HIROSHIMA MATH. J.
17 (1987), 535–560

Nonlinear equations on a Lie group

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(Received January 19, 1987)

Introduction

In their paper [11, 12], I. Hauser and F. J. Ernst proved in the affirmative the Geroch conjecture [2, 3] that all stationary axisymmetric solutions of the Einstein field equations can be essentially derived from the Minkowski space metric by means of Kinnersley-Chitre transformations. In the subsequent paper [13], they extended this result to the case of N Abelian gauge fields interacting with gravitation which contains as special cases the Einstein field equations ($N=0$) and the Einstein-Maxwell field equations ($N=1$). We call their equations the Hauser-Ernst equations.

The purpose of this paper is to give a slightly different formulation of the Hauser-Ernst equations in such a way that the theory of real semisimple Lie groups can be easily applied to it; replacing the real simple group $SU(N+1, 1)$ in their formulation by more general real simple Lie groups, we shall prove our version of the "generalized Geroch conjecture". Our formulation is based on the framework of the theory of homogeneous spaces of Lie groups. The most striking is the analogy with concepts of finite (or infinite) dimensional generalized flag manifolds, parabolic subgroups, the Bruhat (or rather the Birkhoff) decomposition and so on (cf. [14]). In particular we give a new proof, using the Birkhoff decomposition in place of the homogeneous Hilbert problem.

The paper is organized as follows.

In Section 1 we introduce a certain topology into a group of analytic loops on a Lie group; provided with this topology the analytic loop group becomes a topological group. Theorem 1.6 implies that for an arbitrary linear algebraic group any loop sufficiently near the unit loop has a Birkhoff decomposition.

In Section 2 we follow the idea of Hauser-Ernst to define the action of the analytic loop group.

In Section 3, with the aid of a certain involutive real automorphism of $GL(N, \boldsymbol{C})$, we put the Hauser-Ernst equations into simple formulas, from which we can easily deduce various group theoretical properties of solutions. Theorem 3.2 describes the dependency of solutions on their axial values.

Finally, in Section 4 we start with constructing a "seed solution" which plays the same role as the Minkowski space metric in the theory of Hauser-Ernst. Theorem 4.6 assures that our loop group generates solutions. The results of this section show that any solution can be obtained by applying a certain element of our loop group to the seed solution.

The author heartily thanks Prof. K. Okamoto for his helpful advice and encouragement.

1. Analytic loop groups

We start with some definitions. Fix a positive real number R and the circle $S_R=\{s\in \boldsymbol{C};\ |s|=R\}$. For a complex manifold M, we denote by $\mathscr{A}_R^M$ or $\mathscr{A}^M$ the set of all mappings from S_R to M such that they extend to holomorphic mappings from neighborhoods of S_R to M. Let L be a finite dimensional vector space over $\boldsymbol{C}$. Naturally $\mathscr{A}^L$ becomes a vector space over $\boldsymbol{C}$. We introduce a topology on $\mathscr{A}^L$. Take a norm $\|\ \|$ on L. For a formal Laurent series $f=\sum_{n\in \boldsymbol{Z}} f_n s^n$ with $f_n\in L$, we set

$$\|f\|_s = \sum_{n\in \boldsymbol{Z}} \|f_n\| s^n.$$

This is also a formal Laurent series. If we can substitute R into s, we denote its value by $\|f\|_R$. Then $\mathscr{A}^L$ is made into a normed space by $\|\ \|_R$, which is not complete. We notice that this topology of $\mathscr{A}^L$ is independent of a norm on L. For a positive real number b, we set $D_b^L=\{X\in L;\ \|X\|<b\}$ and $\mathscr{D}_b^L=\{f\in\mathscr{A}^L;\ \|f\|_R<b\}$. If F is a holomorphic mapping from D_b^L to $\boldsymbol{C}$, the composition $F(f)=F\circ f$ defines a continuous map from $\mathscr{D}_b^L$ to $\mathscr{A}^{\boldsymbol{C}}$, which we denote by F, too.

Let G be a complex Lie group and $\mathfrak{g}$ be its Lie algebra. We take a basis $\{e_i\}$ of $\mathfrak{g}$. For $x\in\mathfrak{g}$, we put $\|x\|=\max_i |x_i|$ if $x=\sum x_i e_i$ with $x_i\in \boldsymbol{C}$. We always use the norm on $\mathfrak{g}$ of this type. We define a family of subsets of $\mathscr{A}^G$: $\mathscr{F}=\{\exp \mathscr{D}_b^{\mathfrak{g}};\ b>0\}$. It is easy to check that $\mathscr{F}$ becomes a fundamental system of neighborhoods around the unit of $\mathscr{A}^G$. Thus we have the following

LEMMA 1.1. *$\mathscr{A}^G$ is made into a topological group.*

It is clear that for a vector group L, the previous topology on $\mathscr{A}^L$ coincides with the topology defined now. From now on we treat $\mathscr{A}^G$ as a topological group given in Lemma 1.1.

PROPOSITION 1.2. *Let H and G be complex Lie groups.* (1) *If f: $H\to G$ is a holomorphic homomorphism, then f: $\mathscr{A}^H\to\mathscr{A}^G$ is a continuous homomorphism.* (2) *If H is a closed complex Lie subgroup of G, then $\mathscr{A}^H$ is a closed subgroup of $\mathscr{A}^G$.*

PROOF. (1) follows from the definition of the topology. We show (2). If $g\in\mathscr{A}^G\setminus\mathscr{A}^H$, there exists $s_0\in S_R$ such that $g(s_0)\in G\setminus H$. Then for any $g'\in\mathscr{A}^G$ sufficiently near g, we have $g'(s_0)\in G\setminus H$. This means that $\mathscr{A}^H$ is a closed subset of $\mathscr{A}^G$. Let $\mathfrak{h}$ be the Lie algebra of H. Make a basis of $\mathfrak{g}$ by extending a basis of $\mathfrak{h}$. For a sufficiently small $b>0$, we know that $\exp \mathscr{D}_b^{\mathfrak{h}} = (\exp \mathscr{D}_b^{\mathfrak{g}}) \cap \mathscr{A}^H$. So $\mathscr{A}^H$ is a closed subgroup. □

We try to describe the topology of $\mathscr{A}^G$ for a linear algebraic group G in a simple way. For an integer $N>0$, let $M(N, \boldsymbol{C})$ denote the space of all square matrices of size N whose entries are complex numbers. We use the norm $\|x\| = \max_i \sum_j |x_{ij}|$ for $x=(x_{ij}) \in M(N, \boldsymbol{C})$. Let $M(N, \mathscr{A}^G)$ denote $\mathscr{A}^{M(N,\boldsymbol{C})}$ and let $GL(N, \mathscr{A}^{\boldsymbol{C}}) = \{g \in M(N, \mathscr{A}^{\boldsymbol{C}});\ \det g$ is an invertible element of $\mathscr{A}^{\boldsymbol{C}}\}$. Employ the topology of $M(N, \mathscr{A}^{\boldsymbol{C}})$ defined by $\| \ \|_R$. Then $GL(N, \mathscr{A}^{\boldsymbol{C}})$ is open in $M(N, \mathscr{A}^{\boldsymbol{C}})$ and becomes a topological group in a natural way.

LEMMA 1.3. *$\mathscr{A}^{GL(N,\boldsymbol{C})} = GL(N, \mathscr{A}^{\boldsymbol{C}})$ as topological groups.*

PROOF. Of course $\mathscr{A}^{GL(N,\boldsymbol{C})} = GL(N, \mathscr{A}^{\boldsymbol{C}})$ as abstract groups. For $X \in M(N, \mathscr{A}^{\boldsymbol{C}})$, $\|\exp X - 1\|_R \le \exp \|X\|_R - 1$. We put $\log Y = -\sum_{n>0} (1-Y)^n/n$ for $Y \in M(N, \mathscr{A}^{\boldsymbol{C}})$ so that $\|1-Y\|_R < 1$. Then $Y = \exp(\log Y)$. Hence the identity map: $\mathscr{A}^{GL(N,\boldsymbol{C})} \to GL(N, \mathscr{A}^{\boldsymbol{C}})$ is a homeomorphism. □

COROLLARY 1.4. *Let G be a linear algebraic group. Then $\mathscr{A}^G$ is a closed subgroup of $GL(N, \mathscr{A}^{\boldsymbol{C}})$ for a certain integer $N>0$.*

Next we refer to the Birkhoff decomposition. We begin with some notations. Let G be a linear algebraic group and be embedded in $GL(N, \boldsymbol{C})$. The complex projective line P^1 is $\boldsymbol{C} \cup \{\infty\}$, the one-point compactification of $\boldsymbol{C}$. Let $D_R^+ = \{s \in \boldsymbol{C};\ |s| \le R\}$ and $D_R^- =$ the closure of $P^1 \setminus D_R^+$. For an open subset U of P^1, we set $\mathscr{O}(U, G) =$ the space of all holomorphic mappings from U to G. $\{\mathscr{O}(U, G);\ U \supset D_R^+\}$ makes a direct system in a natural way. We denote its direct limit by $\mathscr{P}^G = \mathscr{P}_R^G$. For an open subset U which contains D_R^-, we set $\mathscr{N}(U, G) = \{g \in \mathscr{O}(U, G);\ g(\infty) = 1\}$. Similarly we define the direct limit $\mathscr{N}^G = \mathscr{N}_R^G$. We consider $\mathscr{P}^G$ and $\mathscr{N}^G$ as abstract subgroups of $\mathscr{A}^G$ through the inclusion maps $S_R \to D_R^\pm$.

LEMMA 1.5. *$\mathscr{N}^G$ and $\mathscr{P}^G$ are closed in $\mathscr{A}^G$.*

PROOF. Let G be a Zariski closed subgroup of $GL(N, \boldsymbol{C})$. For a formal Laurent series $g = \sum_{n \in \boldsymbol{Z}} g_n s^n$ with $g_n \in M(N, \boldsymbol{C})$, we set $g^+ = \sum_{n \ge 0} g_n s^n$ and $g^- = \sum_{n<0} g_n s^n$. If $f \in \mathscr{P}^G$ and $g \in \mathscr{A}^G$, then $\|f - g\|_R = \|f - g^+\|_R + \|g^-\|_R$. So if g lies in the closure of $\mathscr{P}^G$, then we have $g^- = 0$. That is to say, g is a holomorphic mapping from a neighborhood of D_R^+ to $M(N, \boldsymbol{C})$. Also we know that there exists $f \in \mathscr{P}^G$ such that $|\det f(s) - \det g(s)| < |\det g(s)|$ for all $s \in S_R$. By the argument principle,

the number of zero points of $\det g$ in $D_R^+ =$

the number of zero points of $\det f$ in $D_R^+ = 0$.

Hence $g(D_R^+)\subset GL(N, \boldsymbol{C})$. This implies that g belongs to $\mathscr{P}^G$. For $\mathscr{N}^G$, the same argument is valid. □

The following theorem plays an important role in our proof of the main result.

THEOREM 1.6. *The product map*: $\mathscr{N}^G\times\mathscr{P}^G\to\mathscr{A}^G$ *is an open embedding as topological spaces.*

PROOF. Step 1. The product map is injective.

For n, $m\in\mathscr{N}^G$ and p, $q\in\mathscr{P}^G$, we assume that $np=mq$. Then $n^{-1}m=pq^{-1}$ on S_R. So we can extend $n^{-1}m$ to a holomorphic mapping from P^1 to G. Therefore $n^{-1}m$ is a constant mapping. Notice that $n^{-1}m$ takes the value 1 at $s=\infty$. Hence $n=m$ and $p=q$.

Step 2. The theorem holds for $G=GL(N, \boldsymbol{C})$.

Let $f=\sum f_n s^n$ (resp. $F=\sum F_n s^n$) be a formal Laurent series whose coefficients are N-square complex matrices (resp. real numbers). If $\|f_n\|\le F_n$ for all $n\in\boldsymbol{Z}$, then we write $f\ll F$. Take $g\in\mathscr{A}^G$ such that $\|1-g\|_R<1/2$. We put $a=1-g$ and $A=A_s=\|a\|_s$ and note that $\|g^-\|_s\ll A$. Set $h\cdot a=(ha)^-$ for $h\in M(N, \mathscr{A}^{\boldsymbol{C}})$. We consider this as a right action of a.

(1.1) We set $f=-\sum_{m\ge 0} g^-(\cdot a)^m$. Then $f\ll A/(1-A)$ and $f=f_-$.

Since $\sum_{m\ge k} g^-(\cdot a)^m\ll AA^k$, we obtain $(fg)^-=f-f\cdot a=-g^-$. We set $c=1+f$. Then $cg=g^+ +(fg)^+$. Hence cg is a holomorphic function on a neighborhood of D_R^+.

$$(1.2)\qquad cg-1=g-1+fg\ll A+A(1+A)/(1-A)=2A/(1-A).$$

By (1.1) and (1.2) we can find a neighborhood $\mathscr{U}$ of 1 in $\mathscr{A}^G$ with the property: For any $g\in\mathscr{U}$, we can define $c\in M(N, \mathscr{A}^{\boldsymbol{C}})$ by the above process and then $|1-\det c(s)|<1$ and $|1-\det c(s)\det g(s)|<1$ for all $s\in S_R$.

For $g\in\mathscr{U}$, we set $g_{\mathrm{n}}=c^{-1}$ and $g_{\mathrm{p}}=cg$. Then (1.1) and (1.2) imply the following estimate.

$$(1.3)\qquad \|g_{\mathrm{p}}-1\|_R<2A_R/(1-A_R)\quad\text{and}\quad \|g_{\mathrm{n}}-1\|_R<A_R/(1-2A_R).$$

Clearly $g_{\mathrm{n}}\in\mathscr{N}^G$, $g_{\mathrm{p}}\in\mathscr{P}^G$ and $g=g_{\mathrm{n}}g_{\mathrm{p}}$. This shows that $\mathscr{U}$ is contained in $\mathscr{N}^G\mathscr{P}^G$. So $\mathscr{N}^G\mathscr{U}\mathscr{P}^G=\mathscr{N}^G\mathscr{P}^G$ and $\mathscr{N}^G\mathscr{P}^G$ is open in $\mathscr{A}^G$.

We prove that the inverse of the product map is continuous. By Step 1, for $g\in\mathscr{N}^G\mathscr{P}^G$, we set

$$g=g_{\mathrm{n}}g_{\mathrm{p}}\quad\text{with}\quad g_{\mathrm{n}}\in\mathscr{N}^G\quad\text{and}\quad g_{\mathrm{p}}\in\mathscr{P}^G.$$

Fix $h=h_{\mathrm{n}}h_{\mathrm{p}}\in\mathscr{N}^G\mathscr{P}^G$. For any $g\in\mathscr{N}^G\mathscr{P}^G$ sufficiently near h, we can assume that $h_{\mathrm{n}}^{-1}gh_{\mathrm{p}}^{-1}$ is in $\mathscr{U}$. Since $(h_{\mathrm{n}}^{-1}gh_{\mathrm{p}}^{-1})_{\mathrm{n}}=h_{\mathrm{n}}^{-1}g_{\mathrm{n}}$ and $(h_{\mathrm{n}}^{-1}gh_{\mathrm{p}}^{-1})_{\mathrm{p}}=g_{\mathrm{p}}h_{\mathrm{p}}^{-1}$, we have $h_{\mathrm{n}}-g_{\mathrm{n}}=h_{\mathrm{n}}(1-(h_{\mathrm{n}}^{-1}gh_{\mathrm{p}}^{-1})_{\mathrm{n}})$ and $h_{\mathrm{p}}-g_{\mathrm{p}}=(1-(h_{\mathrm{n}}^{-1}gh_{\mathrm{p}}^{-1})_{\mathrm{p}})h_{\mathrm{p}}$. Thus (1.3) completes Step 2.

Let G be a Zariski closed subgroup of $GL=GL(N, \boldsymbol{C})$. Then the following diagram is commutative.

$$\begin{array}{ccc} \mathscr{N}^G\mathscr{P}^G & \xrightarrow{\text{the inclusion}} & \mathscr{N}^{GL}\mathscr{P}^{GL} \\ \big\downarrow & & \big\downarrow \text{the inverse map} \\ \mathscr{N}^G\times\mathscr{P}^G & \longrightarrow & \mathscr{N}^{GL}\times\mathscr{P}^{GL} \end{array}$$

So the product map $\mathscr{N}^G\times\mathscr{P}^G\to\mathscr{A}^G$ is an into-homeomorphism. The theorem is proved if we show that $\mathscr{N}^G\mathscr{P}^G$ is open in $\mathscr{A}^G$.

Step 3. If G has no nontrivial rational character, the theorem is true.

It is enough to show that $\mathscr{N}^{GL}\mathscr{P}^{GL}\cap\mathscr{A}^G$ is contained in $\mathscr{N}^G\mathscr{P}^G$. From the assumption, it follows that there exist a rational GL-module V and an element v of V such that $G=\{k\in GL;\ kv=v\}$. If $n\in\mathscr{N}^{GL}$, $p\in\mathscr{P}^{GL}$ and $np\in\mathscr{A}^G$, then we have $p(s)v=n(s)^{-1}v$ for all $s\in S_R$. Hence $p(s)v$ extends to a holomorphic map from P^1 to V. Since $n(\infty)=1$, we have $pv=n^{-1}v=v$. Thus n and p belong to $\mathscr{A}^G$.

Step 4. Let $1\to F\to G\to H\to 1$ be an exact sequence of linear algebraic groups. If the theorem is true for F and H, then it is also true for G.

F is a Zariski closed normal subgroup of G and H is the quotient group G/F. Let κ be the canonical surjection $G\to G/F$. Let $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of F, G and H respectively. We understand $\mathfrak{g}=\mathfrak{f}+\mathfrak{h}$ as vector spaces. Fix a basis of $\mathfrak{g}$ which is a union of bases of $\mathfrak{f}$ and $\mathfrak{h}$. We define a norm $\|\ \|$ on $\mathfrak{g}$ as above. A real number $c>0$ is taken so as to satisfy the following conditions:

(i) The exponential maps $D_c^{\mathfrak{g}}\to G$, $D_c^{\mathfrak{f}}\to F$ and $D_c^{\mathfrak{h}}\to H$ are into-homeomorphisms.

(ii) $F\cap\exp D_c^{\mathfrak{g}}=\exp D_c^{\mathfrak{f}}$.

(iii) $\exp\mathscr{D}_{\jmath}^{\mathfrak{f}}\subset\mathscr{N}^F\mathscr{P}^F$ and $\exp\mathscr{D}_c^{\mathfrak{h}}\subset\mathscr{N}^H\mathscr{P}^H$.

For (iii), we use the assumption of Step 4. Next we take $b>0$ such that $b<c$ and the followings hold:

(iv) We set $j(\kappa(\exp X))=\exp X$ for $X\in D_b^{\mathfrak{h}}$. Then j is a local cross section from the subset $\kappa(\exp D_b^{\mathfrak{h}})$ of H to the subset $\exp D_b^{\mathfrak{g}}$ of G.

(v) $(\exp\mathscr{D}_b^{\mathfrak{g}})(\exp\mathscr{D}_b^{\mathfrak{g}})(\exp\mathscr{D}_b^{\mathfrak{g}})\cap\mathscr{A}^F$ is contained in $\mathscr{A}^F\mathscr{P}^F$.

In addition, we choose $0<a<b$ such that $(\exp\mathscr{D}_a^{\mathfrak{h}})_{\mathrm{n}}$ and $(\exp\mathscr{D}_a^{\mathfrak{h}})_{\mathrm{p}}$ are contained in $\exp\mathscr{D}_a^{\mathfrak{h}}$. For $g\in\exp\mathscr{D}_a^{\mathfrak{g}}$, we set $h=\kappa(g)$, $w=j(h_{\mathrm{n}})$ and $x=j(h_{\mathrm{p}})$. If a is sufficiently small, we can assume that $h_{\mathrm{p}}(D_R^+)$ and $h_{\mathrm{n}}(D_R^-)$ are contained in $\exp D_b^{\mathfrak{h}}$. Then $w\in\mathscr{N}^G\cap\exp\mathscr{D}_b^{\mathfrak{g}}$ and $x\in\mathscr{P}^G\cap\exp\mathscr{D}_b^{\mathfrak{g}}$. By (v), we write $(wx)^{-1}g=yz$ with $y\in\mathscr{N}^F$ and $z\in\mathscr{P}^F$. Note that $g=w(xyx^{-1})xz$, $xyx^{-1}\in\mathscr{A}^F$ and $xz\in\mathscr{P}^G$.

Then "$g \to xyx^{-1}$" defines a continuous map from $\exp \mathscr{D}_a^{\mathfrak{g}}$ to $\mathscr{A}^F$, which sends 1 to 1. Therefore if $a>0$ is sufficiently small, this map sends $\exp \mathscr{D}_a^{\mathfrak{g}}$ into $\mathscr{N}^F \mathscr{P}^F$. Thus $\exp \mathscr{D}_a^{\mathfrak{g}}$ is contained in $\mathscr{N}^G \mathscr{P}^G$.

Step 5. We recall basic exact sequences for linear algebraic groups.

$$1 \longrightarrow [\text{alg. torus}] \longrightarrow [\text{reductive gr.}] \longrightarrow [\text{semisimple gr.}] \longrightarrow 1.$$

$$1 \longrightarrow [\text{unipotent gr.}] \longrightarrow [\text{linear alg. gr.}] \longrightarrow [\text{reductive gr.}] \longrightarrow 1.$$

We can apply Steps 3 and 4 to these sequences. □

COROLLARY 1.7 (cf. Shubin [16]). *Let U be a neighborhood of S_R, M be an analytic manifold and Y be an analytic map $U \times M \to G$. If $Y(\cdot, m)$ belongs to $\mathscr{N}^G \mathscr{P}^G$ for any $m \in M$, then $Y(\cdot,\cdot)_{\mathrm{n}}$ and $Y(\cdot,\cdot)_{\mathrm{p}}$ are analytic on $U \times M$.*

COROLLARY 1.8. *Let χ be a rational character of G and g be an element of $\mathscr{A}^G$ so that $\chi(g)=1$. For $h \in \mathscr{A}^G$ and $Y, Z \in \mathscr{N}^G$, if ghY equals hZ in $\mathscr{A}^G/\mathscr{P}^G$, then we have $\chi(Y)=\chi(Z)$.*

PROOF. Take an element p of $\mathscr{P}^G$ such that $ghY=hZp$ in $\mathscr{A}^G$. Then $\chi(ghY)=\chi(h)\chi(Y)=\chi(h)\chi(Z)\chi(p)$. Hence $\chi(Z)^{-1}\chi(Y)$ defines a holomorphic function on P^1. Clearly this function equals 1. □

In the remainder of this section we treat $\mathscr{A}^G$ in a geometric method. First we explain a basic device. We take affine open subsets of $P^1 = \boldsymbol{C} \cup \{\infty\}$ as follows: $U_0 = P^1 \smallsetminus \{\infty\}$ and $U_1 = P^1 \smallsetminus \{0\}$. The unit circle S_1 is contained in $U_0 \cap U_1 = \boldsymbol{C} \smallsetminus \{0\}$. $\mathscr{A}^G$ is to be defined for S_1. For an element g of $\mathscr{A}^G$, we assign a holomorphic principal G-boundle over P^1 in the following way: Choose open subsets $W_0 \subset U_0$ and $W_1 \subset U_1$ such that $P^1 = W_0 \cup W_1$ and g is defined on $W_0 \cap W_1$. The assigned bundle is to be $P_g = (W_0 \times G \cup W_1 \times G)/\sim$, where "$\sim$" is the glueing so that for $(w_0, a_0) \in W_0 \times G$ and $(w_1, a_1) \in W_1 \times G$, $(w_0, a_0) \sim (w_1, a_1)$ means that $w_0 = w_1$ in $W_0 \cap W_1 \subset W_0$ and $a_1 = g(w_0)a_0$.

For example, "$G = \boldsymbol{C}^\times$, $g = g(s) = s$" is assigned to the associated principal bundle of the tautological line bundle. Here "s" denotes the standard coordinate of $\boldsymbol{C}^\times$.

We notice that for $g, h \in \mathscr{A}^G$, P_g and P_h are isomorphic as holomorphic G-bundles iff g lies in $G\mathscr{N}^G h \mathscr{P}^G$. The set of all isomorphic classes of holomorphic principal G-bundles over P^1 is denoted by $H^1(P^1, G)$. Let G be a connected reductive linear algebraic group over $\boldsymbol{C}$ and T be a maximal torus of G. $W(T)$ denotes the Weyl group of G associated with T. An element of $H^1(P^1, T)$ is represented by a 1-parameter subgroup of T through the above assignment $\mathscr{A}^T \to H^1(P^1, T)$. In fact, the abelian group $\mathrm{Hom}(\boldsymbol{C}^\times, T)$ is isomorphic to $H^1(P^1, T)$. So the Weyl group $W(T)$ acts on $H^1(P^1, T)$ naturally. Grothendieck's classification theorem shows that the canonical map $H^1(P^1, T)/W(T) \to H^1(P^1, G)$

is bijective [5, Théorème 1.1, Corollaire 1]. Therefore, any $g \in \mathscr{A}^G$ is written in the form $g=qhp$ with $q \in G\mathscr{N}^G$, $h \in \mathrm{Hom}(\boldsymbol{C}^\times, T)$ and $p \in \mathscr{P}^G$. We call $g=qhp$ a Birkhoff decomposition of g. Here we apply Grothendieck's theorem to a connected complex semisimple Lie group G of adjoint type, *i.e.* G with a trivial center. We denote the root system of the pair (G, T) by $R=R(G, T)$. It is well known that $\mathrm{Hom}(T, \boldsymbol{C}^\times)$ is a free abelian group with a basis which consists of simple roots. All coroots of R make a root system $R^\vee$. Through the natural dual pairing $\mathrm{Hom}(T, \boldsymbol{C}^\times)\times \mathrm{Hom}(\boldsymbol{C}^\times, T)\to \boldsymbol{Z}$, we identify $\mathrm{Hom}(\boldsymbol{C}^\times, T)$ with the weight lattice of $R^\vee$, which is performed in the abstract root theory. We call elements of this lattice "coweights". Using these terminologies, we have for a connected complex semisimple Lie group G of adjoint type,

$$H^1(P^1, G) = \{\text{the dominant coweights of } R(G, T)\} \text{ as sets.}$$

Next we give another specialization of Grothendieck's theorem (cf. Birkhoff [1]).

$$H^1(P^1, GL(N, \boldsymbol{C})) = \boldsymbol{Z}^{(N)} = \{(m_1,\dots, m_N)\in \boldsymbol{Z}^N;\ m_1\geq\cdots\geq m_N\}$$

as sets. Note that an N-tuple $(m_1,\dots, m_N)$ represents a 1-parameter subgroup $\mathrm{diag}(s^{m_1},\dots, s^{m_N})$ for $s\in \boldsymbol{C}^\times$.

DEFINITION 1.9. Let P and Q be holomorphic principal bundles over a complex manifold. If there exists a holomorphic 1-parameter family of principal bundles $\{P_t;\ t$ in a nonempty open subset of $\boldsymbol{C}\}$ such that $P_t=P$ except for a finite number of t and $P_{t_0}=Q$ for some t_0, then we call Q an elementary deformation of P. If there exists a chain of elementary deformations from P to Q, we say that Q is a deformation of P and write $P\geq Q$.

In $H^1(P^1, GL(N, \boldsymbol{C}))$, the partial ordering "$\geq$" is described as combinatorics. Gohberg and Krein introduced an ordering on $\boldsymbol{Z}^{(N)}$: For $\boldsymbol{m}=(m_i)$ and $\boldsymbol{k}=(k_i)\in \boldsymbol{Z}^{(N)}$,

$$\boldsymbol{m}\leq\boldsymbol{k} \quad \text{iff} \quad \textstyle\sum_i m_i = \sum_i k_i \quad \text{and}$$

$$\textstyle\sum_{m_i\geq p}(m_i-p) \leq \sum_{k_i\geq p}(k_i-p) \qquad \text{for any} \quad p\in \boldsymbol{Z}.$$

LEMMA 1.10. *For any $\boldsymbol{m}\in \boldsymbol{Z}^{(N)}$, Let $P_{\boldsymbol{m}}$ denote the assigned principal $GL(N, \boldsymbol{C})$-bundle. Then for $\boldsymbol{m}$ and $\boldsymbol{k}\in \boldsymbol{Z}^{(N)}$, $P_{\boldsymbol{m}}\geq P_{\boldsymbol{k}}$ iff $\boldsymbol{m}\leq\boldsymbol{k}$.*

PROOF. The "only if" part is an immediate consequence of Gohberg and Krein [4]. But there is a simple explanation in our situation. Let $E_{\boldsymbol{m}}$ be the associated $\boldsymbol{C}^N$-bundle to $P_{\boldsymbol{m}}$ and $E^*_{\boldsymbol{m}}$ be its dual bundle. Let H be the hyperplane bundle over P^1. The tautological line bundle L is the dual bundle of H. For any $p\in \boldsymbol{Z}$, put $E^*_{\boldsymbol{m}}(-p)=E^*_{\boldsymbol{m}}\otimes L^p=\sum_i H^{m_i-p}$ (a direct sum). Then the dimension of the 0-th cohomology group $H^0(P^1, E^*_{\boldsymbol{m}}(-p))$ is $\sum_{m_i\geq p}(m_i-p+1)$. Hence if P_k is a deformation of P_m, then by the semicontinuity of the cohomology, we know that $\sum_{m_i \geq p}(m_i-p+1) \leq \sum_{k_i \geq p}(k_i-p+1)$ for any $p \in Z$. This implies that $\sum_{m_i>q}(m_i-q) \leq \sum_{k_i>q}(k_i-q)$ for any $q \in Z$. Also we notice that the Chern number $c_1(E_m^*)=\sum_i m_i$ is a topological invariant.

Next we prove the "if" part. Suppose that $m \neq k$. Tensoring L^p for some large interger p, we can assume that all m_i and k_i are strictly positive. An elementary argument of combinatorics shows that there exist $n=(n_i) \in Z^{(N)}$ and integers $1 \leq a < b \leq N$ such that $n_i=k_i$ for $i \neq a, b$, $n_a=k_a-1$, $n_b=k_b+1$ and $m \leq n < k$. So it is enough to give a proof for $N=2$. For $t \neq \pm 1 \in C$ and an integer $r>1$, we set

$$g_t = \begin{bmatrix} s^r & ts^{r-1} \\ ts & 1 \end{bmatrix} \text{and}$$

$$h_t = \begin{bmatrix} 1 & 0 \\ -t^{-1}s^{-r+1} & 1 \end{bmatrix} g_t = \begin{bmatrix} s^r & ts^{r-1} \\ (t-t^{-1})s & 0 \end{bmatrix}.$$

Let E_t be the C^2-bundle defined by g_t. We remark that h_t defines the same bundle as E_t for $t \neq 0$.

$$\text{For } f_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad f_1 = \begin{bmatrix} t \\ 0 \end{bmatrix}, \quad h_t f_0 = s^{r-1} f_1 .$$

$$\text{For } f_0' = \begin{bmatrix} 1 \\ -t^{-1}s \end{bmatrix} \quad \text{and} \quad f_1' = \begin{bmatrix} 0 \\ t-t^{-1} \end{bmatrix}, \quad h_t f_0' = s f_1' .$$

Therefore $E_t = L^{r-1}+L$ for $t \neq 0, \pm 1$. Clearly $E_0=L^r+L^0$ and $(r-1, 1) \leq (r, 0)$ in $Z^{(2)}$. As seen above, L^r+L^0 is an elementary deformation of $L^{r-1}+L$. □

Let $\{g_t\}$ be a holomorphic family of elements of $\mathscr{A}^G$ and $g_t=q_t h_t p_t$ be a Birkhoff decomposition. If q_t and p_t are holomorphic in t, then h_t is constant in t, as h_t lies in a discrete space $\mathrm{Hom}(C^{\times}, T)$. Hence g_t's are assigned to the same principal bundle. The converse of this observation is also true for local parameters, cf. Röhrl [15, Lemma 4.1, Corollary 2]. From this point of view, it is an interesting problem to ask whether a holomorphic family of elements of $\mathscr{A}^G$ defines isomorphic bundles.

DEFINITION 1.11. An element g of $\mathscr{A}^G$ is said to be rigid if there exists a real number $\varepsilon>0$ such that any element of $\{h \in \mathscr{A}^G;\ |g(s)-h(s)|<\varepsilon$ for all $s \in S_1\}$ defines the same holomorphic principal bundle as g does. The bundle defined by such g is called rigid. We denote the set of all isomorphic classes of rigid principal G-bundles by $\mathscr{R}(G)$.

We will list up all rigid bundles for connected simple linear algebraic groups over $\boldsymbol{C}$. For that purpose the next lemma is useful.

LEMMA 1.12. *Let $G \to G'$ be a covering homomorphism of complex Lie groups. For an element g_0 of $\mathscr{A}^G$, we denote its projection to $\mathscr{A}^{G'}$ by g_0'. Then g_0 is rigid iff g_0' is rigid.*

PROOF. In the category of topoloical spaces, the natural map $\mathrm{Map}(S_1, G) \to \mathrm{Map}(S_1, G')$ is a local homeomorphism. Let $g_0 \in \mathscr{A}^G$ be rigid and $h' \in \mathscr{A}^{G'}$ be sufficiently near g_0'. Here we use "near" with respect to the compact open topology. Let $h \in \mathrm{Map}(S_1, G)$ be a lift of h' sufficiently near g_0. Then $h \in \mathscr{A}^G$. Hence there exist $n \in G\mathscr{N}^G$ and $p \in \mathscr{P}^G$ such that $ng_0p = h$. Clearly $n'g_0'p' = h'$. So g_0' is rigid.

Conversely we assume that g_0' is rigid. Let $h \in \mathscr{A}^G$ be sufficiently near g_0. Then the projection h' of h is sufficiently near g_0'. By the assumption, we can write $h' = n'g_0'p'$ for some $n' \in G'\mathscr{N}^{G'}$ and $p' \in \mathscr{P}^{G'}$. Since n' is a map from a disk to G', n' has a lift $n \in G\mathscr{N}^G$. Similarly p' has a lift $p \in \mathscr{P}^G$. Then $(ng_0p)' = h'$. So there exists $n_0 \in G$ such that $n_0ng_0p = h$. Thus g_0 is rigid. □

PROPOSITION 1.13. (i) *An element $\boldsymbol{m}$ of $\boldsymbol{Z}^{(N)}$ defines a rigid principal $GL(N, \boldsymbol{C})$-bundle iff $\boldsymbol{m}$ is of the form $(m_1, \dots, m_1, m_1 - 1, \dots, m_1 - 1)$.*

(ii) *Let $G = PGL(N, \boldsymbol{C}) = GL(N, \boldsymbol{C})/\boldsymbol{C}^\times$. Fix a maximal torus T_N of $GL(N, \boldsymbol{C})$ consisting of diagonal matrices. Then $\{\mathrm{diag}(s, \dots, s, 1, \dots, 1) \in \mathrm{Hom}(\boldsymbol{C}^\times, T_N/\boldsymbol{C}^\times); 0 \leq \text{the number of } s < N\}$ is exactly assigned to $\mathscr{R}(G)$.*

PROOF. (i) is an immediate consequence fo Lemma 1.10 and Gohberg and Krein [4].

(ii) We set $Y = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \in GL(2, \boldsymbol{C}) \right\}$. Then the canonical projection $Y \to PGL(2, \boldsymbol{C})$ is an open immersion. Hence for an integer $r \geq 0$, a 1-parameter subgroup $\mathrm{diag}(s^r, 1)$ is assigned to a rigid principal $GL(2, \boldsymbol{C})$-bundle iff $r = 0$ or 1.

Let $g = \mathrm{diag}(s^{m_1}, \dots, s^{m_{N-1}}, 1) \in \mathrm{Hom}(\boldsymbol{C}^\times, T_N/\boldsymbol{C}^\times)$ be assigned to a rigid principal G-bundle. Then $x = \mathrm{diag}(s^{m_1}, s^{m_2}) \in \mathrm{Hom}(\boldsymbol{C}^\times, GL(2, \boldsymbol{C}))$ corresponds to a rigid principal $GL(2, \boldsymbol{C})$-bundle. Otherwise, we have an element h of $\mathscr{A}^{GL(2,\boldsymbol{C})}$ which is sufficiently near x and assigned to a different bundle from P_x. Let $h = n\,\mathrm{diag}(s^{k_1}, s^{k_2})p$ be a Birkhoff decomposition. Then $\{k_1, k_2\} \neq \{m_1, m_2\}$ and $h \times \mathrm{diag}(s^{m_3}, \dots, s^{m_{N-1}}, 1) \in \mathrm{Hom}(\boldsymbol{C}^\times, T_N/\boldsymbol{C}^\times)$ is sufficiently near g. Clearly this leads to a contradiction. Similarly we see that $\mathrm{diag}(s^{m_i}, s^{m_j})$ is rigid for all $1 \leq i < j \leq N$. Here we understand $m_N = 0$. Thus $|m_i - m_j| = 0$ or 1. If we assume that $m_1 \geq \cdots \geq m_{N-1} \geq 0$, then g gets the form stated in the proposition.

Let $g = \mathrm{diag}(s, \dots, s, 1, \dots, 1)$. Since $g(s)$ lies in an affine open subset of $PGL(N, \boldsymbol{C})$, we can treat g as an element of $\mathscr{A}^{GL(N,\boldsymbol{C})}$. Evidently g is rigid in

$\mathscr{A}^{GL(N,C)}$ and so in $\mathscr{A}^{PGL(N,C)}$. $\square$

COROLLARY 1.14. *Let G be the c-fold covering of $PGL(N, \boldsymbol{C})$. Then* $\{\mathrm{diag}(s,\dots, s, 1,\dots, 1)\in \mathrm{Hom}(\boldsymbol{C}^{\times}, T_N/\boldsymbol{C}^{\times})$; *the number of s is smaller than N and devided by $c\}$ naturally represents $\mathscr{R}(G)$.*

PROOF. Using Lemma 1.12, we shall find rigid 1-parameter subgroups of $PGL(N, \boldsymbol{C})$ such that they have lifts in $\mathrm{Hom}(\boldsymbol{C}^{\times}, G)$. Let $g=\mathrm{diag}(s,\dots, s, 1,\dots, 1)$ be a 1-parameter subgroup of $GL(N, \boldsymbol{C})$. We assume that $\det g=s^a$ for some $0<a<N$. Let $y=y(s)$ be a branch of $s^{a/N}$ such that $y(1)=1$. Then gy^{-1} defines a 1-parameter subgroup of G iff $y(e^{2\pi i})=e^{2\pi i a/N}$ equals $(e^{2\pi i c/N})^k$ for some integer $0<k<N/c$. This condition implies that $a=ck$. $\square$

LEMMA 1.15. *For $SO(4, \boldsymbol{C})$, there exists only one isomorphic class of nontrivial rigid principal bundles. Let $T=SO(2, \boldsymbol{C})\times SO(2, \boldsymbol{C})$ be a maximal torus of $SO(4, \boldsymbol{C})$ and f be the standard generator*

$$\begin{bmatrix} (s+s^{-1})/2 & -(s-s^{-1})/2i \\ (s-s^{-1})/2i & (s+s^{-1})/2 \end{bmatrix} \quad \text{of} \quad \mathrm{Hom}(\boldsymbol{C}^{\times}, SO(2, \boldsymbol{C})).$$

Then the element $1_2\times f$ of $\mathrm{Hom}(\boldsymbol{C}^{\times}, T)$ *corresponds to the nontrivial rigid bundle.*

PROOF. We have a 2-fold covering $SO(4, \boldsymbol{C})\to SO(3, \boldsymbol{C})\times SO(3, \boldsymbol{C})$. Hence we pick up rigid $SO(3, \boldsymbol{C})\times SO(3, \boldsymbol{C})$-bundles which lift to $SO(4, \boldsymbol{C})$-bundles. The possibility of lifting of loops can be judged by their restrictions on compact real forms. So we work on two coverings of compact Lie groups

$$Spin(4) = Sp(1) \times Sp(1) \xrightarrow[sp]{} SO(4) \longrightarrow SO(3) \times SO(3).$$

Here sp denotes the spin representation. Let $\boldsymbol{H}$ be the field of quaternion numbers and $Sp(1)=\{a\in \boldsymbol{H}$; the norm of a is $1\}$. For $x\in \boldsymbol{R}^4=\boldsymbol{H}$ and $a, b\in Sp(1)$, we set $sp(a\times b)x=axb^{-1}$. Considering $SO(3)=Sp(1)/\{\pm 1\}$, we have naturally the second covering. Take the standard basis $\{1, i, j, k\}$ of $\boldsymbol{H}$. Using this, we get a representation of $sp(a\times b)$ by a matrix.

We notice that a 1-parameter subgroup of $SO(3)\times SO(3)$ is described with a branched lift to $Sp(1)\times Sp(1)$. For $s=e^{i\theta}$, the restrictions of rigid 1-parameter subgroups of $SO(3, \boldsymbol{C})\times SO(3, \boldsymbol{C})$ are 1×1, $1\times s^{1/2}$, $s^{1/2}\times 1$ and $s^{1/2}\times s^{1/2}$. Also

$$f = f(e^{i\theta}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Their images by sp are $1_2\times 1_2$, $f^{-1/2}\times f^{1/2}$, $f^{1/2}\times f^{-1/2}$ and $1_2\times f$. Clearly $1_2\times 1_2$

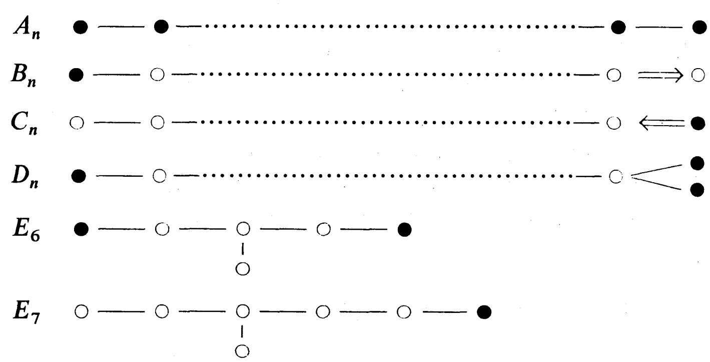
and $1_2 \times f$ are loops in $SO(4)$ and the others are not. □

PROPOSITION 1.16. *For $SO(N, \boldsymbol{C})$, $N \geq 3$, we have a unique isomorphic class of nontrivial rigid principal bundles, which is defined by the 1-parameter subgroup $f \times 1_{N-2}$. Here f is the standard generator of* Hom $(\boldsymbol{C}^\times, SO(2, \boldsymbol{C}))$.

PROOF. For $N=3$ and 4, the proposition has been shown, and hence we assume $N \geq 5$. Let k be the integral part of $N/2$ and $T = SO(2, \boldsymbol{C}) \times \cdots \times SO(2, \boldsymbol{C})$ (k pieces) be a maximal torus of $SO(N, \boldsymbol{C})$. Suppose that a 1-parameter subgroup $g = f^{m_1} \times \cdots \times f^{m_k}$ of T defines a nontrivial rigid $SO(N, \boldsymbol{C})$-bundle. $SO(4, \boldsymbol{C}) \times SO(N-4, \boldsymbol{C})$ is a subgroup of $SO(N, \boldsymbol{C})$, which has the maximal torus T in common. Hence $f^{m_1} \times f^{m_2}$ is a rigid 1-parameter subgroup of $SO(4, \boldsymbol{C})$. So $\{m_1, m_2\} = \{0, 0\}$, $\{1, 0\}$ or $\{0, -1\}$. The others $\{m_i, m_j\}$ are similar. With the action of the Weyl group, we can assume that $m_1 \geq \cdots \geq m_k$. Then g equals $f \times 1_{N-2}$ or $1_{N-2} \times f^{-1}$, each of which defines the same bundle.

Next we prove that $g = f \times 1_{N-2}$ defines a rigid bundle. Let $\{g_t\}$ be a holomorphic family of elements in $\mathscr{A}^{SO(N, \boldsymbol{C})}$ such that $g_0 = g$. As g_t is an element of $\mathscr{A}^{GL(N, \boldsymbol{C})}$, it defines a $\boldsymbol{C}^N$-bundle E_t. We notice that $E_0 = L + L^{-1} + \boldsymbol{C}^{N-2}$ (a direct sum). Let t be sufficiently near 0. By Lemma 1.10, we known that E_t equals E_0 or the trivial bundle. Evidently g_t is nontrivial in $\pi_1 SO(N, \boldsymbol{C})$. So $E_t = E_0$. That is to say, g_t's define the same $GL(N, \boldsymbol{C})$-bundle. Therefore, from Grothendieck [5, Proposition 3.1], it follows that g_t's define the same $O(N, \boldsymbol{C})$-bundle. Let $g_t = n_t g p_t$ be a Birkhoff decomposition in $\mathscr{A}^{O(N, \boldsymbol{C})}$. Note that $(1_{N-1} \times (-1)) g (1_{N-1} \times (-1)) = g$. Hence we can take n_t and p_t in $\mathscr{A}^{SO(N, \boldsymbol{C})}$. □

PROPOSITION 1.17. *Let G be a connected complex simple Lie group of adjoint type. Then $\mathscr{R}(G)$ consists of the trivial bundle and the bundles defined by the coweights corresponding to the roots ● in the Dynkin diagram. Here the word "bundle" means a holomorphic principal bundle over a complex projective line. There exists no root ● in E_8, F_4 and G_2. The Dynkin diagrams of A_n, B_n, C_n, E_6 and E_7 are as follows.*



PROOF. $G=A_n=PGL(n+1, \boldsymbol{C})$: We can choose simple roots such that diag$(s,\dots, s, 1,\dots, 1)$'s in Proposition 1.13 are made into fundamental coweights.

$G=B_n=SO(2n+1, \boldsymbol{C})$: Let h be the nontrivial rigid 1-parameter subgroup given in Proposition 1.16. If we take a suitable simple system of roots $\{a_1,\dots, a_n\}$, then its Dynkin diagram is $\underset{a_1}{\circ}$ —— $\circ$ ——$\cdots$, $\langle a_1, h\rangle=1$ and $\langle a_i, h\rangle=0$ for $i\geq 2$.

$G=C_n=Sp(n, \boldsymbol{C})/\{\pm 1\}$: Let T' be a maximal torus of $Sp(n, \boldsymbol{C})$ consisting of diagonal matrices. Naturally $Sp(1, \boldsymbol{C})\times Sp(n-1, \boldsymbol{C})$ is considered as a subgroup of $Sp(n, \boldsymbol{C})$, which contains T'. An element of Hom $(\boldsymbol{C}^\times, T')$ is factorized into a 1-parameter subgroup of $Sp(1, \boldsymbol{C})$ and that of $Sp(n-1, \boldsymbol{C})$. Hence an inductive argument shows that a rigid $Sp(n, \boldsymbol{C})$-bundle is trivial. Note that $Sp(n-1, \boldsymbol{C})=(1_2\times Sp(n-1, \boldsymbol{C}))/\{\pm 1\}$. We now get an exact sequence:

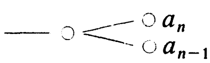
$$1 \longrightarrow Sp(n-1, \boldsymbol{C}) \longrightarrow (Sp(1, \boldsymbol{C})\times Sp(n-1, \boldsymbol{C}))/\{\pm 1\} \longrightarrow PSp(1, \boldsymbol{C}) \longrightarrow 1.$$

Let $T=T'/\{\pm 1\}$. Elements of Hom$(\boldsymbol{C}^\times, T)$ are represented by their branched lifts to $Sp(n, \boldsymbol{C})$. Let $g\in$ Hom$(\boldsymbol{C}^\times, T)$ be rigid and x be the projection of g to $PSp(1, \boldsymbol{C})$. If x has a lift $x'\in$ Hom$(\boldsymbol{C}^\times, Sp(1, \boldsymbol{C}))$, then we have $(x'\times 1_{2n-2})^{-1}g=1_2\times y$, where y is an element of Hom$(\boldsymbol{C}^\times, Sp(n-1, \boldsymbol{C}))$. Since g has a lift in Hom$(\boldsymbol{C}^\times, Sp(n, \boldsymbol{C}))$, g defines the trivial bundle by Lemma 1.12. We now assume that x has no lift in Hom$(\boldsymbol{C}^\times, Sp(1, \boldsymbol{C}))$. Treat g as a branched lift to $Sp(n, \boldsymbol{C})$ and factorize g into $x\times y$, where y is a branched lift of an element of Hom$(\boldsymbol{C}^\times, PSp(n-1, \boldsymbol{C}))$. Then $g(e^{2\pi i})=-1$ since g is nontrivial. We put $h=$diag$(s, s^{-1})\in$ Hom$(\boldsymbol{C}^\times, SL(2, \boldsymbol{C}))$ and understand $Sp(1, \boldsymbol{C})=SL(2, \boldsymbol{C})$. Let $x=n^{-1}h^{m/2}b^{-1}$ be a Birkhoff decomposition in $\mathscr{A}^{PSp(1,\boldsymbol{C})}$. We notice that n and b are lifted to $Sp(1, \boldsymbol{C})$. If n and b also denote their lifts, then the following is well defined: $nxb=h^{m/2}$ in $Sp(1, \boldsymbol{C})$. Hence $(n\times 1_{2n-2})g(b\times 1_{2n-2})=h^{m/2}\times y$. This implies that x is a nontrivial rigid 1-parameter subgroup of $PSp(1, \boldsymbol{C})$. So we can assume that $g=h^{1/2}\times\cdots\times h^{1/2}$ (n pieces). Clearly $h^{1/2}=$diag$(s, 1)$ in $PGL(2, \boldsymbol{C})$. Hence $h^{1/2}\times\cdots\times h^{1/2}$ is rigid in $\mathscr{A}^{PGL(2n,\boldsymbol{C})}$ and so in $\mathscr{A}^{PSp(n,\boldsymbol{C})}$. It is easy to find a simple system $\{a_1,\dots, a_n\}$ for $R(PSp(n, \boldsymbol{C}), T)$ such that its Dynkin diagram is $\cdots \circ \Longleftarrow \underset{a_n}{\circ}$, $\langle a_n, g\rangle=1$ and $\langle a_i, g\rangle=0$ for $i<n$.

$G=D_n=PSO(2n, \boldsymbol{C})/\{\pm 1\}$: Let $T'=SO(2, \boldsymbol{C})\times\cdots\times SO(2, \boldsymbol{C})$ be a maximal torus of $SO(2n, \boldsymbol{C})$ and $T=T'/\{\pm 1\}$. An element of Hom$(\boldsymbol{C}^\times, T)$ is written with its branched lift to $SO(2n, \boldsymbol{C})$. Let f be the standard generator of Hom$(\boldsymbol{C}^\times, SO(2, \boldsymbol{C}))$. Then $g\in$ Hom$(\boldsymbol{C}^\times, T)$ has the form $f^{m_1/2}\times\cdots\times f^{m_n/2}$. Taking the action of the Weyl group into account, we may assume that $m_1\geq\cdots\geq m_{n-1}\geq |m_n|$. Note that $SO(2n-4, \boldsymbol{C})=(1_4\times SO(2n-4, \boldsymbol{C}))/\{\pm 1\}$ and observe the next exact sequence:

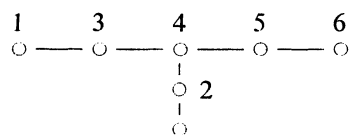
$$1 \longrightarrow SO(2n-4, \boldsymbol{C}) \longrightarrow (SO(4, \boldsymbol{C})\times SO(2n-4, \boldsymbol{C}))/\{\pm 1\} \longrightarrow PSO(4, \boldsymbol{C}) \longrightarrow 1.$$

Let g be a nontrivial rigid element of $\mathrm{Hom}(\boldsymbol{C}^{\times}, T)$ and $x\times y$ be a branched lift of g to $SO(4, \boldsymbol{C})\times SO(2n-4, \boldsymbol{C})$. If x lies in $\mathrm{Hom}(\boldsymbol{C}^{\times}, SO(4, \boldsymbol{C}))$, then $g=(x\times 1_{2n-4})(1_4\times y)$ belongs to $\mathrm{Hom}(\boldsymbol{C}^{\times}, SO(2n, \boldsymbol{C}))$. Therefore $g=f\times 1_2\times\cdots\times 1_2$. Moreover, for a simple system $\{a_1,\dots, a_n\}$ of $R(PSO(2n, \boldsymbol{C}), T)$, its Dynkin diagram is $\underset{a_1}{\circ}$ —— ○ ——$\cdots$, $\langle a_1, g\rangle=1$ and $\langle a_i, g\rangle=0$ for $i>1$. If x does not lie in $\mathrm{Hom}(\boldsymbol{C}^{\times}, SO(4, \boldsymbol{C}))$, then x is a nontrivial rigid 1-parameter subgroup of $PSO(4, \boldsymbol{C})$. Therefore x is $f^{1/2}\times f^{1/2}$ or $f^{1/2}\times f^{-1/2}$. Hence g equals $g_n=f^{1/2}\times\cdots\times f^{1/2}$ or $g_{n-1}=f^{1/2}\times\cdots\times f^{1/2}\times f^{-1/2}$. We notice that $f\sim\mathrm{diag}(s, s^{-1})$ in $SL(2, \boldsymbol{C})$ and $f^{1/2}\sim\mathrm{diag}(s, 1)$ in $PGL(2, \boldsymbol{C})$, where "$\sim$" means "is conjugate to". Therefore $g_n\sim\mathrm{diag}(s,\dots, s, 1,\dots, 1)\sim g_{n-1}$ in $PGL(2n, \boldsymbol{C})$. Thus g_n and g_{n-1} are rigid in $\mathscr{A}^{PGL(2n,\boldsymbol{C})}$ and so in $\mathscr{A}^{PSO(2n,\boldsymbol{C})}$. We can easily choose a simple system $\{a_1,\dots, a_n\}$ such that its Dynkin diagram is



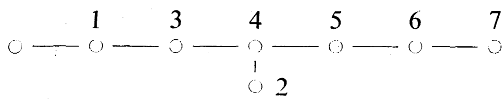
$\langle a_n, g_n\rangle=1$, $\langle a_i, g_n\rangle=0$ for $i\neq n$, $\langle a_{n-1}, g_{n-1}\rangle=1$ and $\langle a_i, g_{n-1}\rangle=0$ for $i\neq n-1$.

$G=E_6$: Fix a maximal torus T and a system of simple roots such that its Dynkin diagram is as below, where an integer "i" indicates a simple root a_i.



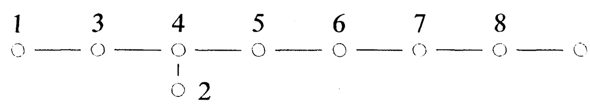
Let a_7 be the negative maximal root , $-(a_1+2a_2+2a_3+3a_4+2a_5+a_6)$. Let $\{g_i\in\mathrm{Hom}(\boldsymbol{C}^{\times}, T);\ i=1,\dots, 6\}$ be the dual basis of $\{a_i;\ i=1,\dots, 6\}$. Let H be a connected semisimple subgroup of E_6 such that H contains T and $\{a_1, a_3, a_4, a_5, a_6\}\cup\{a_7\}$ becomes a system of simple roots. Put $T'=T/(\text{the center of } H)$. Let $\{h_i\in\mathrm{Hom}(\boldsymbol{C}^{\times}, T);\ i=1, 3,\dots, 7\}$ be the dual basis of $\{a_i;\ i=1, 3,\dots, 7\}$. Naturally $\mathrm{Hom}(\boldsymbol{C}^{\times}, T)\subset\mathrm{Hom}(\boldsymbol{C}^{\times}, T')\subset\mathrm{Hom}(\boldsymbol{C}^{\times}, T)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$. Then $h_1=g_1-g_2/2$, $h_3=g_3-g_2$, $h_4=g_4-3g_2/2$, $h_5=g_5-g_2$, $h_6=g_6-g_2/2$ and $h_7=-g_2/2$. We note that $H/(\text{the center of } H)=A_5\times A_1$ and $\mathscr{R}(H)=\{0, h_1-h_7, h_3, h_4-h_7, h_5, h_6-h_7\}$. $W=W(T)$ denotes the Weyl group of E_6. Then $h_1-h_7=g_1$, $Wh_3=Wg_6$, $W(h_4-h_7)=Wg_2$, $Wh_5=Wg_1$ and $h_6-h_7=g_6$. Hence $\mathscr{R}(H)\subset\{0, g_1, g_2, g_6\}$. Let K be a connected semisimple subgroup of E_6 such that T is a maximal torus of K and $\{a_1, a_2, a_3, a_4, a_7\}\cup\{a_6\}$ is a system of simple roots. Put $T^{\sim}=T/(\text{the center of } K)$. Then $\{k_1=g_1-g_5/2, k_2=g_2-g_5, k_3=g_3-g_5, k_4=g_4-3g_5/2, k_7=-g_5/2\}\cup\{k_6=g_6-g_5/2\}$ is a basis of $\mathrm{Hom}(\boldsymbol{C}^{\times}, T^{\sim})$. Note that $K/(\text{the center of } K)$ equals $A_5\times A_1$. Thus $\mathscr{R}(K)=\{0, k_1-k_6, k_2, k_3, k_4-k_6, k_7-k_6\}$. Also $W(k_1-k_6)=Wg_6$, $Wk_2=Wg_6$, $Wk_3=Wg_1$, $W(k_4-k_6)=W(g_5+g_6)$ and $W(k_7-k_6)=Wg_1$. Therefore $\mathscr{R}(E_6)\subset\{0, g_1, g_6, g_5+g_6\}$. From the form of the maximal root, it follows that $\mathrm{Ad}(g_1)$ and $\mathrm{Ad}(g_6)$ are represented by diagonal matrices, whose entries are s, s^{-1} or 1. We notice that g_1 and g_6 are different in $\pi_1 E_6$. Hence Lemma 1.10 implies that $\mathscr{R}(E_6)=\{0, g_1, g_6\}$.

$G=E_7$: Let T be a maximal torus and a simple system of roots be as follows:

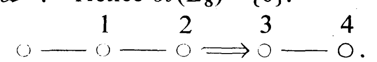


Let a_i be the simple root indicated by "i" for $i=1,\dots,7$ and a_8 be the negative maximal root, $-(2a_1+2a_2+3a_3+4a_4+3a_5+2a_6+a_7)$. Let $\{g_i \in \mathrm{Hom}\,(\boldsymbol{C}^\times, T);\ i=1,\dots,7\}$ be the dual basis of $\{a_i;\ i=1,\dots,7\}$. Let H be a connected semisimple subgroup of E_7 such that H contains T and a system of simple roots is $\{a_i;\ i=8, 1, 3,\dots,7\}$. Let $\{h_i \in \mathrm{Hom}\,(\boldsymbol{C}^\times, T');\ i=8, 1, 3,\dots,7\}$ be the dual basis of $\{a_i;\ i=8, 1, 3,\dots,7\}$, where T' denotes T/(the center of H). Then $\mathscr{R}(H)=\{0, h_1=g_1-g_2, h_4=g_4-2g_2, h_6=g_6-g_2\}$. Let $W=W(T)$ denote the Weyl group of E_7. It is easy to see that $Wh_1=Wg_7$, $Wh_4=Wg_1$ and $Wh_6=Wg_7$. For $\{a_8\} \cup \{a_i;\ i=2,\dots,7\}$, we take the subgroup K as above and put $T^{\sim}=T$/(the center of K). Let $\{k_i \in \mathrm{Hom}\,(\boldsymbol{C}^\times, T^{\sim});\ i=8, 2,\dots,7\}$ be the dual basis of $\{a_i;\ i=8, 2,\dots,7\}$. Then $k_8=-g_1/2$. So g_1 is not rigid in $\mathscr{A}^K$ since K/(the center of K)$=A_1\times D_6$. We know that g_7 is rigid by the form of $\mathrm{Ad}\, g_7$. Thus $\mathscr{R}(E_7)=\{0, g_7\}$.

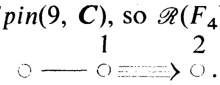
$G=E_8$: Let T be a maximal torus and a simple system of roots $\{a_i;\ i=1,\dots,8\}$ be as follows:



Let a_9 be the negative maximal root, $-(2a_1+3a_2+4a_3+6a_4+5a_5+4a_6+3a_7+2a_8)$. Let $\{g_i \in \mathrm{Hom}\,(\boldsymbol{C}^\times, T);\ i=1,\dots,8\}$ be the dual basis of $\{a_i;\ i=1,\dots,8\}$. Let H denote the connected semisimple subgroup corresponding to $\{a_i;\ i=2,\dots,9\}$. We notice that H/(the center of H) equals D_8. Let $\{h_i \in \mathrm{Hom}\,(\boldsymbol{C}^\times, T$/(the center of H)); $i=2,\dots,9\}$ be the dual basis of $\{a_i;\ i=2,\dots,9\}$. Then $\mathscr{R}(H)=\{0, h_3=g_3-2g_1\}$. For $W=W(T)$, the Weyl group of E_8, we know that $Wh_3=Wg_8$. Similarly, for $\{a_i;\ i=1,\dots,7\} \cup \{a_9\}$, we get the subgroup K and the dual basis $\{k_i \in \mathrm{Hom}\,(\boldsymbol{C}^\times, T$/(the center of K)); $i=1,\dots,7, 9\}$. Since $k_9=-g_8/2$, g_8 is not rigid in $\mathscr{A}^K$. Hence $\mathscr{R}(E_8)=\{0\}$.

$G=F_4$: 

. Let H be the connected semisimple subgroup corresponding to {the negative maximal root, a_1, a_2, a_3}. Then H is equal to $Spin(9, \boldsymbol{C})$, so $\mathscr{R}(F_4)=\{0\}$.

$G=G_2$: 

. Let H be the connected semisimple subgroup corresponding to {the negative maximal root, a_1}. Then $H=SL(3, \boldsymbol{C})$. Thus $\mathscr{R}(G_2)=\{0\}$. □

2. A family of flat connections

Let $Y_1 = Y_1(x, y)$ be a function in an open subset of $\boldsymbol{R}^2$ with values in $M(N, \boldsymbol{C})$. We try to rewrite the next equation in some geometric form.

$$y(\partial_x^2 + \partial_y^2)Y_1 - [\partial_x Y_1, \partial_y Y_1] = 0. \tag{2.1}$$

Let $*$ be the standard Hodge operator on $\boldsymbol{R}^2$. That is to say, $*dx = dy$, $*dy = -dx$, $*1 = dx \wedge dy$ and $*(dx \wedge dy) = 1$. Let s denote a complex parameter. We put $z = x + iy$ and $\bar{z} = x - iy$. For a smooth function Y_1, we set

$$\Omega(Y_1) = 2^{-1}\{(s-z)^{-1}(1+i*)dY_1 + (s-\bar{z})^{-1}(1-i*)dY_1\}. \tag{2.2}$$

Note that $(s-z)(1+i*) + (s-\bar{z})(1-i*) = 2(s-x-y*)$ and

$$\Omega(Y_1) = (s-z)^{-1}(s-\bar{z})^{-1}(s-x-y*)dY_1. \tag{2.3}$$

LEMMA 2.1. *Let d be the exterior derivative on $\boldsymbol{R}^2$. Then* (2.1) *is equivalent to the following equation for $\Omega = \Omega(Y_1)$:*

$$d\Omega + \Omega \wedge \Omega = 0. \tag{2.4}$$

PROOF. For 1-forms a and b on $\boldsymbol{R}^2$ with complex matrix coefficients, $*a \wedge b = -a \wedge *b$. Note that $(1-i*)dz = 0$, $(1+i*)d\bar{z} = 0$. So $dz \wedge (1+i*)dY_1 = 0$ and $d\bar{z} \wedge (1-i*)dY_1 = 0$. Then we have

$$d\Omega = 2^{-1}\{(s-z)^{-1}id*dY_1 - (s-\bar{z})^{-1}id*dY_1\} = -(s-z)^{-1}(s-\bar{z})^{-1}yd*dY_1.$$

For 1-forms, $(1+i*)(1-i*) = 0$, $(1+i*)(1+i*) = 2(1+i*)$ and $(1-i*)(1-i*) = 2(1-i*)$, since $** = -1$. So we have

$$(1+i*)dY_1 \wedge (1+i*)dY_1 = (1-i*)dY_1 \wedge (1-i*)dY_1 = 0.$$

$$(1+i*)dY_1 \wedge (1-i*)dY_1 = 2dY_1 \wedge (1-i*)dY_1, \tag{2.5}$$

$$(1-i*)dY_1 \wedge (1+i*)dY_1 = 2dY_1 \wedge (1+i*)dY_1. \tag{2.6}$$

$$\Omega \wedge \Omega = 4^{-1}(s-z)^{-1}(s-\bar{z})^{-1}\{(2.5)+(2.6)\} = (s-z)^{-1}(s-\bar{z})^{-1}dY_1 \wedge dY_1.$$

Therefore,

$$d\Omega + \Omega \wedge \Omega = (s-z)^{-1}(s-\bar{z})^{-1}\{-y(\partial_x^2 + \partial_y^2)Y_1 + [\partial_x Y_1, \partial_y Y_1]\}dx \wedge dy. \quad \square$$

A 1-form Ω on $\boldsymbol{R}^2$ with coefficients in $M(N, \boldsymbol{C})$ defines a connection $D = d + \Omega$. Then the curvature of D equals $DD = d\Omega + \Omega \wedge \Omega$, and hence (2.4) means that the connection is flat. Let U be a nonempty open subset of $P^1 \times \boldsymbol{R}^2$. For a fixed real number $\delta > 0$, we set $L = \{s \in P^1 = \boldsymbol{C} \cup \{\infty\};\ |s| > \delta\}$. Let p denote the natural projection from $P^1 \times \boldsymbol{R}^2$ to P^1. We assume that U contains $L \times (0, 0)$

and that $pU=L$. Let $\Omega_s=\Omega(s, x, y)$ be a smooth 1-form on U which does not have terms ds and $d\bar{s}$. Also we assume that Ω_s is holomorphic in s. In a word, Ω_s is a holomorphic family of 1-forms.

LEMMA 2.2. *Suppose that $D_s=d+\Omega_s$ is a flat connection and $p^{-1}s \cap U$ is 1-connected for all $s \in L$. Then there exists a smooth mapping $Y_s=Y(s, x, y)$ from U to $GL(N, \boldsymbol{C})$ such that Y_s is holomorphic in s and $Y_s^{-1}dY_s=\Omega_s$.*

Moreover, if $\Omega(\infty,\cdot,\cdot)=0$, then we can take $Y(\infty,\cdot,\cdot)=1$.

PROOF. For a fixed $s \in L$, we have a smooth mapping $Y_s=Y_s(x, y)$ such that $Y_s^{-1}dY_s=\Omega_s$ since $p^{-1}s \cap U$ is 1-connected. Since we have assumed that $L \times (0, 0)$ is contained in U, we can take $Y_s(0, 0)=1$. For any $(s_1, x_1, y_1) \in U$, we take a smooth curve z_t, $0 \leq t \leq 1$, in $\boldsymbol{R}^2$ such that $z_0=(0, 0)$, $z_1=(x_1, y_1)$ and $(s_1, z_t) \in U$ for all t. Then $(d/dt)Y_s(z_t)=Y_s(z_t)\Omega_s(z_t)$ for any s sufficiently near s_1. By the elementary theory of ordinary differential equations, we see that $Y(s, x, y)=Y_s(x, y)$ is holomorphic in s. If $\Omega_\infty=0$, then $Y_\infty=$a constant. □

DEFINITION 2.3. Let R and δ be positive real numbers. We define $D(\delta)$, $\mathscr{A}_{R,\delta}$, $\mathscr{N}_{R,\delta}$ and $\mathscr{P}_{R,\delta}$ as follows.

$$D(\delta) = \{(x, y) \in \boldsymbol{R}^2;\ x^2+y^2<\delta^2\}.$$

$\mathscr{A}_{R,\delta}$ denotes the set of all mappings from $S_R \times D(\delta)$ to $GL(N, \boldsymbol{C})$ such that they are restrictions of analytic mappings in s, x and y.

$\mathscr{N}_{R,\delta}=\{Y \in \mathscr{A}_{R,\delta};\ Y$ can be extended to an analytic mapping from $D_R^- \times D(\delta)$ to $GL(N, \boldsymbol{C})$ and $Y(\infty, x, y)=1\}$.

$\mathscr{P}_{R,\delta}=\{X \in \mathscr{A}_{R,\delta};\ X$ can be extended to an analytic mapping from $D_R^+ \times D(\delta)$ to $GL(N, \boldsymbol{C})\}$.

For $Y_s \in \mathscr{N}_{R,\delta}$, we set $\Omega_s=Y_s^{-1}dY_s$ and consider the following property.

(2.7) $\Omega_s+i{*}\Omega_s$ and $\Omega_s-i{*}\Omega_s$ are extended on P^1 meromorphically in s. Their poles are of order at most one. Their locations are just $x+iy$ and $x-iy$ respectively.

LEMMA 2.4. *Let $Y_s=1+\sum_{n>0} Y_n s^{-n}$ with $Y_n=Y_n(x, y)$ be the Laurent expansion of $Y_s \in \mathscr{N}_{R,\delta}$. If Y_s has the property (2.7), then $Y_s^{-1}dY_s$ equals $\Omega(Y_1)$ for $\Omega(Y_1)$ given in (2.2).*

PROOF. Let $z=x+iy$ and $\bar{z}=x-iy$. Our assumption means that $\Omega_s+i{*}\Omega_s=(s-z)^{-1}A$ and $\Omega_s-i{*}\Omega_s=(s-\bar{z})^{-1}B$, where A and B are 1-forms independent of s. For c and $s \in \boldsymbol{C}$ such that $|s|>|c|$, we have $(s-c)^{-1}=s^{-1}\sum_{n\geq 0}(c/s)^n$. In the equations $(1+i{*})dY_s=(s-z)^{-1}Y_sA$ and $(1-i{*})dY_s=(s-\bar{z})^{-1}Y_sB$, we compare the coefficients of s^{-1}. Then we see that $(1+i{*})dY_1=A$ and $(1-i{*})dY_1=B$. Hence $\Omega_s=2^{-1}\{(s-z)^{-1}(1+i{*})+(s-\bar{z})^{-1}(1-i{*})\}dY_1$. □

Next we state some invariance of the property (2.7), which is basic for our treatment of nonlinear equations.

LEMMA 2.5. *For* Y^0, $Y \in \mathcal{N}_{R,\delta}$, $X \in \mathcal{P}_{R,\delta}$ *and* $g \in \mathcal{A}_R^{GL(N,C)}$, *we assume that* $gY^0X = Y$. *If* Y^0 *has the property* (2.7), *then so does* Y.

PROOF. We notice that $Y^{-1}dY = X^{-1}dX + X^{-1}(Y^0)^{-1}dY^0X$, since $dg=0$. So $Y^{-1}dY$ is extended to D_R^+ meromorphically. Since X and X^{-1} are holomorphic on D_R^+, the poles of $Y^{-1}dY$ are contained in the set of poles of $(Y^0)^{-1}dY^0$. □

3. The nonlinear equations

Let σ be a fixed involutive real automorphism of $GL(N, \boldsymbol{C})$. We assume that σ is a conjugation of inner type or a holomorphic automorphism of outer type. Put $G = \{g \in GL(N, \boldsymbol{C}); \sigma g = g\}$. We extend σ to an automorphism of $\mathcal{A}_{R,\delta}$ as follows: For $g \in \mathcal{A}_{R,\delta}$, we set $(\sigma g)(s, x, y) = \sigma(g(\bar{s}, x, y))$ or $\sigma(g(s, x, y))$, respectively.

For $Z = Z_s = Z(s, x, y) \in \mathcal{A}_{R,\delta}$, we study the following properties:

(3.1) $\qquad \Omega = Z^{-1}dZ$ has the property (2.7).

(3.2) $\qquad (\sigma Z)^{-1}Z$ and $(\sigma Z)^{-1}dZ$ are polynomials of s.

For any $a \in \mathrm{Hom}(\boldsymbol{C}^\times, GL(N, \boldsymbol{C}))$, we define a parabolic subgroup $P_a = \{g \in GL(N, \boldsymbol{C}); \lim_{s\to 0} a(s)ga(s)^{-1}$ converges in $GL(N, \boldsymbol{C})\}$.

DEFINITION 3.1. An element a of $\mathrm{Hom}(\boldsymbol{C}^\times, GL(N, \boldsymbol{C}))$ is said to be *nice* (with respect to σ) if a and σa are commutative and if a and $(\sigma a)^{-1}a$ are polynomials of degree at most 1. Moreover, if G is isomorphic to $Sp(N/2, \boldsymbol{C})$, then we require that $\{v \in \boldsymbol{C}^N; a(s)v = v$ for all $s \in \boldsymbol{C}^\times\}$ has an even dimension.

THEOREM 3.2. *Let* Y *be an element of* $\mathcal{N}_{R,\delta}$ *and* a *be a nice element of* $\mathrm{Hom}(\boldsymbol{C}^\times, GL(N, \boldsymbol{C}))$. *Let* Q_a *denote the parabolic subgroup opposite to* P_a. *We assume*:

(3.3) $\quad a(s)Y(s, x, 0)a(s-x)^{-1}$ *is a function independent of* s.

(3.4) *the above function takes values in an open orbit of* $G\backslash GL(N, \boldsymbol{C})/Q_a$.

Furthermore let $Y_s = 1 + \sum_{n>0} Y_n s^{-n}$ *be the Laurent expansion of* Y.

If $Z = a(s)Y_s$ *has the properties* (3.1–2), *then* Y *is determined by* $Y_1(x, 0)$ *and* $\partial_y Y_1(x, 0)$.

Moreover if $\partial_y Y_1(x, 0) = 0$, *then* Y *is an even function with respect to* y.

For its proof, we give some preliminaries. The niceness of a implies that there exists a basis $\{e_i;\ i=1,\dots,N\}$ of C^N such that $a(s)$ and $(\sigma a)^{-1}(s)$ are represented by the following diagonal matrices:

$$a(s) = \mathrm{diag}(s1_p,\ 1_p,\ 1_r) \quad \text{and} \quad (\sigma a)^{-1}(s) = \mathrm{diag}(1_p,\ s1_p,\ 1_r).$$

Here $p+p+r=N$. Fix such a basis. Now we identify $\mathrm{Hom}_C(C^N, C^N)$ with $M(N, C)$ through this basis.

If σ is a conjugation of inner type, then there is J in $GL(N, C)$ such that $\sigma g = \mathrm{Ad}\,J^{-1}({}^*g^{-1})$ for $g \in GL(N, C)$. Hence $*$ denotes the hermitian conjugation on $M(N, C)$. Since $g=\sigma(\sigma g)=\mathrm{Ad}\,(J^{-1}{}^*J)g$, we see ${}^*J=cJ$ for some $c \in C$. We note that $J={}^*({}^*J)=c\bar{c}J$. For $b=c^{1/2}$, we have ${}^*(bJ)=bJ$. Hence we can assume that $\sigma g=\mathrm{Ad}\,J^{-1}({}^*g^{-1})$ and ${}^*J=J$. We define an hermitian form as follows: $(w_1, w_2)={}^*w_1w_2$ for $w_1, w_2 \in C^N$. Here we understand that elements of C^N are column vectors with respect to the basis e_i. Note that ${}^*a(s)J=J(\sigma a)(\bar{s})^{-1}$ and ${}^*(\sigma a)(s)J=Ja(\bar{s})^{-1}$. Therefore we have $(Je_i, a(s)e_j)=(J(\sigma a)(\bar{s})^{-1}e_i, e_j)$ and $(Je_i, (\sigma a)(s)e_j)=(Ja(\bar{s})^{-1}e_i, e_j)$.

If σ is a holomorphic automorphism of outer type, then there is $J \in GL(N, C)$ such that $\sigma g=\mathrm{Ad}\,J^{-1}({}^tg^{-1})$ for $g \in GL(N, C^t)$. Here ^{t} denotes the transpose on $M(N, C)$. "$\sigma^2=1$" implies that ${}^tJ=cJ$ for some $c \in C$. Since ${}^t({}^tJ)=J$, we have $c^2=1$. We define a symmetric bilinear form as follows: $(w_1, w_2)={}^tw_1w_2$ for $w_1, w_2 \in C^N$. Then $(Je_i, a(s)e_j)=(J\sigma a(s)^{-1}e_i, e_j)$ and $(Je_i, \sigma a(s)e_j)=(Ja(s)^{-1}e_i, e_j)$.

Hereafter, we use the notation $\dagger$ for $*$ or ^{t} according as σ is a conjugation of inner type or a holomorphic automorphism of outer type. Then we have

$$J = \begin{bmatrix} 0 & \boldsymbol{j} & 0 \\ c^\dagger\boldsymbol{j} & 0 & 0 \\ 0 & 0 & \boldsymbol{j}_1 \end{bmatrix},$$

for a $p \times p$ matrix $\boldsymbol{j}$ and an $r \times r$ matrix $\boldsymbol{j}_1$ such that ${}^\dagger\boldsymbol{j}_1=c\boldsymbol{j}_1$ with $c^2=1$. Put

$$J_0 = \begin{bmatrix} 0 & \boldsymbol{j} & 0 \\ c^\dagger\boldsymbol{j} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{j}_1 \end{bmatrix}.$$

Then $J=J_0+J_1$ and $\sigma g=\mathrm{Ad}\,J^{-1}({}^\dagger g^{-1})$ for $g \in GL(N, C)$.

LEMMA 3.3. *Let a and Y be as in Theorem 3.2. For $Z=aY$, we set $M=J(\sigma Z)^{-1}Z$ and $\Omega=Z^{-1}dZ$. Then*

$$M = sJ_0 + J_1 + J_0 Y_1 + {}^{\dagger}Y_1 J_0,$$
$$M\Omega = J_0 dY_1$$
$$= s^{-1}\{sJ_0 dY_1 + (x - y*)J_0 dY_1 + (J_1 + J_0 Y_1 + {}^{\dagger}Y_1 J_0)dY_1\}.$$

In particular,

$$y*J_0 dY_1 = (xJ_0 + J_1 + J_0 Y_1 + {}^{\dagger}Y_1 J_0)dY_1. \tag{3.5}$$

PROOF. Since we assume (3.2), M and $M\Omega$ are polynomials of s. We notice that $a(\sigma a)^{-1} = \operatorname{diag}(s1_{2p}, 1_r)$. Hence,

$$M = {}^{\dagger}YJ(\sigma a)^{-1}aY = {}^{\dagger}Y(sJ_0 + J_1)Y$$
$$= (1 + {}^{\dagger}Y_1 s^{-1})(sJ_0 + J_1)(1 + Y_1 s^{-1}) + O(s^{-2})$$
$$= sJ_0 + J_1 + {}^{\dagger}Y_1 J_0 + J_0 Y_1.$$
$$M\Omega = {}^{\dagger}Y(sJ_0 + J_1)dY$$
$$= (1 + {}^{\dagger}Y_1 s^{-1})(sJ_0 + J_1)s^{-1}dY_1 + O(s^{-2}) = J_0 dY_1.$$

Let $z = x + iy$ and $\bar{z} = x - iy$. Since we assume (3.1), by Lemma 2.4, we obtain

$$\Omega = 2^{-1}\{(s - z)^{-1}(1 + i*) + (s - \bar{z})^{-1}(1 - i*)\}dY_1$$
$$= 2^{-1}s^{-1}\textstyle\sum_{n\geq 0}\{(z/s)^n(1 + i*) + (\bar{z}/s)^n(1 - i*)\}dY_1$$
$$= 2^{-1}s^{-1}\{2 + (z/s)(1 + i*) + (\bar{z}/s)(1 - i*) + \cdots\}dY_1$$

Hence

$$M\Omega = s^{-1}\{sJ_0 + J_1 + {}^{\dagger}Y_1 J_0 + J_0 Y_1\}\{1 + s^{-1}(x - y*) + \cdots\}dY_1$$
$$= s^{-1}\{sJ_0 + J_1 + {}^{\dagger}Y_1 J_0 + J_0 Y_1 + (x - y*)J_0 + \cdots\}dY_1. \quad \square$$

We put

$$Y_1 = \begin{bmatrix} H & B \\ L & K \end{bmatrix}, \text{where } H \text{ is a square matrix of size } 2p,$$

$$\boldsymbol{j}_0 = \begin{bmatrix} 0 & \boldsymbol{j} \\ c^{\dagger}\boldsymbol{j} & 0 \end{bmatrix}, \text{ a square matrix of size } 2p, \text{ and}$$

$$h = x\boldsymbol{j}_0^{-1} + H\boldsymbol{j}_0^{-1} + \boldsymbol{j}_0^{-1}\,{}^{\dagger}H - B\boldsymbol{j}_1^{-1}\,{}^{\dagger}B.$$

LEMMA 3.4. *Retain the above notations. Then*

$$h\boldsymbol{j}_0 dH = y*dH, \quad h\boldsymbol{j}_0 dB = y*dB. \tag{3.6}$$

$$dL = -\boldsymbol{j}_1^{-1}\,{}^{\dagger}B\boldsymbol{j}_0 dH, \quad dK = -\boldsymbol{j}_1^{-1}\,{}^{\dagger}B\boldsymbol{j}_0 dB. \tag{3.7}$$

PROOF. We rewrite (3.5) as follows:

$$(xj_0+j_0H+{}^{\dagger}Hj_0)dH + j_0BdL = yj_0*dH.$$
$$(xj_0+j_0H+{}^{\dagger}Hj_0)dB + j_0BdK = yj_0*dB.$$
$${}^{\dagger}Bj_0dH + j_1dL = 0, \quad {}^{\dagger}Bj_0dB + j_1dK = 0.$$

So (3.7) is clear. It is easy to show (3.6) as well. □

Put $W=(H, B)$, a $p\times N$ matrix. We define W_n, H_n, B_n, and h_n by the Taylor expansions $W=\sum_{n\geq 0} W_n y^n$ with $W_n=W_n(x)=(H_n, B_n)$ and $h=\sum_{n\geq 0} h_n y^n$ with $h_n=h_n(x)$. Then a simple calculation gives

(3.8) $$\partial_x W_{k-2} = \sum_{m+n=k} h_m j_0 n W_n,$$

(3.9) $$-kW_k = \sum_{m+n=k} h_m j_0 \partial_x W_n,$$

(3.10) $$h_k = H_k j_0^{-1} + j_0^{-1}\,{}^{\dagger}H_k - \sum_{m+n=k} B_m j_1^{-1}\,{}^{\dagger}B_n.$$

LEMMA 3.5. $h_0=\begin{bmatrix} h^0 & 0 \\ 0 & 0 \end{bmatrix}$, *where h^0 is a nonsingular $p\times p$ matrix.*

PROOF. From (3.3), it follows that $Y(s, x, 0)=1+s^{-1}Y_1(x, 0)$ and

$$Y_1(x, 0) = \begin{bmatrix} x1_p & u & v \\ 0 & 01_p & 0 \\ 0 & 0 & 01_r \end{bmatrix}.$$

Here u and v are some functions of x. Since $a(s)=\mathrm{diag}(s1_p, 1_{p+r})$, Q_a is a subgroup of $GL(N, \boldsymbol{C})$ consisting of all matrices whose shapes are $\left[\begin{array}{c|c} * & 0 \\ \hline * & *** \\ * & *** \end{array}\right]$, where 0 is the $p\times(p+r)$ zero matrix. All linear subspaces of dimension $q=p+r$ in $\boldsymbol{C}^N$ make the Grassmann variety $Gr(q, N)$. Let V^a be the subspace generated by e_i for $i=p+1,\dots, N$. The mapping from $GL(N, \boldsymbol{C})/Q_a$ to $Gr(q, N)$, which is defined by the correspondence $gQ_a\to gV^a$, is an isomorphism. Put $J(w_1, w_2)={}^{\dagger}w_1Jw_2$ for $w_1, w_2\in \boldsymbol{C}^N$. Then $J(\cdot,\cdot)$ is a hermitian, symmetric or symplectic form on $\boldsymbol{C}^N$. We note that $G=\{g\in GL(N, \boldsymbol{C});\ {}^{\dagger}gJg=J\}$. Hence for $V\in Gr(q, N)$, V belongs to an open orbit in $G\backslash GL(N, \boldsymbol{C})/Q_a$ iff $J(\cdot,\cdot)$ is nondegenerate on V.

Let V be the element of $Gr(q, N)$ corresponding to $a(s)Y(s, x, 0)a(s-x)^{-1}$. Then the restriction of $J(\cdot,\cdot)$ on V is represented by the matrix

$$J_V=\begin{bmatrix} {}^{\dagger}u & 1_p & 0 \\ {}^{\dagger}v & 0 & 1_r \end{bmatrix}\begin{bmatrix} 0 & j & 0 \\ c^{\dagger}j & 0 & 0 \\ 0 & 0 & j_1 \end{bmatrix}\begin{bmatrix} u & v \\ 1_p & 0 \\ 0 & 1_r \end{bmatrix}.$$

Note that $\begin{bmatrix} 1_p & -c^\dagger v\boldsymbol{j}_1 \\ 0 & 1_r \end{bmatrix} J_V = \begin{bmatrix} {}^\dagger u\boldsymbol{j} + c^\dagger \boldsymbol{j} u - c^\dagger \boldsymbol{j} v \boldsymbol{j}_1^{-1} v\boldsymbol{j} & 0 \\ {}^\dagger v\boldsymbol{j} & \boldsymbol{j}_1 \end{bmatrix}$.

Therefore V belongs to an open orbit in $G\backslash GL(N, \boldsymbol{C})/Q_a$ iff $\det (c^\dagger \boldsymbol{j}^{-1\dagger} u - u\boldsymbol{j}^{-1} - v\boldsymbol{j}_1^{-1\dagger} v) \neq 0$. By (3.10), we easily obtain

$$h_0 = \begin{bmatrix} c^\dagger \boldsymbol{j}^{-1\dagger} u - u\boldsymbol{j}^{-1} - v\boldsymbol{j}_1^{-1\dagger} v & 0 \\ 0 & 0 \end{bmatrix}. \quad \square$$

PROOF OF THEOREM 3.2. First we show that the above W_k, $k \geq 2$, are determined by W_0 and W_1 by induction on k. We set

$$W_k = \begin{bmatrix} \alpha_k & \beta_k & \gamma_k \\ \delta_k & \varepsilon_k & \zeta_k \end{bmatrix}.$$

Here α_k, β_k, δ_k and ε_k are $p \times p$ matrices and γ_k and ζ_k are $p \times r$ matrices. By (3.8), we have $h_0 \boldsymbol{j}_0 W_k = \partial_x W_{k-2} - \sum_{1 \leq n < k} h_{k-n} \boldsymbol{j}_0 n W_n$. Using Lemma 3.5, we see that

$$h_0 \boldsymbol{j}_0 W_k = \begin{bmatrix} h^0 \boldsymbol{j} \delta_k & h^0 \boldsymbol{j} \varepsilon_k & h^0 \boldsymbol{j} \zeta_k \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence δ_k, ε_k and ζ_k are determined by W_n, $n < k$. We read (3.9) as follows: $-kW_k = h_k \boldsymbol{j}_0 \partial_x W_0 + (\cdots)$. Here the part $(\cdots)$ is to be determined by induction. By (3.10), $h_k = H_k \boldsymbol{j}_0^{-1} + \boldsymbol{j}_0^{-1\dagger} H_k - B_0 \boldsymbol{j}_1^{-1\dagger} B_k - B_k \boldsymbol{j}_1^{-1\dagger} B_0 + (\cdots)$. Hence

$$h_k = \begin{bmatrix} ? & \alpha_k c^\dagger \boldsymbol{j}^{-1} \\ ? & ? \end{bmatrix} + (\cdots).$$

Note that

$$\boldsymbol{j}_0 \partial_x W_0 = \begin{bmatrix} 0 & 0 & 0 \\ -c^\dagger \boldsymbol{j} & c^\dagger \boldsymbol{j} \partial_x u & c^\dagger \boldsymbol{j} \partial_x v \end{bmatrix}.$$

Therefore we obtain $-k\alpha_k = -\alpha_k + (\cdots)$, $-k\beta_k = \alpha_k \partial_x u + (\cdots)$ and $-k\gamma_k = \alpha_k \partial_x v + (\cdots)$. So α_k, β_k and γ_k are determined by induction.

By (3.7), we know that L and K are determined up to additive constants. Thus $Y_1 = Y_1(x, y)$ is determined completely. From (3.1) and Lemma 2.4, it follows that $Y^{-1}dY$ is unique. For $Y' \in \mathcal{N}_{R,\delta}$, if $(Y')^{-1}dY' = Y^{-1}dY$, then we have $d(Y'Y^{-1}) = 0$. That is to say, $Y' = gY$ for some $g \in \mathscr{A}_R^{GL(N,\boldsymbol{C})}$. In addition, assuming $Y'(s, x, 0) = Y(s, x, 0)$, we get $g = 1$. Thus the first half of Theorem 3.2 is proved.

Clearly, the properties (3.1–2) are invariant under the transformation "$y \to$

$-y$". Hence, if $Z(s, x, y)$ has the properties (3.1–2), then so does $Z(s, x, -y)$. Thus, if $\partial_y Y_1(x, 0)=0$, the first half of Theorem 3.2 implies $Y(s, x, y)=Y(s, x, -y)$. □

Let $\mathscr{G}_R^\sigma=\{g \in \mathscr{P}_R^{GL(N,C)} \subset \mathscr{A}_R^{GL(N,C)};\ \sigma g=g\}$. We remark that the properties (3.1–2) have an apparent invariance under the left action of $\mathscr{G}_R^\sigma$. If $Z \in \mathscr{A}_{R,\delta}$ has the properties (3.1–2) and g is an element of $\mathscr{G}_R^\sigma$, then gZ also has the properties (3.1–2).

Let $a, a' \in \mathrm{Hom}(\boldsymbol{C}^\times, GL(N, \boldsymbol{C}))$ be polynomials of degree at most 1. For Y and $Y' \in \mathscr{N}_{R,\delta}$, we assume $aY=a'Y'$. Then a simple argument implies that $a=a'$ and $Y=Y'$.

In some physical context [9, 10], the conditions (3.3–4) appear naturally.

Definition 3.6. Let a be a nice element of $\mathrm{Hom}(\boldsymbol{C}^\times, GL(N, \boldsymbol{C}))$ and Y be in $\mathscr{N}_{R,\delta}$. If Y and a satisfy the conditions (3.3–4), then we say that (Y, a) *is in the Hauser-Ernst gauge.*

4. The generation of solutions

We start with seeking a solution. Freely we employ the notations in Section 3. We suppose that σ is represented as in the proof of Theorem 3.2.

Lemma 4.1. *Let u be a constant $p \times p$ matrix such that* $\det(c^\dagger \boldsymbol{j} u+{}^\dagger u \boldsymbol{j}) \neq 0$. *Let* $w=-2^{-1}\boldsymbol{j}^{-1}(u\boldsymbol{j}^{-1}+c^\dagger \boldsymbol{j}^{-1}\,{}^\dagger u)^{-1}$. We set

$$\xi=\begin{bmatrix}(1+2uw)/s\\ 2w\\ 0\end{bmatrix},\quad \delta=\begin{bmatrix}u/s\\ 1_p\\ 0\end{bmatrix}\quad \text{and}\quad \gamma=\begin{bmatrix}0\\ 0\\ 1_r\end{bmatrix},$$

where ξ and δ are $N \times p$ matrices and γ is an $N \times r$ matrix. Let $\eta=2\delta w$ and $\tau=s\{(1-x/s)^2+(y/s)^2\}^{1/2}$. Here we understand that x/s and y/s are sufficiently small and that $1^{1/2}=1$. Let

$$Y(s, x, y)=[(x-s)\eta+\tau\xi,\ \delta,\ \gamma]$$

and $a(s)=\mathrm{diag}(s1_p, 1_{p+r})$. Then $Z=aY$ has the properties (3.1–2) *and (Y, a) is in the Hauser-Ernst gauge.*

Proof. We notice that

$$Y_1(x, y)=\begin{bmatrix}-x1_p & u & 0\\ y^2 w & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Let $\Omega=\tau^{-2}(s-x-y*)dY_1$. Then $\tau^2 Y\Omega=\tau^2[\eta dx+\xi d\tau, 0, 0]=\tau^2 dY$. Since $Y^{-1}dY=\Omega$, (2.3) implies (3.1). We put $\alpha=\tau(1+2uw)+2(x-s)uw$ and $\beta=2\tau w+2(x-s)w$. Then

$$^{\dagger}ZJZ=\begin{bmatrix} ^{\dagger}\beta c^{\dagger}\boldsymbol{j}\alpha+{}^{\dagger}\alpha\boldsymbol{j}\beta & ^{\dagger}\beta c^{\dagger}\boldsymbol{j}u+{}^{\dagger}\alpha\boldsymbol{j} & 0 \\ c^{\dagger}\boldsymbol{j}\alpha+{}^{\dagger}u\boldsymbol{j}\beta & c^{\dagger}\boldsymbol{j}u+{}^{\dagger}u\boldsymbol{j} & 0 \\ 0 & 0 & \boldsymbol{j}_1 \end{bmatrix} \quad \text{and}$$

$$^{\dagger}ZJdZ=\begin{bmatrix} ^{\dagger}\beta c^{\dagger}\boldsymbol{j}d\alpha+{}^{\dagger}\alpha\boldsymbol{j}d\beta & 0 & 0 \\ c^{\dagger}\boldsymbol{j}d\alpha+{}^{\dagger}u\boldsymbol{j}d\beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We notice that $\boldsymbol{j}w=c^{\dagger}w^{\dagger}\boldsymbol{j}$ and $(u+c^{\dagger}\boldsymbol{j}^{-1\dagger}u\boldsymbol{j})w=-1/2$. A direct calculation shows that

$$^{\dagger}\beta c^{\dagger}\boldsymbol{j}\alpha+{}^{\dagger}\alpha\boldsymbol{j}\beta=2y^2\boldsymbol{j}w, \quad c^{\dagger}\boldsymbol{j}\alpha+{}^{\dagger}u\boldsymbol{j}\beta=(s-x)c^{\dagger}\boldsymbol{j},$$

$$^{\dagger}\beta c^{\dagger}\boldsymbol{j}d\alpha+{}^{\dagger}\alpha\boldsymbol{j}d\beta=2\boldsymbol{j}wydy \quad \text{and} \quad c^{\dagger}\boldsymbol{j}d\alpha+{}^{\dagger}u\boldsymbol{j}d\beta=-c^{\dagger}\boldsymbol{j}dx.$$

Thus Z has the property (3.2). It is easy to see that (Y, a) is in the Hauser-Ernst gauge. □

DEFINITION 4.2. For (Y, a), which is in the Hauser-Ernst gauge, we define a function $\alpha(Y)(x)=a(s)Y(s, x, 0)a(s-x)^{-1}Q_a$. This takes its values in $GL(N, \boldsymbol{C})/Q_a$.

LEMMA 4.3. *Let (Y^0, a) and (Y, a) with $Y^0, Y\in\mathcal{N}_{R,\delta}$ be in the Hauser-Ernst gauge. We assume that $gaY^0X=aY$ for $g=g(s)\in\mathcal{P}_R^{GL(N,\boldsymbol{C})}$ and $X\in\mathcal{P}_{R,\delta}$. Then $\alpha(Y)(x)=g(x)\alpha(Y^0)(x)$ in $GL(N, \boldsymbol{C})/Q_a$.*

PROOF. We are assuming that $(a(s)Y(s, x, 0))^{-1}g(s)a(s)Y^0(s, x, 0)$ for $|s|\leq R$ is holomorphic in s. Computing its residue at $s=x$, we can deduce that $(a(s)Y(s, x, 0))^{-1}g(x)a(s)Y^0(s, x, 0)$ belongs to Q_a. □

LEMMA 4.4. *Let $a\in\mathrm{Hom}\,(\boldsymbol{C}^{\times}, GL(N, \boldsymbol{C}))$ be nice. For $g\in\mathcal{G}_R^{\sigma}$, $X\in\mathcal{P}_{R,\delta}$ and $Y^0, Y\in\mathcal{N}_{R,\delta}$, we assume that $gaY^0X=aY$ and that aY^0 has the properties* (3.1–2). *Then aY also has the properties* (3.1–2).

PROOF. We put $Z=gaY^0$. Clearly Z has the properties (3.1–2). We recall that $Y(s, x, y)$ for $|s|\geq R$ and $X(s, x, y)$ for $|s|\leq R$ have no pole. Since $(aY)^{-1}d(aY)=X^{-1}Z^{-1}dZX+X^{-1}dX$, aY has the property (3.1).

$$\sigma(aY)^{-1}aY=(\sigma X)^{-1}(\sigma Z)^{-1}ZX.$$

$$\sigma(aY)^{-1}d(aY)=(\sigma X)^{-1}(\sigma Z)^{-1}(dZ)X+(\sigma X)^{-1}(\sigma Z)^{-1}ZdX.$$

Hence it is easy to see that aY has the property (3.2). □

Let $\mathscr{H}_{R,a}=\{g\in\mathscr{G}_R^q;\ g(s)$ belongs to P_a for $|s|\leq R\}$. We notice that the solution given in Lemma 4.1 is an even function with respect to y. We now introduce a special class of solutions.

DEFINITION 4.5. $\mathscr{S}_{R,\delta,a}=\{0\in\mathscr{N}_{R,\delta};\ aY$ has the properties (3.1–2), (Y, a) is in the Hauser-Ernst gauge and $Y(s, x, y)=Y(s, x, -y)\}$.

THEOREM 4.6. *Let $Y^0\in\mathscr{S}_{R,\delta,a}$ and $g\in\mathscr{H}_{R,a}$. Then for some $\varepsilon>0$, there exists $Y\in\mathscr{S}_{R,\varepsilon,a}$ such that $gaY^0=aY$ in $\mathscr{A}_{R,\varepsilon}/\mathscr{P}_{R,\varepsilon}$.*

PROOF. We notice that $a(s)^{-1}g(s)a(s)Y^0(s, x, 0)$ lies in $\mathscr{N}_R^P\mathscr{P}_R^P$ as a function of s. Here P denotes the parabolic subgroup defined by a. By Theorem 1.6 and Corollary 1.7, we can find a real number $\varepsilon>0$ such that $a^{-1}gaY^0$ belongs to $\mathscr{N}_{R,\varepsilon}\mathscr{P}_{R,\varepsilon}$. So we set $a^{-1}gaY^0=YX$ with $Y\in\mathscr{N}_{R,\varepsilon}$ and $X\in\mathscr{P}_{R,\varepsilon}$. Clearly Y and X are expanded in power series of y^2. From the explicit form of $a(s)Y(s, x, 0)$, it follows that (Y, a) is in the Hauser-Ernst gauge. By Lemma 4.4, we know that Y belongs to $\mathscr{S}_{R,\varepsilon,a}$. □

DEFINITION 4.7. Let $a\in\mathrm{Hom}(\boldsymbol{C}^\times, GL(N, \boldsymbol{C}))$ be nice. Put $V^a=\{v\in\boldsymbol{C}^N;\ av=v\}$ and $q=\dim V^a$. We define $\mathscr{O}_{\delta,a}=\{f(x);\ -\delta<x<\delta,\ f$ is analytic and takes its values in an open orbit of $G\backslash Gr(q, N)\}$.

PROPOSITION 4.8. *Let Y^0, $Y\in\mathscr{S}_{R,\delta,a}$ and $g\in\mathscr{G}_R^q$. If $g(x)\alpha(Y^0)(x)=\alpha(Y)(x)$ in $\mathscr{O}_{\delta,a}$, then there exists a positive real number ε such that $gaY^0=aY$ in $\mathscr{A}_{R,\varepsilon}/\mathscr{P}_{R,\varepsilon}$.*

PROOF. Working on $GL(N, \boldsymbol{C})$, we assume that $(a(s)Y(s, x, 0))^{-1}g(x)a(s)Y^0(s, x, 0)$ belongs to Q_a. As seen in the proof of Lemma 4.3, $(a(s)Y(s, x, 0))^{-1}\cdot g(s)a(s)Y^0(s, x, 0)$ has no pole in $|s|\leq R$. By Theorem 1.6, we obtain a real number $\varepsilon>0$ such that $(aY)^{-1}gaY^0$ lies in $\mathscr{N}_{R,\varepsilon}\mathscr{P}_{R,\varepsilon}$. So we can put $gaY^0=aYY'X$ with $Y'\in\mathscr{N}_{R,\varepsilon}$ and $X\in\mathscr{P}_{R,\varepsilon}$. Observe that Y' and X are even functions with respect to y and that $Y'(s, x, 0)=1$. Because YY' is in the Hauser-Ernst gauge, Lemma 4.4 implies that YY' belongs to $\mathscr{S}_{R,\varepsilon,a}$. By Theorem 3.2, we know that $YY'=Y$. □

PROPOSITION 4.9. *Let $a\in\mathrm{Hom}(\boldsymbol{C}^\times, GL(N, \boldsymbol{C}))$ be nice and f be in $\mathscr{O}_{\delta,a}$. Then there exist positive real numbers R and ε, $g\in G$ and $Y\in\mathscr{S}_{R,\varepsilon,b}$ for $b=gag^{-1}$ such that $\alpha(Y)(x)=b(s)Y(s, x, 0)V^b$ equals $f(x)$ in $Gr(q, N)$.*

PROOF. If $b=gag^{-1}$ and $Y\in\mathscr{S}_{R,\varepsilon,b}$, then $gV^a=V^b$ and $g^{-1}Yg\in\mathscr{S}_{R,\varepsilon,a}$. Therefore we suppose that f has an analytic representative in P_a. Then there exists $h(s)\in\mathscr{H}_{R,a}$ for some $0<R\leq\delta$ such that $h(x)f(0)=f(x)$ in $Gr(q, N)$. By

Lemma 4.1, we get a solution Y^0 such that $\alpha(Y^0)(x)=f(0)$. Now the proposition follows from Theorem 4.6. □

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