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On the structure of connection coefficients for hypergeometric systems

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Introduction

In this paper we shall be concerned with a connection problem for the so-called hypergeometric system of linear differential equations

$$(0.1)\qquad (t-B)\frac{dX}{dt} = AX \quad (t\in \boldsymbol{C}),$$

where X is an n-dimensional column vector, A is an n by n constant matrix and B is an n by n diagonal matrix. This is a Fuchsian system with regular singularities at diagonal elements of B and infinity in the whole complex t-plane.

The global study of (0.1) was initiated by K. Okubo [9], who investigated an effective method of algebraic computation of the monodromy group for (0.1) without accessory parameters, together with the reduction of every single Fuchsian differential equation to (0.1) ([10], see also [7]). R. Schäfke [12] and W. Balser-W. B. Jurkat-D. A. Lutz [1] cleared up the relation between connection coefficients of (0.1) and the Stokes multipliers of the Birkhoff system of linear differential equations

$$z\frac{dY}{dz} = \{-(1+A)+Bz\}Y,$$

which has a regular singularity at $z=0$ and an irregular singularity of rank 1 at $z=\infty$, through the Laplace transformation $Y(z)=\int X(t)e^{zt}dt$.

Recently M. Kohno [5] has shown that the connection problem for (0.1) can be solved by a global analysis of the system of linear difference equations

$$(0.2)\qquad (B-\lambda)(z+1)G(z+1) = (z-A)G(z) \quad (z\in \boldsymbol{C})$$

which gives the coefficients in power series solutions of (0.1). In [6] he has also analyzed a case when there appear logarithmic solutions at finite singularities and has shown the global Frobenius theorem. By means of the method of [5], the author [14] (see also [13]) has analyzed completely (0.1) in the case when A is diagonalizable and has only two distinct eigenvalues, and has verified the following results:

(i) Principal solutions of (0.2) in the right half z-plane give the solutions of

(0.1) which are holomorphic everywhere in the finite complex t-plane except at only one singular point;

(ii) The non-holomorphic solution of (0.1) near a finite singularity is given by a solution of (0.2) which has zeros in the left half z-plane; and

(iii) Connection coefficients between such solutions as stated in (i) and (ii) and a fundamental set of solutions near infinity are expressed explicitly in terms of solutions of a certain (appropriately determined) hypergeometric system of dimension $n-1$.

The purpose of this paper is to prove such results (i), (ii) and (iii) in a more general case when there appear logarithmic solutions not only at finite singularities but also at infinity. Moreover, we consider especially a new connection problem between solutions of (0.1) near finite singularities.

In (0.1) let us assume that B has multiple eigenvalues, i.e.,

$$B = \mathrm{diag}\,[\overbrace{\lambda_1,\dots,\lambda_1}^{n_1}, \overbrace{\lambda_2,\dots,\lambda_2}^{n_2},\dots, \overbrace{\lambda_p,\dots,\lambda_p}^{n_p}]$$
$$(\lambda_k \neq \lambda_l\ (k\neq l),\ n_k \geqq 1,\ n_1+n_2+\cdots+n_p=n).$$

We write A in the $(n_1, n_2,\dots, n_p)$-partitioned form

$$A = [A_{jk}] \quad (\text{i.e., } A_{jk} \text{ is an } n_j \text{ by } n_k \text{ matrix})$$

and denote the distinct eigenvalues of A by μ_l $(l=1,\dots, q)$. For simplicity we suppose that A_{kk} $(k=1,\dots, p)$ consist of only one Jordan canonical block, i.e.,

$$A_{kk} = \nu_k + J(n_k) \quad (k=1,\dots, p)$$

and the Jordan canonical form of A has only one Jordan block for each μ_l $(l=1,\dots, q)$, i.e., A is similar to

$$(\mu_1+J(m_1)) \oplus (\mu_2+J(m_2)) \oplus\cdots\oplus (\mu_q+J(m_q))$$
$$(m_l\geqq 1,\ m_1+m_2+\cdots+m_q=n),$$

where $J(m)$ denotes the m-dimensional shifting matrix, i.e.,

$$J(m) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \in M_m(\boldsymbol{C}).$$

Moreover, we assume the following:

[A_0] No three λ_k lie on a straight line.

[A_1] None of the quantities ν_k $(k=1,\dots, p)$ and $\mu_l-\mu_k$ $(l, k=1,\dots, q,$ $l\neq k)$ is an integer.

[A_2] None of the quantities μ_l $(l=1,\dots,q)$ and $\mu_l-\nu_k$ $(l=1,\dots,q,\ k=1,\dots,p)$ is an integer.

The condition [A_2] is related to the reducibility of (0.1) (see [10] and [1]). In more general cases when A_{kk} $(k=1,\dots,p)$ consist of several Jordan canonical blocks or the Jordan canonical form of A has several Jordan blocks for each μ_l $(l=1,\dots,q)$ (with some generic conditions corresponding to [$A_0 \sim A_2$]), we only need a slight modification of the consideration which will be stated below.

Our development proceeds along the following line. In Section 1 we are concerned with local solutions of (0.1) near singularities. In Section 2 we investigate the system (0.2) through the Mellin transformation. In Section 3 we analyze the Barnes integral representations of solutions of (0.1) and clear up the structure of connection coefficients between solutions of (0.1) near a finite singularity and near infinity. In Section 4, using the Barnes integral representations obtained in §3, we investigate the connection coefficients between solutions of (0.1) near finite singularities.

§ 1. Local solutions

Since (0.1) is a Fuchsian system with regular singularities $t=\lambda_k$ $(k=1,\dots,p)$ and ∞, (0.1) has convergent series solutions at each singularity. As to solutions near the finite singularities it is sufficient to consider them at one singular point, for instance, λ_p. Changing the variable t for $t'=t-\lambda_p$, we may assume that $\lambda_p=0$. Moreover, without loss of generality, we may assume that

$$\arg\lambda_1 < \arg\lambda_2 < \cdots < \arg\lambda_{p-1} < \arg\lambda_1 + 2\pi \tag{1.1}$$

by the assumption [A_0]. Hereafter we denote ν_p and n_p by ν' and n', respectively.

Near $t=0$ there exist n' non-holomorphic solutions of (0.1) of the form

$$\hat{X}_j(t) = \sum_{j'=0}^{j} \frac{1}{(j-j')!} (\log t)^{j-j'} t^{\nu'} \hat{x}_{j'}(t) \qquad (j=0, 1,\dots, n'-1), \tag{1.2}$$

where

$$\hat{x}_j(t) = \sum_{m=0}^{\infty} \hat{G}_j(m) t^m \qquad (j=0, 1,\dots, n'-1) \tag{1.3}$$

which are convergent for $|t|<R$, $R=\min\{|\lambda_k|;\ k=1,\dots,p-1\}$. The coefficient vectors $\hat{G}_j(m)$ $(m\geqq 0, j=0, 1,\dots, n'-1)$ are characterized as solutions of the systems of linear difference equations

$$(1.4)_j \qquad \begin{cases} B(m+1+\nu')\hat{G}_j(m+1) \\ \quad = (m+\nu'-A)\hat{G}_j(m) - B\hat{G}_{j-1}(m+1) + \hat{G}_{j-1}(m) \quad (m\geqq 0) \\ B\nu'\hat{G}_j(0) = -B\hat{G}_{j-1}(0), \quad \hat{G}_j(0) \neq 0 \quad (j=0, 1,\dots, n'-1), \end{cases}$$

where $\hat{G}_{-1}(m)\equiv 0$. Besides, there exist $n-n'$ holomorphic solutions of (0.1) whose coefficient vectors are characterized as a solution of the system of linear difference equations with ν' replaced by 0 in $(1.4)_0$, i.e., (0.2) with $\lambda=0$.

On the other hand, near $t=\infty$ there exist n linearly independent solutions of (0.1) of the form

$$Y^{lr}(t) = \sum_{r'=0}^{r} \frac{1}{(r-r')!} (\log t)^{r-r'} t^{\mu_l} y^{lr'}(t) \qquad (1.5)$$

$$(l=1,\dots, q,\ r=0,\ 1,\dots,\ m_l-1),$$

where

$$y^{lr}(t) = \sum_{s=0}^{\infty} H^{lr}(s) t^{-s} \quad (l=1,\dots, q,\ r=0,\ 1,\dots,\ m_l-1) \qquad (1.6)$$

which are convergent for $|t|>R'$, $R'=\max\{|\lambda_k|;\ k=1,\dots, p-1\}$. The coefficient vectors $H^{lr}(s)$ are determined by the systems of linear difference equations

$$\begin{aligned}(s-\mu_l+A)H^{lr}(s) &= B(s-1-\mu_l)H^{lr}(s-1)\\ &\quad + H^{l,r-1}(s) - BH^{l,r-1}(s-1)\\ &\quad (s\geqq 1,\ r=0,\ 1,\dots,\ m_l-1)\end{aligned} \qquad (1.7)_r$$

subject to the initial conditions

$$(A-\mu_l)H^{lr}(0) = H^{l,r-1}(0) \quad (r=0,\ 1,\dots,\ m_l-1), \qquad (1.8)_r$$

where $H^{l,-1}(s)\equiv 0$ and $H^{l0}(0)\neq 0$ $(l=1,\dots, q)$.

Now, for each l $(l=1,\dots, q)$ we define

$$H^l(\mu;\ 0) = \sum_{r=0}^{m_l-1} (\mu-\mu_l)^r H^{lr}(0) \qquad (1.9)$$

and for $s\geqq 1$

$$H^l(\mu;\ s) = (s-\mu+A)^{-1}B(s-1-\mu)H^l(\mu;\ s-1)$$

inductively, where μ is a complex parameter. We here observe that $H^l(\mu;\ s)$ $(s\geqq 0)$ are holomorphic at $\mu=\mu_l$ by the assumption $[A_1]$. We define the differential operators ∂_μ^r $(r=0,\ 1,\ 2,\dots)$ by

$$\partial_\mu^r = \frac{1}{r!}\frac{\partial^r}{\partial\mu^r} \qquad (r=0,\ 1,\ 2,\dots).$$

Then we have

$$H^{lr}(s) = \partial_\mu^r[H^l(\mu;\ s)]_{\mu=\mu_l} \quad (s\geqq 0,\ r=0,\ 1,\dots,\ m_l-1).$$

Therefore, observing that

$$\frac{1}{r!}(\log t)^r t^{\mu_l} = \partial_\mu^r[t^\mu]_{\mu=\mu_l} \quad (r=0,\ 1,\ 2,\ldots)$$

and using the Leibniz rule

$$\partial_\mu^r[fg] = \sum_{r'=0}^{r} \partial_\mu^{r-r'}[f]\partial_\mu^{r'}[g],$$

we have

$$Y^{lr}(t) = \sum_{s=0}^{\infty} \partial_\mu^r[H^l(\mu;\ s)t^{\mu-s}]_{\mu=\mu_l} \quad (r=0,\ 1,\ldots,\ m_l-1).$$

Moreover, one can interchange the differentiation and the summation to obtain

$$Y^{lr}(t) = \partial_\mu^r[\sum_{s=0}^{\infty} H^l(\mu;\ s)t^{\mu-s}]_{\mu=\mu_l} \quad (r=0,\ 1,\ldots,\ m_l-1).$$

These facts are well known as the Frobenius theorem. The $\hat{G}_j(m)$ are also given by differentiation with respect to a parameter, which will be discussed in §2.2.

§2. Solutions of the system of difference equations

In order to obtain the Barnes integral representations of the solutions of (0.1) near $t=0$, we shall first consider (0.2) with $\lambda=0$, which is rewritten in the form

$$(z-A)^{-1}B\cdot(z+1)G(z+1) = G(z), \tag{2.1}$$

where z is a complex variable.

Hereafter we use the following notation:

$$\Gamma_f(z) = \prod_{i=1}^{m}(\Gamma(z-\zeta_i))^{r_i}$$

for a polynomial $f(z)=\prod_{i=1}^{m}(z-\zeta_i)^{r_i}$, which obviously satisfies

$$\Gamma_f(z+1) = f(z)\Gamma_f(z).$$

We also denote the *minimal polynomial* of A by $\varphi(z)$, i.e.,

$$\varphi(z) = \prod_{l=1}^{q}(z-\mu_l)^{m_l}$$

and define

$$N = \deg\varphi - 2.$$

Putting

$$G(z) = \frac{\Gamma_\varphi(z)}{\Gamma(z+1)}\bar{G}(z), \tag{2.2}$$

we can transform (2.1) into

$$\varphi(z)(z-A)^{-1}B\bar{G}(z+1) = \bar{G}(z). \tag{2.3}$$

We here put

$$\hat{A}(z) = \varphi(z)(z-A)^{-1}$$

which is a polynomial matrix of the form

$$\hat{A}(z) = z^{N+1} - (\textstyle\sum_{l=1}^{q} m_l\mu_l - A)z^N + \cdots. \tag{2.4}$$

Writing $\bar{G}(z)$ and $\hat{A}(z)$ in the $(n-n', n')$-partitioned form

$$\bar{G}(z) = \begin{bmatrix} \bar{G}^1(z) \\ \bar{G}^2(z) \end{bmatrix} \quad \text{and} \quad \hat{A}(z) = \begin{bmatrix} \hat{A}^{11}(z) & \hat{A}^{12}(z) \\ \hat{A}^{21}(z) & \hat{A}^{22}(z) \end{bmatrix},$$

respectively, and defining

$$\tilde{B} = \operatorname{diag}[\overbrace{\tilde{\lambda}_1,\dots,\tilde{\lambda}_1}^{n_1}, \overbrace{\tilde{\lambda}_2,\dots,\tilde{\lambda}_2}^{n_2},\dots, \overbrace{\tilde{\lambda}_{p-1},\dots,\tilde{\lambda}_{p-1}}^{n_{p-1}}],$$

$$\tilde{\lambda}_k = \lambda_k^{-1} \quad (k=1,\dots,p-1),$$

we can rewrite (2.3) as

$$\hat{A}^{11}(z)\tilde{B}^{-1}\bar{G}^1(z+1) = \bar{G}^1(z) \tag{2.5}$$

and

$$\hat{A}^{21}(z)\tilde{B}^{-1}\bar{G}^1(z+1) = \bar{G}^2(z). \tag{2.6}$$

Namely (0.2) with $\lambda=0$ is reducible to (2.5), since $\bar{G}^2(z)$ can be determined by solving (2.5). We observe that (2.5) has the dimension $n-n'$ which is equal to the number of holomorphic solutions of (0.1) near $t=0$. We hereafter denote $n-n'$ by n''.

2.1. Principal solutions in the right half plane

We here investigate the principal solutions of (2.1), i.e., (2.5) in the right half z-plane by means of the Mellin transformation. For that purpose we first have to make some preparations.

Let α_i $(i=1,\dots,N)$ be constants such that

$$\alpha_i \not\equiv \alpha_{i'} \pmod 1 \quad (i, i'=1,\dots,N,\ i\neq i'), \tag{2.7}$$

$$\alpha_i \not\equiv \mu_l \pmod 1 \quad (i=1,\dots,N,\ l=1,\dots,q), \tag{2.8}$$

(2.9) $$\alpha_i \not\equiv \nu_k \pmod 1 \quad (i=1,\dots,N,\ k=1,\dots,p),$$

(2.10) $$\alpha_i \not\equiv 0 \pmod 1 \quad (i=1,\dots,N),$$

and define

$$\psi(z) = \prod_{i=1}^{N}(z-\alpha_i).$$

LEMMA 1. *$\hat{A}(z)/\psi(z)$ is developed as*

(2.11) $$\frac{1}{\psi(z)}\hat{A}(z) = z - A_0 - \sum_{i=1}^{N}\frac{1}{z-\alpha_i}A_i,$$

where

(2.12) $$A_0 = \sum_{l=1}^{q} m_l\mu_l - \sum_{i=1}^{N}\alpha_i - A$$

and

(2.13) $$A_i = -\prod_{j\neq i}(\alpha_i-\alpha_j)^{-1}\cdot\varphi(\alpha_i)(\alpha_i-A)^{-1} \quad (i=1,\dots,N).$$

PROOF. By (2.4) and (2.7), it is obvious that $\hat{A}(z)/\psi(z)$ is developable in terms of a partial fraction of the form (2.11). Comparing

$$\prod_{i=1}^{N}(z-\alpha_i)\{z-A_0-\sum_{i=1}^{N}(z-\alpha_i)^{-1}A_i\} = z^{N+1} - (\sum_{i=1}^{N}\alpha_i+A_0)z^N + \cdots$$

with (2.4), we obtain (2.12). Moreover, from

$$A_i = -\lim_{z\to\alpha_i}(z-\alpha_i)\hat{A}(z)/\psi(z)$$

together with (2.8), we have (2.13). □

We write A_i ($i=0, 1,\dots, N$) in the (n'', n')-partitioned form

$$A_i = \begin{bmatrix} A_i^{11} & A_i^{12} \\ A_i^{21} & A_i^{22} \end{bmatrix} \quad (i=0, 1,\dots, N),$$

and define

$$\tilde{A}_i = \tilde{B}A_i^{11}\tilde{B}^{-1} \quad (i=0, 1,\dots, N).$$

Putting

$$\tilde{G}^1(z) = \frac{1}{\Gamma_\psi(z)}\tilde{G}(z)$$

and using Lemma 1, we can transform (2.5) into

$$\left(z - \tilde{A}_0 - \sum_{i=1}^{N}\frac{1}{z-\alpha_i}\tilde{A}_i\right)\tilde{G}(z+1) = \tilde{B}\tilde{G}(z).$$

We introduce $\tilde{G}^i(z)$ $(i=1,\dots, N)$ by

$$\tilde{G}^i(z) = \frac{1}{z-1-\alpha_i}\tilde{G}(z) \quad (i=1,\dots, N) \tag{2.14}$$

to obtain

$$(z-\tilde{A}_0)\tilde{G}(z+1) - \textstyle\sum_{i=1}^{N} \tilde{A}_i\tilde{G}^i(z+1) = \tilde{B}\tilde{G}(z). \tag{2.15}$$

We consequently obtain

$$(z-\mathbf{A})\mathbf{G}(z+1) = \mathbf{B}\mathbf{G}(z), \tag{2.16}$$

where

$$\mathbf{A} = \begin{bmatrix} \tilde{A}_0 & \tilde{A}_1\cdots\tilde{A}_N & \\ \tilde{I} & \alpha_1\tilde{I} & \\ \vdots & & \ddots & \\ \tilde{I} & & & \alpha_N\tilde{I} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \tilde{B} & & & \\ & 0\tilde{I} & & \\ & & \ddots & \\ & & & 0\tilde{I} \end{bmatrix},$$

$\tilde{I}$ being the n''-dimensional identity matrix, and

$$\mathbf{G}(z) = \begin{bmatrix} \tilde{G}(z) \\ \tilde{G}^1(z) \\ \vdots \\ \tilde{G}^N(z) \end{bmatrix}.$$

All preparations having been made, we now apply the Mellin transformation

$$\mathbf{G}(z) = \frac{1}{\Gamma(z-\rho)}\int t^{z-\rho-1}\mathbf{W}(t)dt,$$

where ρ is a complex parameter, to (2.16) and then obtain the system of linear differential equations

$$(t-\mathbf{B})\frac{d\mathbf{W}}{dt} = (\rho-1-\mathbf{A})\mathbf{W} \tag{2.17}$$

which is a hypergeometric system of dimension $(N+1)n''$. For the determination of arguments of $t-\tilde{\lambda}_k$ and t, we introduce cut planes

$$\mathscr{D} = \boldsymbol{C}\setminus\bigcup_{l=1}^{p-1}\{\tau\tilde{\lambda}_l;\ \tau\geqq 1\}$$

and

$$\mathscr{D}_k = \mathscr{D}\setminus\{\tau\tilde{\lambda}_k;\ \tau\leqq 0\} \quad (k=1,\dots, p-1).$$

Then, for $t\in\mathscr{D}_k$ we can define the values of $\arg(t-\tilde{\lambda}_k)$ and $\arg t$ so that

$$\arg\tilde{\lambda}_k - 2\pi < \arg(t-\tilde{\lambda}_k) < \arg\tilde{\lambda}_k$$

and

$$\arg\tilde{\lambda}_k - \pi < \arg t < \arg\tilde{\lambda}_k + \pi$$

($\arg\tilde{\lambda}_k = -\arg\lambda_k$), respectively ($k=1,\dots,p-1$). For each k ($k=1,\dots,p-1$), let $\mathbf{W}_{kh}(\rho;t)$ ($h=0,1,\dots,n_k-1$) be solutions of (2.17) in $\mathscr{D}_k$ which are developed as

$$\mathbf{W}_{kh}(\rho;t) = \tilde{\lambda}_k^\rho \sum_{h'=0}^{h} \partial_\nu^{h-h'}[(e^{\pi i}(t-\tilde{\lambda}_k)/\tilde{\lambda}_k)^{\rho-1+\nu}]_{\nu=\tilde{\nu}_k}\mathbf{w}_{kh'}(\rho;t),$$

$$\mathbf{w}_{kh}(\rho;t) = \sum_{m=0}^{\infty}\mathbf{C}_{kh}(\rho;m)(t-\tilde{\lambda}_k)^m \quad (h=0,1,\dots,n_k-1)$$

near $t=\tilde{\lambda}_k$, where

$$\tilde{\nu}_k = \nu_k + \sum_{i=1}^{N}\alpha_i - \sum_{l=1}^{q} m_l\mu_l .$$

The coefficients $\mathbf{C}_{kh}(\rho;m)$ are vectors of the form

$$\mathbf{C}_{kh}(\rho;m) = \sum_{h'=0}^{h}\partial_\nu^{h-h'}\left[\frac{1}{\Gamma(\rho+\nu+m)}\right]_{\nu=\tilde{\nu}_k}\hat{\mathbf{C}}_{kh'}(m)$$

$$(m\geqq 0,\ h=0,1,\dots,n_k-1),$$

where $\hat{\mathbf{C}}_{kh}(m)$ ($h=0,1,\dots,n_k-1$) are the vectors determined uniquely by the systems of linear difference equations

$$\begin{cases} (\mathbf{B}-\tilde{\lambda}_k)\hat{\mathbf{C}}_{kh}(m+1) = (m+\tilde{\nu}_k+\mathbf{A})\hat{\mathbf{C}}_{kh}(m) + \hat{\mathbf{C}}_{k,h-1}(m) \\ \qquad\qquad (m\geqq 0,\ \hat{\mathbf{C}}_{k,-1}(m)\equiv 0) \\ \hat{\mathbf{C}}_{kh}(0) = e_{(N+1)n''}(n_1+\cdots+n_{k-1}+h+1) \\ \qquad\qquad (h=0,1,\dots,n_k-1), \end{cases}$$

$e_m(s)$ being the s-th unit m-vector (cf. §1). Using these $\mathbf{W}_{kh}(\rho;t)$, we define

$$\mathbf{G}_{kh}(z) = \frac{1}{\Gamma(z-\rho)}\int_0^{\tilde{\lambda}_k} t^{z-\rho-1}\mathbf{W}_{kh}(\rho;t)dt \tag{2.18}$$

$$(k=1,\dots,p-1,\ h=0,1,\dots,n_k-1)$$

for $\mathrm{Re}(z-\rho)>0$, where the path of integration is the straight line from 0 to $\tilde{\lambda}_k$ (note that $\arg\tilde{\lambda}_k \not\equiv \arg\tilde{\lambda}_l \pmod{2\pi}$ ($k\neq l$) by the assumption [A_0]). The parameter ρ is selected so that $\mathrm{Re}(\rho+\tilde{\nu}_k)>0$ (for every $k=1,\dots,p-1$) and $\rho\in\mathscr{E}$, where

$$\mathscr{E} = \{z;\ \mathrm{Re}\,z>\mathrm{Re}\,\alpha_i+1 \text{ and } z\not\equiv\alpha_i \pmod 1\ (i=1,\dots,N)\}.$$

Concerning these $\mathbf{G}_{kh}(z)$, we have the following

PROPOSITION 1. (i) *For each* k, h $(k=1,\dots, p-1, h=0, 1,\dots, n_k-1)$, $\mathbf{G}_{kh}(z)$ *is holomorphic and is a solution of* (2.16) *in* $\mathrm{Re}(z-\rho)>0$. *Therefore* $\mathbf{G}_{kh}(z)$ *is analytically continued into* $C\setminus\{\alpha_i+1-s;\ i=1,\dots, N,\ s=0, 1, 2,\dots\}$ *and satisfies* (2.16). *The points* α_i+1-s $(i=1,\dots, N,\ s=0, 1, 2,\dots)$ *are simple poles of* $\mathbf{G}_{kh}(z)$.

(ii) *For* $z\in\mathscr{E}$,

$$\mathbf{G}_{kh}(z) = \lim_{t\to 0,\, t\in\mathscr{D}_k} \mathbf{W}_{kh}(z;\, t) \tag{2.19}$$

holds $(k=1,\dots, p-1,\ h=0, 1,\dots, n_k-1)$. *Therefore* $\mathbf{G}_{kh}(z)$ *does not depend on* ρ.

(iii) *As* $z\to\infty$, $|\arg z|<\pi/2+\varepsilon$ *for sufficiently small* $\varepsilon>0$, *the asymptotic expansion*

$$\mathbf{G}_{kh}(z) \sim \mathbf{F}_{kh}(z)$$

holds, where $\mathbf{F}_{kh}(z)$ *is a formal solution of* (2.16) *of the form*

$$\mathbf{F}_{kh}(z) = \tilde{\lambda}_k^z \Gamma(z)^{-1} \sum_{h'=0}^{h} \partial_\nu^{h-h'} [z^{-\nu}]_{\nu=\tilde{\nu}_k} \sum_{s=0}^{\infty} \mathbf{f}_{kh'}(s) z^{-s},$$

$$\mathbf{f}_{kh}(0) = e_{(N+1)n''}(n_1+\cdots+n_{k-1}+h+1)$$

$$(k=1,\dots, p-1,\ h=0, 1,\dots, n_k-1).$$

PROOF. (i) The system (2.17) has n'' linearly independent solutions for each of the exponents 0 and $\rho-1-\alpha_i$ $(i=1,\dots, N)$ at $t=0$ (note that $\rho\not\equiv\alpha_i$ (mod 1) and (2.7) holds). Since $\mathrm{Re}(\rho-1-\alpha_i)>0$ $(i=1,\dots, N)$, we have

$$\mathbf{W}_{kh}(\rho;\, t) = O(1) \quad \text{as} \quad t \longrightarrow 0, \quad t\in\mathscr{D}_k.$$

Therefore $\mathbf{G}_{kh}(z)$ is well defined and holomorphic for $\mathrm{Re}(z-\rho)>0$. By a simple calculation we can easily see that $\mathbf{G}_{kh}(z)$ satisfies (2.16) in $\mathrm{Re}(z-\rho)>0$. The desired analytic continuation of $\mathbf{G}_{kh}(z)$ can be obtained by means of the equation (2.16), i.e., (2.15) and (2.14). We here observe that the first n'' components of $\mathbf{G}_{kh}(z)$ are holomorphic at $z=\alpha_i+1$ $(i=1,\dots, N)$.

(ii) For $t\in\mathscr{D}_k$, $\mathrm{Re}(z-\rho)>0$, we define

$$\mathbf{G}_{kh}(z;\, t) = \frac{1}{\Gamma(z-\rho)} \int_t^{\tilde{\lambda}_k} (\tau-t)^{z-\rho-1} \mathbf{W}_{kh}(\rho;\, \tau) d\tau,$$

where the path of integration is a curve in $\mathscr{D}_k$ from t to $\tilde{\lambda}_k$ and $\arg(\tau-t)$ is taken continuously along the path of integration with $\arg(\tilde{\lambda}_k-t)=\arg(t-\tilde{\lambda}_k)+\pi$ at the endpoint $\tilde{\lambda}_k$. For $|t-\tilde{\lambda}_k|$ sufficiently small, we obtain

$$\mathbf{G}_{kh}(z;\, t) = \mathbf{W}_{kh}(z;\, t)$$

by termwise integration. Moreover, since both $\mathbf{G}_{kh}(z;\, t)$ and $\mathbf{W}_{kh}(z;\, t)$ are holomorphic in $\mathscr{D}_k$, this formula is valid for every $t \in \mathscr{D}_k$, $\mathrm{Re}\,(z-\rho)>0$. Hence, for $\mathrm{Re}\,(z-\rho)>0$ we have

$$\mathbf{G}_{kh}(z) = \lim_{t\to 0} \mathbf{G}_{kh}(z;\, t) = \lim_{t\to 0} \mathbf{W}_{kh}(z;\, t).$$

Furthermore, combining the facts that

$$\frac{\partial}{\partial t} \mathbf{W}_{kh}(z;\, t) = -\, \mathbf{W}_{kh}(z-1;\, t),$$

which is verified by termwise differentiation and analytic continuation, and

$$\lim_{t\to 0,\, t\in\mathscr{D}_k} t\mathbf{W}_{kh}(z-1;\, t) = 0 \qquad \text{for} \quad z \in \mathscr{E}$$

with (2.17), we can easily see that $\lim_{t\to 0} \mathbf{W}_{kh}(z;\, t)$ satisfies (2.15) and (2.14) for $z \in \mathscr{E}$. Hence $\lim_{t\to 0} \mathbf{W}_{kh}(z;\, t)$ is holomorphic in $z \in \mathscr{E}$. Therefore, we have (2.19) for $z \in \mathscr{E}$ ($\cup \{z;\, \mathrm{Re}\,(z-\rho)>0\}$) (cf. [12]).

(iii) We apply the following

LEMMA 2. *Let $g(t)$ satisfy the following two conditions:*
(a) *$g(t)$ is holomorphic in the sector $\{t;\, |\arg t|<\theta\}$, and*

$$g(t) = \textstyle\sum_{s=0}^{\infty} a_s t^s \quad (|t|<\hat{R})\,;$$

(b) *$g(t)=O(e^{bt})$ for some b as $t\to\infty$ in the sector $\{t;\, |\arg t|<\theta\}$.*
Then

$$G(z) = \int_0^\infty e^{-zt}(\log t)^m t^{\sigma-1} g(t)dt \quad (\mathrm{Re}\,\sigma>0)$$

has an asymptotic expansion of the form

$$G(z) \sim \textstyle\sum_{k=0}^{m} \binom{m}{k} (\log z^{-1})^k z^{-\sigma} \sum_{s=0}^{\infty} \Gamma^{(m-k)}(\sigma+s) a_s z^{-s}$$

as $z\to\infty$, $|\arg(z-b)| \leqq \pi/2+\theta-\varepsilon$ $(\varepsilon>0)$.

PROOF OF LEMMA 2. The asymptotic expansion for $|\arg(z-b)| \leqq \pi/2-\varepsilon$ $(\varepsilon>0)$ can be proven in a standard manner by expressing $g(t)$ as a finite part of the expansion plus an error term whose Laplace integral can be easily estimated. Then, rotating the path of integration, we obtain the desired asymptotic expansion in the sector stated above (see [11, Chap. 4 §3 and Chap. 9 §1]). □

Changing the variable t for τ by $t=\tilde{\lambda}_k e^{-\tau}$ in (2.18), we have

$$\mathbf{G}_{kh}(z) = \tilde{\lambda}_k^{z-\rho}\frac{1}{\Gamma(z-\rho)}\int_0^\infty e^{-(z-\rho)\tau}\mathbf{W}_{kh}(\rho;\tilde{\lambda}_k e^{-\tau})d\tau$$
$$= \tilde{\lambda}_k^{z}\frac{1}{\Gamma(z-\rho)}\sum_{h'=0}^{h}\frac{1}{(h-h')!}\times$$
$$\times\int_0^\infty e^{-(z-\rho)\tau}(\log\tau)^{h-h'}\tau^{\rho-1+\tilde{\nu}_k}\hat{\mathbf{w}}_{kh'}(\rho;\tau)d\tau,$$

where

$$\hat{\mathbf{w}}_{kh}(\rho;\tau) = \sum_{h'=0}^{h}\partial_\nu^{h-h'}[\tau^{-(\rho-1+\nu)}]_{\nu=\tilde{\nu}_k}\tilde{\lambda}_k^{-\rho}\mathbf{W}_{kh'}(\rho;\tilde{\lambda}_k e^{-\tau})$$
$$= \sum_{h'=0}^{h}\partial_\nu^{h-h'}[((1-e^{-\tau})/\tau)^{\rho-1+\nu}]_{\nu=\tilde{\nu}_k}\mathbf{w}_{kh'}(\rho;\tilde{\lambda}_k e^{-\tau})$$

which satisfies the conditions in Lemma 2 with sufficiently small $\theta>0$ ($k=1,\dots, p-1$, $h=0, 1,\dots, n_k-1$). Therefore, writing

$$\hat{\mathbf{w}}_{kh}(\rho;\tau) = \sum_{s=0}^{\infty}\hat{\mathbf{f}}_{kh}(\rho;s)\tau^s \quad \text{at} \quad \tau=0,$$

we obtain an asymptotic expansion of the form

$$\mathbf{G}_{kh}(z) \sim \tilde{\lambda}_k^{z}\frac{1}{\Gamma(z-\rho)}\sum_{h'=0}^{h}\partial_\nu^{h-h'}[(z-\rho)^{-(\rho+\nu)}]_{\nu=\tilde{\nu}_k}\sum_{s=0}^{\infty}\mathbf{f}_{kh'}(\rho;s)(z-\rho)^{-s}$$

as $z\to\infty$, $|\arg z|<\pi/2+\varepsilon$ for sufficiently small $\varepsilon>0$, where

$$\mathbf{f}_{kh}(\rho;s) = \sum_{h'=0}^{h}\partial_\nu^{h-h'}[\Gamma(\rho+\nu+s)]_{\nu=\tilde{\nu}_k}\hat{\mathbf{f}}_{kh'}(\rho;s) \quad (s\geqq 0)$$
$$(k=1,\dots,p-1,\ h=0,1,\dots,n_k-1).$$

Then, developing again the right hand side of the above formula in terms of z^{-1}, we obtain the required asymptotic expansions. As to the initial vectors $\mathbf{f}_{kh}(0)$, we have

$$\mathbf{f}_{kh}(0) = \mathbf{f}_{kh}(\rho;0) = \sum_{h'=0}^{h}\partial_\nu^{h-h'}[\Gamma(\rho+\nu)]_{\nu=\tilde{\nu}_k}\hat{\mathbf{w}}_{kh'}(\rho;0)$$
$$= \sum_{h'=0}^{h}\partial_\nu^{h-h'}[\Gamma(\rho+\nu)]_{\nu=\tilde{\nu}_k}\mathbf{w}_{kh'}(\rho;\tilde{\lambda}_k)$$
$$= \hat{\mathbf{C}}_{kh}(0) = e_{(N+1)n''}(n_1+\cdots+n_{k-1}+h+1)$$
$$(k=1,\dots,p-1,\ h=0,1,\dots,n_k-1).$$

This completes the proof of (iii) (cf. [4]). □

We denote by $\tilde{G}_{kh}(z)$ the n''-vector which consists of the first n'' components of $\mathbf{G}_{kh}(z)$ ($k=1,\dots,p-1$, $h=0,1,\dots,n_k-1$). Besides, we define

$$\bar{G}_{kh}^{1}(z) = \frac{1}{\Gamma_\psi(z)}\tilde{G}_{kh}(z) \quad (k=1,\dots,p-1,\ h=0,1,\dots,n_k-1),$$

which are entire solutions of (2.5) (note that $\tilde{G}_{kh}(z)$ are holomorphic at $z=\alpha_i+1$

($i=1,\dots, N$)). Moreover, we denote by $G_{kh}(z)$ the solution of (0.2) with $\lambda=0$ constructed by $\bar{G}^1_{kh}(z)$ with (2.6) and (2.2) ($k=1,\dots, p-1$, $h=0, 1,\dots, n_k-1$), which are the principal solutions in the right half z-plane. Namely $G_{kh}(z)$ has the asymptotic exapnsion

$$(2.20) \qquad G_{kh}(z) \sim F_{kh}(z) \quad \text{as} \quad z \longrightarrow \infty, \quad |\arg z| < \pi/2 + \varepsilon \quad (\varepsilon>0),$$

where $F_{kh}(z)$ is a formal solution of (0.2) with $\lambda=0$ of the form

$$F_{kh}(z) = \lambda_k^{-z} \sum_{h'=0}^{h} \partial_\nu^{h-h'} [z^{-\nu-1}]_{\nu=\nu_k} \sum_{s=0}^{\infty} f_{kh'}(s) z^{-s},$$

$$f_{kh}(0) = e_n(n_1+\cdots+n_{k-1}+h+1)$$

$$(k=1,\dots, p-1,\ h=0, 1,\dots, n_k-1).$$

These $G_{kh}(z)$ will be used for the Barnes integral representations of the holomorphic solutions of (0.1) near $t=0$ in §3.1.

REMARK 1. According to the general theory of difference equations (e.g. [2, pp. 270–271]), a solution of (0.2) with $\lambda=0$ which has the asymptotic expansion (2.20) is uniquely determined. Hence $G_{kh}(z)$, and also $\bar{G}^1_{kh}(z)$, ($k=1,\dots, p-1$, $h=0, 1,\dots, n_k-1$) are independent of α_i ($i=1,\dots, N$).

2.2. Solution having zeros in the left half plane

In this section, in order to construct solutions of $(1.4)_j$ ($j=0, 1,\dots, n'-1$), we consider a particular solution of (2.5) which has zeros in the left half z-plane. In view of their asymptotic behavior in the right half z-plane, we construct these solutions by means of a linear combination of the principal solutions with constant coefficients.

We first calculate the determinant of $[\bar{G}^1_{kh}(z)]=[\bar{G}^1_{10}(z)\cdots\bar{G}^1_{1,n_1-1}(z)\cdots \bar{G}^1_{p-1,0}(z)\cdots\bar{G}^1_{p-1,n_{p-1}-1}(z)]$. It is obvious that $\det[\bar{G}^1_{kh}(z)]$ satisfies the difference equation

$$\det \hat{A}^{11}(z)\cdot \det \tilde{B}^{-1}\cdot \det[\bar{G}^1_{kh}(z+1)] = \det[\bar{G}^1_{kh}(z)].$$

Since

$$(2.21) \qquad \begin{bmatrix} \hat{A}^{11}(z) & \hat{A}^{12}(z) \\ 0 & \hat{I} \end{bmatrix} (z-A) = \begin{bmatrix} \varphi(z)\tilde{I} & 0 \\ * & z-A_{pp} \end{bmatrix},$$

$\hat{I}$ being the n'-dimensional identity matrix, we have

$$\det \hat{A}^{11}(z) = \varphi(z)^{n''}\cdot \det(z-A_{pp})/\det(z-A)$$

$$= (z-\nu')^{n'} \cdot \prod_{l=1}^{q}(z-\mu_l)^{(n''-1)m_l}.$$

Therefore, denoting the last polynomial by $d(z)$, we obtain

$$\det[\bar{G}^1_{kh}(z)] = q(z)\cdot\prod_{k=1}^{p-1}\tilde{\lambda}_k^{n_k z}/\Gamma_d(z),$$

where $q(z)$ is a periodic function with period 1. By virtue of the asymptotic expansions of $\tilde{G}_{kh}(z)$ (i.e., $\mathbf{G}_{kh}(z)$) and the Stirling formula, we can easily see that

$$q(z)\sim 1 \quad \text{as} \quad z\longrightarrow\infty, \quad |\arg z|<\pi/2+\varepsilon \quad (\varepsilon>0).$$

Hence $q(z)$ is indeed a constant and is equal to 1. Thus we have

$$\det[\bar{G}^1_{kh}(z)] = \prod_{k=1}^{p-1}\tilde{\lambda}_k^{n_k z}/\Gamma_d(z)$$

(cf. [10, Chap. 2 §2] and [6, Theorem 2]).

Let us define $\gamma_{kh}(\nu)$ $(k=1,\dots,p-1,\ h=0,1,\dots,n_k-1)$ as the solution of the system of linear equations

$$\begin{aligned}&\sum_{k=1}^{p-1}\sum_{h=0}^{n_k-1}\gamma_{kh}(\nu)\bar{G}^1_{kh}(\nu+1)\\ &= -\frac{\Gamma(\nu+1)}{\Gamma_\varphi(\nu+1)}\tilde{B}\begin{bmatrix}A_{1p}\\ \vdots\\ A_{p-1,p}\end{bmatrix}\sum_{j=0}^{n'-1}(\nu-\nu')^j e_{n'}(j+1),\end{aligned}$$

where ν is a complex parameter. Since $\det[\bar{G}^1_{kh}(\nu'+1)]\neq 0$, $\gamma_{kh}(\nu)$ are uniquely determined for every ν in a neighborhood of ν'. Using these $\gamma_{kh}(\nu)$, we define a solution $\bar{G}^1(\nu;z)$ of (2.5) by

$$\bar{G}^1(\nu;z) = \sum_{k=1}^{p-1}\sum_{h=0}^{n_k-1}\gamma_{kh}(\nu)\bar{G}^1_{kh}(z).$$

Then, $\bar{G}^1(\nu;\nu)$ has a zero of order n' at $\nu=\nu'$. In fact, since

$$\hat{A}^{11}(z)\begin{bmatrix}A_{1p}\\ \vdots\\ A_{p-1,p}\end{bmatrix} = \hat{A}^{12}(z)(z-A_{pp})$$

from the (1, 2)-block of (2.21), we have

$$\begin{aligned}\bar{G}^1(\nu;\nu) &= \hat{A}^{11}(\nu)\tilde{B}^{-1}\bar{G}^1(\nu;\nu+1)\\ &= -\frac{\Gamma(\nu+1)}{\Gamma_\varphi(\nu+1)}\hat{A}^{12}(\nu)(\nu-A_{pp})\sum_{j=0}^{n'-1}(\nu-\nu')^j e_{n'}(j+1)\\ &= -(\nu-\nu')^{n'}\frac{\Gamma(\nu+1)}{\Gamma_\varphi(\nu+1)}\hat{A}^{12}(\nu)e_{n'}(n')\end{aligned}$$

(note that $\nu'\neq -1, -2,\dots$). Moreover, since we have

$$\bar{G}^1(\nu;\nu-s) = \hat{A}^{11}(\nu-s)\tilde{B}^{-1}\bar{G}^1(\nu;\nu-s+1)$$

by (2.5), $\bar{G}^1(\nu;\nu-s)$ $(s=1,2,\dots)$ also have a zero of order n' at $\nu=\nu'$. Hence

$\bar{G}^1(v'; z)$ has zeros at $z=v'-s$ $(s=0, 1, 2,\dots)$.

Now let us define solutions of $(1.4)_j$ $(j=0, 1,\dots, n'-1)$. We denote by $G(v; z)$ the solution of (0.2) with $\lambda=0$ constructed by $\bar{G}^1(v; z)$ with (2.6) and (2.2), and define

$$\hat{G}_j(z) = \partial_v^j[G(v; z+v)]_{v=v'} \tag{2.22}$$

$$= \sum_{k=1}^{p-1} \sum_{h=0}^{n_k-1} \sum_{j'=0}^{j} \frac{1}{(j-j')!} \gamma_{kh}^{(j-j')}(v') \frac{1}{j'!} G_{kh}^{(j')}(z+v')$$

$$(j=0, 1,\dots, n'-1).$$

Then, $\hat{G}_j(z)$ $(j=0, 1,\dots, n'-1)$ satisfy

$$B\cdot(z+1+v')\hat{G}_j(z+1) = (z+v'-A)\hat{G}_j(z) - B\hat{G}_{j-1}(z+1) + \hat{G}_{j-1}(z)$$

$$(j=0, 1,\dots, n'-1, \hat{G}_{-1}(z)\equiv 0).$$

Moreover, since $G(v; v-s)$ $(s=1, 2,\dots)$ have a zero of order n' at $v=v'$, $\hat{G}_j(z)$ $(j=0, 1,\dots, n'-1)$ vanish at $z=-s$ $(s=1, 2,\dots)$. As to $\hat{G}_j(0)$, we have

$$\hat{G}_j(0) = e_n(n''+j+1) \quad (j=0, 1,\dots, n'-1),$$

since

$$(v-A)G(v; v) = B\cdot(v+1)G(v; v+1)$$

$$= (v-A) \sum_{j=0}^{n'-1} (v-v')^j e_n(n''+j+1) - (v-v')^{n'} e_n(n),$$

and hence

$$G(v; v) = \sum_{j=0}^{n'-1} (v-v')^j e_n(n''+j+1) - (v-v')^{n'}(v-A)^{-1} e_n(n)$$

hold. Therefore, $\hat{G}_j(z)$ $(j=0, 1,\dots, n'-1)$ are solutions of $(1.4)_j$ $(j=0, 1,\dots, n'-1)$, which will be used for the Barnes integral representations of the non-holomorphic solutions $\hat{X}_j(t)$ $(j=0, 1,\dots, n'-1)$ of (0.1) near $t=0$ in §3.2. In addition, it will be shown in §4 that for each k $(k=1,\dots, p-1)$, $\gamma_{kh}(v)$ $(h=0, 1,\dots, n_k-1)$ give the connection coefficients between $\hat{X}_j(t)$ $(j=0, 1,\dots, n'-1)$ and non-holomorphic solutions of (0.1) near $t=\lambda_k$.

§ 3. The Barnes integral representations

3.1. The Barnes integrals for holomorphic solutions

We consider the Barnes integral

$$X_{kh}(t) = -\frac{1}{2\pi i}\int_{\mathscr{C}} G_{kh}(z)p(z)t^z dz$$

$$(k=1,\dots, p-1, h=0, 1,\dots, n_k-1),$$

where

$$p(z) = \pi e^{-\pi i z}/\sin \pi z$$

and the path of integration $\mathscr{C}$ is a Barnes contour running along the straight line $z = -ia$ from $+\infty - ia$ to $0 - ia$, a curve from $0 - ia$ to $0 + ia$ and the straight line $z = ia$ from $0 + ia$ to $+\infty + ia$ such that the points $z = m$ $(m = 0, 1, 2, \dots)$ lie to the right of $\mathscr{C}$ and the points $z = \mu_l - s$ $(s = 0, 1, 2, \dots, l = 1, \dots, q)$ lie to the left of $\mathscr{C}$. The constant a is taken as $a > \max\{|\mathrm{Im}\,\mu_l|;\ l = 1, \dots, q\}$. By virtue of the asymptotic behavior of $G_{kh}(z)$, the above integral is absolutely convergent for $|t| < |\lambda_k|$ and is equal to the sum of residues at $z = m$ $(m = 0, 1, 2, \dots)$, i.e.,

$$X_{kh}(t) = \sum_{m=0}^{\infty} G_{kh}(m) t^m \quad (|t| < |\lambda_k|),$$

which is a holomorphic solution of (0.1) near $t = 0$.

Now let ξ be an arbitrary negative number not equal to $\mathrm{Re}(\mu_l - s)$ $(s = 0, 1, 2, \dots, l = 1, \dots, q)$. We take the poisitive integers N_l $(l = 1, \dots, q)$ such that

$$-(N_l + 1) < \xi - \mathrm{Re}\,\mu_l < -N_l \quad (l = 1, \dots, q).$$

Replacing the path $\mathscr{C}$ by the rectilinear contour $\mathscr{L} = \mathscr{L}(\xi)$ which runs first from $+\infty - ia$ to $\xi - ia$, next from $\xi - ia$ to $\xi + ia$ and finally from $\xi + ia$ to $+\infty + ia$, we obtain

$$X_{kh}(t) = -\frac{1}{2\pi i}\int_{\mathscr{L}} G_{kh}(z) p(z) t^z dz - \sum \mathrm{Res}\,[G_{kh}(z) p(z) t^z],$$

where the summation covers all poles in the bounded domain encircled by $\mathscr{L}$ and the curve from $-ia$ to ia of $\mathscr{C}$. Since, by (2.2), $G_{kh}(z)$ has poles of order m_l at $z = \mu_l - s$ $(s = 0, 1, 2, \dots, l = 1, \dots, q)$ and has zeros of order 1 at $z = -s$ $(s = 1, 2, \dots)$, the integrand has poles of order m_l only at $z = \mu_l - s$ $(s = 0, 1, \dots, N_l, l = 1, \dots, q)$ in that domain. Hence we obtain

$$\begin{aligned}
&\sum \mathrm{Res}\,[G_{kh}(z) p(z) t^z] \\
&= \sum_{l=1}^{q} \sum_{s=0}^{N_l} \mathrm{Res}\,[G_{kh}(z) p(z) t^z : z = \mu_l - s] \\
&= \sum_{l=1}^{q} \sum_{s=0}^{N_l} \partial_z^{m_l - 1}[(z - \mu_l + s)^{m_l} G_{kh}(z) p(z) t^z]_{z = \mu_l - s} \\
&= \sum_{l=1}^{q} \sum_{r=0}^{m_l - 1} \partial_\mu^{m_l - 1 - r}[p(\mu)]_{\mu = \mu_l} \sum_{r'=0}^{r} \partial_\mu^{r - r'}[t^\mu]_{\mu = \mu_l} \sum_{s=0}^{N_l} H_{kh}^{lr'}(s) t^{-s},
\end{aligned}$$

where

$$H_{kh}^{lr}(s) = \partial_z^r[(z - \mu_l + s)^{m_l} G_{kh}(z)]_{z = \mu_l - s}$$
$$(s = 0, 1, 2, \dots, l = 1, \dots, q, r = 0, 1, \dots, m_l - 1).$$

We here observe that for each l $(l = 1, \dots, q)$, $H_{kh}^{lr}(s)$ $(r = 0, 1, \dots, m_l - 1)$ satisfy the systems $(1.8)_r$ and $(1.7)_r$ $(r=0, 1,\dots, m_l-1)$. This fact is easily verified by applying ∂_z^r $(r=0, 1,\dots, m_l-1)$ together with the Leibniz rule to

$$(-z+A)\cdot(z-\mu_l+s)^{m_l}G_{kh}(z) = -B(z+1)\cdot(z+1-\mu_l+s-1)^{m_l}G_{kh}(z+1)$$

and letting z tend to μ_l-s (note that $G_{kh}(z)$ is holomorphic at $z=\mu_l+1$) (cf. [14, p. 233]). In particular, we have

$$(A-\mu_l)^{m_l}H_{kh}^{l,m_l-1}(0) = 0 \quad (l=1,\dots, q).$$

Since $H^{lr}(0)$ $(r=0, 1,\dots, m_l-1)$ (which are given in §1) form a basis of the kernel of $(A-\mu_l)^{m_l}$ $(l=1,\dots, q)$, there uniquely exist the constants η_{kh}^{lr} $(l=1,\dots, q,$ $r=0, 1,\dots, m_l-1)$ such that

$$H_{kh}^{l,m_l-1}(0) = \textstyle\sum_{r=0}^{m_l-1} \eta_{kh}^{l,m_l-1-r} H^{lr}(0) \quad (l=1,\dots, q).$$

Then, multiplying this formula by $(A-\mu_l)^{m_l-1-r}$ and taking account of $(1.8)_r$, we obtain

$$H_{kh}^{lr}(0) = \textstyle\sum_{r'=0}^{r} \eta_{kh}^{l,r-r'} H^{lr'}(0) \qquad (3.1)$$
$$(l=1,\dots, q,\ r=0, 1,\dots, m_l-1).$$

Therefore, we have

$$H_{kh}^{lr}(s) = \textstyle\sum_{r'=0}^{r} \eta_{kh}^{l,r-r'} H^{lr'}(s) \quad (s\geqq 0)$$
$$(l=1,\dots, q,\ r=0, 1,\dots, m_l-1),$$

since the right hand sides of these formulas also satisfy $(1.7)_r$ $(r=0, 1,\dots, m_l-1)$. We thus obtain

$$X_{kh}(t) = \textstyle\sum_{l=1}^{q} \sum_{r=0}^{m_l-1} T_{kh}^{l,m_l-1-r} \sum_{r'=0}^{r} \partial_\mu^{r-r'}[t^\mu]_{\mu=\mu_l} \sum_{s=0}^{N_l} H^{lr'}(s)t^{-s}$$
$$-\frac{1}{2\pi i}\int_{\mathscr{L}} G_{kh}(z)p(z)t^z dz \quad (|t|<|\lambda_k|),$$

where

$$T_{kh}^{lr} = -\textstyle\sum_{r'=0}^{r} \partial_\mu^{r-r'}[p(\mu)]_{\mu=\mu_l}\, \eta_{kh}^{lr'}$$
$$(l=1,\dots, q,\ r=0, 1,\dots, m_l-1).$$

We now estimate the last term. Observing that

$$\begin{cases} p(\xi+ia+ye^{i\theta}) = O(1) \\ p(\xi-ia+ye^{-i\theta}) = O(e^{-2\pi\sin\theta\cdot y}) \end{cases} \quad \text{as} \quad y \longrightarrow +\infty \qquad (3.2)$$

uniformly in $\theta\in[0, \pi/2]$, and using the asymptotic behavior of $G_{kh}(z)$, we can see in a standard manner that the integral

$$\int_{\xi-i\infty}^{\xi+i\infty} G_{kh}(z)p(z)t^z dz$$

is the analytic continuation of the above integral to a sector

$$S'_k = \{t;\ \arg\lambda_k+\varepsilon' \leqq \arg t \leqq \arg\lambda_k+2\pi-\varepsilon'\} \quad (\varepsilon'>0).$$

Moreover, we have

$$\left\| \int_{\xi-i\infty}^{\xi+i\infty} G_{kh}(z)p(z)t^z dz \right\| < K|t|^{\xi} \qquad \text{for} \quad t\in S'_k,$$

where $\|\cdot\|$ means the maximum norm of a vector and K is a positive constant independent of t (but depending on ξ and ε') (see [3, Lemma 6 and Lemma 6a]). Hence $X_{kh}(t)$ is analytically continued into $C\setminus\{\tau\lambda_k;\ \tau\geqq 1\}$ and

$$X_{kh}(t) = \sum_{l=1}^{q} \sum_{r=0}^{m_l-1} T_{kh}^{l,m_l-1-r}\, Y^{lr}(t) + o(t^{-\xi'}) \qquad (t\to\infty,\ t\in S'_k)$$

holds, where ξ' is an arbitrary positive number. Since $t=\infty$ is a regular singularity of (0.1), we actually obtain

$$X_{kh}(t) = \sum_{l=1}^{q} \sum_{r=0}^{m_l-1} T_{kh}^{l,m_l-1-r} Y^{lr}(t) \qquad \text{for} \quad t\in S_k,$$

where $S_k=\{t;\ \arg\lambda_k<\arg t<\arg\lambda_k+2\pi\}$. In order to obtain the connection formula in another sector, we anew take $e^{2\pi i\delta z}G_{kh}(z)$ $(\delta\in Z)$ instead of $G_{kh}(z)$. Then we have the connection formula with T_{kh}^{lr} replaced by $T_{kh}^{lr}(\delta)$ in the above for $t\in S_k(\delta)$, where

$$T_{kh}^{lr}(\delta) = -\sum_{r'=0}^{r} \partial_{\mu}^{r-r'} [e^{2\pi i\delta\mu}p(\mu)]_{\mu=\mu_l}\, \eta_{kh}^{lr'} \qquad (l=1,\dots,q,\ r=0,1,\dots,m_l-1)$$

and

$$S_k(\delta) = \{t;\ \arg\lambda_k-2\pi\delta<\arg t<\arg\lambda_k-2\pi\delta+2\pi\} \qquad (\delta\in Z,\ k=1,\dots,p-1). \tag{3.3}$$

Actually, for each l $(l=1,\dots,q)$, η_{kh}^{lr} $(r=0,1,\dots,m_l-1)$ are determined successively from η_{kh}^{l0}. Let $n_0=n_0(l)$, $1\leqq n_0\leqq n''$, be a number such that the n_0-th component of $H^{l0}(0)$ is not zero. (Note that if an n-vector H whose the first n'' components are all zero satisfies $(A-\mu_l)H=0$, then $H=0$ by virtue of the assumption [A_2].) We define

$$\eta_{kh}^{l}(\mu) = [(\mu-\mu_l)^{m_l}G_{kh}(\mu)]_{n_0}/[H^l(\mu;\ 0)]_{n_0},$$

where $[\ \ ]_{n_0}$ denotes the n_0-th component. Then, we have

$$\eta_{kh}^{lr} = \partial_\mu^r[\eta_{kh}^l(\mu)]_{\mu=\mu_l} \quad (r=0, 1,\dots, m_l-1),$$

since

$$[(\mu-\mu_l)^{m_l}G_{kh}(\mu)]_{n_0}-\{\textstyle\sum_{r=0}^{m_l-1}(\mu-\mu_l)^r\eta_{kh}^{lr}\}[H^l(\mu;\,0)]_{n_0}$$

has a zero of order m_l at $\mu=\mu_l$ by (1.9) and (3.1). Hence, by using the Leibniz rule, we obtain

$$(3.4)\qquad T_{kh}^{lr}(\delta) = -\,\partial_\mu^r[e^{2\pi i\delta\mu}p(\mu)\eta_{kh}^l(\mu)]_{\mu=\mu_l} \qquad (r=0, 1,\dots, m_l-1).$$

Observing from Proposition 1 (ii) that $\eta_{kh}^l(\mu)$ has a representation of the form

$$\eta_{kh}^l(\mu) = \frac{\Gamma(\mu+1-\mu_l)^{m_l}\Gamma_{\varphi_l}(\mu)}{\Gamma(\mu+1)\Gamma_\psi(\mu)}\lim_{t\to 0}[\mathbf{W}_{kh}(\mu;\,t)]_{n_0}/[H^l(\mu;\,0)]_{n_0},$$

where $\varphi_l(\mu)=\varphi(\mu)/(\mu-\mu_l)^{m_l}$ and α_i $(i=1,\dots, N)$ are taken so that $\mu_l\in\mathscr{E}$, we can conclude that the connection coefficients between $X_{kh}(t)$ and $Y^{lr}(t)$ $(l=1,\dots, q$, $r=0, 1,\dots, m_l-1)$ are given explicitly by the solution $\mathbf{W}_{kh}(\rho;\,t)$ of (2.17) (with the factor consisting of exponential- and Γ-functions).

3.2. The Barnes integrals for non-holomorphic solutions

For the non-holomorphic solutions of (0.1) we first consider the Barnes integrals

$$\hat{x}_j(t) = -\frac{1}{2\pi i}\int_{\hat{\mathscr{C}}}\hat{G}_j(z)p(z)t^z dz \qquad (j=0, 1,\dots, n'-1),$$

where the path of integration $\hat{\mathscr{C}}$ is a Barnes contour similar to $\mathscr{C}$ stated in the beginning of §3.1 (note that the integrands have poles at $z=m$ $(m=0, 1, 2,\dots)$ and $z=-\nu'+\mu_l-s$ $(s=0, 1, 2,\dots, l=1,\dots, q)$). Observing that the asymptotic expansions for $G_{kh}^{(j)}(z)$ $(j=1, 2,\dots)$ are given by differentiating that for $G_{kh}(z)$, i.e., (2.20), we can see that these integrals are absolutely convergent for $|t|<R$ and

$$\hat{x}_j(t) = \textstyle\sum_{m=0}^\infty \hat{G}_j(m)t^m \quad (|t|<R, j=0, 1,\dots, n'-1)$$

hold. Moreover, since

$$\begin{aligned}&\operatorname{Res}[\hat{G}_j(z)p(z)t^z\colon z=-\nu'+\mu_l-s]\\ &= \operatorname{Res}[\partial_\nu^j[G(\nu;\,z+\nu)]_{\nu=\nu'}p(z)t^z\colon z=-\nu'+\mu_l-s]\\ &= \partial_\nu^j[\operatorname{Res}[G(\nu;\,z+\nu)p(z)t^z\colon z=-\nu+\mu_l-s]]_{\nu=\nu'}\\ &= \partial_\nu^j[t^{-\nu}\operatorname{Res}[G(\nu;\,z)p(z-\nu)t^z\colon z=\mu_l-s]]_{\nu=\nu'},\end{aligned}$$

we obtain

$$\hat{x}_j(t) = \partial_\nu^j [t^{-\nu} \sum_{l=1}^{q} \sum_{r=0}^{m_l-1} \hat{T}^{l,m_l-1-r}(\nu) \times \\ \times \sum_{r'=0}^{r} \partial_\mu^{r-r'} [t^\mu]_{\mu=\mu_l} \sum_{s=0}^{\hat{N}_l} H^{lr'}(s) t^{-s}]_{\nu=\nu'} \\ - \frac{1}{2\pi i} \int_{\hat{\mathscr{L}}} \hat{G}_j(z) p(z) t^z dz \quad (|t| < R),$$

where we have replaced $\hat{\mathscr{C}}$ by the rectilinear contour $\hat{\mathscr{L}} = \hat{\mathscr{L}}(\hat{\xi})$ for a negative number $\hat{\xi}$. Besides, in the above, $\hat{N}_l$ $(l=1,\dots, q)$ are positive integers similar to N_l in §3.1 and

$$\hat{T}^{lr}(\nu) = - \sum_{k=1}^{p-1} \sum_{h=0}^{n_k-1} \sum_{r'=0}^{r} \partial_\mu^{r-r'} [p(\mu-\nu)]_{\mu=\mu_l} \gamma_{kh}(\nu) \eta_{kh}^{lr'} \\ (l=1,\dots, q,\ r=0, 1,\dots, m_l-1).$$

Therefore, the non-holomorphic solutions

$$\hat{X}_j(t) = \sum_{j'=0}^{j} \partial_\nu^{j-j'} [t^\nu]_{\nu=\nu'} \hat{x}_{j'}(t) \quad (j=0, 1,\dots, n'-1)$$

of (0.1) near $t=0$ have a representation of the form

$$\hat{X}_j(t) = \sum_{l=1}^{q} \sum_{r=0}^{m_l-1} \partial_\nu^j [\hat{T}^{l,m_l-1-r}(\nu)]_{\nu=\nu'} \times \\ \times \sum_{r'=0}^{r} \partial_\mu^{r-r'} [t^\mu]_{\mu=\mu_l} \sum_{s=0}^{\hat{N}_l} H^{lr'}(s) t^{-s} + \hat{I}_j(t) \\ (|t| < R,\ j=0, 1,\dots, n'-1),$$

where

$$\hat{I}_j(t) = - \frac{1}{2\pi i} \sum_{j'=0}^{j} \partial_\nu^{j-j'} [t^\nu]_{\nu=\nu'} \int_{\hat{\mathscr{L}}} \hat{G}_{j'}(z) p(z) t^z dz \\ = - \frac{1}{2\pi i} \sum_{k=1}^{p-1} \sum_{h=0}^{n_k-1} \sum_{j'=0}^{j} \partial_\nu^{j-j'} [\gamma_{kh}(\nu) t^\nu]_{\nu=\nu'} \frac{1}{j'!} \int_{\hat{\mathscr{L}}} G_{kh}^{(j')}(z+\nu') p(z) t^z dz \\ (j=0, 1,\dots, n'-1)$$

which are analytically continued into the sector

$$\bigcap_{k=1}^{p-1} S_k = \{t;\ \arg \lambda_{p-1} < \arg t < \arg \lambda_1 + 2\pi\}$$

(note (1.1)) and have the following estimates:

$$\| \hat{I}_j(t) \| < \hat{K} |t|^{\hat{\xi} + \mathrm{Re}\,\nu' + \varepsilon} \quad \text{for} \quad t \in \bigcap_{k=1}^{p-1} S'_k \\ (j=0, 1,\dots, n'-1),$$

$\hat{K}$ and ε being positive constants. Hence, $\hat{X}_j(t)$ $(j=0, 1,\dots, n'-1)$ are also analytically continued into $\bigcap_{k=1}^{p-1} S_k$ and satisfy the connection formulas

$$\hat{X}_j(t) = \sum_{l=1}^{q} \sum_{r=0}^{m_l-1} \hat{T}_j^{l,m_l-1-r} Y^{lr}(t) \quad (t \in \cap_{k=1}^{p-1} S_k)$$
$$(j=0, 1,\dots, n'-1),$$

where

$$\hat{T}_j^{lr} = \partial_\nu^j [\hat{T}^{lr}(\nu)]_{\nu=\nu'}$$
$$= -\partial_\nu^j \partial_\mu^r [\sum_{k=1}^{p-1} \sum_{h=0}^{n_k-1} p(\mu-\nu)\gamma_{kh}(\nu)\eta_{kh}^l(\mu)]_{\nu=\nu',\mu=\mu_l}$$
$$(j=0, 1,\dots, n'-1,\ l=1,\dots, q,\ r=0, 1,\dots, m_l-1).$$

In order to obtain the analytic continuation of $\hat{X}_j(t)$ into another sector, we anew take

$$\hat{G}_j(z) = \partial_\nu^j [\sum_{k=1}^{p-1} \sum_{h=0}^{n_k-1} \gamma_{kh}(\nu) e^{2\pi i \delta_k z} G_{kh}(z+\nu)]_{\nu=\nu'},$$

where δ_k $(k=1,\dots, p-1)$ are suitable integers, instead of (2.22). Then we obtain the connection formulas with $\hat{T}_j^{lr}$ replaced by $\hat{T}_j^{lr}(\delta_1,\dots, \delta_{p-1})$ in the above for $t \in \cap_{k=1}^{p-1} S_k(\delta_k)$, where

(3.5) $\hat{T}_j^{lr}(\delta_1,\dots, \delta_{p-1})$
$$= -\partial_\nu^j \partial_\mu^r [\sum_{k=1}^{p-1} \sum_{h=0}^{n_k-1} e^{2\pi i \delta_k(\mu-\nu)} p(\mu-\nu)\gamma_{kh}(\nu)\eta_{kh}^l(\mu)]_{\nu=\nu',\mu=\mu_l}$$
$$(j=0, 1,\dots, n'-1,\ l=1,\dots, q,\ r=0, 1,\dots, m_l-1).$$

3.3. Connection problem for (2.17)

In order to obtain another characterization of $\hat{T}_j^{lr}$ and $\gamma_{kh}(\nu)$, we shall consider the connection problem between solutions of (2.17) near $t=\infty$ and near $t=0$. In this section we assume that $\rho \not\equiv \nu'$ (mod 1) and $\rho \not\equiv \alpha_i$ (mod 1) $(i=1,\dots, N)$.

Let $\mathbf{V}_j(\rho; t)$ $(j=0, 1,\dots, n'-1)$ be solutions of (2.17) near $t=\infty$ for the exponent $-(\rho-1-\nu')$ of the form

$$\mathbf{V}_j(\rho; t) = \sum_{j'=0}^{j} \partial_\nu^{j-j'}[t^{\rho-1-\nu}]_{\nu=\nu'} \mathbf{v}_{j'}(\rho; t),$$
$$\mathbf{v}_j(\rho; t) = \sum_{s=0}^{\infty} \mathbf{K}_j(\rho; s) t^{-s},$$
$$\mathbf{K}_j(\rho; s) = \sum_{j'=0}^{j} \partial_\nu^{j-j'}[\Gamma(s+1+\nu-\rho)]_{\nu=\nu'} \hat{\mathbf{K}}_{j'}(s)$$
$$(j=0, 1,\dots, n'-1).$$

The $\hat{\mathbf{K}}_j(s)$ $(j=0, 1,\dots, n'-1)$ are the vectors determined uniquely by the systems of linear difference equations

$$(3.6)_j \qquad (s+\nu'-\mathbf{A})\hat{\mathbf{K}}_j(s) = \mathbf{B}\hat{\mathbf{K}}_j(s-1) - \hat{\mathbf{K}}_{j-1}(s) \quad (s \geqq 1)$$
$$(j=0, 1,\dots, n'-1,\ \hat{\mathbf{K}}_{-1}(s) \equiv 0)$$

subject to the initial conditions

$$(3.7)_j \qquad (\nu'-\mathbf{A})\hat{\mathbf{K}}_j(0) = -\hat{\mathbf{K}}_{j-1}(0) \quad (j=0, 1,\dots, n'-1, \hat{\mathbf{K}}_{-1}(0)=0).$$

Observing that $(3.6)_0$ is equivalent to (2.16), we here define

$$\hat{\mathbf{K}}_j(s) = \partial_\nu^j[\mathbf{G}(\nu; s+1+\nu)]_{\nu=\nu'} \quad (j=0, 1,\dots, n'-1),$$

where

$$\mathbf{G}(\nu; s+1+\nu) = \sum_{k=1}^{p-1}\sum_{h=0}^{n_k-1}\gamma_{kh}(\nu)\mathbf{G}_{kh}(s+1+\nu).$$

Then these $\hat{\mathbf{K}}_j(s)$ $(j=0, 1,\dots, n'-1)$ satisfy $(3.7)_j$ $(j=0, 1,\dots, n'-1)$ as well as $(3.6)_j$ $(j=0, 1,\dots, n'-1)$, since $\mathbf{G}(\nu; \nu-m)$ $(m=0, 1, 2,\dots)$ have a zero of order n' at $\nu=\nu'$ (note (2.14) and (2.9)). (This guarantees the existence of $\mathbf{V}_j(\rho; t)$ $(j=0, 1,\dots, n'-1)$.)

As to the solutions near $t=0$, we define for the exponent 0

$$\mathbf{U}_{kh}(\rho; t) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\mathbf{G}_{kh}(\rho-m)t^m$$

$$(k=1,\dots, p-1, h=0, 1,\dots, n_k-1)$$

and for the exponent $\rho-1-\alpha_i$

$$\hat{\mathbf{U}}_{kh}^i(\rho; t) = t^{\rho-1-\alpha_i}\sum_{m=0}^{\infty}\Gamma(\alpha_i+1-\rho-m)\mathbf{R}_{kh}^i(m)t^m$$
$$(i=1,\dots, N, k=1,\dots, p-1, h=0, 1,\dots, n_k-1),$$

where

$$\mathbf{R}_{kh}^i(m) = \operatorname{Res}[\mathbf{G}_{kh}(z)\colon z=\alpha_i+1-m] \quad (m=0, 1, 2,\dots)$$
$$(i=1,\dots, N, k=1,\dots, p-1, h=0, 1,\dots, n_k-1).$$

By virtue of the equation (2.16) for $\mathbf{G}_{kh}(z)$ we can easily see that these $\mathbf{U}_{kh}(\rho; t)$ and $\hat{\mathbf{U}}_{kh}^i(\rho; t)$ also satisfy (2.17) (cf. §1).

Defining

$$\mathscr{S}_k(\delta) = \{t\in\mathscr{D};\ \arg\tilde{\lambda}_k+2\pi\delta-2\pi<\arg t<\arg\tilde{\lambda}_k+2\pi\delta\}$$
$$(\delta\in\mathbf{Z}, k=1,\dots, p-1),$$

we have the following

PROPOSITION 2. *Let δ_k $(k=1,\dots, p-1)$ be integers such that $\bigcap_{k=1}^{p-1}\mathscr{S}_k(\delta_k)\neq\emptyset$. Then $\mathbf{V}_j(\rho; t)$ $(j=0, 1,\dots, n'-1)$ are analytically continued into $\bigcap_{k=1}^{p-1}\mathscr{S}_k(\delta_k)$ and satisfy the connection formulas*

$$\mathbf{V}_j(\rho;\, t)$$
$$= -\sum_{k=1}^{p-1}\sum_{h=0}^{n_k-1}\partial_\nu^j[e^{2\pi i\delta_k(\rho-\nu)}p(\rho-\nu)\gamma_{kh}(\nu)]_{\nu=\nu'}\mathbf{U}_{kh}(\rho;\, t)$$
$$-\sum_{i=1}^{N}\sum_{k=1}^{p-1}\sum_{h=0}^{n_k-1}\partial_\nu^j[e^{2\pi i\delta_k(\alpha_i-\nu)}p(\alpha_i-\nu)\gamma_{kh}(\nu)]_{\nu=\nu'}\hat{\mathbf{U}}^i_{kh}(\rho;\, t)$$
$$(j=0,\, 1,\ldots,\, n'-1)$$

for $t\in\bigcap_{k=1}^{p-1}\mathscr{S}_k(\delta_k)$.

PROOF. We consider the Barnes integrals

$$\mathbf{v}_j(\rho;\, t) = -\frac{1}{2\pi i}\int_{\tilde{\mathscr{C}}}\partial_\nu^j[\Gamma(z+1+\nu-\rho)\mathbf{G}(\nu;\, z+1+\nu)]_{\nu=\nu'}p(z)t^{-z}dz$$
$$(j=0,\, 1,\ldots,\, n'-1),$$

where

$$\mathbf{G}(\nu;\, z) = \sum_{k=1}^{p-1}\sum_{h=0}^{n_k-1}\gamma_{kh}(\nu)e^{2\pi i\delta_k(z-\nu)}\,\mathbf{G}_{kh}(z) \tag{3.8}$$

and the path of integration $\tilde{\mathscr{C}}$ is a Barnes contour such that the points $z=s$ $(s=0,$ $1, 2,\ldots)$ lie to the right of $\tilde{\mathscr{C}}$ and the points $z=\rho-1-\nu'-m$ $(m=0, 1, 2,\ldots)$ and $z=\alpha_i-\nu'-m$ $(m=0, 1, 2,\ldots, i=1,\ldots, N)$ lie to the left of $\tilde{\mathscr{C}}$. These points are poles of the integrands. By an analysis similar to §§3.1–3.2, we obtain

$$\mathbf{v}_j(\rho;\, t) = \sum_{s=0}^{\infty}\mathbf{K}_j(\rho;\, s)t^{-s} \qquad \text{for} \quad |t| > 1/R,$$

and for $t\in\bigcap_{k=1}^{p-1}\mathscr{S}_k(\delta_k)$, $|t|<1/R'$,

$$\mathbf{v}_j(\rho;\, t) = -\sum_{m=0}^{\infty}\partial_\nu^j\left[\frac{(-1)^m}{m!}\,p(\rho-\nu)\mathbf{G}(\nu;\, \rho-m)t^{m-\rho+1+\nu}\right]_{\nu=\nu'}$$
$$-\sum_{i=1}^{N}\sum_{m=0}^{\infty}\partial_\nu^j[\Gamma(\alpha_i+1-\rho-m)p(\alpha_i-\nu)\mathbf{R}^i(\nu;\, m)t^{m-\alpha_i+\nu}]_{\nu=\nu'}$$
$$(j=0,\, 1,\ldots,\, n'-1),$$

where

$$\mathbf{R}^i(\nu;\, m) = \operatorname{Res}[\mathbf{G}(\nu;\, z)\colon z=\alpha_i+1-m]$$
$$(m=0,\, 1,\, 2,\ldots,\, i=1,\ldots,\, N).$$

Hence, $\mathbf{V}_j(\rho;\, t)$ $(j=0, 1,\ldots, n'-1)$ are analytically continued into $\bigcap_{k=1}^{p-1}\mathscr{S}_k(\delta_k)$, and for $t\in\bigcap_{k=1}^{p-1}\mathscr{S}_k(\delta_k)$, $|t|<1/R'$,

$$\mathbf{V}_j(\rho;\, t) = -\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\cdot\partial_\nu^j[p(\rho-\nu)\mathbf{G}(\nu;\, \rho-m)]_{\nu=\nu'}\cdot t^m$$
$$-\sum_{i=1}^{N}\sum_{m=0}^{\infty}\Gamma(\alpha_i+1-\rho-m)\cdot\partial_\nu^j[p(\alpha_i-\nu)\mathbf{R}^i(\nu;\, m)]_{\nu=\nu'}\cdot t^{m+\rho-1-\alpha_i}$$
$$(j=0,\, 1,\ldots,\, n'-1)$$

hold. Noting that $\mathbf{G}(\nu; z)$ is given by (3.8) in the above, we have the desired connection formulas. □

Using this proposition, for $\rho\in\mathscr{E}$ we have

$$\mathbf{V}_j(\rho; t) \longrightarrow -\sum_{k=1}^{p-1}\sum_{h=0}^{n_k-1}\partial_\nu^j[e^{2\pi i\delta_k(\rho-\nu)}p(\rho-\nu)\gamma_{kh}(\nu)]_{\nu=\nu'}\mathbf{G}_{kh}(\rho)$$
$$(j=0, 1,\dots, n'-1)$$

as $t\to 0$, $t\in\bigcap_{k=1}^{p-1}\mathscr{S}_k(\delta_k)$. Hence, by combining this with (3.5), we can conclude that the connection coefficients $\hat{T}_j^{lr}(\delta_1,\dots,\delta_{p-1})$ between $\hat{X}_j(t)$ $(j=0, 1,\dots, n'-1)$ and $Y^{lr}(t)$ $(l=1,\dots, q,\ r=0, 1,\dots, m_l-1)$ are also given explicitly by the solutions of (2.17) with α_i $(i=1,\dots, N)$ so that $\mu_l\in\mathscr{E}$ as follows:

$$(3.9)\quad \hat{T}_j^{lr}(\delta_1,\dots,\delta_{p-1})$$
$$=\partial_\mu^r\left[\frac{\Gamma(\mu+1-\mu_l)^{m_l}\Gamma_{\varphi_l}(\mu)}{\Gamma(\mu+1)\Gamma_\psi(\mu)}\lim_{t\to 0}[\mathbf{V}_j(\mu; t)]_{n0}/[H^l(\mu; 0)]_{n0}\right]_{\mu=\mu_l}$$
$$(j=0, 1,\dots, n'-1,\ l=1,\dots, q,\ r=0, 1,\dots, m_l-1),$$

where t is let tend to 0 in the sector $\bigcap_{k=1}^{p-1}\mathscr{S}_k(\delta_k)$.

We summarize all results derived in §3 in the following

THEOREM 1. *Let $Y^{lr}(t)$ $(l=1,\dots, q,\ r=0, 1,\dots, m_l-1)$ be solutions of* (0.1) *of the form* (1.5) *with* (1.6) *near $t=\infty$, and let $S_k(\delta)$ $(\delta\in\mathbf{Z},\ k=1,\dots, p-1)$ be sectors defined by* (3.3).

(i) *For each k, h $(k=1,\dots, p-1,\ h=0, 1,\dots, n_k-1)$, the holomorphic solution $X_{kh}(t)$ of* (0.1) *near $t=0$ which is characterized by $X_{kh}(0)=G_{kh}(0)$, where $G_{kh}(z)$ is a solution of* (0.2) *with $\lambda=0$ characterized by the asymptotic expansion* (2.20), *is holomorphic in $\mathbf{C}\setminus\{\tau\lambda_k;\ \tau\geqq 1\}$ and satisfies the connection formula*

$$X_{kh}(t)=\sum_{l=1}^{q}\sum_{r=0}^{m_l-1}T_{kh}^{l,m_l-1-r}(\delta)Y^{lr}(t)\qquad (t\in S_k(\delta)).$$

The coefficients $T_{kh}^{lr}(\delta)$ $(\delta\in\mathbf{Z},\ l=1,\dots, q,\ r=0, 1,\dots, m_l-1)$ are given by (3.4).

(ii) *Let δ_k $(k=1,\dots, p-1)$ be integers such that $\bigcap_{k=1}^{p-1}S_k(\delta_k)\neq\phi$. The non-holomorphic solutions $\hat{X}_j(t)$ $(j=0, 1,\dots, n'-1)$ of* (0.1) *near $t=0$ of the form* (1.2) *with* (1.3), *where $\hat{G}_j(0)=e_n(n_1+\cdots+n_{p-1}+j+1)$ $(j=0, 1,\dots, n'-1)$, are analytically continued into the sector $\bigcap_{k=1}^{p-1}S_k(\delta_k)$ and satisfy the connection formulas*

$$\hat{X}_j(t)=\sum_{l=1}^{q}\sum_{r=0}^{m_l-1}\hat{T}_j^{l,m_l-1-r}(\delta_1,\dots,\delta_{p-1})Y^{lr}(t)$$
$$(t\in\bigcap_{k=1}^{p-1}S_k(\delta_k),\ j=0, 1,\dots, n'-1).$$

The coefficients $\hat{T}_j^{lr}(\delta_1,\dots,\delta_{p-1})$ $(j=0, 1,\dots, n'-1,\ l=1,\dots, q,\ r=0, 1,\dots, m_l-1)$ *are given by* (3.5) (*or* (3.9)).

(iii) $X_{kh}(t)$ $(k=1,\dots, p-1,\ h=0, 1,\dots, n_k-1)$ *and* $\hat{X}_j(t)$ $(j=0, 1,\dots, n'-1)$ *form a fundamental set of solutions of* (0.1).

The statement (iii) follows immediately from the fact that $\det[\bar{G}^1_{kh}(0)]\neq 0$.

§4. Connection coefficients between solutions near finite singularities

In this section we consider the connection coefficients between solutions of (0.1) near $t=0$ and near another finite singularity. These coefficients, of course, can be calculated in terms of the connection coefficients between solutions near finite singularities and near infinity. However, in order to clear up the meaning of $\gamma_{kh}(v)$ for (0.1), we here investigate them directly.

We consider particular solutions $\tilde{X}_j(t)$ $(j=0, 1,\dots, n'-1)$ of (0.1) defined by

$$\tilde{X}_j(t) = \hat{X}_j(t) - \textstyle\sum_{k=1}^{p-1}\sum_{h=0}^{n_k-1} \partial_v^j[\gamma_{kh}(v)]_{v=v'}X_{kh}(t)$$
$$(j=0, 1,\dots, n'-1)$$

(cf. [8, §5]). We first observe that $\hat{X}_j(t)$ $(j=0, 1,\dots, n'-1)$ have a representation of the form

$$\hat{X}_j(t) = -\frac{1}{2\pi i}\int_{\hat{\mathscr{C}}} \partial_v^j[G(v;\, z+v)t^{z+v}]_{v=v'}p(z)dz$$
$$= -\frac{1}{2\pi i}\int_{\hat{\mathscr{C}}'} \partial_v^j[G(v;\, z)p(z-v)]_{v=v'}t^z dz$$
$$(|t|<R,\ j=0, 1,\dots, n'-1),$$

where $\hat{\mathscr{C}}'$ is the contour obtained by removing $\hat{\mathscr{C}}$ by v'. This can be verified by the change of variables and the integration by parts. Therefore, by virtue of the Cauchy theorem, we have

$$\tilde{X}_j(t) = -\frac{1}{2\pi i}\int_{\mathscr{C}^*} \partial_v^j[G(v;\, z)\{p(z-v)-p(z)\}]_{v=v'}t^z dz \tag{4.1}$$
$$(|t|<R,\ j=0, 1,\dots, n'-1),$$

where the path $\mathscr{C}^*$ is a Barnes contour running from $+\infty-ia$ to $+\infty+ia$ such that the points $z=m$ $(m=0, 1, 2,\dots)$ and $z=v'+m$ $(m=0, 1, 2,\dots)$ lie to the right of $\mathscr{C}^*$, and the points $z=\mu_l-s$ $(s=0, 1, 2,\dots,\ l=1,\dots, q)$ lie to the left of $\mathscr{C}^*$. The constant a is taken as $a>\max\{|\mathrm{Im}\, v'|, |\mathrm{Im}\,\mu_l|\ (l=1,\dots, q)\}$.

Now, in (4.1) we replace the path $\mathscr{C}^*$ by the contour $\mathscr{B}$ which runs first along the imaginary axis from $-i\infty$ to $-ia$, next along $\mathscr{C}^*$ from $0-ia$ to $0+ia$ and finally along the imaginary axis from $+ia$ to $+i\infty$. Then, noting that

$$p(z-v) - p(z) = p(z-v)p(-z)/p(-v),$$

and

$$p^{(j)}(\pm ia + ye^{\pm i\theta} - v')p(\mp ia - ye^{\pm i\theta}) = O(e^{-2\pi\sin\theta\cdot y}) \tag{4.2}$$

as $y \to +\infty$ uniformly in $\theta \in [0, \pi/2]$ ($j=0, 1, 2,\dots$), we can easily see that the integral obtained by replacing $\mathscr{C}^*$ by $\mathscr{B}$ in (4.1) is the analytic continuation of $\tilde{X}_j(t)$ into the sector

$$S = \{t;\ \arg\lambda_{p-1} - 2\pi < \arg t < \arg\lambda_1 + 2\pi\}$$

(compare (4.2) with (3.2)). Namely, $\tilde{X}_j(t)$ ($j=0, 1,\dots, n'-1$) are holomorphic in S and have the following representations:

$$\tilde{X}_j(t) = -\frac{1}{2\pi i}\int_{\mathscr{B}} \partial_v^j[G(v;\, z)p(z-v)/p(-v)]_{v=v'}p(-z)t^z dz$$
$$(t \in S,\ j=0, 1,\dots, n'-1).$$

For each l ($l=1,\dots, q$), we denote by $X^l_{kh}(t)$ ($k=1,\dots, l-1, l+1,\dots, p$, $h=0, 1,\dots, n_k-1$) and $\hat{X}^l_j(t)$ ($j=0, 1,\dots, n_l-1$) the holomorphic and the non-holomorphic solutions of (0.1) near $t=\lambda_l$ which are defined in a way similar to $X_{kh}(t)$ and $\hat{X}_j(t)$ near $t=0$, respectively. In addition, we denote by $G^l_{kh}(z)$ ($k=1,\dots, l-1, l+1,\dots, p$, $h=0, 1,\dots, n_k-1$) the principal solutions of (0.2) with $\lambda=\lambda_l$ in the right half z-plane defined in a way similar to $G_{kh}(z)$ for $\lambda=0$. Namely, $G^l_{kh}(z)$ is characterized by the asymptotic expansion

$$G^l_{kh}(z) \sim (\lambda_k-\lambda_l)^{-z}\textstyle\sum_{h'=0}^{h}\partial_v^{h-h'}[z^{-v-1}]_{v=v_k}\sum_{s=0}^{\infty}f^l_{kh'}(s)z^{-s}$$

as $z\to\infty$, $|\arg z| < \pi/2+\varepsilon$ ($\varepsilon>0$), where

$$f^l_{kh}(0) = e_n(n_1+\cdots+n_{k-1}+h+1) \quad (k\neq l,\ h=0, 1,\dots, n_k-1),$$

and $X^l_{kh}(t)$ is characterized by the expansion

$$X^l_{kh}(t) = \textstyle\sum_{m=0}^{\infty} G^l_{kh}(m)(t-\lambda_l)^m \quad \text{at} \quad t=\lambda_l$$
$$(k\neq l,\ h=0, 1,\dots, n_k-1).$$

As for $\hat{X}^l_j(t)$, we have

$$\hat{X}^l_j(t) = \textstyle\sum_{j'=0}^{j}\partial_v^{j-j'}[(t-\lambda_l)^v]_{v=v_l}\,\hat{x}^l_{j'}(t)$$
$$(j=0, 1,\dots, n_l-1)$$

near $t=\lambda_l$, where $\hat{x}^l_j(t)$ ($j=0, 1,\dots, n_l-1$) are holomorphic at $t=\lambda_l$ and

$$\hat{x}^l_j(\lambda_l) = e_n(n_1+\cdots+n_{l-1}+j+1) \quad (j=0, 1,\dots, n_l-1).$$

Concerning these solutions we have the following

THEOREM 2. *If λ_l satisfies*

$$\arg\lambda_{p-1} - \pi < \arg\lambda_l < \arg\lambda_1 + \pi, \tag{4.3}$$

then we have

$$X^l_{pj}(t) = \sum_{j'=0}^{j} \partial_\nu^{j-j'}[\lambda_l^{-\nu}\Gamma(-\nu)]_{\nu=\nu'}\hat{X}_{j'}(t)$$
$$- \sum_{k=1}^{p-1}\sum_{h=0}^{n_k-1} \partial_\nu^j[\lambda_l^{-\nu}\Gamma(-\nu)\gamma_{kh}(\nu)]_{\nu=\nu'}X_{kh}(t)$$
$$(j=0, 1,\dots, n'-1)$$

for $t\in\{t;\ |t|<R\}\setminus\{\tau\lambda_l;\ \tau\leqq 0\}$, *where* $\arg t\in(\arg\lambda_l-\pi,\ \arg\lambda_l+\pi)$.

PROOF. We first observe that each $\tilde{X}_j(t)$ $(j=0, 1,\dots, n'-1)$ can be expressed by a linear combination of $X^l_{pj}(t)$ $(j=0, 1,\dots, n'-1)$ (note that (4.3) holds). In order to determine the coefficients we calculate the asymptotic expansion for

$$\partial_t^m[\tilde{X}_j(t)]_{t=\lambda_l} \tag{4.4}$$
$$= -\frac{1}{2\pi i}\int_{\mathscr{B}} \partial_\nu^j\left[G(\nu;\ z)\frac{p(z-\nu)}{p(-\nu)}\right]_{\nu=\nu'} p(-z)\frac{\Gamma(-z+m)}{\Gamma(-z)}\lambda_l^z dz\cdot\frac{(-\lambda_l)^{-m}}{\Gamma(m+1)}$$

as $m\to\infty$ $(j=0, 1,\dots, n'-1)$. Let β be an arbitrary positive number not equal to $\mathrm{Re}\,\nu'+s$ $(s=0, 1, 2,\dots)$, and let M be a positive integer such that

$$M < \beta - \mathrm{Re}\,\nu' < M+1.$$

Observing that the points $z=s$ $(s=0, 1,\dots, m-1)$ are no longer poles of the integrand in (4.4), we deform the path $\mathscr{B}$ in (4.4) with $m>\beta$ to be the straight line from $\beta-i\infty$ to $\beta+i\infty$ plus a closed curve which circles the points $z=\nu'+s$ $(s=0, 1,\dots, M)$ in the negative direction. Then we have

$$\partial_t^m[\tilde{X}_j(t)]_{t=\lambda_l}$$
$$= (-\lambda_l)^{-m}\sum_{s=0}^{M}\partial_\nu^j\left[G(\nu;\ \nu+s)\frac{\Gamma(m-\nu-s)}{\Gamma(-\nu-s)\Gamma(m+1)}\lambda_l^{\nu+s}\right]_{\nu=\nu'}$$
$$-\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\partial_\nu^j[G(\nu;\ z)p(z-\nu)/p(-\nu)]_{\nu=\nu'}\cdot$$
$$\cdot\Gamma(z+1)\Gamma(m-z)(e^{\pi i}\lambda_l)^z dz\cdot(-\lambda_l)^{-m}/\Gamma(m+1).$$

By virtue of the condition (4.3) and the inequality

$$|\Gamma(x+iy)| \leqq \Gamma(x) \qquad \text{for} \quad x>0, \quad y\in\boldsymbol{R},$$

we can easily estimate the last term, which we denote by $\tilde{I}_j(m)$, as follows:

$$\|\tilde{I}_j(m)\| < \tilde{K}|-\lambda_l|^{-m}\Gamma(m-\beta)/\Gamma(m+1) \qquad \text{for} \quad m > \beta,$$

where $\tilde{K}$ is a positive constant independent of m (but depending on β). This implies an asymptotic expansion of the form

$$\partial_t^m[\tilde{X}_j(t)]_{t=\lambda_l}$$
$$\sim (-\lambda_l)^{-m}\sum_{j'=0}^{j} \partial_\nu^{j-j'}[m^{-\nu-1}]_{\nu=\nu'} \sum_{s=0}^{\infty} \chi_{j'}(s)m^{-s}$$

as $m\to\infty$, where

$$\chi_j(0) = \partial_\nu^j[G(\nu;\, \nu)\lambda_l^\nu/\Gamma(-\nu)]_{\nu=\nu'}$$
$$= \sum_{j'=0}^{j} \partial_\nu^{j-j'}[\lambda_l^\nu/\Gamma(-\nu)]_{\nu=\nu'}e_n(n_1+\cdots+n_{p-1}+j'+1)$$
$$(j=0,\, 1,\dots,\, n'-1).$$

Comparing this with the asymptotic expansions of $G_{pj}^l(m)$ $(j=0,\, 1,\dots,\, n'-1)$, we have

$$\tilde{X}_j(t) = \sum_{j'=0}^{j} \partial_\nu^{j-j'}[\lambda_l^\nu/\Gamma(-\nu)]_{\nu=\nu'}X_{pj'}^l(t)$$
$$(j=0,\, 1,\dots,\, n'-1)$$

for $t\in \boldsymbol{C}\setminus\{\tau\lambda_l;\, \tau\leqq 0\}$. Multiplying both sides of this formula by $\partial_\nu^{j''-j}[\lambda_l^{-\nu}\cdot \Gamma(-\nu)]_{\nu=\nu'}$ and summing them over j, $0\leqq j\leqq j''$, we obtain the desired connection formulas $(j''=0,\, 1,\dots,\, n'-1)$. □

In general, taking

$$G(\nu;\, z) = \sum_{k=1}^{p-1} \sum_{h=0}^{n_k-1} \gamma_{kh}(\nu)e^{2\pi i\delta_k(z-\nu)}G_{kh}(z) \quad (\delta_k\in \boldsymbol{Z}),$$

we have

THEOREM 2′. *Let* δ_k $(k=1,\dots,\, p-1)$ *be integers such that* $\bigcap_{k=1}^{p-1} S_k(\delta_k)= \{t;\, \theta_1<\arg t<\theta_2\}\neq\emptyset$.

(i) *The solutions of* (0.1) *defined by*

$$\tilde{X}_j(\delta_1,\dots,\, \delta_{p-1};\, t)$$
$$= \hat{X}_j(t) - \sum_{k=1}^{p-1} \sum_{h=0}^{n_k-1}\partial_\nu^j[e^{-2\pi i\delta_k\nu}\gamma_{kh}(\nu)]_{\nu=\nu'}X_{kh}(t)$$
$$(j=0,\, 1,\dots,\, n'-1)$$

are holomorphic in the sector $\{t;\, \theta_1-2\pi<\arg t<\theta_2\}$.

(ii) *If* λ_l *satisfies*

$$\theta_1 < \arg\lambda_l - 2\pi\delta_0 + \pi < \theta_2$$

for some integer δ_0, *then we have*

$$X^l_{pj}(t) = \sum_{j'=0}^{j} \partial_\nu^{j-j'}[e^{2\pi i\delta_0\nu}\lambda_l^{-\nu}\Gamma(-\nu)]_{\nu=\nu'}\hat{X}_{j'}(t)$$
$$- \sum_{k=1}^{p-1}\sum_{h=0}^{n_k-1} \partial_\nu^j[e^{2\pi i(\delta_0-\delta_k)\nu}\lambda_l^{-\nu}\Gamma(-\nu)\gamma_{kh}(\nu)]_{\nu=\nu'}X_{kh}(t)$$
$$(j=0, 1,\dots, n'-1)$$

for $t\in\{t;\ |t|<R\}\setminus\{\tau\lambda_l;\ \tau\leqq 0\}$, *where* $\arg t\in(\arg\lambda_l-2\pi\delta_0-\pi,\ \arg\lambda_l-2\pi\delta_0+\pi)$.

As an immediate consequence of this theorem, we have

COROLLARY. *If* $\lambda_k\notin Q_{l,l'}=\{\tau(\tau'\lambda_l+(1-\tau')\lambda_{l'});\ \tau\leqq 0,\ 0\leqq\tau'\leqq 1\}$ *for every* $k\neq l,\ l',\ p$, *then we have*

$$X^l_{pj}(t) = \sum_{j'=0}^{j} \partial_\nu^{j-j'}[(e^{2\pi i\delta}\lambda_{l'}/\lambda_l)^\nu]_{\nu=\nu'}X^{l'}_{pj'}(t)$$
$$(j=0, 1,\dots, n'-1)$$

for $t\in\mathbf{C}\setminus Q_{l,l'}$, *where* δ *is the integer such that* $|\arg\lambda_{l'}-\arg\lambda_l+2\pi\delta|<\pi$.

REMARK 2. In the case when $\lambda_{k'}, \lambda_{k''},\dots\in Q_{l,l'}$, by repeated application of Theorem 2′, we can see that $X^l_{pj}(t)$ $(j=0, 1,\dots, n'-1)$ are represented by means of a linear combination of $X^{l'}_{pj}(t)$, $X^{l'}_{k'h}(t)$, $X^{l'}_{k''h}(t),\dots$ in which the coefficients are expressed in terms of $\gamma_{k'h}(\nu)$, $\gamma_{k''h}(\nu),\dots$ (cf. [1, §3]).

As to the non-holomorphic solutions, defining

$$P'_l = \cup_{\rho\in(0,1)}\{t;\ |t-\rho\lambda_l|<r\}\setminus\{\tau\lambda_l;\ \tau\leqq 0\}$$

and

$$P_l = P'_l\setminus\{\tau\lambda_l;\ \tau\geqq 1\} \quad (l=1,\dots, p-1),$$

where $r=r(l)$ is a positive number such that $\lambda_k\notin P_l$ for every $k=1,\dots, p$, we have

THEOREM 3. *For each* l $(l=1,\dots, p-1)$, *we have*

$$\hat{X}_j(t) = \sum_{h=0}^{n_l-1}\partial_\nu^j\partial_\mu^{n_l-1-h}[c_l(\nu, \mu)]_{\nu=\nu',\mu=\nu_l}\hat{X}^l_h(t) + \Phi^l_j(t)$$
$$(j=0, 1,\dots, n'-1)$$

for $t\in P_l$, *where*

$$c_l(\nu, \mu) = (e^{\pi i}\lambda_l)^{-\mu}\Gamma(-\mu)\sum_{h=0}^{n_l-1}\gamma_{lh}(\nu)(\mu-\nu_l)^{n_l-1-h},$$

(4.5) $$\arg\lambda_l - \pi < \arg t < \arg\lambda_l + \pi,$$

(4.6) $$\arg\lambda_l < \arg(t-\lambda_l) < \arg\lambda_l + 2\pi$$

and $\Phi^l_j(t)$ $(j=0, 1,\dots, n'-1)$ *are holomorphic in* P'_l.

PROOF. In Theorem 2′ we take δ_k $(k=1,\dots, p-1)$ so that $\arg\lambda_l+\pi\in(\theta_1, \theta_2)$

($\delta_0=0$). Then we have $\delta_l=0$, since $\lambda_l e^{\pi i}\in\bigcap_{k=1}^{p-1} S_k(\delta_k)\subset S_l(\delta_l)$. Therefore, for $t\in P_l$ with (4.5) we have

$$\sum_{j'=0}^{j}\partial_\nu^{j-j'}[\lambda_l^{-\nu}\Gamma(-\nu)]_{\nu=\nu'}\hat{X}_{j'}(t)$$
$$=\sum_{h=0}^{n_l-1}\partial_\nu^{j}[\lambda_l^{-\nu}\Gamma(-\nu)\gamma_{lh}(\nu)]_{\nu=\nu'}X_{lh}(t)+\tilde{\Phi}_j^l(t)$$
$$(j=0, 1,\dots, n'-1),$$

and hence

$$\hat{X}_j(t)=\sum_{h=0}^{n_l-1}\partial_\nu^{j}[\gamma_{lh}(\nu)]_{\nu=\nu'}X_{lh}(t)+\hat{\Phi}_j^l(t) \quad (j=0, 1,\dots, n'-1), \tag{4.7}$$

where $\tilde{\Phi}_j^l(t)$ and $\hat{\Phi}_j^l(t)$ ($j=0, 1,\dots, n'-1$) are holomorphic in P_l'. On the other hand, in the theorem for λ_l corresponding to Theorem 2′ for 0, we take δ_0 and δ_k ($k\neq l$) so that $\arg(-\lambda_l)-2\pi\delta_0+\pi=\arg\lambda_l+2\pi\in(\theta_1, \theta_2)$. Then we have

$$X_{lh}(t)=\sum_{h'=0}^{h}\partial_\mu^{h-h'}[(e^{\pi i}\lambda_l)^{-\mu}\Gamma(-\mu)]_{\mu=\nu_l}\hat{X}_{h'}^l(t)+\Psi_h^l(t) \quad (h=0, 1,\dots, n_l-1) \tag{4.8}$$

for $t\in P_l$ with (4.6), where $\Psi_h^l(t)$ ($h=0, 1,\dots, n_l-1$) are holomorphic in P_l'. Substituting (4.8) for $X_{lh}(t)$ in (4.7), we thus obtain the desired connection formulas. □

Comparing Theorems with Propositions, if $(N+1)n''<n$, we can consider our procedure to be a reduction of connection problems for hypergeometric systems. Moreover, applying again our method to (2.17), we can think of the hierarchy of connection coefficients in some cases. In the case when $n''=1$ (i.e., $n'=n-1$), for example, (2.17) has the dimension $n-1$. Moreover, the quantity corresponding to n'' in the connection problem for (2.17) is also equal to 1. Hence we can obtain the successive reduction of connection problems and the hierarchy of connection coefficients in this case. See also [13] and [14].

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