, 0]; X) \to X$ defined by

$$\text{(12)} \qquad F(t, \psi) := \int_{-r}^{0} k(-\tau)g(\psi(\tau))d\tau$$

for $t \in [0, T]$ and $\psi \in C_A([-r, 0]; X)$. Nevertheless, since $U = C_A([-r, 0]; X)$ is *not* open in $C([-r, 0]; X)$ in general, Theorem 7 can not be applied to (11).

Thus, our main goal here is to show that, using the additional properties of F due to its specific form, we can prove a local existence result concerning strong solutions of (11). Since it is very clear what a strong solution of (11) ought to be, we do not give an explicit definition of this concept.

We begin with the following definition to make a precise statement of our local existence result.

DEFINITION 4. A mapping $g\colon D(A) \to X$ is called *A-demiclosed* if for each $T > 0$ the following conditions are satisfied.

(Ad_1) For each $y \in C_A([0, T]; X)$ such that there exists $f \in L^2([0, T]; X)$, $f(t) \in Ay(t)$ for a.a. $t \in]0, T[$, the mapping $t \to g(y(t))$ is strongly measurable from $[0, T]$ into X.

(Ad_2) If (y_n) is a sequence in $C_A([0, T]; X)$, $y \in C_A([0, T]; X)$, $f_n(t) \in Ay_n(t)$ for each $n \in N$ and for a.a. $t \in]0, T[$, $f(t) \in Ay(t)$ for a.a. $t \in]0, T[$, and if in addition

$$\begin{aligned} y_n &\longrightarrow y && \text{strongly in } C([0, T]; X), \\ f_n &\longrightarrow f && \text{weakly in } L^2([0, T]; X), \\ g(y_n) &\longrightarrow \tilde{g} && \text{weakly in } L^2([0, T]; X), \end{aligned}$$

then $\tilde{g}(t) = g(y(t))$ for a.a. $t \in]0, T[$.

(Ad_3) There exists a monotone nondecreasing function $m\colon R_+ \to R_+$ such that

$$\|g(u)\| \leqq m(\|A^0 u\|)$$

for each $u \in D(A)$, where $\|A^0 u\| := \inf\{\|v\|;\ v \in Au\}$.

We then impose the following general assumptions.

(H_1) X is a real Banach space whose dual X^* is uniformly convex, and A: $D(A) \subset X \to 2^X$ is an m-accretive operator such that $(I_d + \lambda A)^{-1}$ is compact for each $\lambda > 0$.

(H_2) k: $[0, r] \to \mathscr{L}(X)$ is a mapping of class C^1, where $\mathscr{L}(X)$ is the Banach space (endowed with the usual sup-norm) of all linear, bounded operators from X into itself.

(H_3) g: $D(A) \to X$ is an A-demiclosed mapping.

THEOREM 8. *Assume that* (H_1), (H_2) *and* (H_3) *are satisfied. Then, for each* $\varphi \in C_A([-r, 0]; X)$ *with* $\varphi(0) \in D(A)$ *and* $g(\varphi) \in L^2([-r, 0]; X)$, *there exists* $T_0 = T_0(\varphi) \in]0, T[$, *such that the problem* (11) *has least at one strong solution defined on* $[0, T_0]$.

PROOF. The idea of proof consists in showing that a suitably defined operator has at least one fixed point whose existence will imply the existence of at least one strong solution of (11) defined on $[0, T_0]$ with $T_0 \in]0, T]$.

To begin with, let $h > 0$ and $S \in]0, \min\{T, r\}]$, and define

$$K_h^S := \{f; f \in W^{1,2}([0, S]; X), \|f\|_{1,2,S} \leqq h, f(0) = F(0, \varphi)\}$$

where $\|\cdot\|_{1,2,S}$ is the usual norm on $W^{1,2}([0, S]; X)$, i.e.,

$$\|f\|_{1,2,S} := \left[\int_0^S \left(\|f(t)\|^2 + \left\|\frac{df}{dt}(t)\right\|^2 \right) dt\right]^{1/2}$$

for each $f \in W^{1,2}([0, S]; X)$ and F is given by (12).

Now, for each $f \in K_h^S$, we write u^f for the unique strong solution of the problem

$$(13) \qquad \begin{cases} \dfrac{du^f}{dt}(t) + Au^f(t) \ni f(t), & 0 \leqq t \leqq S \\ u^f(0) = \varphi(0), \end{cases}$$

and define the operator Q: $D(Q) \subset K_h^S \to W^{1,2}([0, S]; X)$ by

$$(Qf)(t) := F(t, \hat{u}_t^f)$$

for $f \in K_h^S$ and $t \in [0, S]$. Here $\hat{u}^f$: $[-r, S] \to X$ is defined by (10) with T_0 replaced by S and F is given by (12); hence

$$D(Q) := \{f; f \in K_h^S, t \to F(t, \hat{u}_t^f) \text{ belongs to } W^{1,2}[(0, S]; X)\}.$$

We shall prove that for some suitably chosen $h > 0$ and S in $]0, \min\{T, r\}]$

the operator Q has at least one fixed point $f\in K_h^S$ which by means of the correspondence $f\to\hat{u}^f$ defines a strong solution of (11) on $[0, S]$.

To this aim, we show first that for $h>0$ —large enough— and for $S\in]0, \min\{T, r\}]$ —small enough— the operator Q is defined on all of K_h^S, i.e., $D(Q)=K_h^S$, and maps K_h^S into itself. We prove this with the aid of the next simple lemma.

LEMMA 3. *For each $h>0$ and each $S\in]0, \min\{T, r\}]$, there exists a constant $C(S, h)>0$ such that*

$$\|A^0u^f(t)\| \leqq C(S, h)$$

for each $f\in K_h^S$ and for a.a. $t\in]0, S[$, where u^f is the unique strong solution of (13). *In addition, for $h>0$ and $0<S\leqq S'\leqq\min\{T, r\}$ we have*

$$C(S, h)\leqq C(S', h).$$

PROOF OF LEMMA 3. First we recall that, in view of [4, Theorem 2.2, p. 131], for each h and S as above and each $f\in K_h^S$ the problem (13) has a unique strong solution u^f satisfying

$$\left\|\frac{du^f}{dt}(t)\right\|\leqq\|(A\varphi(0)-f(0))^0\| + \int_0^t\left\|\frac{df}{ds}(s)\right\|ds$$

for a.a. $t\in]0, S[$, where $\|(A\varphi(0)-f(0))^0\|=\inf\{\|v-f(0)\|;\ v\in A\varphi(0)\}$. Then, taking into account the relation $f(0)=F(0, \varphi)$, we have

$$\begin{aligned}
\|A^0u^f(t)\| &\leqq \left\|\frac{du^f}{dt}(t)\right\| + \|f(t)\|\\
&\leqq \|(A\varphi(0)-F(0, \varphi))^0\|\\
&\quad+\int_0^t\left\|\frac{df}{ds}(s)\right\|ds+\left\|F(0, \varphi)+\int_0^t\frac{df}{ds}(s)ds\right\|\\
&\leqq \|(A\varphi(0)-F(0, \varphi))^0\| + \|F(0, \varphi)\| + 2\int_0^t\left\|\frac{df}{ds}(s)\right\|ds\\
&\leqq \|(A\varphi(0)-F(0, \varphi))^0\| + \|F(0, \varphi)\| + 2\left(\int_0^S\left\|\frac{df}{ds}(s)\right\|^2ds\right)^{1/2}\left(\int_0^S ds\right)^{1/2}\\
&= \|(A\varphi(0)-F(0, \varphi))^0\| + \|F(0, \varphi)\| + 2hS^{1/2} =: C(S, h),
\end{aligned}$$

and this completes the proof.

PROOF OF THEOREM 8 — continued. In view of (H_2) there exists $c_0>0$ such that

$$\|k(s)\|_{\mathscr{L}(X)}\leqq c_0,\quad \left\|\frac{dk}{ds}(s)\right\|_{\mathscr{L}(X)}\leqq c_0 \tag{14}$$

for each $s \in [0, S]$. Put

(15) $$c_1 := \left(\int_{-r}^{0} \|g(\varphi(s))\|^2 ds \right)^{1/2},$$

which is finite by hypothesis. In view of (14), (15), (Ad_1), (Ad_3) and Lemma 3, we have

$$\begin{aligned}\left\| \int_{t-r}^{t} k(t-s) g(\hat{u}^f(s)) ds \right\| &\leqq c_0 \int_{t-r}^{0} \|g(\varphi(s))\| ds + c_0 \int_{0}^{t} \|g(u^f(s))\| ds \\ &\leqq c_0 \left(\int_{-r}^{0} \|g(\varphi(s))\|^2 ds \right)^{1/2} \left(\int_{-r}^{0} ds \right)^{1/2} + c_0 \int_{0}^{t} m(\|A^0 u^f(s)\|) ds \\ &\leqq c_0 \cdot c_1 \cdot r^{1/2} + c_0 \cdot m(C(S, h)) \cdot S,\end{aligned}$$

for $h > 0$, $S \in]0, \min\{T, r\}]$, $f \in K_h^S$, and $t \in [0, S]$. Thus we conclude that

(16) $$\int_0^S \left\| \int_{t-r}^{t} k(t-s) g(\hat{u}^f(s)) ds \right\|^2 dt \leqq S[c_0 \cdot c_1 \cdot r^{1/2} + c_0 \cdot m(C(S, h)) \cdot S]^2$$

for $h > 0$, $S \in]0, \min\{T, r\}]$ and $f \in K_h^S$. Using the same arguments as above, we also deduce

(17) $$\int_0^S \left\| \int_{t-r}^{t} \frac{dk}{dt} (t-s) g(\hat{u}^f(s)) ds \right\|^2 dt \leqq S[c_0 \cdot c_1 \cdot r^{1/2} + c_0 \cdot m(C(S, h)) \cdot S]^2$$

for $h > 0$, $S \in]0, \min\{T, r\}]$ and $f \in K_h^S$.

At this point, we recall that

$$(Qf)(t) := F(t, \hat{u}_t^f) = \int_{t-r}^{t} k(t-s) g(\hat{u}^f(s)) ds$$

for $f \in K_h^S$ and $t \in [0, S]$. Hence, if $S \in]0, \min\{T, r\}]$ then we have $g(\hat{u}^f(t-r)) = g(\varphi(t-r))$ for each $t \in [0, S]$, and consequently

$$\frac{d}{dt}(Qf)(t) = k(0) g(u^f(t)) - k(r) g(\varphi(t-r)) + \int_{t-r}^{t} \frac{dk}{dt}(t-s) g(\hat{u}^f(s)) ds$$

for each $f \in K_h^S$ and for a.a. $t \in]0, S[$.

Now, applying (14), (15), (16), (17) (Ad_3) and Lemma 3 and a standard argument, we obtain

(18) $$\|Qf\|_{1_{2,S}}^2 \leqq c_0^2 \cdot c_1^2 + H(S. h)$$

for $h > 0$, $S \in]0, \min\{T, r\}]$ and $f \in K_h^S$, where $\lim_{S \downarrow 0} H(S, h) = 0$ for each $h > 0$. But (18) shows that for each $h > c_0 \cdot c_1$ we may choose S in $]0, \min\{T, r\}]$ so that $c_0^2 \cdot c_1^2 + H(S, h) \leqq h^2$. Thus, for $h > 0$ and $S \in]0, \min\{T, r\}]$ as above, Q is defined on all of K_h^S and maps K_h^S into itself.

In order to complete the proof, we need the following simple but useful fixed point theorem due to Arino, Gautier and Penot [1].

THEOREM 9. *Let E be a metrizable locally convex topological vector space and let K be a weakly compact convex subset of E. Then any weakly sequentially continuous map Q: $K \to K$ has a fixed point.*

For the proof of Theorem 9 the reader is referred to [1, Theorem 1, p. 274].

PROOF OF THEOREM 8—continued. Since K_h^S is bounded, closed and convex in $W^{1,2}([0, S]; X)$ and X is reflexive —being the predual of a uniformly convex Banach space— it readily follows that K_h^S is weakly compact in $W^{1,2}([0, S]; X)$. Thus, in view of Theorem 9, it suffices to show that Q is weakly sequentially continuous from K_h^S into itself. On the other hand, a simple topological argument together with the fact that K_h^S is weakly compact shows that Q is weakly sequentially continuous from K_h^S into itself if and only if the graph of Q is weakly sequentially closed in $K_h^S \times K_h^S$. Therefore we shall prove that the graph of Q is weakly sequentially closed in $K_h^S \times K_h^S$.

To this aim, let $((f_n, q_n))$ be a sequence in graph (Q) such that

$$\lim f_n = f \quad \text{and} \quad \lim q_n = q \quad \text{weakly in} \quad W^{1,2}([0, S]; X).$$

We denote by (u^n) the sequence (u^{f_n}) and observe that, by Theorem 6, we have

$$\lim u^n = u^f \quad \text{strongly in} \quad C([0, S]; X).$$

Let (v_n) be a sequence in $L^2([0, S]; X)$ defined by

$$v_n(t) := -\frac{du^n}{dt}(t) + f_n(t)$$

for each $n \in N$ and for a.a. $t \in]0, S[$. Since $v_n(t) \in Au^n(t)$ for each $n \in N$ and for a.a. $t \in]0, S[$, Lemma 3 ensures that (v_n) is bounded in $L^2([0, S]; X)$ (in fact (v_n) is bounded in $L^\infty([0, S]; X)$). Hence we may assume without loss of generality that

$$\lim v_n = v \quad \text{weakly in} \quad L^2([0, S]; X).$$

At this point, we define the operator $\mathscr{A}$: $D(\mathscr{A}) \subset L^2([0, S]; X) \to 2^{L^2([0,S];X)}$ by

$$\mathscr{A}u := \{v;\ v \in L^2([0, S]; X),\ v(t) \in Au(t) \text{ for a.a. } t \in]0, S[\}$$

where

$$D(\mathscr{A}) = \{u;\ u \in L^2([0, S]; X),\ u(t) \in D(A) \text{ for a.a. } t \in]0, S[$$
$$\text{and there exists } v \in L^2([0, S]; X) \text{ such that}$$
$$v(t) \in Au(t) \text{ for a.a. } t \in]0, S[\}.$$

It is a simple exercise to prove that $\mathscr{A}$ is m-accretive in $L^2([0, S]; X)$. In addition, since the dual of $L^2([0, S]; X)$ is $L^2([0, S]; X^*)$ —see [8, Theorem 8.20.5, p. 607]— and X^* is uniformly convex, it follows that $L^2([0, S]; X^*)$ is uniformly convex too. See [27, Theorem 4.2 and Remark 4.7, p. 365]. Hence [4, Proposition 3.5, p. 75] implies that $\mathscr{A}$ is demiclosed. Therefore $v \in \mathscr{A}u^f$, or equivalently, $v(t) \in Au^f(t)$ for a.a. $t \in]0, S[$. Now, by Lemma 3 and (Ad_3) we may assume without loss of generality (by extracting a subsequence if necessary) that

$$\lim g(u^n) = g_0 \quad \text{weakly in} \quad L^2([0, S]; X).$$

From (Ad_2) it then follows that $g_0(t) = g(u^f(t))$ for a.a. $t \in]0, S[$.

Finally, noting that

$$q_n(t) = \int_{t-r}^{t} k(t-s)g(\hat{u}^n(s))ds$$

for $n \in N$ and $t \in [0, S]$, we easily conclude that

$$q(t) = \int_{t-r}^{t} k(t-s)g(\hat{u}^f(s))ds = (Qf)(t)$$

for each $t \in [0, S]$. Therefore the graph of Q is weakly sequentially closed in $K_h^S \times K_h^S$, and thus Theorem 9 implies that Q has at least one fixed point $f \in K_h^S$ which by means of $f \to \hat{u}^f$ defines a strong solution of (11) on $[0, S]$, thereby completing the proof.

REMARK 4. As seen from the proof of Theorem 8, we may allow g to depend on s. It is also obvious that we can treat the case where k need not be a convolution kernel.

6. Continuation of the solutions

In this section we state without proof several results concerning the continuation of the solutions of (1) and (11).

THEOREM 10. *Let $A: D(A) \subset X \to 2^X$ be an m-accretive operator which generates a semigroup $\{S(t);\ t \geq 0\}$ on $D(A)$ such that $S(t)$ is compact for each $t > 0$, and let $F: [0, +\infty[\times C([-r, 0]; X) \to X$ be a continuous mapping. If F maps bounded subsets of $[0, +\infty[\times C([-r, 0]; X)$ into bounded subsets of X, then* *each integral solution u of* (1) *can be continued to the right up to a noncontinuable one* $\tilde{u}: [-r, T_{\max}[\to D(A)$ *where either* $T_{\max} = +\infty$, *or* $T_{\max} < +\infty$ *and in the latter case*

$$\lim_{t \uparrow T_{\max}} \|\tilde{u}(t)\| = +\infty.$$

REMARK 5. The interesting fact in Theorem 10 is that in the case where $T_{\max} < +\infty$ we have not merely $\limsup_{t \uparrow T_{\max}} \|\tilde{u}(t)\| = +\infty$ as mentioned in [9, Theorem 4.2, p. 7], but the formula $\lim_{t \uparrow T_{\max}} \|\tilde{u}(t)\| = +\infty$. The proof of Theorem 10 which follows exactly the same lines of [25, Theorem 3.2.2, p. 127] is inspired by [17, Theorem 6.1, p. 233].

Using the same arguments as in [9, Theorem 4.2, p. 8] we may prove

THEOREM 11. *Let* $A: D(A) \subset X \to 2^X$ *be an m-accretive operator which generates a semigroup* $\{S(t);\, t \geqq 0\}$ *on* $\overline{D(A)}$ *such that* $S(t)$ *is compact for each* $t > 0$, *and let* $F: [0, +\infty[\times C([-r, 0]; X) \to X$ *be a continuous mapping. If there exists two locally integrable functions* k_1, k_2 *such that*

$$\|F(t, \varphi)\| \leqq k_1(t) \|\varphi\|_{C([-r,0];X)} + k_2(t)$$

for each $(t, \varphi) \in [0, +\infty[\times C([-r, 0]; X)$, *then each integral solution of* (1) *can be continued to the right up to an integral solution defined on* $[0, +\infty[$.

Concerning the problem (11) we have

THEOREM 12. *Assume that* (H_1), (H_2) *and* (H_3) *are satisfied with* $T = +\infty$, *and let* $u: [0, S[\to D(A)$ *be a strong solution of* (11). *Then u is noncontinuable as a strong solution if and only if the mapping* $t \to \|A^0 u(t)\|$ *is unbounded in the* L^∞*-norm on every neighbourhood of S.*

The proof of Theorem 12 is very similar to that of [15, Theorem VI, p. 288].

7. Examples and applications

Our aim here is to illustrate the abstract existence theory developed in Sections 4 and 5 by some examples of functional partial differential equations.

We begin with an example related to Theorem 7. Let Ω be a bounded domain in R^n, $n \geqq 1$, with smooth boundary Γ, and consider the following functional partial differential equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta\beta(u) = F(t, u_t) \quad \text{for a.a. } (t, x) \in]0, T[\times \Omega \\ (19) \quad &u(t, x) = 0 \quad \text{for a.a. } (t, x) \in]0, T[\times \Gamma \\ &u(s, x) = \varphi(s, x) \quad \text{for each } s \in [-r, 0] \text{ and for a.a. } x \in \Omega, \end{aligned}$$

where $\beta: R \to R$ is a given nondecreasing function with $\beta(0)=0$, $F: [0, T] \times C([-r, 0]; L^1(\Omega)) \to L^1(\Omega)$ is a continuous mapping, and $\varphi \in C([-r, 0]; L^1(\Omega))$.

THEOREM 13. *Assume that* $\beta \in C^1(R-\{0\}; R)$, $\beta(0)=0$, *and there exist* $c>0$ *and* $p>(n-2)/n$ *if* $n>2$, $p>0$ *if* $n \leqq 2$, *such that*

$$\beta'(u) \geqq c|u|^{p-1}$$

for each $u \in R-\{0\}$. *Assume, in addition, that* F *is continuous from* $[0, T] \times C([-r, 0]; L^1(\Omega))$ *into* $L^1(\Omega)$. *Then, for each* $\varphi \in C([-r, 0]; L^1(\Omega))$, *there exists* $T_0 = T_0(\varphi) \in]0, T]$ *such that the problem* (19) *has at least one solution defined on* $[0, T_0]$.

PROOF. Take $X = L^1(\Omega)$ and observe that (19) can be rewritten in the form (1) with F as above and $A: D(A) \subset X \to 2^X$ defined by $Au := \{-\Delta\beta(u)\}$ for each $u \in D(A)$, where

$$D(A) = \{u \in L^1(\Omega);\ \beta(u) \in W_0^{1,1}(\Omega),\ \Delta\beta(u) \in L^1(\Omega)\}.$$

It is well known that A is m-accretive in $L^1(\Omega)$. See [5]. On the other hand, by [2, Proposition 3.1, p. 12], A generates a semigroup $\{S(t);\ t \geqq 0\}$ on $\overline{D(A)}$ such that $S(t)$ is compact for each $t>0$. Hence Theorem 7 can be applied to obtain the desired assertion. Q. E. D.

Next, we analyse two examples related to Theorem 8. First, let us consider the problem

$$
(20)\quad
\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \int_{t-r}^{t} \tilde{k}(t-s)\, \frac{\partial}{\partial x}\Big(\tilde{g}\Big(\frac{\partial u}{\partial x}(s, x)\Big)\Big)\, ds, \\
&\qquad\qquad \text{for a.a. } (t, x) \in]0, T[\times]0, 1[, \\
&\frac{\partial u}{\partial x}(t, 0) \in \rho(u(t, 0)),\ -\frac{\partial u}{\partial x}(t, 1) \in \rho(u(t, 1)), \\
&\qquad\qquad \text{for a.a. } t \in]0, T[, \\
&u(s, x) = \varphi(s, x), \quad \text{for each } s \in [-r, 0] \text{ and a.a. } x \in]0, 1[.
\end{aligned}
$$

THEOREM 14. *Let* $\tilde{k}: [0, r] \to R$ *and* $\tilde{g}: R \to R$ *be functions of class* C^1 *and let* $\rho: D(\rho) \subset R \to 2^R$ *be a maximal monotone operator with* $0 \in \rho(0)$. *Then, for each* $\varphi \in C([-r, 0]; L^2([0, 1]))$ *satisfying* $\varphi(t) \in H^2([0, 1])$ *for* $t \in [-r, 0]$, $(\partial\varphi/\partial x)(t, 0) \in \rho(\varphi(t, 0))$, $-(\partial\varphi/\partial x)(t, 1) \in \rho(\varphi(t, 1))$ *for* $t \in [-r, 0]$ *and* $\partial^2\varphi/\partial x^2 \in L^2([-r, 0]; L^2([0, 1]))$, *there exists* $T_0 = T_0(\varphi)$ *in* $]0, T]$ *such that the problem* (20) *has at least one strong solution defined on* $[0, T_0]$.

PROOF. Take $X = L^2([0, 1])$ and observe that (20) can be rewritten in the form (11) with A, k and g defined as follows:

$$Au := \left\{ -\frac{d^2u}{dx^2} \right\} \in 2^X$$

for each $u \in D(A) = \left\{ u \in H^2([0, 1]);\ \frac{du}{dx}(0) \in \rho(u(0)),\ -\frac{du}{dx}(1) \in \rho(u(1)) \right\}$,

$$k(s) := \tilde{k}(s) I_d \in \mathscr{L}(X)$$

for each $s \in [0, r]$, where I_d is the identity on X, and

$$g(u)(x) := \frac{d}{dx}\left(\tilde{g}\left(\frac{du}{dx}(x) \right) \right) \in X$$

for each $u \in D(A)$ and for a.a. $x \in]0, 1[$.

Clearly $(I_d + \lambda A)^{-1}$ is compact for each $\lambda > 0$. On the other hand, a simple argument together with the fact that $\tilde{g}$ is of class C^1 shows that g is A-demiclosed. Therefore Theorem 8 can be applied to obtain the desired assertion.

Finally, let us consider the following problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \int_{t-r}^{t} \tilde{k}(t-s) \frac{\partial}{\partial x}\left(\tilde{g}\left(\frac{\partial u}{\partial x}(s, x) \right) \right) ds,$$

$$\text{for a.a. } (t, x) \in]0, T[\times]0, 1[,$$

(21) $$u(t, 0) = u(t, 1) = 0, \quad \text{for a.a. } t \in]0, T[,$$

$$u(s, x) = \varphi_0(s, x), \quad \frac{\partial u}{\partial t}(s, x) = \varphi_1(s, x)$$

$$\text{for each } s \in [-r, 0] \text{ and for a.a. } x \in]0, 1[,$$

where $\tilde{k}$ and $\tilde{g}$ are as in the previous example, $\varphi_0 \in C([-r, 0]; H_0^1([0, 1]))$ and $\varphi_1 \in C([-r, 0]; L^2([0, 1]))$.

THEOREM 15. *Let $\tilde{k}: [0, r] \to R$ and $\tilde{g}: R \to R$ be functions of class C^1. Then, for each pair of $\varphi_0 \in C([-r, 0]; H_0^1([0, 1]))$ and $\varphi_1 \in C([-r, 0]; L^2([0, 1]))$ satisfying $\varphi_0(s) \in H^2([0, 1])$ for $s \in [-r, 0]$, $\varphi_1(0) \in H_0^1([0, 1])$ and $\partial^2 \varphi_0 / \partial x^2 \in L^2([-r, 0]; L^2([0, 1]))$, there exists $T_0 = T_0(\varphi_0, \varphi_1) \in]0, T]$ such that the problem* (21) *has at least one strong solution defined on* $[0, T_0]$.

PROOF. First of all, we observe that (21) can be rewritten as a first-order system of the form

$$\frac{\partial u}{\partial t} - v = 0, \quad \text{for a.a. } (t, x) \in]0, T[\times]0, 1[,$$

$$\frac{\partial v}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \int_{t-r}^{t} \tilde{k}(t-s) \frac{\partial}{\partial x}\left(\tilde{g}\left(\frac{\partial u}{\partial x}(s, x)\right)\right) ds,$$

$$\text{for a.a. } (t, x) \in]0, T[\times]0, 1[,$$

$$u(t, 0) = u(t, 1) = 0, \quad \text{for a.a. } t \in]0, T[$$

$$u(s, x) = \varphi_0(s, x), \quad v(s, x) = \varphi_1(s, x),$$

$$\text{for each } s \in [-r, 0] \text{ and for a.a. } x \in]0, 1[.$$

This time, we employ the space

$$X = \left\{\begin{pmatrix} u \\ v \end{pmatrix};\ u \in H_0^1([0, 1]),\ v \in L^2([0, 1])\right\}$$

with the norm $\|\cdot\|$ defined by

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\| := \left(\int_0^1 \left|\frac{du}{dx}(x)\right|^2 ds\right)^{1/2} + \left(\int_0^1 v^2(x) ds\right)^{1/2}$$

for $\begin{pmatrix} u \\ v \end{pmatrix} \in X$. Then X is a real Hilbert space. Obviously, the system mentioned above can be rewritten in the abstract form (11) with A, k, g and φ defined as follows:

$$A\begin{pmatrix} u \\ v \end{pmatrix} := \left\{\begin{pmatrix} -v \\ -d^2u/dx^2 \end{pmatrix}\right\} \in 2^X$$

for each $\begin{pmatrix} u \\ v \end{pmatrix} \in D(A) = \left\{\begin{pmatrix} u \\ v \end{pmatrix} \in X;\ u \in H^2([0, 1]),\ v \in H_0^1([0, 1])\right\}$,

$$k(s) := \begin{pmatrix} 0 \\ \tilde{k}(s)\tilde{I}_d \end{pmatrix} \in \mathscr{L}(X)$$

for each $s \in [0, r]$, $\tilde{I}_d$ being the identity of $L^2([0, 1])$.

$$g\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 \\ (d/dx)\tilde{g}(du/dx) \end{pmatrix} \in X$$

for each $\begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$, and $\varphi \in C([-r, 0]; X)$ is defined by $\varphi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$.

It is a simple exercise to show that A, k and g as defined above satisfy the hypotheses of Theorem 8, and the desired assertion is obtained.

REMARK 6. We may easily verify that the operator A defined in the proof of Theorem 15 generates a semigroup $\{S(t);\ t \geqq 0\}$ on X with $S(t)$ *noncompact* for $t > 0$, even though $(I_d + \lambda A)^{-1}$ is compact for each $\lambda > 0$.

REMARK 7. Both Theorem 7 and Theorem 8 can be suitably reformulated in order to handle infinite delays.

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