HIROSHIMA MATH. J.
19 (1989), 197–202

Removability of polar sets for energy finite harmonic functions on harmonic spaces with adjoint structure

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(Received April 11, 1988)

It is well known that, in the classical potential theory, a polar set is removable for Dirichlet finite harmonic functions (see, e.g., [1]). This result was extended by the author to the case of self-adjoint harmonic spaces in [5]. But, as remarked in Remark 3 in [5], there seems to be no known results of this type in the non-elliptic case, even for solutions of the heat equation.

In the present note, we prove that, on harmonic spaces with adjoint structure, polar sets are removable for energy finite harmonic functions. In case the constant functions are harmonic, the energy coincides with the Dirichlet integral, so that our result implies the removability of heat polar sets for Dirichlet-finite solutions of the heat equation. Also, our proof provides a new proof to the classical result, which is quite different from the known proofs (cf. e.g., [1] and [5]).

A preliminary abridged version of the present paper is given in the APPENDIX of [7].

§ 1. Preliminaries

We consider a $\mathfrak{P}$-harmonic space $(X, \mathscr{H})$ with an adjoint harmonic space $(X, \mathscr{H}^*)$ as defined in [6]. By definition, there exists a Green function $G(x, y)$ associated with the structures $\mathscr{H}$ and $\mathscr{H}^*$, which satisfies conditions (G.0), (G.1), (G*.1), (G.2) and (G*.2) given in [6]. For a non-negative measure μ on X, we write

$$G\mu = \int G(\cdot, y)\, d\mu(y) \quad \text{and} \quad G^*\mu = \int G(x, \cdot)\, d\mu(x).$$

$G\mu$ (resp. $G^*\mu$) is an $\mathscr{H}$-potential (resp. $\mathscr{H}^*$-potential) on X if it is finite on a dense set. We can easily show that this is the case if 1 is $\mathscr{H}^*$-superharmonic (resp. $\mathscr{H}$-superharmonic) and $\mu(X) < +\infty$.

By standard arguments (cf. e.g., [4; § 4], [3; 1.VII and 1.XVII, § 4 and § 5]), we obtain

PROPOSITION 1. *For any open set U in X, the harmonic spaces $(U, \mathscr{H}|_U)$ and $(U, \mathscr{H}^*|_U)$ are mutually adjoint with a Green function $G^U(x, y)$ such that*

$$G(x, y) = G^U(x, y) + h^U(x, y) \quad \textit{for all } x, y \in U,$$

where h^U is a function on $X \times X$ such that $h^U(\cdot, y)$ (resp. $h^U(x, \cdot)$) is $\mathscr{H}$-harmonic (resp. $\mathscr{H}^*$-harmonic) on U for any $y \in U$ (resp. $x \in U$).

As in [6], we assume that the constant function 1 is superharmonic for both $\mathscr{H}$ and $\mathscr{H}^*$.

Let $\sigma: \mathscr{R} \to \mathscr{M}$ (resp. $\sigma^*: \mathscr{R}^* \to \mathscr{M}$) be the measure representation associated with $G(x, y)$ (see [6]), where $\mathscr{R}$ (resp. $\mathscr{R}^*$) is the sheaf of functions which are locally expressible as differences of continuous $\mathscr{H}$-superharmonic (resp. $\mathscr{H}^*$-superharmonic) functions, and $\mathscr{M}$ is the sheaf of signed measures on X. The gradient measure δ_f of $f \in \mathscr{R}(U)$ (U: open $\subset X$) is defined by

$$\delta_f = \tfrac{1}{2}\{2f\sigma(f) - \sigma(f^2) - f^2\sigma(1)\},$$

which is a non-negative measure on U (see [4]). The Dirichlet integral $D_U[f]$ of $f \in \mathscr{R}(U)$ is the total mass of δ_f, namely

$$D_U[f] = \delta_f(U),$$

and the *energy* $E_U[f]$ of $f \in \mathscr{R}(U)$ is given by

$$E_U[f] = D_U[f] + \frac{1}{2}\int_U f^2\, d\sigma(1).$$

Thus, $E_U[f] = D_U[f]$ if $\sigma(1) = 0$ on U. The mapping $f \to E_U[f]^{1/2}$ is a semi-norm on the linear space $\mathscr{R}_E(U) = \{f \in \mathscr{R}(U) \mid E_U[f] < +\infty\}$.

We consider the linear space of energy-finite (Dirichlet-finite, in case $\sigma(1) = 0$) harmonic functions; namely, for an open set U in X, let

$$\mathscr{H}_E(U) = \{u \in \mathscr{H}(U) \mid E_U[u] < +\infty\} = \mathscr{H}(U) \cap \mathscr{R}_E(U).$$

We first establish the following

PROPOSITION 2. *For any $u \in \mathscr{H}_E(U)$, the least $\mathscr{H}$-harmonic majorant v of $|u|$ exists and $v \in \mathscr{H}_E(U)$.*

PROOF. Since $\sigma(u) = 0$, $2\delta_u + u^2\sigma(1) = -\sigma(u^2)$. Hence, $\mu = -\sigma(u^2)$ is a non-negative measure on U and $\mu(U) < +\infty$. It follows that $G^U\mu = \int G^U(\cdot, y)\, d\mu(y)$ is a continuous $\mathscr{H}$-potential on U and $h = u^2 + G^U\mu$ is $\mathscr{H}$-harmonic on U. Since $h \geq u^2$ and $h^{1/2}$ is $\mathscr{H}$-superharmonic (cf. [4; the proof of Lemma 6.2]), it follows that $|u|$ has the least $\mathscr{H}$-harmonic majorant v such that $|u| \leq v \leq h^{1/2}$. Since $h - v^2$ is $\mathscr{H}$-superharmonic and majorized by $h - u^2 = G^U\mu$, we have $h - v^2 = G^U\nu$ with $\nu = -\sigma(v^2)$. Since $G^U\nu \leq G^U\mu$, [6; Lemma 1.3] implies that $\nu(U) \leq \mu(U) < +\infty$. This means that $v \in \mathscr{H}_E(U)$, since $\nu = 2\delta_v + v^2\sigma(1)$.

COROLLARY 1. *If $u \in \mathscr{H}_E(U)$, then $u = u_1 - u_2$ with $u_1, u_2 \in \mathscr{H}_E(U)$, $u_1 \geq 0$ and $u_2 \geq 0$ on U.*

§ 2. Polar sets

Let $\mathscr{P}$ (resp. $\mathscr{P}^*$) be the set of all $\mathscr{H}$-potentials (resp. $\mathscr{H}^*$-potentials) on X and let $\mathscr{P}_C$ (resp. $\mathscr{P}_C^*$) be the subset consisting of all continuous ones in $\mathscr{P}$ (resp. $\mathscr{P}^*$).

A compact set K in X is said to be polar (with respect to $\mathscr{H}$) if there is $p \in \mathscr{P}$ such that $p(x) = +\infty$ for all $x \in K$.

We denote by $\mathscr{C}(X)$ the set of continuous functions on X and by $\mathscr{C}_0(X)$ the set of functions in $\mathscr{C}(X)$ with compact support.

LEMMA 1. *Let K be a compact polar set in X and let $\{V_n\}$ be a sequence of open sets such that $V_n \supset \bar{V}_{n+1}$ and $\bigcap_{n=1}^{\infty} V_n = K$. Then there is a sequence $\{p_n\}$ in $\mathscr{P}_C$ such that $p_n = 1$ on a neighborhood of K,* Supp $\sigma(p_n) \subset V_n$ *for each n and $p_n \downarrow 0$ locally uniformly on $X \backslash K$.*

PROOF. By definition, there is $p \in \mathscr{P}$ such that $p(x) = +\infty$ for all $x \in K$. For each n, let $U_n = \{x \in X \mid p(x) > n\}$, which is an open set containing K. Choose $\varphi_n \in \mathscr{C}_0(X)$ such that $0 \leq \varphi_n \leq 1$ on X, $\varphi_n = 1$ on a neighborhood of K and Supp $\varphi_n \subset U_n \cap V_n$. Since $p/n > \varphi_n$ on X, $R\varphi_n \leq p/n$, where R denotes the reduction operator for $\mathscr{H}$ (cf. [2; pp. 39–40] or [4; § 2–3]). $p_n = R\varphi_n$ belongs to $\mathscr{P}_C$ and Supp $\sigma(p_n) \subset U_n \cap V_n$. Obviously, $p_n = 1$ on a neighborhood of K. We may assume that $\varphi_n \geq \varphi_{n+1}$, so that $p_n \geq p_{n+1}$. Let $p_0 = \lim_{n\to\infty} p_n$. Since p_n is $\mathscr{H}$-harmonic on $X \backslash \bar{V}_n$, p_0 is $\mathscr{H}$-harmonic on $X \backslash K$. Since $p_0 \leq p/m$ for all m, it follows that $p_0 = 0$ on $X \backslash K$. By Dini's lemma, the convergence is locally uniform on $X \backslash K$.

LEMMA 2. *If K is a compact polar set in X and U is an open set containing K, then there is a non-negative measure μ_0 on X such that* Supp $\mu_0 \subset U$, *$G\mu_0$ is finite continuous on $X \backslash K$ and $G\mu_0(x) = +\infty$ for $x \in K$.*

PROOF. Choose $\{p_n\}$ as in the above lemma with $\{V_n\}$ such that $V_1 \subset U$ and let $\mu_n = \sigma(p_n)$. Then, $p_n = G\mu_n$ and Supp $\mu_n \subset V_n \subset U$. We can choose a subsequence $\{p_{n_j}\}$ such that $\sum_{j=1}^{\infty} p_{n_j}$ converges locally uniformly on $X \backslash K$. Then, $p_0 = \sum_{j=1}^{\infty} p_{n_j}$ is finite continuous on $X \backslash K$ and $p_0(x) = +\infty$ for $x \in K$. Let $\nu_m = \sum_{j=1}^{m} \mu_{n_j}$. For each $y \in K$, there is $z_y \in X \backslash K$ such that $G(z_y, y) > 0$; for, otherwise $G(x, y) = 0$ for all $x \in X \backslash K$, and since K has no interior point it would follow that $G(x, y) = 0$ for all $x \in X$, which is absurd. By continuity, there is an open neighborhood W_y of y such that $\alpha_y \equiv \inf_{w \in W_y} G(z_y, w) > 0$. Then

$$\nu_m(W_y) \leq \frac{1}{\alpha_y} \int G(z_y, w)\, d\nu_m(w) \leq \frac{1}{\alpha_y} p_0(z_y) < +\infty .$$

Since K is compact, there are $y_1, \dots, y_l \in K$ such that $W \equiv W_{y_1} \cup \cdots \cup W_{y_l} \supset K$. Then $\{\nu_m(W)\}$ is bounded. Since $\operatorname{Supp} \mu_n \subset W$ for sufficiently large n, it follows that $\{\nu_m\}$ is vaguely convergent, so that $\mu_0 = \sum_{j=1}^{\infty} \mu_{n_j}$ is a non-negative measure on X with $\operatorname{Supp} \mu_0 \subset U$. Obviously, $G\mu_0 = p_0$.

LEMMA 3. *If K is a compact polar set and if $G^*\nu$ ($\nu \geq 0$) is bounded in a neighborhood of K, then $\nu(K) = 0$.*

PROOF. Suppose $G^*\nu$ is bounded on $U \supset K$ and let $M = \sup_U G^*\nu$. Let μ_0 be the measure given in the above lemma for K and U. Since $G\mu_0(x) = +\infty$ for $x \in K$, for any $\varepsilon > 0$ we have

$$\nu(K) \leq \varepsilon \int G\mu_0 \, d\nu = \varepsilon \int G^*\nu \, d\mu_0 \leq \varepsilon M \mu_0(X) .$$

Since $\mu_0(X) < +\infty$, it follows that $\nu(K) = 0$.

PROPOSITION 3. *If K is a compact polar set in X and U is an open set containing K, then there exists a sequence $\{p_n\}$ in $\mathscr{P}_C$ such that $p_n = 1$ on a neighborhood of K, $\operatorname{Supp} \sigma(p_n) \subset U$ for each n, $p_n \downarrow 0$ locally uniformly on $X \setminus K$ and $\sigma(p_n)(X) \to 0$ ($n \to \infty$).*

PROOF. It suffices to show that $\sigma(p_n)(X) \to 0$ ($n \to \infty$) for $\{p_n\}$ given in Lemma 1 with $\{V_n\}$ such that $V_1 \subset U$. We may assume that U is relatively compact. Choose $\psi \in \mathscr{C}_0(X)$ such that $0 \leq \psi \leq 1$ on X and $\psi = 1$ on U. Let R^* denote the reduction operator for $\mathscr{H}^*$ and let $q = R^*\psi$. Then $q \in \mathscr{P}_C^*$, so that $q = G^*\lambda$ with $\lambda = \sigma^*(q)$. Since q is bounded, $\lambda(K) = 0$ by the above lemma. Hence

$$\sigma(p_n)(X) = \sigma(p_n)(U) = \int q \, d\sigma(p_n) = \int p_n \, d\lambda = \int_{X \setminus K} p_n \, d\lambda .$$

Since $p_n \downarrow 0$ on $X \setminus K$ and $\operatorname{Supp} \lambda$ is compact, Lebesgue's convergence theorem implies $\int_{X \setminus K} p_n \, d\lambda \to 0$ ($n \to \infty$). Thus $\sigma(p_n)(X) \to 0$.

REMARK 1. Proposition 3 means that if K is a compact polar set, then $c_0(K) = 0$ for the capacity c_0 defined in [7]. Conversely, using the arguments as in the proof of Lemma 2 (taking a subsequence such that $\nu_{n_j}(X) < 2^{-j}$, say), we can show that $c_0(K) = 0$ implies that K is polar.

In [7], we have shown that $c_0(K) = c_0^*(K)$. Hence we have

COROLLARY 2. *If K is a compact polar set in X and U is an open set containing K, then there is a sequence $\{q_n\}$ in $\mathscr{P}_C^*$ such that $q_n = 1$ on a neighborhood of K, $\operatorname{Supp} \sigma^*(q_n) \subset U$ for each n, $q_n \downarrow 0$ locally uniformly on $X \setminus K$ and $\sigma^*(q_n)(X) \to 0$ ($n \to \infty$).*

LEMMA 4. *Let K be a compact polar set in X and suppose $p \in \mathscr{P}$ is $\mathscr{H}$-harmonic on $X \setminus K$. Then there exists a non-negative measure μ on X such that* $\operatorname{Supp} \mu \subset K$ *and* $p = G\mu$.

PROOF. As in the proof of Lemma 2, for each $y \in K$ there is $x_y \in X \setminus K$ such that $G(x_y, y) > 0$. Choose a relatively compact open neighborhood W_y of x_y such that $\overline{W_y} \cap K = \varnothing$, and choose $\varphi_y \in \mathscr{C}(X)$ such that $\varphi_y = 0$ on a neighborhood of x_y, $\varphi_y = 1$ on $X \setminus \overline{W_y}$ and $0 \leq \varphi_y \leq 1$ in X. Put $q_y = R^*(G(x_y, \cdot)\varphi_y)$. Then, $q_y \in \mathscr{P}_C^*$, $\operatorname{Supp} \sigma^*(q_y) \subset \overline{W_y}$ and $q_y(y) = G(x_y, y) > 0$. By continuity, there is an open neighborhood V_y of y such that $q_y > 0$ on V_y. Choose $y_1, \dots, y_k \in K$ such that $V_{y_1} \cup \cdots \cup V_{y_k} \supset K$, and let $q_K = q_{y_1} + \cdots + q_{y_k}$. Then, $q_K \in \mathscr{P}_C^*$, $\operatorname{Supp} \sigma^*(q_K) \cap K = \varnothing$ and $q_K(z) > 0$ for all $z \in K$. Put $\alpha = \inf_K q_K$ and $V = \{y \in X \mid q_K(y) > \alpha/2\}$. Then V is an open set containing K. Choose a sequence $\{U_n\}$ of relatively compact open sets such that $V \supset U_n \supset \bar{U}_{n+1} \supset K$ and $\bigcap_{n=1}^{\infty} U_n = K$, and then choose $\psi_n \in \mathscr{C}(X)$ such that $\psi_n = 1$ on $X \setminus U_n$, $\psi_n = 0$ on U_{n+1} and $0 \leq \psi_n \leq 1$ on X. Put $p_n = R(\psi_n p)$. Then $p_n \in \mathscr{P}_C$ and $p_n = p$ on $X \setminus U_n$. Since p is $\mathscr{H}$-harmonic on $X \setminus \bar{U}_n$, we see that $\operatorname{Supp} \sigma(p_n) \subset \bar{U}_n \subset V$. Put $\mu_n = \sigma(p_n)$ and $\nu_K = \sigma^*(q_K)$. Then

$$\mu_n(X) = \mu_n(V) \leq \frac{2}{\alpha} \int q_K \, d\mu_n = \frac{2}{\alpha} \int p_n \, d\nu_K \leq \frac{2}{\alpha} \int p \, d\nu_K .$$

Since $\operatorname{Supp} \nu_K$ is compact and disjoint from K, p is bounded on $\operatorname{Supp} \nu_K$. Hence, $\{\mu_n(X)\}$ is bounded, so that there exists a non-negative measure μ on X such that a subsequence $\{\mu_{n_j}\}$ of $\{\mu_n\}$ converges vaguely to μ. Then $\operatorname{Supp} \mu \subset K$ and $p_{n_j}(x) \to G\mu(x)$ $(j \to \infty)$ for any $x \in X \setminus K$. Hence $p = G\mu$ on $X \setminus K$. Since K is polar, it follows that $p = G\mu$ on X.

§ 3. Removability theorem

THEOREM. *Let K be a compact polar set in X. Then K is removable with respect to $\mathscr{H}_E$; namely, for any $u \in \mathscr{H}_E(X \setminus K)$ there exists $\tilde{u} \in \mathscr{H}(X)$ such that $\tilde{u}|_{X \setminus K} = u$.*

PROOF. By Corollary 1, we may assume that $u \geq 0$. By [2; Theorem 6.2.1], there exists an $\mathscr{H}$-superharmonic function $\tilde{u}$ on X such that $\tilde{u}|_{X \setminus K} = u$. Then $\tilde{u} = h + p$ with $h \in \mathscr{H}(X)$ and $p \in \mathscr{P}$. It suffices to show that $p = 0$.

Since p is $\mathscr{H}$-harmonic on $X \setminus K$, Lemma 4 implies that $p = G\mu$ with a non-negative measure μ such that $\operatorname{Supp} \mu \subset K$. Let V be any relatively compact open set containing K. Since $E_V[h] < \infty$, we see that $E_{V \setminus K}[p] < +\infty$. Choose $\varphi \in \mathscr{C}(X)$ such that $\varphi = 0$ on a neighborhood of K, $\varphi = 1$ on a neighborhood W of $X \setminus V$ and $0 \leq \varphi \leq 1$ on X. Put $p' = R(p\varphi)$. Then $p' \in \mathscr{P}_C$, $p' = p$ on W and $\operatorname{Supp} \sigma(p')$ is compact (in fact, contained in $V \setminus K$). Thus,

$$E_W[p] = E_W[p'] \le E_X[p'] \le \int p' \, d\sigma(p') < +\infty$$

by [6; Theorem 3.1]. Hence $E_{X \setminus K}[p] < +\infty$.

Put $\nu = 2\delta_p + p^2\sigma(1)$ on $X \setminus K$. Since $\nu(X \setminus K) = E_{X \setminus K}[p] < +\infty$, ν can be regarded as a non-negative measure on X and we see that $G\nu \in \mathscr{P}$. Since p is $\mathscr{H}$-harmonic on $X \setminus K$, $\nu = -\sigma(p^2)$ on $X \setminus K$. Thus, $(G\nu + p^2)|_{X \setminus K} \in \mathscr{H}(X \setminus K)$. Again by [2; Theorem 6.2.1] and Lemma 4, there exists a non-negative measure ν_0 such that $\operatorname{Supp} \nu_0 \subset K$ and $G\nu + p^2 = h' + G\nu_0$ on $X \setminus K$ with $h' \in \mathscr{H}(X)$. Since p is bounded outside a compact set (cf. e.g., [4; Proposition 2.5]), we see that $h' = 0$. By Corollary 2, there is a sequence $\{\lambda_n\}$ of non-negative measures on X such that $G^*\lambda_n \in \mathscr{P}_C^*$, $G^*\lambda_n = 1$ on a neighborhood of K, $\operatorname{Supp} \lambda_n$ is compact for each n and $\lambda_n(X) \to 0$ $(n \to \infty)$. Since $p^2 \le G\nu_0$ on $X \setminus K$ and $\lambda_n(K) = 0$ (by Lemma 3), we have

$$\int p^2 \, d\lambda_n \le \int G\nu_0 \, d\lambda_n = \int G^*\lambda_n \, d\nu_0 = \nu_0(K) < +\infty .$$

Hence

$$\mu(X) = \mu(K) = \int G^*\lambda_n \, d\mu = \int p \, d\lambda_n \le \left(\int p^2 \, d\lambda_n \right)^{1/2} \lambda_n(X)^{1/2}$$
$$\le \nu_0(K)^{1/2} \lambda_n(X)^{1/2} \to 0 \qquad (n \to \infty) .$$

Thus, $\mu = 0$, and hence $p = 0$, q.e.d.

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