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Unbounded nonoscillatory solutions of nonlinear ordinary differential equations of arbitrary order

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1. Introduction

Consider the differential equation

$$(1.1) \qquad y^{(n)} + \sigma f(t, y, y', \dots, y^{(n-1)}) = 0,$$

where $n \geqq 2$, $\sigma = +1$ or -1, and f: $[0, \infty) \times \boldsymbol{R}^n \to \boldsymbol{R}$ is a continuous function such that

$$(1.2) \qquad y_0 f(t, y_0, y_1, \dots, y_{n-1}) \geqq 0 \qquad \text{for} \quad (t, y_0, y_1, \dots, y_{n-1}) \in [0, \infty) \times \boldsymbol{R}^n.$$

Let $\mathcal{N}$ denote the set of all nonoscillatory solutions of (1.1), that is, those solutions which are defined in some neighborhood of infinity and are eventually positive or negative. We denote by $\mathcal{N}_k$, $0 \leqq k \leqq n$, the set of all $y \in \mathcal{N}$ satisfying the inequalities

$$(1.3)_k \qquad \begin{cases} y(t)y^{(i)}(t) > 0, & t \geqq T_y, \quad 0 \leqq i \leqq k-1, \\ (-1)^{i-k} y(t)y^{(i)}(t) \geqq 0, & t \geqq T_y, \quad k \leqq i \leqq n \end{cases}$$

for $T_y > 0$ sufficiently large. Such an $\mathcal{N}_k$ is often referred to as a Kiguradze class for (1.1). Of basic importance is the fact [4, 5] that, under condition (1.2), every nonoscillatory solution $y \in \mathcal{N}$ of (1.1) falls into one and only one Kiguradze class $\mathcal{N}_k$ with k such that

$$(1.4) \qquad n \not\equiv k \pmod 2 \quad \text{if} \quad \sigma = +1, \quad \text{and} \quad n \equiv k \pmod 2 \quad \text{if} \quad \sigma = -1;$$

in other words, $\mathcal{N}$ has the following decomposition:

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{n-1} && \text{for} \quad \sigma = +1 \quad \text{and} \quad n \quad \text{even}, \\ \mathcal{N} &= \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{n-1} && \text{for} \quad \sigma = +1 \quad \text{and} \quad n \quad \text{odd}, \\ \mathcal{N} &= \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_n && \text{for} \quad \sigma = -1 \quad \text{and} \quad n \quad \text{even}, \\ \mathcal{N} &= \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_n && \text{for} \quad \sigma = -1 \quad \text{and} \quad n \quad \text{odd}. \end{aligned}$$

Note that (1.4) is equivalent to $(-1)^{n-k-1}\sigma = 1$.

The study of Kiguradze classes has been one of the central problems in the qualitative theory of higher order ordinary differential equations; the reader is referred to Foster and Grimmer [1], Lovelady [10, 11] and Naito [12] for results concerning $\mathcal{N}_k$, $0<k<n$, and to Hartman and Wintner [2], Kiguradze [4, 5, 6], Kiguradze and Kvinikadze [7] and Kvinikadze [9] for results concerning $\mathcal{N}_0$ and $\mathcal{N}_n$.

In what follows k is assumed to be an integer such that $0<k<n$ and $(-1)^{n-k-1}\sigma=1$, and our attention is restricted to the classes $\mathcal{N}_k$ for such k's. If $y\in\mathcal{N}_k$, then, in view of $(1.3)_k$, there exist positive constants c_1, c_2 and T such that

$$c_1 t^{k-1} \leqq |y(t)| \leqq c_2 t^k \quad \text{for} \quad t \geqq T; \tag{1.5}$$

more precisely, exactly one of the following three cases holds:

$$\lim_{t\to\infty} y(t)/t^k = \text{const} \neq 0; \tag{1.6}$$

$$\lim_{t\to\infty} y(t)/t^k = 0 \quad \text{and} \quad \lim_{t\to\infty} y(t)/t^{k-1} = +\infty \quad \text{or} \quad -\infty; \tag{1.7}$$

$$\lim_{t\to\infty} y(t)/t^{k-1} = \text{const} \neq 0. \tag{1.8}$$

The conditions (1.6)–(1.8) may be rewritten, respectively, as

$$\lim_{t\to\infty} y^{(k)}(t) = \text{const} \neq 0; \tag{1.6'}$$

$$\lim_{t\to\infty} y^{(k)}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} y^{(k-1)}(t) = +\infty \quad \text{or} \quad -\infty; \tag{1.7'}$$

$$\lim_{t\to\infty} y^{(k-1)}(t) = \text{const} \neq 0. \tag{1.8'}$$

The set of all $y\in\mathcal{N}_k$ satisfying (1.6), (1.7) or (1.8) is denoted, respectively, by $\mathcal{N}_k[\max]$, $\mathcal{N}_k[\text{int}]$ or $\mathcal{N}_k[\min]$; thus

$$\mathcal{N}_k = \mathcal{N}_k[\max] \cup \mathcal{N}_k[\text{int}] \cup \mathcal{N}_k[\min].$$

The classes $\mathcal{N}_k[\max]$, $\mathcal{N}_k[\min]$, $0<k<n$, have been extensively studied in the literature. However, it seems to us that very little is known about the intermediate classes $\mathcal{N}_k[\text{int}]$. As far as we are aware, the only references concerning the existence of members of $\mathcal{N}_k[\text{int}]$ are Heidel [3], the authors [8] and Naito [12], in which some special cases of (1.1) are considered.

The objective of this paper is to propose a systematic study of the classes $\mathcal{N}_k[\text{int}]$, $0<k<n$, for *sublinear* equations of the form (1.1). We first present sufficient conditions which guarantee the existence of members of $\mathcal{N}_k[\text{int}]$ for (1.1). We then show that there is a class of equations of the form (1.1) for which the situation $\mathcal{N}_k[\text{int}]\neq\emptyset$ can be completely characterized. Our results can be applied to the equation

$$y^{(n)} + \sigma p(t)|y|^\gamma \operatorname{sgn} y = 0, \quad 0<\gamma<1, \tag{1.9}$$

where p: $[0, \infty)\to[0, \infty)$ is continuous. In particular, Theorem 2 implies that (1.9) has a solution $y(t)$ satisfying (1.7) if and only if

$$\int^{\infty} p(t)t^{n-k-1+k\gamma}dt < \infty \quad \text{and} \quad \int^{\infty} p(t)t^{n-k+(k-1)\gamma}dt = \infty.$$

2. Conditions for $\mathcal{N}_k[\mathrm{int}]\neq\emptyset$

We first give a sufficient condition under which there exists a solution $y(t)$ of (1.1) satisfying (1.7), i.e., $\mathcal{N}_k[\mathrm{int}]\neq\emptyset$ for (1.1).

THEOREM 1. *Let k be an integer such that $0<k<n$ and $(-1)^{n-k-1}\sigma=1$. In addition to (1.2) suppose that there exist continuous functions f_*, f^*: $[0, \infty)^{k+1}\to[0, \infty)$ and ψ_*, ψ^*: $[0, \infty)^{n-k}\to(0, \infty)$ such that*

$$\begin{aligned}(2.1)\qquad & f_*(t, |y_0|,\dots, |y_{k-1}|)\psi_*(|y_k|,\dots, |y_{n-1}|) \\ &\leqq |f(t, y_0,\dots, y_{n-1})| \leqq f^*(t, |y_0|,\dots, |y_{k-1}|)\psi^*(|y_k|,\dots, |y_{n-1}|) \\ &\qquad\qquad \textit{for} \quad (t, y_0,\dots, y_{n-1})\in[0, \infty)\times \boldsymbol{R}^n.\end{aligned}$$

Suppose moreover that $f_*(t, z_0,\dots, z_{k-1})$ and $f^*(t, z_0,\dots, z_{k-1})$ are nondecreasing in z_i, $0\leqq i\leqq k-1$, $\psi_*(z_k,\dots, z_{n-1})$ is nonincreasing in z_i, $k\leqq i\leqq n-1$, and $\psi^*(z_k,\dots, z_{n-1})$ is nondecreasing in z_i, $k\leqq i\leqq n-1$.

Then, $\mathcal{N}_k[\mathrm{int}]$ is nonempty for (1.1) if

$$(2.2)\qquad \int_0^{\infty} t^{n-k-1}f^*(t, at^k,\dots, at)dt < \infty \quad \textit{for some} \quad a > 0$$

and

$$(2.3)\qquad \int_0^{\infty} t^{n-k}f_*(t, bt^{k-1},\dots, b)dt = \infty \quad \textit{for all} \quad b > 0.$$

PROOF. Let c, $0<c<a$, be fixed. Choose $T>0$ large enough so that

$$\int_T^{\infty} t^{n-i-1}f^*(t, c(t+1)^k,\dots, c(t+1))dt \leqq \frac{c}{\psi^*(c,\dots, c)} \quad \text{for} \quad k\leqq i\leqq n-1.$$

Denote by Y the set of all $y\in C^{n-1}[T, \infty)$ satisfying the following inequalities

$$(2.4)\qquad \begin{cases} \dfrac{ct^{k-i-1}}{(k-i-1)!} \leqq y^{(i)}(t) \leqq \dfrac{ct^{k-i-1}}{(k-i-1)!} + \dfrac{ct^{k-i}}{(k-i)!}, & 0\leqq i\leqq k-1, \\ 0\leqq(-1)^{i-k}y^{(i)}(t)\leqq c, \quad k\leqq i\leqq n-1, \end{cases}$$

for $t\geqq T$ and define the integro-differential operator $\mathcal{F}$ by

(2.5)

$$\mathcal{F}y(t) = \frac{ct^{k-1}}{(k-1)!} + \int_T^t \frac{(t-s)^{k-1}}{(k-1)!}\int_s^{\infty}\frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, y(r),\dots, y^{(n-1)}(r))drds$$

for $t \geqq T$. $\mathscr{F}$ is well defined on Y, since $y \in Y$ implies $|y^{(i)}(t)| \leqq c(t+1)^{k-i}$, $0 \leqq i \leqq k-1$, and $|y^{(i)}(t)| \leqq c$, $k \leqq i \leqq n-1$, for $t \geqq T$. Noting that

$$(\mathscr{F}y)^{(i)}(t) = \frac{ct^{k-i-1}}{(k-i-1)!} + \int_T^t \frac{(t-s)^{k-i-1}}{(k-i-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, y(r), \dots, y^{(n-1)}(r))drds \quad \text{for} \quad 0 \leqq i \leqq k-1,$$

$$(\mathscr{F}y)^{(i)}(t) = (-1)^{i-k} \int_t^\infty \frac{(s-t)^{n-i-1}}{(n-i-1)!} f(s, y(s), \dots, y^{(n-1)}(s))ds \quad \text{for} \quad k \leqq i \leqq n-1,$$

we see that for $y \in Y$ and $t \geqq T$

$$\begin{aligned}
&(\mathscr{F}y)^{(i)}(t) \\
&\leqq \frac{ct^{k-i-1}}{(k-i-1)!} + \int_T^t \frac{(t-s)^{k-i-1}}{(k-i-1)!} ds \cdot \int_T^\infty \frac{(r-T)^{n-k-1}}{(n-k-1)!} f^*(r, c(r+1)^k, \dots, c(r+1))dr \cdot \psi^*(c, \dots, c) \\
&\leqq \frac{ct^{k-i-1}}{(k-i-1)!} + \frac{ct^{k-i}}{(k-i)!}, \quad 0 \leqq i \leqq k-1,
\end{aligned}$$

and

$$\begin{aligned}
&(-1)^{i-k}(\mathscr{F}y)^{(i)}(t) \\
&\leqq \int_T^\infty \frac{(s-T)^{n-i-1}}{(n-i-1)!} f^*(s, c(s+1)^k, \dots, c(s+1))ds \cdot \psi^*(c, \dots, c) \\
&\leqq c, \quad k \leqq i \leqq n-1,
\end{aligned}$$

which shows that $\mathscr{F}y \in Y$. Therefore $\mathscr{F}$ maps Y, a closed convex subset of $C^{n-1}[T, \infty)$, into itself. Furthermore it is easy to verify that $\mathscr{F}$ is continuous and $\mathscr{F}Y$ is relatively compact in the topology of $C^{n-1}[T, \infty)$. Therefore $\mathscr{F}$ has a fixed point $y \in Y$ by the Schauder-Tychonoff theorem. Differentiating the integro-differential equation $y(t) = \mathscr{F}y(t)$, $t \geqq T$, we see that $y = y(t)$ is a solution of equation (1.1) on $[T, \infty)$. That $\lim_{t\to\infty} y^{(k)}(t) = 0$ is a consequence of the equality

$$y^{(k)}(t) = \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, y(s), \dots, y^{(n-1)}(s))ds, \quad t \geqq T.$$

From (2.3) and

$$\begin{aligned}
y^{(k-1)}(t) &= c + \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, y(r), \dots, y^{(n-1)}(r))drds \\
&\geqq c + \psi_*(c, \dots, c) \cdot \int_T^t \frac{(r-T)^{n-k}}{(n-k)!} f_*\left(r, \frac{cr^{k-1}}{(k-1)!}, \dots, c\right) dr,
\end{aligned}$$

it follows that $\lim_{t\to\infty} y^{(k-1)}(t)=\infty$. Thus $y=y(t)$ is a positive solution of (1.1) belonging to $\mathcal{N}_k[\mathrm{int}]$.

Similarly, it can be shown that the mapping $\mathcal{F}$ defined by (2.5) with c replaced by $-c$ has a fixed element in the set consisting of all $y\in C^{n-1}[T, \infty)$ satisfying for $t\geqq T$

$$\begin{cases} -\dfrac{ct^{k-i-1}}{(k-i-1)!}-\dfrac{ct^{k-i}}{(k-i)!}\leqq y^{(i)}(t)\leqq -\dfrac{ct^{k-i-1}}{(k-i-1)!}, & 0\leqq i\leqq k-1, \\ -c\leqq (-1)^{i-k}y^{(i)}(t)\leqq 0, \quad k\leqq i\leqq n-1. \end{cases}$$

This fixed point gives a negative solution of (1.1) belonging to $\mathcal{N}_k[\mathrm{int}]$. The proof of Theorem 1 is complete.

In the next theorem a necessary and sufficient condition for $\mathcal{N}_k[\mathrm{int}]\neq\emptyset$ is given for equation (1.1) with stronger nonlinearity.

THEOREM 2. *Let k be an integer such that $0<k<n$ and $(-1)^{n-k-1}\sigma=1$. Suppose that the function f in* (1.2) *satisfies* (1.2) *and* (2.1) *with monotone functions f_*, f^*, ψ_*, ψ^* as described in Theorem* 1. *Suppose moreover that there exist a constant $M>0$ and continuous functions ω_*: $(0, 1]\to(0, \infty)$ and ω^*: $[1, \infty)\to(0, \infty)$ such that*

$$f^*(t, z_0,\dots, z_{k-1})\leqq Mf_*(t, z_0,\dots, z_{k-1}) \quad \textit{for} \quad (t, z_0,\dots, z_{k-1})\in(0, \infty)^{k+1}; \tag{2.6}$$

$$f_*(t, \xi z_0,\dots, \xi z_{k-1})\geqq \omega_*(\xi)f_*(t, z_0,\dots, z_{k-1}) \quad \textit{for}\ (t, z_0,\dots, z_{k-1}, \xi)\in(0, \infty)^{k+1}\times(0, 1]; \tag{2.7}$$

$$\int_0^1 \frac{d\xi}{\omega_*(\xi)}<\infty; \tag{2.8}$$

$$f^*(t, \xi z_0,\dots, \xi z_{k-1})\leqq \omega^*(\xi)f^*(t, z_0,\dots, z_{k-1}) \quad \textit{for} \quad (t, z_0,\dots, z_{k-1}, \xi)\in(0, \infty)^{k+1}\times[1, \infty); \tag{2.9}$$

$$\limsup_{\xi\to\infty}\frac{\omega^*(\xi)}{\xi}<\infty. \tag{2.10}$$

Then, $\mathcal{N}_k[\mathrm{int}]$ is nonempty for (1.1) *if and only if both* (2.2) *and* (2.3) *are satisfied.*

The "if" part follows from Theorem 1. To prove the "only if" part the following lemma is needed.

LEMMA. *If $y\in\mathcal{N}_k[\mathrm{int}]$ for* (1.1), *then*
(i) *there is a $c>0$ such that*

(2.11) $$\frac{|y(t)|}{t^{k-1}} \geqq c\frac{|y^{(i)}(t)|}{t^{k-i-1}} \quad \textit{for all large } t, \quad 0 \leqq i \leqq k, \quad \textit{and}$$

(ii) $|y(t)|/t^{k-1}$ *is nondecreasing for all large* t.

PROOF. Without loss of generality we may suppose that $y(t)$ is eventually positive, so that there is $T>0$ such that for $t \geqq T$

(2.12) $$y^{(i)}(t) > 0, \quad 0 \leqq i \leqq k-1, \quad \text{and} \quad (-1)^{i-k}y^{(i)}(t) \geqq 0, \quad k \leqq i \leqq n.$$

As in [4, the proof of Lemma 2], $y(t)$ satisfies

$$y^{(k-i)}(t) \geqq \frac{1}{i}(t-T)y^{(k-i+1)}(t), \quad t \geqq T, \quad 1 \leqq i \leqq k,$$

and in particular

$$y(t) \geqq \frac{1}{k!}(t-T)^i y^{(i)}(t), \quad t \geqq T, \quad 1 \leqq i \leqq k.$$

Thus, if c is a constant with $0<c<1/k!$, then

$$\frac{y(t)}{t^{k-1}} - c\frac{y^{(i)}(t)}{t^{k-i-1}} \geqq t^{1-k}\left[y(t) - \frac{1}{k!}(t-T)^i y^{(i)}(t)\right] \geqq 0, \quad 0 \leqq i \leqq k,$$

for large t, proving the truth of (2.11). To verify (ii), it is enough to show that

(2.13) $$y'(t)t - (k-1)y(t) \geqq 0 \quad \text{for large} \quad t,$$

since $(y(t)/t^{k-1})' = t^{-k}[y'(t)t-(k-1)y(t)]$. If $k=1$, then (2.13) is obvious by (2.12). If $k>1$, then, since $y^{(k-1)}(t)$ is nondecreasing, the mean value theorem for $y^{(k-2)}(t)$ implies

$$y^{(k-2)}(t) \leqq y^{(k-2)}(T) + y^{(k-1)}(t)(t-T), \quad t \geqq T.$$

Integrating the above inequality $k-2$ times from T to t, we obtain after some manipulations

$$(k-1)y(t) \leqq \sum_{j=0}^{k-2}\frac{k-j-1}{j!}y^{(j)}(T)(t-T)^j + y'(t)(t-T), \quad t \geqq T,$$

which yields

$$t^{2-k}[y'(t)t-(k-1)y(t)] \geqq t^{2-k}\left(-\sum_{j=0}^{k-2}\frac{k-j-1}{j!}y^{(j)}(T)(t-T)^j + Ty'(t)\right)$$

for $t \geqq T$. The right hand side of the above tends to ∞ as $t \to \infty$, because $\lim_{t\to\infty} y'(t)/t^{k-2} = \infty$ for any eventually positive $y \in \mathcal{N}_k[\text{int}]$. Therefore (2.13) is satisfied, and so the proof of Lemma is complete.

THE PROOF OF THE "ONLY IF" PART OF THEOREM 2. Let $y \in \mathcal{N}_k[\text{int}]$ for (1.1). We first show that (2.2) holds. We may assume that (2.12) holds for $t \geqq T$. Then, the latter of (2.12) implies that $(-1)^{i-k} y^{(i)}(t) \leqq c_0$, $t \geqq T$, $k \leqq i \leqq n-1$, for some $c_0 > 0$. Combining this fact with the equation

$$y^{(k)}(t) = \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, y(s), \dots, y^{(n-1)}(s)) ds, \quad t \geqq T, \tag{2.14}$$

which follows easily from (1.1), we have

$$y^{(k)}(t) \geqq c_1 \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f_*(s, y(s), \dots, y^{(k-1)}(s)) ds, \quad t \geqq T, \tag{2.15}$$

where $c_1 = \psi_*(c_0, \dots, c_0) > 0$. Therefore, repeated integration of (2.15) over $[T, t]$ shows that

$$\begin{aligned} y^{(i)}(t) &\geqq c_1 \int_T^t \frac{(t-s)^{k-i-1}}{(k-i-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f_*(r, y(r), \dots, y^{(k-1)}(r)) dr ds \\ &= c_1 \int_T^t \int_T^r \frac{(t-s)^{k-i-1}(r-s)^{n-k-1}}{(k-i-1)!(n-k-1)!} f_*(r, y(r), \dots, y^{(k-1)}(r)) ds dr \\ &\quad + c_1 \int_t^\infty \int_T^t \frac{(t-s)^{k-i-1}(r-s)^{n-k-1}}{(k-i-1)!(n-k-1)!} f_*(r, y(r), \dots, y^{(k-1)}(r)) ds dr \\ &\geqq c_2 (t-T)^{k-i-1} \int_T^t (r-T)^{n-k} f_*(r, y(r), \dots, y^{(k-1)}(r)) dr \\ &\quad + c_2 (t-T)^{k-i} \int_t^\infty (r-T)^{n-k-1} f_*(r, y(r), \dots, y^{(k-1)}(r)) dr, \\ &\qquad\qquad t \geqq T, \quad 0 \leqq i \leqq k-1, \end{aligned}$$

where $c_2 = c_1[(n-1)(k-1)!(n-k-1)!]^{-1} > 0$; in particular

$$y^{(i)}(t) \geqq c_2 (t-T)^{k-i} \int_t^\infty (r-T)^{n-k-1} f_*(r, y(r), \dots, y^{(k-1)}(r)) dr, \quad t \geqq T, \quad 0 \leqq i \leqq k-1. \tag{2.16}$$

If $T_2 > T$ is taken so large that

$$\xi(t) = \int_t^\infty (r-T)^{n-k-1} f_*(r, y(r), \dots, y^{(k-1)}(r)) dr \leqq 1 \qquad \text{for} \quad t \geqq T_2,$$

then the hypotheses on f_* and (2.16) imply

$$\begin{aligned} &f_*(t, y(t), \dots, y^{(k-1)}(t)) \\ &\quad \geqq f_*(t, c_2(t-T)^k \xi(t), \dots, c_2(t-T)\xi(t)) \\ &\quad \geqq f_*(t, c_2(t-T)^k, \dots, c_2(t-T)) \omega_*(\xi(t)), \quad t \geqq T_2, \end{aligned}$$

so that $\xi(t)$ satisfies

$$(2.17) \quad -\frac{\xi'(t)}{\omega_*(\xi(t))} \geqq (t-T)^{n-k-1} f_*(t, c_2(t-T)^k, \dots, c_2(t-T)), \quad t \geqq T_2.$$

Integration of (2.17) yields

$$\int_{\xi(t)}^{\xi(T_2)} \frac{d\xi}{\omega_*(\xi)} \geqq \int_{T_2}^{t} (s-T)^{n-k-1} f_*(s, c_2(s-T)^k, \dots, c_2(s-T))ds, \quad t \geqq T_2,$$

which, in view of (2.8), implies

$$\int_{T_2}^{\infty} (s-T)^{n-k-1} f_*(s, c_2(s-T)^k, \dots, c_2(s-T))ds < \infty.$$

Because of (2.6) this means that (2.2) holds.

Our final task is to prove (2.3). Suppose to the contrary that (2.3) does not hold. By (2.6) this may be restated as

$$(2.18) \qquad \int_0^{\infty} t^{n-k} f^*(t, bt^{k-1}, \dots, b)dt < \infty \quad \text{for some} \quad b > 0.$$

We now integrate (2.14) to get

$$\begin{aligned} y(t) &= \sum_{j=0}^{k-1} \frac{y^{(j)}(T)}{j!}(t-T)^j + \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \\ &\qquad\qquad \cdot f(r, y(r), \dots, y^{(n-1)}(r))drds \\ &\leqq \sum_{j=0}^{k-1} \frac{y^{(j)}(T)}{j!}(t-T)^j \\ &\quad + \frac{\psi^*(c_0, \dots, c_0)}{(k-1)!(n-k-1)!} \int_T^t \int_T^r \\ &\qquad\qquad (t-s)^{k-1}(r-s)^{n-k-1} f^*(r, y(r), \dots, y^{(k-1)}(r))dsdr \\ &\quad + \frac{\psi^*(c_0, \dots, c_0)}{(k-1)!(n-k-1)!} \int_t^{\infty} \int_T^t \\ &\qquad\qquad (t-s)^{k-1}(r-s)^{n-k-1} f^*(r, y(r), \dots, y^{(k-1)}(r))dsdr, \quad t \geqq T. \end{aligned}$$

Using the inequalities

$$\int_T^r (t-s)^{k-1}(r-s)^{n-k-1}ds \leqq (t-T)^{k-1}(r-T)^{n-k}, \quad T \leqq r \leqq t,$$

$$\int_T^t (t-s)^{k-1}(r-s)^{n-k-1}ds \leqq (t-T)^k(r-T)^{n-k-1}, \quad T \leqq t \leqq r,$$

in the above, we obtain

$$\begin{aligned}(2.19)\quad y(t) \leqq & \sum_{j=0}^{k-1} \frac{y^{(j)}(T)}{j!}(t-T)^j \\ & + c_3(t-T)^{k-1}\int_T^t (r-T)^{n-k} f^*(r, y(r),\dots, y^{(k-1)}(r))dr \\ & + c_3(t-T)^k \int_t^\infty (r-T)^{n-k-1} f^*(r, y(r),\dots, y^{(k-1)}(r))dr, \quad t \geqq T,\end{aligned}$$

for a positive constant c_3. Using the hypotheses on f^*, Lemma and the fact that $\lim_{t\to\infty} y(t)/t^{k-1} = \infty$, we see that

$$\begin{aligned}(2.20)\quad & f^*(r, y(r),\dots, y^{(k-1)}(r)) \\ & = f^*\left(r, \frac{1}{bc}\cdot c\frac{y(r)}{r^{k-1}}\cdot br^{k-1}, \frac{1}{bc}\cdot c\frac{y'(r)}{r^{k-2}}\cdot br^{k-2},\dots, \frac{1}{bc}\cdot c\, y^{(k-1)}(r)\cdot b\right) \\ & \leqq f^*\left(r, \frac{y(r)}{bcr^{k-1}}\cdot br^{k-1}, \frac{y(r)}{bcr^{k-1}}\cdot br^{k-2},\dots, \frac{y(r)}{bcr^{k-1}}\cdot b\right) \\ & \leqq f^*\left(r, \frac{y(t)}{bct^{k-1}}\cdot br^{k-1}, \frac{y(t)}{bct^{k-1}}\cdot br^{k-2},\dots, \frac{y(t)}{bct^{k-1}}\cdot b\right) \\ & \leqq f^*(r, br^{k-1}, br^{k-2},\dots, b)\,\omega^*\left(\frac{y(t)}{bct^{k-1}}\right)\end{aligned}$$

for sufficiently large r and t, $r \leqq t$. From (2.19) and (2.20) it follows that, if T is chosen large enough,

$$\begin{aligned}(2.21)\quad 1 \leqq & \sum_{j=0}^{k-1} \frac{y^{(j)}(T)}{j!}\frac{t^j}{y(t)} \\ & + c_3\frac{t^{k-1}}{y(t)}\,\omega^*\left(\frac{y(t)}{bct^{k-1}}\right)\int_T^t r^{n-k} f^*(r, br^{k-1}, br^{k-2},\dots, b)dr \\ & + c_3\frac{t^k}{y(t)}\int_t^\infty r^{n-k-1} f^*(r, y(r), y'(r),\dots, y^{(k-1)}(r))dr, \quad t \geqq T.\end{aligned}$$

Since $t^j/y(t) \to 0$ as $t\to\infty$, $0 \leqq j \leqq k-1$, and $\frac{t^{k-1}}{y(t)}\omega^*\left(\frac{y(t)}{bct^{k-1}}\right)$ is bounded by (2.10), and

$$\int_T^\infty r^{n-k} f^*(r, br^{k-1}, br^{k-2},\dots, b)dr \to 0 \quad \text{as} \quad T\to\infty$$

by (2.18), we see from (2.21) that there are $c_4 > 0$ and $T' \geqq T$ such that

$$(2.22)\qquad y(t) \leqq c_4 t^k \int_t^\infty r^{n-k-1} f^*(r, y(r),\dots, y^{(k-1)}(r))dr, \quad t \geqq T'.$$

Put

$$(2.23)\qquad z(t) = \int_t^\infty r^{n-k-1} f^*(r, y(r),\dots, y^{(k-1)}(r))dr, \quad t \geqq T'.$$

Since, as in (2.20),

$$\begin{aligned} &f^*(r, y(r),\dots, y^{(k-1)}(r)) \\ &\leqq f^*(r, br^{k-1},\dots, b)\omega^*\left(\frac{y(r)}{bcr^{k-1}}\right) \\ &\leqq K\frac{y(r)}{bcr^{k-1}} f^*(r, br^{k-1},\dots, b), \quad r \geqq T', \end{aligned}$$

where $K=\sup_{\xi\geqq 1}\omega^*(\xi)/\xi>0$, we have from (2.22) and (2.23)

$$\begin{aligned} z(t) &\leqq \frac{c_4K}{bc}\int_t^\infty r^{n-k-1}\cdot\frac{r^k z(r)}{r^{k-1}}\cdot f^*(r, br^{k-1},\dots, b)dr \\ &\leqq \frac{c_4K}{bc} z(t)\int_t^\infty r^{n-k} f^*(r, br^{k-1},\dots, b)dr, \quad t \geqq T', \end{aligned}$$

and consequently,

$$1 \leqq \frac{c_4K}{bc}\int_t^\infty r^{n-k} f^*(r, br^{k-1},\dots, b)dr, \quad t \geqq T'.$$

However this is a contradiction, since the right hand side tends to 0 as $t\to\infty$. Thus we conclude that (2.3) is satisfied. The proof of Theorem 2 is complete.

EXAMPLE. Let k be such that $0<k<n$ and $(-1)^{n-k-1}\sigma=1$, and consider the equation

$$(2.24)\qquad y^{(n)} + \sigma p(t)|y|^{\gamma_0}|y'|^{\gamma_1}\cdots|y^{(k-1)}|^{\gamma_{k-1}}\operatorname{sgn} y = 0,$$

where $\sigma=+1$ or -1, $\gamma_0\geqq 0$, $\gamma_1\geqq 0,\dots, \gamma_{k-1}\geqq 0$ are constants such that

$$(2.25)\qquad \gamma_0 + \gamma_1 + \cdots + \gamma_{k-1} < 1,$$

and p: $[0, \infty)\to[0, \infty)$ is continuous. The function

$$f(t, y_0, y_1,\dots, y_{n-1}) = p(t)|y_0|^{\gamma_0}|y_1|^{\gamma_1}\cdots|y_{k-1}|^{\gamma_{k-1}}\operatorname{sgn} y_0$$

satisfies all the hypotheses of Theorem 2 with

$$\begin{aligned} &f_* = f^* = p(t)z_0^{\gamma_0}z_1^{\gamma_1}\cdots z_{k-1}^{\gamma_{k-1}}, \quad \psi_* = \psi^* \equiv 1, \\ &\omega_* = \omega^* = \xi^{\gamma_0+\gamma_1+\cdots+\gamma_{k-1}}. \end{aligned}$$

Conditions (2.2) and (2.3) reduce, respectively, to

$$(2.26)\qquad \int_0^\infty p(t)t^{n-k-1+\gamma_0 k+\gamma_1(k-1)+\cdots+\gamma_{k-1}}dt < \infty$$

and

$$\int_0^\infty p(t)t^{n-k+\gamma_0(k-1)+\gamma_1(k-2)+\cdots+\gamma_{k-2}}dt = \infty, \tag{2.27}$$

and so (2.26) plus (2.27) is necessary and sufficient for (2.24) to have a (nonoscillatory) solution belonging to $\mathcal{N}_k[\text{int}]$. Notice that (2.26) and (2.27) are compatible by the sublinear condition (2.25). When specialized to the case $\gamma_1 = \cdots = \gamma_{k-1} = 0$, our result asserts that $\mathcal{N}_k[\text{int}]$ is nonempty for the equation

$$y^{(n)} + \sigma p(t)|y|^\gamma \operatorname{sgn} y = 0, \quad 0 \leqq \gamma < 1, \tag{2.28}$$

if and only if

$$\int_0^\infty p(t)t^{n-k-1+\gamma k}dt < \infty \tag*{$(2.29)_k$}$$

and

$$\int_0^\infty p(t)t^{n-k+\gamma(k-1)}dt = \infty. \tag*{$(2.30)_k$}$$

It should be noted that an integer k for which both $(2.29)_k$ and $(2.30)_k$ hold, if one exists, is unique, that is, for equation (2.28) one and only one of the intermediate classes $\mathcal{N}_k[\text{int}]$, $0<k<n$, may have a member. In fact, this follows from the observation that $(2.29)_k$ implies $(2.29)_{k+1}$, that $(2.30)_k$ implies $(2.30)_{k-1}$ and that $(2.29)_k$ holds if and only if $(2.30)_{k+1}$ does not hold.

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