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Abstract quasi-linear equations of evolution in nonreflexive Banach spaces

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Introduction

In [7] T. Kato established the existence of classical solutions to the abstract quasi-linear equations of the form

$$(CP) \qquad \begin{aligned} & du(t)/dt + A(t, u(t))u(t) = f(t, u(t)), \quad 0 \le t \le T, \\ & u(0) = a, \end{aligned}$$

and applied his theory to a wide variety of problems from mathematical physics. In his theory two reflexive Banach spaces X and Y are used in such a way that Y is continuously and densely embedded in X, the solution $u(t)$ of (CP) lies in some open subset W of Y and $du(t)/dt$ is found in X. The reflexivity of X (and hence that of Y) is essential for the theory, and it is important from the theoretical point of view to eliminate this restriction for the spaces X and Y. Furthermore, in order to apply the theory to partial differential equations in suitable function spaces, it is required to extend this theory to the case of nonreflexive Banach spaces.

The first purpose of the present paper is to establish an existence theorem for the classical solutions of (CP) in a pair of general Banach spaces $X \supset Y$. We shall show that the solutions are continuous in Y-norm and continuously differentiable in X. In order to construct such solutions, we employ the following type of difference approximation of (CP):

$$(D) \qquad \begin{aligned} & \frac{u_\Delta(t) - u_k}{t - t_k} + A(t_k, u_k)u_\Delta(t) = f(t_k, u_k), \quad t_k \le t \le t_{k+1}, \quad 0 \le k \le N-1, \\ & u_\Delta(0) = a, \end{aligned}$$

where $\Delta: 0 = t_0 < t_1 < \cdots < t_N = T$ is a partition of the interval $[0, T]$ and $u_k = u_\Delta(t_k)$. This approach is one of the main features of our argument. In case the operators $A(t, w)$ and $f(t, w)$ are independent of t and of the form $A(w)$ and $f(w)$, it was proved in [14] that for any initial value $a \in W$, one finds a $T > 0$ such that the solution u_Δ of (D) exists in W, and that $\{u_\Delta\}$ converges in X to a unique weak solution (in the sense of [13]) of (CP) as $|\Delta| = \max(t_k - t_{k-1}) \to 0$. In this t-independent case the proof of the convergence of $\{u_\Delta(t)\}$

was straightforward since a general convergence theorem for the difference approximation of nonlinear evolution equations formulated for quasi-dissipative operators can be applied to our case. (The result due to Y. Kobayashi [10] was employed in the previous paper [14].) Similar results were obtained for "*quasinonlinear*" equations by Crandall and Souganidis [1], and in their subsequent paper [2] it was proved that $\{u_\Delta\}$ converges in Y-norm provided X is reflexive. However, the methods used in these works do not seem to be effective enough to prove the same results even for the case in which the operators $A(t, w)$ depend on $t \in [0, T]$ in the sense of Kato [7].

The second purpose of this paper is to extend the results of [2] and [14] to the case in which the operators $A(t, w)$ and $f(t, w)$ depend on $t \in [0, T]$. It will be seen that our assumption on the t-dependence of $A(t, w)$ is weaker than that of [7]. The significance of our results here is that the reflexivity assumption for X (and hence Y) is eliminated. These results are stated in Section 4.

Consider the sequence of Cauchy problems

$$(CP^n) \qquad \begin{aligned} &du^n(t)/dt + A^n(t, u^n(t))u^n(t) = f^n(t, u^n(t)), \qquad 0 \le t \le T, \\ &u^n(0) = a^n, \end{aligned}$$

where $n \in \bar{\boldsymbol{N}} \equiv \boldsymbol{N} \cup \{\infty\}$ and $\boldsymbol{N} = \{1, 2, \dots\}$. Kato showed in [7; Theorem 7] the uniform convergence of $\{u^n(t)\}_{n \in \boldsymbol{N}}$ (to $u^\infty(t)$) in Y-norm under appropriate assumptions. Our third purpose is to give a simple proof of this convergence result by applying our existence theorem. Consider the set $c(X)$ of all convergent sequences $\boldsymbol{x} = \{x^n\}$ in X. $c(X)$ is a Banach space with respect to the norm $\|\boldsymbol{x}\|_{c(X)} \equiv \sup_n \|x^n\|$. The sequence of Cauchy problems (CP^n) can be regarded as a single Cauchy problem

$$(\boldsymbol{CP}) \qquad \begin{aligned} &d\boldsymbol{u}(t)/dt + \boldsymbol{A}(t, \boldsymbol{u}(t))\boldsymbol{u}(t) = \boldsymbol{f}(t, \boldsymbol{u}(t)), \qquad 0 \le t \le T, \\ &\boldsymbol{u}(0) = \boldsymbol{a}, \end{aligned}$$

in the Banach space $c(X)$. The uniform convergence of $\{u^n(t)\}$ in Y is equivalent to the existence of the classical solution $\boldsymbol{u}(t)$ of $(\boldsymbol{CP})$. Indeed, we can apply our results obtained in Section 4 to $(\boldsymbol{CP})$ and find the solution $\boldsymbol{u}(t)$. It should be noted that $c(X)$ is nonreflexive even if X is reflexive, and so one cannot apply previous results (for instance [13]) to $(\boldsymbol{CP})$. These results are stated in Section 5 along with some other related results.

Other applications and further extensions of our results will be published elsewhere.

§1. Basic hypotheses and main results

We consider two real Banach spaces X and Y with norms $\|\cdot\|$ and $\|\cdot\|_Y$, respectively. The operator norm of a bounded linear operator A on Y to X is denoted by $\|A\|_{Y,X}$. However, we write $\|\cdot\|$ (or $\|\cdot\|_Y$) for the operator norm $\|\cdot\|_{X,X}$ (or $\|\cdot\|_{Y,Y}$) for brevity in notation. The symbol $\boldsymbol{B}(X)$ (or $\boldsymbol{B}(Y)$) denotes the set of all bounded linear operators on X (or Y) into itself.

In this paper we put one condition on the Banach spaces X and Y, four conditions on the operators $A(t, w)$ and three conditions on the nonlinear operators $f(t, w)$. First we impose the following condition on the pair (X, Y):

(X) Y is continuously and densely embedded in X. There is a linear isometry S of Y onto X. There is a nonvoid open subset W of Y.

It should be noted that under assumption (X), X is reflexive if and only if Y is reflexive.

Let X^* be the dual space of X. The value of $x^* \in X^*$ at $x \in X$ is denoted by $\langle x, x^* \rangle$. The duality mapping of X is denoted by F, i.e.,

$$F(x) = \{x^* \in X^*; \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \qquad \text{for } x \in X .$$

For the operators $A(t, w)$ we assume $(A1)$ through $(A3)$ below.

$(A1)$ There are positive numbers T_0 and α such that for each $t \in [0, T_0]$ and $w \in W$, $-A(t, w)$ is the infinitesimal generator of a (C_0)-semigroup $\{\exp[-sA(t, w)]\}_{s \geq 0}$ on X satisfying $\|\exp[-sA(t, w)]\| \leq e^{\alpha s}$ for $s \geq 0$. The domain $D(A(t, w))$ of $A(t, w)$ includes Y.

Under assumption $(A1)$, the set $\{\lambda > \alpha\}$ is included in the resolvent set of $-A(t, w)$, and

$$\|[1 + hA(t, w)]^{-1}\| \leq (1 - h\alpha)^{-1} \tag{1.1}$$

for each $h > 0$ with $h\alpha < 1$, $t \in [0, T_0]$ and $w \in W$.

The next assumption is concerned with the restriction of $A(t, w)$ to Y, which is a bounded linear operator on Y to X by the closed graph theorem.

$(A2)$ There is a positive number μ_1 such that

$$\|A(t, w) - A(t, z)\|_{Y,X} \leq \mu_1 \|w - z\|$$

for all $t \in [0, T_0]$ and $w, z \in W$. For each $w \in W$ and $y \in Y$, $A(t, w)y$ is a continuous function from $[0, T_0]$ into X.

From assumption $(A2)$ it follows that for any bounded subset B of W there is a $K_B > 0$ such that

$$\|A(t, w)\|_{Y,X} \leq K_B \tag{1.2}$$

for all $t \in [0, T_0]$ and $w \in B$.

($A3$) For each $t \in [0, T_0]$ and $w \in W$, there is a bounded linear operator $B(t, w)$ on X into itself such that

$$SA(t, w)S^{-1} = A(t, w) + B(t, w),$$

and $\|B(t, w)\| \leq \lambda_1$ for some positive number λ_1 which is independent of $t \in [0, T_0]$ and $w \in W$.

We next make two hypotheses on the operators $f(t, w)$.

($f1$) For each $t \in [0, T_0]$ and $w \in W$, $f(t, w) \in Y$ and

$$\|f(t, w)\|_Y \leq \lambda_2$$

for some positive number λ_2 which is independent of $t \in [0, T_0]$ and $w \in W$.

($f2$) There is a positive number μ_2 such that

$$\|f(t, w) - f(t, z)\| \leq \mu_2 \|w - z\|$$

for each $t \in [0, T_0]$, $w \in W$ and $z \in W$. For each $w \in W$, $f(t, w)$ is an X-valued continuous function on $[0, T_0]$.

Throughout this paper the above conditions (X) through ($f2$) are always assumed. Under these assumptions, $\exp[-sA(t, w)](Y) \subset Y$ for each $s \geq 0$, $t \in [0, T_0]$ and $w \in W$, and the restriction of $\{\exp[-sA(t, w)]\}_{s \geq 0}$ to Y is a (C_0)-semigroup on Y such that

$$\|\exp[-sA(t, w)]\|_Y \leq e^{(\alpha + \lambda_1)s}$$

for each $s \geq 0$, $t \in [0, T_0]$ and $w \in W$. Therefore, we have

$$\begin{aligned} \|[1 + hA(t, w)]^{-1}\|_Y &= \|[1 + h(A(t, w) + B(t, w))]^{-1}\| \\ &\leq (1 - h(\alpha + \lambda_1))^{-1} \end{aligned} \tag{1.3}$$

for each $h > 0$ with $h(\alpha + \lambda_1) < 1$, $t \in [0, T_0]$ and $w \in W$. See [5].

The existence of the solution $u_\Delta(t)$ of the difference scheme (D) (as introduced in the Introduction) follows from assumptions (X) through ($f2$) above. Let $u_\Delta(0) = u_0 = a$. We begin by defining $u_\Delta(t)$ on the first subinterval $[0, t_1]$:

$$u_\Delta(t) = [1 + tA(0, a)]^{-1}a + t[1 + tA(0, a)]^{-1}f(0, a)$$

for $t \in [0, t_1]$. Suppose that $u_k \in W$ is given. Then $u_\Delta(t)$ on $[t_k, t_{k+1}]$ is defined by

$$\begin{aligned} u_\Delta(t) = {} & [1 + (t - t_k)A(t_k, u_k)]^{-1}u_k \\ & + (t - t_k)[1 + (t - t_k)A(t_k, u_k)]^{-1}f(t_k, u_k). \end{aligned} \tag{1.4}$$

This is well defined for t with $(t-t_k)\alpha<1$. Here and after we always assume that

$$|\Delta|\beta<1/2\,,$$

where $|\Delta|=\max_{1\le k\le N}(t_k-t_{k-1})$ and $\beta=\alpha+\lambda_1$. We write

$$A_k=A(t_k,u_k)\,,\quad B_k=B(t_k,u_k)\quad\text{and}\quad f_k=f(t_k,u_k)\quad\text{for } k=0,1,\dots,N\,,\text{ and}$$

$$J_k=[1+(t_{k+1}-t_k)A_k]^{-1}\,,\quad R_k=[1+(t_{k+1}-t_k)(A_k+B_k)]^{-1}$$
$$\text{and}\qquad h_k=t_{k+1}-t_k\qquad\text{for } k=0,1,\dots,N-1\,,$$

for simplicity in notation. Now, we have:

PROPOSITION 1.1. *Let $a\in W$. Choose $\phi\in W$ and $r>0$ such that $a\in B(\phi,r)$ and $\bar{B}(\phi,r)\subset W$, where $B(\phi,r)=\{y\in Y;\ \|y-\phi\|_Y<r\}$ and $\bar{B}(\phi,r)$ is the closure of $B(\phi,r)$ in Y. Let $T\in(0,T_0]$ be sufficiently small so that*

$$\inf_{z\in Y}\{e^{2\beta T}\|a-\phi\|_Y+(1+e^{2\beta T})\|S\phi-z\|+M_zTe^{2\beta T}\}<r\,,$$

where $M_z=K_{B(\phi,r)}\|z\|_Y+\lambda_1\|z\|+\lambda_2$ and $K_{B(\phi,r)}$ is given in (1.2). Then the solution $u_\Delta(t)$ of (D) exists in $B(\phi,r)$ $(\subset W)$ for any partition $\Delta=\{t_k\}_{k=0}^N$ of $[0,T]$.

PROOF. $u_\Delta(0)=a$ is given and belongs to $B(\phi,r)$. Suppose that $\{u_\Delta(t);\ 0\le t\le t_k\}$ is contained in $B(\phi,r)$ for some k. Then $u_\Delta(t)$ on $[t_k,t_{k+1}]$ is given by (1.4) and belongs to Y. We will show that $u_\Delta(t)\in B(\phi,r)$ for $t_k\le t\le t_{k+1}$. Multiplying both sides of (D) by S and using the relation in (A3), we have

$$Su_\Delta(t)=[1+(t-t_k)(A_k+B_k)]^{-1}S\{u_k+(t-t_k)f_k\}\,.$$

Therefore, by (1.3), (1.2), (A3), (f1) and the relation $(1-t)^{-1}\le e^{2t}$ for $0\le t\le 1/2$, we have

$$\begin{aligned}\|Su_\Delta(t)-z\|&\le\|[1+(t-t_k)(A_k+B_k)]^{-1}\{Su_k-z\}\|\\&\quad+\|[1+(t-t_k)(A_k+B_k)]^{-1}z-z\|\\&\quad+(t-t_k)\|[1+(t-t_k)(A_k+B_k)]^{-1}Sf_k\|\\&\le e^{2\beta(t-t_k)}\|Su_k-z\|+M_z(t-t_k)e^{2\beta(t-t_k)}\end{aligned}$$

for any $z\in Y$ and $t_k\le t\le t_{k+1}$. By induction it follows that

$$\begin{aligned}\|Su_\Delta(t)-z\|&\le e^{2\beta(t-t_0)}\|Sa-z\|+M_z\textstyle\sum_{j=0}^k h_je^{2\beta(t-t_j)}\\&\le e^{2\beta T}\|Sa-z\|+M_zTe^{2\beta T}\,.\end{aligned}$$

Thus we have

$$\begin{aligned}\|u_\Delta(t) - \phi\|_Y &\le \|Su_\Delta(t) - z\| + \|z - S\phi\| \\ &\le e^{2\beta T}\|Sa - z\| + M_z T e^{2\beta T} + \|z - S\phi\| \\ &\le e^{2\beta T}\|Sa - S\phi\| + (1 + e^{2\beta T})\|S\phi - z\| + M_z T e^{2\beta T}\end{aligned}$$

for any $z \in Y$, which implies that $u_\Delta(t) \in B(\phi, r)$. Q.E.D.

Here and after we fix $a \in W$, $B(\phi, r)$ and $T \in (0, T_0]$ as above, unless otherwise stated.

The convergence of $\{u_\Delta\}$ in X can be obtained under the same assumptions. See Section 2. To obtain the convergence of $\{u_\Delta\}$ in Y, we make additional assumptions:

($A4$) There is a positive number μ_3 such that

$$\|B(t, w) - B(t, z)\| \le \mu_3 \|w - z\|_Y$$

for each $t \in [0, T_0]$, $w \in W$ and $z \in W$. For each $w \in W$, $B(t, w)$: $[0, T_0] \to \boldsymbol{B}(X)$ is strongly measurable.

($f3$) There is a positive number μ_4 such that

$$\|f(t, w) - f(t, z)\|_Y \le \mu_4 \|w - z\|_Y$$

for all $t \in [0, T_0]$, $w \in W$ and $z \in W$. For each $w \in W$, $f(t, w)$: $[0, T_0] \to Y$ is strongly measurable.

(Bf) For each $w \in W$, $B(t, w)$: $[0, T_0] \to \boldsymbol{B}(X)$ and $f(t, w)$: $[0, T_0] \to Y$ are strongly continuous.

Conditions ($A4$), ($f3$) and (Bf) will be used in Sections 4 and 5. The limit of u_Δ can be found under the conditions as mentioned above and it gives the desired solution of (CP). Our main results are stated as follows:

THEOREM A. *Let* (X), ($A1$), ($A2$), ($A3$), ($f1$) *and* ($f2$) *hold. Then there is an X-valued continuous function $u(t)$ on $[0, T]$ with $u(0) = a$* such that $\lim u_\Delta(t) = u(t)$ *in X, uniformly for $t \in [0, T]$. Moreover, the function $u(t)$ is the mild solution of (CP) in the sense that $u(t)$ satisfies*

$$(M) \qquad u(t) = U^u(t, 0)a + \int_0^t U^u(t, s)\mathscr{F}(s, u(s))\, ds$$

for $t \in [0, T]$.

This theorem is proved in Sections 2 and 3 and is formulated in Theorems 2.4 and 3.4. The operators $\mathscr{F}(t, w)$ are defined in the paragraph before Theorem 3.4. The family of bounded linear operators $\{U^u(t, s); 0 \le s \le t \le T\}$ on X is constructed in Lemma 3.3. By Lemma 3.7 and Theorem B below, we will see that $w(t) = U^u(t, s)y$ with $y \in Y$ is the solution of the initial value problem

$$(L) \qquad dw(t)/dt + A(t, u(t))w(t) = 0\,, \quad s \le t \le T, \quad u(s) = y\,.$$

It should be noted that the linear operators $A(t, u(t))$ are not well defined under the assumptions of Theorem A.

THEOREM B. *Let* (X), $(A1)$–$(A4)$, $(f1)$–$(f3)$ *hold. Then there is a sequence* $\{\Delta(m)\}$ *of partitions of* $[0, T]$ *such that* $\lim |\Delta(m)| = 0$ *and that* $\{u_{\Delta(m)}(t)\}$ *converges in* Y, *uniformly on* $[0, T]$, *to the unique classical solution* $u(t)$ *of* (CP).

Theorem B is the existence theorem for the classical solutions and is proved in Section 4. The convergence problem mentioned at the end of the Introduction is treated in Section 5 and the results are summarized in Theorem 5.2.

§2. Convergence of $\{u_\Delta\}$ in X

Throughout this section, we assume that conditions (X), $(A1)$, $(A2)$, $(A3)$, $(f1)$ and $(f2)$ are satisfied. We do not need conditions $(A4)$, $(f1)$ and (Bf).

Let $\Delta: 0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$, and let $u_\Delta(t)$ be the solution of (D) on $[0, T]$. $u_\Delta(t)$ is given by formula (1.4). Note that $\beta|\Delta| < 1/2$ is always assumed. It is easy to see that $u_\Delta(t)$ is continuous on $[0, T]$ into Y, and is continuously differentiable on each interval (t_k, t_{k+1}) $(0 \le k \le N - 1)$ into X. $du_\Delta(t)/dt$ on (t_k, t_{k+1}) is given by

$$(2.1) \qquad du_\Delta(t)/dt = [1 + (t - t_k)A_k]^{-1}\{-A_k u_\Delta(t) + f_k\}\,.$$

In this section, we will show that $\{u_\Delta(t)\}$ converges in X-norm uniformly, as $|\Delta| \to 0$, to a continuous function $u(t)$ in X. Once this is done, it is easy to see that the step functions $\{v_\Delta(t)\}$, defined by

$$v_\Delta(t) = u_{k-1} \quad \text{on } t \in [t_{k-1}, t_k)\,, \quad 1 \le k \le N\,, \quad \text{and} \quad v_\Delta(T) = u_N\,,$$

also converges uniformly to $u(t)$. To prove the convergence of $\{u_\Delta(t)\}$, we prepare some lemmas.

LEMMA 2.1. *For each* $\varepsilon > 0$ *there is a partition* $\Delta(\varepsilon): 0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_{N(\varepsilon)}^\varepsilon = T$ *of* $[0, T]$ *such that*

(*i*) $|\Delta(\varepsilon)| \le \varepsilon$,

(*ii*) $\|u_{\Delta(\varepsilon)}(t) - u_{\Delta(\varepsilon)}(t_k^\varepsilon)\|_Y \le \varepsilon$ *for* $t \in [t_k^\varepsilon, t_{k+1}^\varepsilon]$, $0 \le k \le N(\varepsilon) - 1$, (*iii*) $\|\{A(t'', u_k^\varepsilon)[1 + (t' - t_k^\varepsilon)A(t'', u_k^\varepsilon)]^{-1} - A_k^\varepsilon\}u_{\Delta(\varepsilon)}(t)\| \le \varepsilon$ *for* $t, t', t'' \in [t_k^\varepsilon, t_{k+1}^\varepsilon]$, $0 \le k \le N(\varepsilon) - 1$, *and*

(*iv*) $\|[1 + (t' - t_k^\varepsilon)A(t'', u_k^\varepsilon)]^{-1}f(t, u_k^\varepsilon) - f(t_k^\varepsilon, u_k^\varepsilon)\| \le \varepsilon$ *for* $t, t', t'' \in [t_k^\varepsilon, t_{k+1}^\varepsilon]$, $0 \le k \le N(\varepsilon) - 1$,

where $u_{\Delta(\varepsilon)}(t)$ *is the solution of (D) for* $\Delta(\varepsilon)$, *and* $u_k^\varepsilon = u_{\Delta(\varepsilon)}(t_k^\varepsilon)$ *and* $A_k^\varepsilon = A(t_k^\varepsilon, u_k^\varepsilon)$ *for* $0 \le k \le N(\varepsilon)$.

PROOF. We put $t_0^\varepsilon = 0$ and $u_{\Delta(\varepsilon)}(t_0^\varepsilon) = a$. Inductively, we define $\{t_k^\varepsilon\}$ and $u_{\Delta(\varepsilon)}$ in the following way. Suppose that $\{t_j^\varepsilon; j = 0, 1, \ldots, k\}$ and $\{u_{\Delta(\varepsilon)}(t); 0 \le t \le t_k^\varepsilon\}$ are constructed, and that $t_k^\varepsilon < T$. Then, let $t_{k+1}^\varepsilon \in (t_k^\varepsilon, T]$ be the largest number satisfying (*i*) to (*iv*). By the continuity of the functions in (*i*) to (*iv*), we can choose $t_{k+1}^\varepsilon > t_k^\varepsilon$. We will show that there is an $N(\varepsilon)$ such that $t_{N(\varepsilon)}^\varepsilon = T$. Assume for the contrary that $t_n^\varepsilon < T$ for all $n \in N$. Let $t_\infty = \lim t_n^\varepsilon$. Then it will be shown in Lemma 2.2 that $u_{\Delta(\varepsilon)}(t)$ converges in Y-norm to an element $u_\infty \in \bar{B}(\phi, r)$, as $t \to t_\infty$. We use this fact for the moment. Then we have $t_{k+1}^\varepsilon - t_k^\varepsilon \le \varepsilon/2$ and

$$\|u_{\Delta(\varepsilon)}(t) - u_k^\varepsilon\|_Y \le \|u_{\Delta(\varepsilon)}(t) - u_\infty\|_Y + \|u_\infty - u_k^\varepsilon\|_Y \le \varepsilon/2\,,$$

for sufficiently large k and $t_k^\varepsilon \le t \le t_{k+1}^\varepsilon$. Similarly, the left-hand sides of the inequalities in (*iii*) and (*iv*) are less than $\varepsilon/2$ for sufficiently large k. This contradicts the definition of $\{t_k^\varepsilon\}$. Q.E.D.

Let $\Delta = \{t_n\}$ be a strictly increasing sequence in $[0, T]$ satisfying $|\Delta|\beta < 1/2$: $0 \le t_0 < t_1 < \cdots < t_n < \cdots \le T$. We put $t_\infty = \lim t_n$. Let $\{u_n\}$ be the sequence of elements in $B(\phi, r)$ satisfying

$$\frac{u_{n+1} - u_n}{t_{n+1} - t_n} + A_n u_{n+1} = f_n\,, \quad n = 0, 1, 2, \ldots,$$
$$u_0 = a\,.$$

Then we have:

LEMMA 2.2. *The sequence* $\{u_n\}$ *converges in* Y, *as* $n \to \infty$.

PROOF. Let $k > j > i > 1$ and $y \in Y$. We put

$$U_n = \textstyle\prod_{p=i}^n J_p = J_n \cdot J_{n-1} \cdots J_i\,,$$

for $n \ge i$. Here, the notation J_p is described in Section 1. Then we have

$$\begin{aligned}\|u_{k+1} - u_{j+1}\|_Y &= \|Su_{k+1} - Su_{j+1}\| \\ &\le \|Su_{k+1} - SU_kS^{-1}y\| + \|SU_kS^{-1}y - U_ky\| + \|U_ky - U_jy\| \\ &\quad + \|U_jy - SU_jS^{-1}y\| + \|SU_jS^{-1}y - Su_{j+1}\|\,.\end{aligned}$$

We put I, II, III, II′ and I′ for the terms of the right-hand side of the above inequality. Then we have

$$\mathrm{I} = \|J_k(u_k + h_k f_k) - J_k U_{k-1} S^{-1} y\|_Y \le e^{2\beta h_k} \|u_k - U_{k-1} S^{-1} y\|_Y + \lambda_2 h_k e^{2\beta h_k} .$$

By induction, it follows that

$$\mathrm{I} \le e^{2\beta t_{k+1}} \|u_i - S^{-1} y\|_Y + \lambda_2 (t_{k+1} - t_i) e^{2\beta t_{k+1}} .$$

Similarly, we have

$$\mathrm{I}' \le e^{2\beta t_{j+1}} \|u_i - S^{-1} y\|_Y + \lambda_2 (t_{j+1} - t_i) e^{2\beta t_{j+1}} .$$

Next, since $SU_k S^{-1} = \prod_{p=i}^{k} R_p$, it follows that

$$\begin{aligned} \mathrm{II} &= \|R_k \{\textstyle\prod_{p=i}^{k-1} R_p - U_{k-1} - h_k B_k U_k\} y\| \\ &\le e^{2\beta h_k} \|SU_{k-1} S^{-1} y - U_{k-1} y\| + \lambda_1 h_k e^{2\beta h_k} e^{2\beta (t_{k+1} - t_i)} \|y\| . \end{aligned}$$

Therefore, by induction, we have

$$\mathrm{II} \le \lambda_1 e^{4\beta t_{k+1}} (t_{k+1} - t_i) \|y\| .$$

Similarly, we have

$$\mathrm{II}' \le \lambda_1 e^{4\beta t_{j+1}} (t_{j+1} - t_i) \|y\| .$$

Now, since

$$\begin{aligned} U_k y - U_j y &= \textstyle\sum_{p=j+1}^{k} (U_p y - U_{p-1} y) \\ &= \textstyle\sum_{p=j+1}^{k} J_p \{U_{p-1} y - (1 + h_p A_p) U_{p-1} y\} \\ &= -\textstyle\sum_{p=j+1}^{k} h_p J_p A_p U_{p-1} y , \end{aligned}$$

an estimate of III is given by

$$\begin{aligned} \mathrm{III} &\le K_{B(\phi, r)} \textstyle\sum_{p=j+1}^{k} h_p e^{2\beta (t_{k+1} - t_i)} \|y\|_Y \\ &\le K_{B(\phi, r)} e^{2\beta T} (t_{k+1} - t_j) \|y\|_Y . \end{aligned}$$

Combining these estimates, we have

$$\begin{aligned} \limsup_{k, j \to \infty} \|u_{k+1} - u_{j+1}\|_Y \le 2\{ & e^{2\beta t_\infty} \|u_i - S^{-1} y\|_Y \\ & + \lambda_2 (t_\infty - t_i) e^{2\beta t_\infty} + \lambda_1 e^{4\beta t_\infty} (t_\infty - t_i) \|y\|\} \end{aligned}$$

for every $y \in Y$ and $i > 1$, which implies that

$$\lim_{j, k \to \infty} \|u_{k+1} - u_{j+1}\|_Y = 0 . \qquad \text{Q.E.D.}$$

Let Δ: $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0, T]$. For each $w \in W$ and $t \in [t_k, t_{k+1})$ $(0 \le k \le N - 1)$, we write

$$A_\Delta(t, w) = A(t_k, w)[1 + (t - t_k)A(t_k, w)]^{-1} \qquad \text{and}$$
$$f_\Delta(t, w) = [1 + (t - t_k)A(t_k, w)]^{-1}f(t_k, w),$$

and for each $w \in W$ and $t = T$, we define

$$A_\Delta(T, w) = A(T, w) \qquad \text{and} \qquad f_\Delta(T, w) = f(T, w),$$

for simplicity in notation. By our assumptions, we see that

$$\|A_\Delta(t, w) - A_\Delta(t, z)\|_{Y,X} \le \mu_1 e^{4\beta|\Delta|}\|w - z\| \tag{2.2}$$

for $w, z \in W$ and $t \in [0, T]$,

$$\langle -A_\Delta(t, w)x, x^*\rangle \le \alpha e^{2\alpha|\Delta|}\|x\|^2 \tag{2.3}$$

for $w \in W$, $x \in Y$, $x^* \in F(x)$ and $t \in [0, T]$, and

$$\|A_\Delta(t, w)\|_{Y,X} \le K_{B(\phi,r)}e^{2\beta|\Delta|} \tag{2.4}$$

for $w \in B(\phi, r)$ and $t \in [0, T]$. We put $\mu_1' = \mu_1 e^{4\beta T}$, $\alpha' = \alpha e^{2\alpha T}$ and $K' = K_{B(\phi,r)}e^{2\beta T}$. These are upper bounds for $\mu_1 e^{4\beta|\Delta|}$, $\alpha e^{2\alpha|\Delta|}$ and $K_{B(\phi,r)}e^{2\beta|\Delta|}$, respectively. Similar estimates hold for f_Δ:

$$\|f_\Delta(t, w)\|_Y \le \lambda_2' \qquad \text{for } t \in [0, T], \quad w \in W, \quad \text{and}$$
$$\|f_\Delta(t, w) - f_\Delta(t, z)\| \le \mu_2'\|w - z\| \qquad \text{for} \quad t \in [0, T] \quad \text{and} \quad w, z \in W,$$

where $\lambda_2' = \lambda_2 e^{2\beta T}$ and $\mu_2' = \mu_1\lambda_2 T e^{4\beta T} + \mu_2 e^{2\alpha T}$. Now we have:

LEMMA 2.3. *Let $\varepsilon > 0$ and let $\Delta = \{t_k\}_{k=0}^N$ be a partition of $[0, T]$ satisfying (i) to (iv) of Lemma 2.1. Let $\hat\Delta = \{\hat t_j\}_{j=0}^{\hat N}$ be any partition of $[0, T]$ satisfying*

$$|\hat\Delta| \le \min_{1\le k\le N}(t_k - t_{k-1}). \tag{2.5}$$

Then there is a $C > 0$ such that

$$\|u_\Delta(t) - u_{\hat\Delta}(t)\| \le \varepsilon C$$

for all $t \in [0, T]$.

PROOF. Let $t \in [0, T]$ and $x^* \in F(u_{\hat\Delta}(t) - u_\Delta(t))$. Without loss of generality, we assume that $t \in (t_k, t_{k+1}) \cap (\hat t_j, \hat t_{j+1})$ for some $k \in \{0, 1, \dots, N-1\}$ and $j \in \{0, 1, \dots, \hat N - 1\}$. Then, by (2.1), we have

$$\begin{aligned}(d/dt)\|u_{\hat\Delta}(t) - u_\Delta(t)\|^2 &= -2\langle A_{\hat\Delta}(t, \hat u_j)u_{\hat\Delta}(t) - A_\Delta(t, u_k)u_\Delta(t), x^*\rangle \\ &\quad + 2\langle f_{\hat\Delta}(t, \hat u_j) - f_\Delta(t, u_k), x^*\rangle \\ &\equiv 2(\mathrm{I} + \mathrm{II}),\end{aligned}$$

where $\hat u_j = u_{\hat\Delta}(\hat t_j)$ and $u_k = u_\Delta(t_k)$. We divide I into 6 terms: $\mathrm{I} = \sum_{i=1}^6 (i)$,

$$(1) = -\langle\{A_{\hat{\Delta}}(t, \hat{u}_j) - A_{\hat{\Delta}}(t, u_{\hat{\Delta}}(t))\}u_{\hat{\Delta}}(t), x^*\rangle ,$$
$$(2) = -\langle\{A_{\hat{\Delta}}(t, u_{\hat{\Delta}}(t)) - A_{\hat{\Delta}}(t, u_{\Delta}(t))\}u_{\hat{\Delta}}(t), x^*\rangle ,$$
$$(3) = -\langle A_{\hat{\Delta}}(t, u_{\Delta}(t))\{u_{\hat{\Delta}}(t) - u_{\Delta}(t)\}, x^*\rangle ,$$
$$(4) = -\langle\{A_{\hat{\Delta}}(t, u_{\Delta}(t)) - A_{\hat{\Delta}}(t, u_k)\}u_{\Delta}(t), x^*\rangle ,$$
$$(5) = -\langle A_{\hat{\Delta}}(t, u_k)u_{\Delta}(t) - A(t_k, u_k)u_{\Delta}(t), x^*\rangle \quad \text{and}$$
$$(6) = -\langle\{A(t_k, u_k) - A_{\Delta}(t, u_k)\}u_{\Delta}(t), x^*\rangle .$$

By (2.2), we have

$$(1) \le \mu_1' \|u_{\hat{\Delta}}(t) - \hat{u}_j\| \cdot \|u_{\hat{\Delta}}(t)\|_Y \cdot \|u_{\hat{\Delta}}(t) - u_{\Delta}(t)\| .$$

Since $u_{\hat{\Delta}}(t) \in B(\phi, r)$, we have

$$\|u_{\hat{\Delta}}(t)\|_Y \le r + \|\phi\|_Y \equiv R . \tag{2.6}$$

By the equation

$$u_{\hat{\Delta}}(t) - \hat{u}_j = -(t - \hat{t}_j)[1 + (t - \hat{t}_j)A(\hat{t}_j, \hat{u}_j)]^{-1}\{A(\hat{t}_j, \hat{u}_j)\hat{u}_j - f(\hat{t}_j, \hat{u}_j)\} ,$$

we have

$$\|u_{\hat{\Delta}}(t) - u_j\| \le L|\hat{\Delta}| \le L\varepsilon , \tag{2.7}$$

where $L = e^{2\alpha T}\{K_{B(\phi,r)}R + \lambda_2\|S^{-1}\|_{X,X}\}$. Note that $|t - \hat{t}_j| \le |\hat{\Delta}| \le \varepsilon$, by (2.5). Thus we have

$$(1) \le \mu_1' LR\varepsilon \|u_{\hat{\Delta}}(t) - u_{\Delta}(t)\| .$$

By the same way, we have

$$(2) \le \mu_1' R \|u_{\hat{\Delta}}(t) - u_{\Delta}(t)\|^2 \quad \text{and}$$
$$(4) \le \mu_1' LR\varepsilon \|u_{\hat{\Delta}}(t) - u_{\Delta}(t)\| ,$$

and by (2.3), we have

$$(3) \le \alpha' \|u_{\hat{\Delta}}(t) - u_{\Delta}(t)\|^2 .$$

By (*iii*) of Lemma 2.1, we have

$$(6) \le \varepsilon \|u_{\hat{\Delta}}(t) - u_{\Delta}(t)\| .$$

Now, since $(5) \le J \cdot \|u_{\hat{\Delta}}(t) - u_{\Delta}(t)\|$, with

$$J = \|A_{\hat{\Delta}}(t, u_k)u_{\Delta}(t) - A(t_k, u_k)u_{\Delta}(t)\| ,$$

we consider J. Suppose that $\hat{t}_j \ge t_k$. Put $t' = t - \hat{t}_j + t_k$. Then we have $t - \hat{t}_j = t' - t_k$, $t_k \le t' \le t_{k+1}$ and $t_k \le \hat{t}_j \equiv t'' \le t_{k+1}$, by (2.5). Therefore, by (*iii*) of Lemma 2.1, we have

$$J = \|\{A(t'', u_k)[1 + (t' - t_k)A(t'', u_k)]^{-1} - A(t_k, u_k)\}u_\Delta(t)\| \leq \varepsilon .$$

Next, we consider the case $\hat{t}_j < t_k$. Since

$$\begin{aligned} J \leq & \|\{A_{\hat{\Delta}}(t, u_k) - A_{\hat{\Delta}}(t, u_{k-1})\}u_\Delta(t)\| + \|A_{\hat{\Delta}}(t, u_{k-1})\{u_\Delta(t) - u_k\}\| \\ & + \|A_{\hat{\Delta}}(t, u_{k-1})u_k - A(t_{k-1}, u_{k-1})u_k\| + \|\{A(t_{k-1}, u_{k-1}) - A(t_k, u_{k-1})\}u_k\| \\ & + \|A(t_k, u_{k-1})\{u_k - u_\Delta(t)\}\| + \|\{A(t_k, u_{k-1}) - A(t_k, u_k)\}u_\Delta(t)\| , \end{aligned}$$

we have

$$\begin{aligned} J \leq & (\mu_1' LR + K' + 1 + K_{B(\phi,r)} + \mu_1 LR)\varepsilon \\ & + \|A_{\hat{\Delta}}(t, u_{k-1})u_k - A(t_{k-1}, u_{k-1})u_k\| , \end{aligned}$$

by (2.2), (2.4), (2.6), (2.7) and (*ii*) and (*iii*) of Lemma 2.1. Put $t' = t - \hat{t}_j + t_{k-1}$. Then, since $t - \hat{t}_j = t' - t_{k-1}$ and $t', \hat{t}_j \in [t_{k-1}, t_k]$ by (2.5), the second term of the right-hand side of the above inequality is written as

$$\|\{A(\hat{t}_j, u_{k-1})[1 + (t' - t_{k-1})A(\hat{t}_j, u_{k-1})]^{-1} - A(t_{k-1}, u_{k-1})\}u_\Delta(t_k)\| ,$$

and is less than or equal to ε, by (*iii*) of Lemma 2.1. Thus, in either case, we have

$$J \leq \{(\mu_1 + \mu_1')LR + K' + K_{B(\phi,r)} + 1\}\cdot\varepsilon .$$

Therefore, we have

$$\mathrm{I} \leq \varepsilon C_1 \|u_{\hat{\Delta}}(t) - u_\Delta(t)\| + \omega_1 \|u_{\hat{\Delta}}(t) - u_\Delta(t)\|^2 ,$$

where $\omega_1 = \mu_1' R + \alpha'$ and $C_1 = 3\mu_1' LR + \mu_1 LR + K' + K_{B(\phi,r)} + 2$. An estimate for II is obtained similarly:

$$\mathrm{II} \leq \varepsilon C_2 \|u_{\hat{\Delta}}(t) - u_\Delta(t)\| + \omega_2 \|u_{\hat{\Delta}}(t) - u_\Delta(t)\|^2 ,$$

for some $C_2 > 0$ and $\omega_2 > 0$. Thus we have

$$(d/dt)\|u_{\hat{\Delta}}(t) - u_\Delta(t)\|^2 \leq 2\varepsilon(C_1 + C_2)\|u_{\hat{\Delta}}(t) - u_\Delta(t)\| + 2(\omega_1 + \omega_2)\|u_{\hat{\Delta}}(t) - u_\Delta(t)\|^2 .$$

It follows that

$$\|u_{\hat{\Delta}}(t) - u_\Delta(t)\| \leq \varepsilon\{(C_1 + C_2)/(\omega_1 + \omega_2)\}\{e^{(\omega_1+\omega_2)t} - 1\} . \qquad \text{Q.E.D.}$$

THEOREM 2.4. *There is a continuous function $u(t)$ on $[0, T]$ into X with $u(0) = a$ such that*

$$\lim_{|\Delta|\to 0} u_\Delta(t) = u(t) \qquad \text{in } X ,$$

uniformly in $t \in [0, T]$.

PROOF. Let $\varepsilon > 0$ by an positive number and let $\Delta(\varepsilon) = \{t_k^\varepsilon\}$ be a partition of $[0, T]$ satisfying (*i*) to (*iv*) of Lemma 2.1. We put $\delta = \min_k(t_{k+1}^\varepsilon - t_k^\varepsilon)$.

Then, by Lemma 2.3, for any partitions Δ and Δ' satisfying $|\Delta| < \delta$ and $|\Delta'| < \delta$, we have

$$\|u_\Delta(t) - u_{\Delta'}(t)\| \le \|u_\Delta(t) - u_{\Delta(\varepsilon)}(t)\| + \|u_{\Delta(\varepsilon)}(t) - u_{\Delta'}(t)\| \le 2\varepsilon C$$

for all $t \in [0, T]$. This implies the conclusion of Theorem 2.4. Q.E.D.

REMARK 2.5. $u(t)$ obtained by Theorem 2.4 belongs to the X-closure $\tilde{B}(\phi, r)$ of $B(\phi, r)$. We can prove that $u(t)$ is the unique weak solution of (CP) in the sense of [13].

§3. Convergence of $\{U_\Delta(t, s)\}$ in X and existence of mild solutions

Throughout this section, we assume that conditions (X), $(A1)$, $(A2)$ $(A3)$, $(f1)$ and $(f2)$ are satisfied. Under these assumptions, we will show that $u(t)$ obtained by Theorem 2.4 is the mild solution of (CP), and that under the additional assumptions, $u(t)$ is the unique classical solution of (CP).

A function u: $[0, T] \to X$ is called the *strong solution* of (CP) if it satisfies the following:

(*i*) $u(t)$ is Lipschitz continuous in X and is strongly differentiable in X for *a.e.* $t \in [0, T]$, and

(*ii*) $u(t) \in W$ and satisfies (CP) for *a.e.* $t \in [0, T]$.

Furthermore, if $u(t)$ is continuous in Y then $u(t)$ is called the *classical solution* of (CP). Note that the clasical solution is continuously differentiable in X. The notion of the *mild solution* is described in Theorem 3.4.

Let $\Delta = \{t_k\}_{k=0}^N$ be a partition of $[0, T]$ and let $u_\Delta(t)$ be the solution of (D) for Δ. We denote by Ω a triangle defined by

$$\Omega = \{(t, s); 0 \le s \le t \le T\} .$$

With the aid of $u_\Delta(t)$, we define a family of bounded linear operators $\{U_\Delta(t, s); (t, s) \in \Omega\}$ on X into itself by

$$U_\Delta(t, s) = [1 + (t - s)A_k]^{-1} ,$$

if $t_k \le s \le t \le t_{k+1}$ for some $k \in \{0, 1, \dots, N-1\}$, and

$$U_\Delta(t, s) = [1 + (t - t_k)A_k]^{-1} \dots [1 + (t_{j+1} - s)A_j]^{-1} ,$$

if $t_j \le s \le t_{j+1}$ and $t_k \le t \le t_{k+1}$ for some j, $k \in \{0, 1, \dots, N-1\}$ with $j < k$. Note that $U_\Delta(t, s)$ depends not only on (t, s) and Δ but also on $u_\Delta(t)$. $U_\Delta(t, s)(Y) \subset Y$ and $U_\Delta(t, s)$ is strongly continuous on Ω to $\boldsymbol{B}(Y)$ as well as to

$\boldsymbol{B}(X)$. It follows from (1.1) and (1.3) that

$$\|U_\Delta(t, s)\| \le e^{2(t-s)\alpha} \quad \text{and} \quad \|U_\Delta(t, s)\|_Y \le e^{2(t-s)\beta} \tag{3.1}$$

for $(t, s) \in \Omega$. It is easy to see that

$$U_\Delta(t, t) = 1 \quad \text{and} \quad U_\Delta(t, t_k)\cdot U_\Delta(t_k, s) = U_\Delta(t, s) \tag{3.2}$$

for $0 \le s \le t_k \le t \le T$. For each $y \in Y$ and $(t, s) \in \Omega \backslash \{(t_k, t_j)\}$, we have

$$(\partial/\partial t)U_\Delta(t, s)y = -A^s_\Delta(t, v_\Delta(t))U_\Delta(t, s)y \quad \text{and} \tag{3.3}$$

$$(\partial/\partial s)U_\Delta(t, s)y = U_\Delta(t, s)A^t_\Delta(s, v_\Delta(s))y\,, \tag{3.4}$$

where $v_\Delta(t)$ is the step function defined in Section 2, and for each $w \in W$ and $(t, s) \in \Omega$, we write

$$A^s_\Delta(t, w) = A_\Delta(t, w)\ (= A(t_k, w)[1 + (t - t_k)A(t_k, w)]^{-1})\,,$$
$$A^t_\Delta(s, w) = [1 + (t_{j+1} - s)A(t_j, w)]^{-1}A(t_j, w)\,,$$

if $t_j \le s < t_{j+1}$ and $t_k \le t < t_{k+1}$ for some $j < k$, and

$$A^s_\Delta(t, w) = A(t_k, w)[1 + (t - s)A(t_k, w)]^{-1}\,,$$
$$A^t_\Delta(s, w) = [1 + (t - s)A(t_k, w)]^{-1}A(t_k, w)\,,$$

if $t_k \le s \le t < t_{k+1}$ for some k. (2.2), (2.3) and (2.4) hold true even if A_Δ is replaced by A^t_Δ or A^s_Δ.

LEMMA 3.1. *Let $\varepsilon > 0$, $s \in [0, T)$ and $y \in Y$. Then there is a partition $\Delta = \{t_k\}_{k=0}^N$ of $[0, T]$ such that*

(*o*) $s = t_p$ *for some* $0 \le p < N$,

(*i*) $|\Delta| \le \varepsilon$,

(*ii*) $\|U_\Delta(t, s)y - U_\Delta(t_k, s)y\|_Y \le \varepsilon$ *for* $t_k \le t \le t_{k+1}$, $k = p, \dots, N-1$,

(*iii*) $\|\{A(t'', u_k)[1 + (t' - t_k)A(t'', u_k)]^{-1} - A_k\}U_\Delta(t, s)y\| \le \varepsilon$ *for* $t_k \le t, t', t'' \le t_{k+1}$, $k = p, \cdots, N-1$, *and*

(*iv*) $\|\{A(t'', u_k)[1 + (t' - t_k)A(t'', u_k)]^{-1} - A_k\}y\| \le \varepsilon$ *for* $t_k \le t, t', t'' \le t_{k+1}$, $k = 0, \cdots, p-1$,

where $A_k = A(t_k, u_k)$, $u_k = u_\Delta(t_k)$ and $u_\Delta(t)$ is the solution of (D) for Δ as before.

PROOF. The proof is essentially the same as that of Lemma 2.1. Let $\Delta_1 = \{t_k\}_{k=0}^p$ be a partition of $[0, s]$ satisfying $|\Delta_1| \le \varepsilon$ and (*iv*). Then we can append points $\{t_k\}_{k=p+1}^N$ in $[s, T]$ to Δ_1 to make an expected partition Δ. These can be done by the same way as in the proof of Lemma 2.1. Q.E.D.

The proof of the next lemma is similar to that of Lemma 2.3.

LEMMA 3.2. *Let* $\varepsilon > 0$, $s \in [0, T)$ *and* $y \in Y$, *and let* $\Delta = \{t_k\}_{k=0}^{N}$ *be a partition of* $[0, T]$ *satisfying* (*o*) *to* (*iv*) *of Lemma* 3.1. *Let* $\hat{\Delta} = \{\hat{t}_j\}_{j=0}^{\hat{N}}$ *be any partition of* $[0, T]$ *satisfying*

$$|\hat{\Delta}| \leq \min_{1 \leq k \leq N} (t_k - t_{k-1}).$$

Then there are $C_1 = C_1(\|y\|_Y) > 0$ *and* $C_2 = C_2(\|y\|_Y) > 0$ *such that*

$$\|U_\Delta(t, s)y - U_{\hat{\Delta}}(t, s)y\| \leq \varepsilon C_1 + C_2 \cdot \int_s^t \|v_\Delta(\sigma) - v_{\hat{\Delta}}(\sigma)\| \, d\sigma$$

for $(t, s) \in \Omega$.

PROOF. Let $t \in [s, T]$ and $x^* \in F(U_{\hat{\Delta}}(t, s)y - U_\Delta(t, s)y)$. Suppose that $t \in (t_k, t_{k+1}) \cap (\hat{t}_j, \hat{t}_{j+1})$ for some k and j. Then we have

$$\begin{aligned}(\partial/\partial t)\|U_{\hat{\Delta}}(t,s)y - U_\Delta(t,s)y\|^2 = & -2\langle\{A^s_{\hat{\Delta}}(t, v_{\hat{\Delta}}(t)) - A^s_{\hat{\Delta}}(t, v_\Delta(t))\}U_{\hat{\Delta}}(t, s)y, x^*\rangle \\ & - 2\langle A^s_{\hat{\Delta}}(t, v_\Delta(t))\{U_{\hat{\Delta}}(t, s)y - U_\Delta(t, s)y\}, x^*\rangle \\ & - 2\langle\{A^s_{\hat{\Delta}}(t, v_\Delta(t)) - A_k\}U_\Delta(t, s)y, x^*\rangle \\ & -2\langle\{A_k - A^s_{\hat{\Delta}}(t, v_\Delta(t))\}U_\Delta(t, s)y, x^*\rangle \\ \leq & \ 2\{\mu_1' e^{2\beta T}\|y\|_Y\|v_{\hat{\Delta}}(t) - v_\Delta(t)\| + \varepsilon + J\} \\ & \cdot\|U_{\hat{\Delta}}(t,s)y - U_\Delta(t,s)y\| + 2\alpha'\|U_{\hat{\Delta}}(t,s)y - U_\Delta(t,s)y\|^2,\end{aligned}$$

where $J = \|\{A^s_{\hat{\Delta}}(t, v_\Delta(t)) - A_k\}U_\Delta(t, s)y\|$. Suppose that $t_k \leq \hat{t}_j$. Then $J \leq \varepsilon$ by (*iii*) of Lemma 3.1. Suppose that $\hat{t}_j < t_k$. Then we have

$$\begin{aligned} J \leq & \ \|\{A^s_{\hat{\Delta}}(t, u_k) - A_k\}\cdot\{U_\Delta(t, s)y - U_\Delta(t_k, s)y\}\| \\ & + \|\{A^s_{\hat{\Delta}}(t, u_k) - A^s_{\hat{\Delta}}(t, u_{k-1})\}U_\Delta(t_k, s)y\| \\ & + \|\{A^s_{\hat{\Delta}}(t, u_{k-1}) - A_{k-1}\}U_\Delta(t_k, s)y\| \\ & + \|\{A_{k-1} - A(t_k, u_{k-1})\}U_\Delta(t_k, s)y\| \\ & + \|\{A(t_k, u_{k-1}) - A(t_k, u_k)\}U_\Delta(t_k, s)y\| \\ \leq & \ 2K'\varepsilon + \mu_1'\|u_k - u_{k-1}\|\cdot e^{2\beta T}\|y\|_Y + 2\varepsilon + \mu_1\|u_k - u_{k-1}\|\cdot e^{2\beta T}\|y\|_Y.\end{aligned}$$

Therefore, in any case, we have

$$J \leq 2\varepsilon(K' + \mu_1' L e^{2\beta T}\cdot\|y\|_Y + 1).$$

Integrating the differential inequality, we have

$$\begin{aligned}\|U_{\hat{\Delta}}(t, s)y - U_\Delta(t, s)y\| \leq & \ \mu_1 e^{(\alpha' + 2\beta)T}\|y\|_Y \int_s^t \|v_{\hat{\Delta}}(\sigma) - v_\Delta(\sigma)\| \, d\sigma \\ & + 2\varepsilon e^{\alpha' T}(K' + \mu_1' L e^{2\beta T}\cdot\|y\|_Y + 2)T. \quad \text{Q.E.D.}\end{aligned}$$

LEMMA 3.3. *For each $s \in [0, T)$ and $x \in X$, the limit*

$$\lim_{|\Delta|\to 0} U_\Delta(t, s)x \equiv U^u(t, s)x$$

exists in X uniformly in $t \in [s, T]$. Furthermore, $U^u(t, s)$ satisfies the following:

(*a*) *$U^u(t, s)$ is strongly continuous on Ω to $\boldsymbol{B}(X)$ and $\|U^u(t, s)\| \le e^{\alpha(t-s)}$,*

(*b*) *$U^u(t, t) = 1$ and $U^u(t, \sigma)\cdot U^u(\sigma, s) = U^u(t, s)$ for $0 \le s \le \sigma \le t \le T$, and*

(*c*) *if $u(s_0) \in W$ for some $s_0 \in [0, T]$ then*

$$(\partial/\partial s)U^u(t, s)y|_{s=s_0} = U^u(t, s_0)A(s_0, u(s_0))y \quad \textit{and}$$
$$(\partial/\partial t)U^u(t, s_0)y|_{t=s_0} = -A(s_0, u(s_0))y$$

for $y \in Y$ and $s_0 \le t \le T$.

PROOF. Let $\eta > 0$, $s \in [0, T)$ and $y \in Y$. Since $\lim_{|\Delta|\to 0} u_\Delta(t) = u(t) = \lim_{|\Delta|\to 0} v_\Delta(t)$, there is a $\delta_1 > 0$ such that $\|u(t) - v_\Delta(t)\| < \eta$ for $t \in [0, T]$ and Δ with $|\Delta| \le \delta_1$. We put $\varepsilon = \min(\eta, \delta_1)$ and let $\Delta(\varepsilon) = \{t_k^\varepsilon\}_{k=0}^{N(\varepsilon)}$ be the partition of $[0, T]$ satisfying (*o*) to (*iv*) of Lemma 3.1. Put $\delta = \min_k (t_k^\varepsilon - t_{k-1}^\varepsilon)$. Then, for any partitions Δ and Δ' with $|\Delta| < \delta$ and $|\Delta'| < \delta$, we have

$$\begin{aligned}\|U_\Delta(t, s)y - U_{\Delta'}(t, s)y\| &\le \|U_\Delta(t, s)y - U_{\Delta(\varepsilon)}(t, s)y\| + \|U_{\Delta(\varepsilon)}(t, s) - U_{\Delta'}(t, s)y\| \\ &\le 2\varepsilon C_1 + 4C_2(t-s)\eta \le 2(C_1 + 2C_2 T)\eta\,,\end{aligned}$$

by Lemma 3.2. Therefore, $\lim_{|\Delta|\to 0} U_\Delta(t, s)y$ exists uniformly in $t \in [s, T]$. On the other hand, since Y is dense in X and $\|U_\Delta(t, s)\|$ is uniformly bounded, $\lim_{|\Delta|\to 0} U_\Delta(t, s)x$ exists uniformly in $t \in [s, T]$ for every $x \in X$, and so $U^u(t, s)x \equiv \lim_{|\Delta|\to 0} U_\Delta(t, s)x$ is continuous in $t \in [s, T]$.

Next, by (3.4), we have

$$U_\Delta(t, s)y - U_\Delta(t, s')y = \int_{s'}^{s} U_\Delta(t, \sigma)A_\Delta^t(\sigma, v_\Delta(\sigma))y\, d\sigma \tag{3.5}$$

for $(t, s), (t, s') \in \Omega$ and $y \in Y$. Therefore, we have

$$\|U_\Delta(t, s)y - U_\Delta(t, s')y\| \le |s - s'|\cdot K'e^{2\alpha T}\|y\|_Y\,.$$

Passing to the limit as $|\Delta| \to 0$, we see that $U^u(t, s)y$ is Lipschitz continuous in $s \in [0, t]$ uniformly in t. Therefore, $U^u(t, s)y$ is continuous on Ω to X, and thus $U^u(t, s)x$ is continuous for every $x \in X$. By (3.5) with $s' = s_0$, we have

$$\begin{aligned}&\left\|\frac{1}{s-s_0}\{U_\Delta(t, s)y - U_\Delta(t, s_0)y - \int_{s_0}^{s} U_\Delta(t, \sigma)A_\Delta^t(\sigma, u(s_0))y\, d\sigma\}\right\| \\ &= \left\|\frac{1}{s-s_0}\int_{s_0}^{s} U_\Delta(t, \sigma)\{A_\Delta^t(\sigma, v_\Delta(\sigma)) - A_\Delta^t(\sigma, u(s_0))\}y\, d\sigma\right\| \\ &\le \mu_1' e^{2\alpha T}\|y\|_Y\cdot\left|\frac{1}{s-s_0}\int_{s_0}^{s}\|v_\Delta(\sigma) - u(s_0)\|\, d\sigma\right|.\end{aligned}$$

Passing to the limit as $|\Delta| \to 0$, we obtain

$$\left\| \frac{1}{s - s_0} \{U^u(t, s)y - U^u(t, s_0)y - \int_{s_0}^{s} U^u(t, \sigma)A(\sigma, u(s_0))y\, d\sigma\} \right\|$$
$$\le \mu_1' e^{2\alpha T} \|y\|_Y \cdot \left| \frac{1}{s - s_0} \int_{s_0}^{s} \|u(\sigma) - u(s_0)\|\, d\sigma \right| .$$

Since $u(t)$ is continuous, this implies the first part of (c). To prove the second part of (c), we will show that

$$\|U^u(t, s_0)y - \exp[-(t - s_0)A(s_0, u(s_0))]y\| = o(t - s_0), \quad \text{as } t \downarrow s_0 . \tag{3.6}$$

Once this is done, we have

$$\left\| \frac{U^u(t, s_0)y - y}{t - s_0} + A(s_0, u(s_0))y \right\|$$
$$\le \left\| \frac{U^u(t, s_0)y - \exp[-(t - s_0)A(s_0, u(s_0))]y}{t - s_0} \right\|$$
$$+ \left\| \frac{\exp[-(t - s_0)A(s_0, u(s_0))]y - y}{t - s_0} + A(s_0, u(s_0))y \right\|$$
$$\to 0, \quad \text{as } t \downarrow s_0 ,$$

which implies the second part of (c). To this end, we differentiate $U_\Delta(t, \sigma) \cdot \exp[-(\sigma - s_0)A(s_0, u(s_0))]y$ with respect to σ and then integrate the result over $\sigma \in [s_0, t]$. Then we have

$$\|\exp[-(t - s_0)A_0]y - U_\Delta(t, s_0)y\|$$
$$\le \left\| \int_{s_0}^{t} U_\Delta(t, \sigma)\{A_\Delta^t(\sigma, v_\Delta(\sigma)) - A_\Delta^t(\sigma, u_0)\} \exp[-(\sigma - s_0)A_0]y\, d\sigma \right\|$$
$$+ \left\| \int_{s_0}^{t} U_\Delta(t, \sigma)\{A_\Delta^t(\sigma, u_0) - A_0\} \exp[-(\sigma - s_0)A_0]y\, d\sigma \right\|$$
$$\le \mu_1' e^{(2\alpha+\beta)T} \|y\|_Y \int_{s_0}^{t} \|v_\Delta(\sigma) - u_0\|\, d\sigma$$
$$+ e^{2\alpha T} \int_{s_0}^{t} \|\{A_\Delta^t(\sigma, u_0) - A_0\} \exp[-(\sigma - s_0)A_0]y\|\, d\sigma ,$$

where $u_0 = u(s_0)$ and $A_0 = A(s_0, u_0)$. Passing to the limit as $|\Delta| \to 0$, we have

$$\|\exp[-(t - s_0)A_0]y - U^u(t, s_0)y\|$$
$$\le \mu_1' e^{(2\alpha+\beta)T} \|y\|_Y \int_{s_0}^{t} \|u(\sigma) - u(s_0)\|\, d\sigma$$
$$+ e^{2\alpha T} \int_{s_0}^{t} \|\{A(\sigma, u_0) - A_0\} \exp[-(\sigma - s_0)A_0]y\|\, d\sigma ,$$

which implies (3.6). $\|U^u(t, s)\| \le e^{\alpha(t-s)}$ follows from (1.1), and (*b*) follows from (3.2). Q.E.D.

By ($f2$), there is a unique function $\mathscr{F}(t, w)$ satisfying (*i*) $\mathscr{F}(t, w)$ is defined for $t \in [0, T_0]$ and $w \in \tilde{W}$, where $\tilde{W}$ is the closure of W in X, (*ii*) $\mathscr{F}(t, w) = f(t, w)$ for $t \in [0, T_0]$ and $w \in W$, and (*iii*) $\|\mathscr{F}(t, w) - \mathscr{F}(t, z)\| \le \mu_2 \|w - z\|$ for $t \in [0, T_0]$ and w, $z \in \tilde{W}$, and for each $w \in \tilde{W}$, $\mathscr{F}(t, w)$ is continuous on $[0, T_0]$ to X. With the aid of $\mathscr{F}(t, w)$, we obtain the following:

THEOREM 3.4. *$u(t)$ obtained by Theorem 2.4 is the mild solution of (CP) in the sense that $u(t)$ satisfies*

$$(M) \qquad u(t) = U^u(t, 0)a + \int_0^t U^u(t, s)\mathscr{F}(s, u(s))\, ds$$

for $t \in [0, T]$.

PROOF. For each partition $\Delta = \{t_k\}_{k=0}^N$ of $[0, T]$, we define a family of bounded linear operators $\{\bar{U}_\Delta(t, s); (t, s) \in \Omega\}$ and a function $f_\Delta(s)$ by

$$\bar{U}_\Delta(t, s) = U_\Delta(t, t_j) \qquad \text{if } (t, s) \in \Omega \text{ and } t_j \le s < t_{j+1} \text{ for some } j\,,$$
$$f_\Delta(s) = f_j \quad (= f(t_j, u_\Delta(t_j))) \qquad \text{if } t_j \le s < t_{j+1} \text{ for some } j\,,$$

$U_\Delta(T, T) = 1$ and $f_\Delta(T) = f(T, u_\Delta(T))$. Then $\bar{U}_\Delta(t, s)$ converges strongly to $U^u(t, s)$ in $\boldsymbol{B}(X)$, and $f_\Delta(s)$ converges to $\mathscr{F}(s, u(s))$ in X. On the other hand, we have

$$(3.7) \qquad u_\Delta(t) = U_\Delta(t, 0)a + \int_0^t \bar{U}_\Delta(t, s) f_\Delta(s)\, ds\,,$$

since $u_\Delta(t) = U_\Delta(t, t_k)u_k + (t - t_k)U_\Delta(t, t_k)f_k$ for $t_k \le t < t_{k+1}$, $k = 0, \dots, N-1$. Therefore, taking the limit of both sides of (3.7), as $|\Delta| \to 0$, we obtain (M).

Q.E.D.

REMARK 3.5. By ($A2$), there is a unique family of linear operators $\{\mathscr{A}(t, w); t \in [0, T_0],\ w \in \tilde{W}\}$ satisfying ($A1$) and ($A2$) with $A(t, w)$ and W replaced by $\mathscr{A}(t, w)$ and $\tilde{W}$ respectively, and $A(t, w) = \mathscr{A}(t, w)$ for each $t \in [0, T_0]$ and $w \in W$. Using these operators, we can prove

$$(c') \qquad (\partial/\partial s)U^u(t, s)y = U^u(t, s)\mathscr{A}(s, u(s))y$$

for $(t, s) \in \Omega$ and $y \in Y$. However, we do not use (c') in this paper.

Let $u(t)$ be the strong solution of (CP). Then we have

$$u(t) - U^u(t, 0)a = \int_0^t (\partial/\partial s)U^u(t, s)u(s)\, ds = \int_0^t U^u(t, s)f(s, u(s))\, ds\,.$$

Therefore, *every strong solution is the mild solution.* Conversely, we have:

COROLLARY 3.6. *Let $u(t)$ be the mild solution of (CP). Suppose that $u(t) \in W$ for a.e. $t \in [0, T]$. Then $u(t)$ is the unique strong solution of (CP).*

PROOF. Let $t \in [0, T]$ be the point such that $u(t) \in W$. By (*b*) of Lemma 3.3 and (*M*), we have

$$\begin{aligned} u(\tau) &= U^u(\tau, t)U^u(t, 0)a + \int_0^t U^u(\tau, t)U^u(t, \sigma)\mathscr{F}(\sigma, u(\sigma))\, d\sigma \\ &\quad + \int_t^\tau U^u(\tau, \sigma)\mathscr{F}(\sigma, u(\sigma))\, d\sigma \\ &= U^u(\tau, t)u(t) + \int_t^\tau U^u(\tau, \sigma)\mathscr{F}(\sigma, u(\sigma))\, d\sigma \end{aligned}$$

for $t \leq \tau \leq T$. Therefore, by (*c*) of Lemma 3.3, we have

$$\begin{aligned} \frac{u(\tau) - u(t)}{\tau - t} &= \frac{\{U^u(\tau, t) - 1\}u(t)}{\tau - t} + \frac{1}{\tau - t}\int_t^\tau U^u(\tau, \sigma)\mathscr{F}(\sigma, u(\sigma))\, d\sigma \\ &\to -A(t, u(t))u(t) + f(t, u(t)), \quad \text{as } \tau \downarrow t, \end{aligned}$$

since $f(t, u(t)) = \mathscr{F}(t, u(t))$. Therefore, the strong right derivative $(d/dt)^+u(t)$ exists and satisfies

$$(d/dt)^+u(t) + A(t, u(t))u(t) = f(t, u(t)), \tag{3.8}$$

whenever $u(t) \in W$. By (2.1), $\|du_\Delta(t)/dt\| \leq K'(\|\phi\|_Y + r) + \lambda_2'$. Therefore, $u_\Delta(t)$ is Lipschitz continuous uniformly in Δ, and so $u(t)$ is also Lipschitz continuous. Now, since $\langle u(t), x^*\rangle$ is Lipschitz continuous for every $x^* \in X^*$, we have

$$\begin{aligned} \langle u(t) - a, x^*\rangle &= \int_0^t (d/ds)\langle u(s), x^*\rangle\, ds \\ &= \left\langle \int_0^t (d/ds)^+u(s)\, ds, x^* \right\rangle. \end{aligned}$$

This implies that $u(t)$ is strongly differentiable in X for *a.e.* $t \in [0, T]$ and $(d/dt)u(t) = (d/dt)^+u(t)$. Thus $u(t)$ is the strong solution of (CP) by (3.8). The uniqueness of the strong solution can be proved by usual way. Q.E.D.

Suppose that X is reflexive. Then we have $\bar{B}(\phi, r) = \tilde{B}(\phi, r)$ by Kato's lemma. See [7]. So $u(t) \in W$ for all $t \in [0, T]$. Therefore, $u(t)$ is the strong solution of (CP). However, in the rest of this section, we shall show that $u(t)$ is a classical solution of (CP), if X is reflexive.

LEMMA 3.7. *Suppose that $u(t) \in W$ for all $t \in [0, T]$ and that $B(t, u(t))$: $[0, T] \to \boldsymbol{B}(X)$ is strongly measurable. Then we have*

(a') *$U^u(t, s)(Y) \subset Y$, $U^u(t, s)$ is strongly continuous on Ω to $\boldsymbol{B}(Y)$ and $\|U^u(t, s)\|_Y \le e^{(t-s)\beta}$, and*

(d) *$(\partial/\partial t)U^u(t, s)y = -A(t, u(t))U^u(t, s)y$ for every $y \in Y$ and $(t, s) \in \Omega$.*

PROOF. We employ the method of [3]. Consider the Volterra-type integral equation

$$(V) \qquad W^u(t, s)x = U^u(t, s)x - \int_s^t W^u(t, \sigma)B(\sigma, u(\sigma))U^u(\sigma, s)x \, d\sigma$$

for $(t, s) \in \Omega$ and $x \in X$. It is easy to see that (V) has a unique solution W^u. For each $(t, s) \in \Omega$, $W^u(t, s)$ is a bounded linear operator on X, and W^u is strongly continuous on Ω to $\boldsymbol{B}(X)$. Multiplying both sides of (V) by S^{-1}, we obtain another integral equation

$$(V') \qquad Z^u(t, s)x = S^{-1}U^u(t, s)x - \int_s^t Z^u(t, \sigma)B(\sigma, u(\sigma))U^u(\sigma, s)x \, d\sigma$$

for $(t, s) \in \Omega$ and $x \in X$. (V') has also the unique solution $Z^u = S^{-1}W^u$. Let $\varepsilon > 0$, $s \in [0, T)$ and $y \in Y$, and let $\Delta = \{t_k\}_{k=0}^N$ be a partition of $[0, T]$ satisfying (o) to (iv) of Lemma 3.1. Then we have

$$\begin{aligned}(\partial/\partial\sigma)U^u(t, \sigma)S^{-1}U_\Delta(\sigma, s)y &= U^u(t, \sigma)S^{-1}\{SA(\sigma, u(\sigma))S^{-1} - A_\Delta^s(\sigma, v_\Delta(\sigma))\}U_\Delta(\sigma, s)y \\ &= U^u(t, \sigma)S^{-1}\{A(\sigma, u(\sigma)) - A_\Delta^s(\sigma, v_\Delta(\sigma))\}U_\Delta(\sigma, s)y \\ &\quad + U^u(t, \sigma)S^{-1}B(\sigma, u(\sigma))U_\Delta(\sigma, s)y \\ &\equiv \mathrm{I} + \mathrm{II}\end{aligned} \tag{3.9}$$

for $s \le \sigma \le t \le T$. Suppose that $t_k \le \sigma < t_{k+1}$ for some k. Then by (iii) and (iv) of Lemma 3.1, we have

$$\begin{aligned}\|\mathrm{I}\| &\le e^{2\alpha T}\|S^{-1}\| \cdot \{\|[A(\sigma, u(\sigma)) - A(\sigma, v_\Delta(\sigma))]U_\Delta(\sigma, s)y\| \\ &\quad + \|[A(\sigma, v_\Delta(\sigma)) - A(t_k, v_\Delta(\sigma))]U_\Delta(\sigma, s)y\| \\ &\quad + \|[A(t_k, v_\Delta(\sigma)) - A_\Delta^s(\sigma, v_\Delta(\sigma))]U_\Delta(\sigma, s)y\|\} \\ &\le e^{2\alpha T}\|S^{-1}\|\{\mu_1' e^{2\beta T}\|y\|_Y \cdot \|u(\sigma) - v_\Delta(\sigma)\| + 2\varepsilon\}.\end{aligned}$$

Therefore, integrating both sides of (3.9) and then passing to the limit as $\varepsilon \downarrow 0$, we have

$$S^{-1}U^u(t, s)y - U^u(t, s)S^{-1}y = \int_s^t U^u(t, \sigma)S^{-1}B(\sigma, u(\sigma))U^u(\sigma, s)y \, d\sigma .$$

This implies that $U^u S^{-1}$ is also the solution of (V'), and so $S^{-1}W^u = U^u S^{-1}$. Thus we have $U^u(t, s)(Y) \subset Y$ and $U^u(t, s)$ is strongly continuous on Ω to $\boldsymbol{B}(Y)$. (d) follows from (b) and (c) of Lemma 3.3. Q.E.D.

Let $u(t)$ be the mild solution of (CP). We consider the following condition:

$$(C) \qquad \begin{aligned} &u(t) \in W \text{ for all } t \in [0, T]\,, \quad \text{and} \\ &B(t, u(t))\colon [0, T] \to \boldsymbol{B}(X) \quad \text{and} \quad f(t, u(t))\colon [0, T] \to Y \\ &\text{are strongly measurable}\,. \end{aligned}$$

THEOREM 3.8. *Let $u(t)$ be the mild solution of (CP) and let (C) hold. Then $u(t)$ is the unique classical solution of (CP).*

PROOF. By Lemma 3.7 and (M), $u(t)$ is continuous in Y, since $f(t, u(t))$ is Bochner integrable in Y. Q.E.D.

If X is reflexive, (C) is automatically satisfied. (See [7].) Therefore, we have:

COROLLARY 3.9. *Suppose that X is reflexive. Then (CP) has the unique classical solution $u(t)$.*

§4. Convergence of $\{u_\Delta\}$ in Y and existence of classical solutions

Throughout this section, we assume that conditions (X), $(A1)$ through $(A4)$ and $(f1)$ through $(f3)$ are satisfied. Consider the set $\mathbf{B}$ of all continuous functions $v\colon [0, T] \to Y$ whose values $v(t)$ are contained in $\bar{B}(\phi, r)$. $\mathbf{B}$ is the complete metric space with respect to the metric

$$d(v, w) \equiv \sup_t \|v(t) - w(t)\|_Y \qquad \text{for } v, w \in \mathbf{B}\,.$$

For each $v \in \mathbf{B}$, we define Φv by

$$\begin{aligned}[][\Phi v](t) \equiv{}& S^{-1}U^u(t, 0)Sa \\ &+ \int_0^t S^{-1}U^u(t, s)\{Sf(s, v(s)) - B(s, v(s))Sv(s)\}\, ds\end{aligned}$$

for $0 \le t \le T$, where u is the mild solution of (CP), and $\{U^u(t, s)\}$ is the family of bounded linear operators on X obtained by Lemma 3.3. By $(A4)$ and $(f3)$, $Sf(t, v(t))\colon [0, T] \to X$ and $B(t, v(t))\colon [0, T] \to \boldsymbol{B}(X)$ are strongly measurable. Therefore, we see that $\Phi v\colon [0, T] \to Y$ is continuous.

We here explain the use of the operator Φ in brief. Suppose for the moment that there is a classical solution u of (CP). Then, using the relation in $(A3)$, we see (at least formally) that $v = Su$ is a mild solution of

$$dv(t)/dt + A(t, u(t))v(t) = Sf(t, u(t)) - B(t, u(t))Su(t), \quad 0 \le t \le T,$$

with $v(0) = Sa$. In view of equation (L) in Section 2, we see that

$$\begin{aligned} Su(t) &= U^u(t, 0)Sa + \int_0^t U^u(t, s)\{Sf(s, u(s)) - B(s, u(s))Su(s)\}\, ds \\ &= [S\Phi u](t). \end{aligned}$$

This means that u is a fixed point of Φ.

In the following, we will prove that *Φ has a unique fixed point $\bar{u} \in \mathbf{B}$* and that $\bar{u}$ is the unique classical solution of (CP).

LEMMA 4.1. *There is a $T \in (0, T_0]$ (which is smaller than or equal to that of Proposition 1.1) such that $\Phi v \in \mathbf{B}$ for all $v \in \mathbf{B}$.*

PROOF. We will show that $\|[\Phi v](t) - \phi\|_Y \le r$. By the definition of Φ, we have

$$\begin{aligned} \|[\Phi v](t) - \phi\|_Y &\le \|U^u(t, 0)Sa - S\phi\| \\ &\quad + e^{\alpha T} \int_0^t \|Sf(s, v(s)) - B(s, v(s))Sv(s)\|\, ds \\ &\le \sup_{0 \le t \le T} \|U^u(t, 0)Sa - Sa\| + \|a - \phi\|_Y \\ &\quad + e^{\alpha T}\{\lambda_2 + \lambda_1(\|\phi\|_Y + r)\}T \end{aligned}$$

for all $v \in \mathbf{B}$. Since $\|a - \phi\|_Y < r$, we have $\|[\Phi v](t) - \phi\|_Y < r$ if $T \in (0, T_0]$ is sufficiently small. Q.E.D.

In the rest of this section, we fix T as above.

LEMMA 4.2. *Φ: $\mathbf{B} \to \mathbf{B}$ has a unique fixed point $\bar{u} \in \mathbf{B}$.*

PROOF. We use the contraction mapping theorem. For each $v \in \mathbf{B}$ and $w \in \mathbf{B}$, we have

$$\begin{aligned} \|[\Phi v](t) - [\Phi w](t)\|_Y &\le \int_0^t \|U^u(t, s)S\{f(s, v(s)) - f(s, w(s))\}\|\, ds \\ &\quad + \int_0^t \|U^u(t, s)\{B(s, v(s))Sv(s) - B(s, w(s))Sw(s)\}\|\, ds \\ &\le e^{\alpha T}\{\mu_4 + \mu_3(\|\phi\|_Y + r) + \lambda_1\} \cdot \int_0^t \|v(s) - w(s)\|_Y\, ds. \end{aligned}$$

Put $C \equiv e^{\alpha T}\{\mu_4 + \mu_3(\|\phi\|_Y + r) + \lambda_1\}$. Then, it follows that

$$\|[\Phi^n v](t) - [\Phi^n w](t)\|_Y \le \frac{C^n}{(n-1)!} \int_0^t (t-s)^{n-1} \|v(s) - w(s)\|_Y \, ds$$
$$\le \frac{(Ct)^n}{n!} d(v, w)$$

for $0 \le t \le T$ and $n \in N$. Therefore, we have

$$d(\Phi^n v, \Phi^n w) \le \frac{(CT)^n}{n!} d(v, w)$$

for every $n \in N$. Q.E.D.

In the following, we will prove that $u = \bar{u}$ ($\in \bar{B}(\phi, r) \subset W$). This implies that u is the classical solution. To this end, for each $\Delta = \{t_k\}_{k=0}^N$, we consider the difference equation

$$(W) \quad \begin{aligned} &\frac{w_\Delta(t) - w_k}{t - t_k} + A_k w_\Delta(t) = G(t_k)\,, \qquad t_k < t \le t_{k+1}\,, \qquad 0 \le k \le N-1\,, \\ &w_\Delta(0) = Sa\,, \end{aligned}$$

where $w_k = w_\Delta(t_k)$, $A_k = A(t_k, u_k)$ and

$$G(t) = Sf(t, \bar{u}(t)) - B(t, \bar{u}(t))S\bar{u}(t)$$

for $0 \le t \le T$. The solution $w_\Delta(t)$ of (W) is written as

$$w_\Delta(t) = U_\Delta(t, 0)Sa + \int_0^t \bar{U}_\Delta(t, s)G_\Delta(s)\, ds\,, \tag{4.1}$$

where $G_\Delta(s) = G(t_k)$ if $t_k \le s < t_{k+1}$ for some $0 \le k \le N-1$, and $G_\Delta(T) = G(T)$. The notation $\bar{U}_\Delta(t, s)$ is described in the proof of Theorem 3.4. Note that $w_\Delta(t) \in D(A_k)$ for $t_k < t \le t_{k+1}$, $0 \le k \le N-1$.

LEMMA 4.3. *There is a sequence $\{\Delta(m)\}_{m=1}^\infty$ of partitions of $[0, T]$ such that $\lim_{m\to\infty} |\Delta(m)| = 0$ and that*

$$\lim_{m\to\infty} \sup_t \|w_{\Delta(m)}(t) - S\bar{u}(t)\| = 0\,. \tag{4.2}$$

PROOF. Since $\bar{u} = \Phi\bar{u}$, we have

$$S\bar{u}(t) = U^u(t, 0)Sa + \int_0^t U^u(t, s)G(s)\, ds \tag{4.3}$$

for $0 \le t \le T$. By the well known lemma of Evans [4], there is a sequence of partitions $\Delta(m)$ ($m \in N$) satisfying

$$|\Delta(m)| \le 1/m \quad \text{and} \quad \int_0^T \|G(t) - G_{\Delta(m)}(t)\|\, dt \le 1/m \tag{4.4}$$

for $m \in N$, since $G(t)$ is strongly measurable and bounded in X. By (4.1) and (4.3), we have

$$\|w_{\Delta(m)}(t) - S\bar{u}(t)\| \le \sup_t \|\{U_{\Delta(m)}(t, 0) - U^u(t, 0)\}Sa\| + \int_0^t \|\{\bar{U}_{\Delta(m)}(t, s) - U^u(t, s)\}G(s)\|\, ds + \int_0^t \|\bar{U}_{\Delta(m)}(t, s)\{G(s) - G_{\Delta(m)}(s)\}\|\, ds$$

for $0 \le t \le T$. We put

$$\varphi_m(s) \equiv \sup_{\tau \in [s,T]} \|\{\bar{U}_{\Delta(m)}(\tau, s) - U^u(\tau, s)\}G(s)\|$$

for $0 \le s \le T$ and $m \in N$. Then $\varphi_m(s)$ is the bounded measurable function of $0 \le s \le T$ and satisfies $\lim_{m\to\infty} \varphi_m(s) = 0$ by Lemma 3.3. Therefore, by (4.4) and Lemma 3.3, we have

$$\sup_t \|w_{\Delta(m)}(t) - S\bar{u}(t)\| \le \sup_t \|\{U_{\Delta(m)}(t, 0) - U^u(t, 0)\}Sa\| + \int_0^T \varphi_m(s)\, ds + e^{2\alpha T}/m \to 0\,,$$

as $m \to \infty$. Q.E.D.

LEMMA 4.4. *For every partition $\Delta = \{t_k\}_{k=0}^N$ of $[0, T]$ satisfying $|\Delta| \le (2\beta)^{-1}$, we have*

$$\sup_t \|u_\Delta(t) - S^{-1}w_\Delta(t)\|_Y \le Te^{\omega T}\cdot \varepsilon_\Delta\,,$$

where $\omega > \max\,[2\{\mu_4 + \mu_3(r + \|\phi\|_Y)\}, 4\beta]$, and

$$\varepsilon_\Delta = \{\mu_4 + \lambda_1 + \mu_3(r + \|\phi\|_Y)\}\cdot \max_k\,\{\sup_{t_k \le t \le t_{k+1}} \|w_\Delta(t) - S\bar{u}(t_k)\|\}\,.$$

PROOF. Multiplying both sides of (W) by S^{-1}, and then using the relation $S^{-1}A_k = A_kS^{-1} - S^{-1}B_k$, we have

$$\text{(Z)}\qquad \frac{z_\Delta(t) - z_k}{t - t_k} + A_k z_\Delta(t) = f(t_k, \bar{u}(t_k)) + S^{-1}\{B(t_k, u_k)Sz_\Delta(t) - B(t_k, \bar{u}(t_k))S\bar{u}(t_k)\}$$

for $t_k < t \le t_{k+1}$, $0 \le k \le N - 1$, where $z_\Delta(t) \equiv S^{-1}w_\Delta(t)$ and $z_k \equiv S^{-1}w_k$. Therefore, we have

$$\text{(4.5)}\qquad \begin{aligned} z_\Delta(t) = {} & [1 + (t - t_k)A_k]^{-1}z_k + (t - t_k)[1 + (t - t_k)A_k]^{-1}f(t_k, \bar{u}(t_k)) \\ & + (t - t_k)[1 + (t - t_k)A_k]^{-1}S^{-1}B(t_k, u_k)S\{z_\Delta(t) - \bar{u}(t_k)\} \\ & + (t - t_k)[1 + (t - t_k)A_k]^{-1}S^{-1}\{B(t_k, u_k) - B(t_k, \bar{u}(t_k))\}S\bar{u}(t_k) \end{aligned}$$

for $t_k \le t \le t_{k+1}$, $0 \le k \le N-1$. By (1.4) and (4.5), we have

$$\begin{aligned}\|z_\Delta(t) - u_\Delta(t)\|_Y \le{}& e^{2\beta(t-t_k)}\|z_k - u_k\|_Y + \mu_4(t-t_k)e^{2\beta(t-t_k)}\|u_k - \bar{u}(t_k)\|_Y \\ &+ \lambda_1(t-t_k)e^{2\beta(t-t_k)}\|z_\Delta(t) - \bar{u}(t_k)\|_Y \\ &+ \mu_3(t-t_k)e^{2\beta(t-t_k)}\|u_k - \bar{u}(t_k)\|_Y \cdot \|\bar{u}(t_k)\|_Y .\end{aligned}$$

Therefore, using the inequality

$$\|u_k - \bar{u}(t_k)\|_Y \le \|u_k - z_k\|_Y + \|z_k - \bar{u}(t_k)\|_Y ,$$

we have

$$\begin{aligned}\|z_\Delta(t) - u_\Delta(t)\|_Y \le{}& e^{2\beta(t-t_k)}\{1 + (t-t_k)[\mu_4 + \mu_3(r + \|\phi\|_Y)]\} \cdot \|z_k - u_k\|_Y \\ &+ (t-t_k)e^{2\beta(t-t_k)} \cdot \varepsilon_\Delta .\end{aligned}$$

Now, since $1 + h\{\mu_4 + \mu_3(r + \|\phi\|_Y)\} \le e^{\omega h/2}$ for $h \ge 0$, we have

$$\|z_\Delta(t) - u_\Delta(t)\|_Y \le e^{(t-t_k)\omega}\|z_k - u_k\|_Y + (t-t_k)e^{(t-t_k)\omega} \cdot \varepsilon_\Delta$$

for $t_k \le t \le t_{k+1}$, $0 \le k \le N-1$. It follows that

$$\|z_\Delta(t) - u_\Delta(t)\|_Y \le \textstyle\sum_{j=0}^{k}(t_{j+1} - t_j)e^{\omega(t-t_j)} \cdot \varepsilon_\Delta \le Te^{\omega T} \cdot \varepsilon_\Delta . \qquad \text{Q.E.D.}$$

THEOREM 4.5. *Suppose that conditions* (*X*), (*A*1) *through* (*A*4) *and* (f1) *through* (f3) *are satisfied. Then we have the following*:

(*i*) *There is a sequence* $\{\Delta(m)\}_{m=1}^{\infty}$ *of partitions of* $[0, T]$ *such that* $\lim_{m\to\infty}|\Delta(m)| = 0$ *and that* $\{u_{\Delta(m)}(t)\}$ *converges in Y, uniformly on* $[0, T]$, *to the unique classical solution* $u(t)$ *of* (*CP*).

(*ii*) *In addition to the assumptions above, assume that* (*Bf*) *is satisfied. Then* $\{u_{\Delta(m)}(t)\}$ *converges in Y, uniformly on* $[0, T]$, *for every sequence* $\{\Delta(m)\}_{m=1}^{\infty}$ *satisfying* $\lim_{m\to\infty}|\Delta(m)| = 0$.

PROOF. We take $\{\Delta(m)\}$ as in Lemma 4.3. Then we have

$$\begin{aligned}\sup_t \|u_{\Delta(m)}(t) - \bar{u}(t)\|_Y &= \sup_t \|Su_{\Delta(m)}(t) - S\bar{u}(t)\| \\ &\le \sup_t \|Su_{\Delta(m)}(t) - w_{\Delta(m)}(t)\| + \sup_t \|w_{\Delta(m)}(t) - S\bar{u}(t)\| \\ &\le Te^{\omega T} \cdot \varepsilon_{\Delta(m)} + \sup_t \|w_{\Delta(m)}(t) - S\bar{u}(t)\| \\ &\to 0, \qquad \text{as} \quad m \to \infty ,\end{aligned} \tag{4.6}$$

by Lemma 4.3 and Lemma 4.4. On the other hand, since $\{u_{\Delta(m)}(t)\}$ converges to the mild solution $u(t)$ of (*CP*), we have $u = \bar{u}$. Therefore, $u(t) \in W$ and $u(t)$ is continuous on $[0, T]$ into Y, and thus condition (*C*) is satisfied. From Theorem 3.8 it follows that $u(t)$ is the unique classical solution. This proves (*i*). If (*Bf*) holds, $G(t)$ is continuous on $[0, T]$ into X, and so (4.2) holds for every $\{\Delta(m)\}$ satisfying $\lim_{m\to\infty}|\Delta(m)|=0$. Therefore, (*ii*) follows from (4.6).

Q.E.D.

§ 5. Convergence of $\{u_\Delta^n\}$ in Y

In this section, we consider the sequence of Cauchy problems (CP^n) $(n\in\bar{N}\equiv N\cup\{\infty\})$. Throughout this section, we assume conditions (S) and (C) below.

(S) The operators $\{A^n, f^n\}$ satisfy conditions (X), $(A1)$ through $(A4)$ and $(f1)$ through $(f3)$ uniformly in $n\in\bar{N}$, by which we mean that all the constants T_0, α, μ_1, $\cdots$, μ_4 are independent of $n\in\bar{N}$. X, Y, S and W in condition (X) are common to all (CP^n).

(C) For each $w\in W$ and $y\in Y$,

$$\lim_{n\to\infty}A^n(t,w)y=A^\infty(t,w)y \qquad \text{in } X \text{ uniformly on } [0,T_0]\,, \tag{5.1}$$

$$\lim_{n\to\infty}B^n(t,w)=B^\infty(t,w) \qquad \text{strongly in } \boldsymbol{B}(X) \tag{5.2}$$

for each $t\in[0,T_0]$, where $B^n(t,w)=SA^n(t,w)S^{-1}-A^n(t,w)$,

$$\lim_{n\to\infty}f^n(t,w)=f^\infty(t,w) \qquad \text{in } Y \tag{5.3}$$

for each $t\in[0,T_0]$, and

$$\lim_{n\to\infty}f^n(t,w)=f^\infty(t,w) \qquad \text{in } X \text{ uniformly on } [0,T_0]\,. \tag{5.4}$$

Let $\Delta=\{t_k\}_{k=0}^N$ be a partition of $[0,T]$. Consider the difference approximation (D^n) for (CP^n) of the type (D):

$$(D^n)\qquad \begin{aligned}&\frac{u_\Delta^n(t)-u_k^n}{t-t_k}+A_k^n u_\Delta^n(t)=f_k^n\,,\quad t_k\le t\le t_{k+1}\,,\quad 0\le k\le N-1\,,\\ &u_\Delta^n(0)=a^n\,,\end{aligned}$$

where $u_k^n=u_\Delta^n(t_k)$, $A_k^n=A^n(t_k,u_k^n)$ and $f_k^n=f^n(t_k,u_k^n)$. To study the convergence of $\{u_\Delta^n\}$, we prepare some notions and notations. Let Z be an arbitrary Banach space with the norm $\|\cdot\|_Z$. We denote by $c(Z)$ the set of all *convergent* sequences $z=\{z^n\}$ in Z. $c(Z)$ is a Banach space with the norm $\|z\|_{c(Z)}\equiv\sup_n\|z^n\|_Z\,(<\infty)$ for $z=\{z^n\}\in c(Z)$.

LEMMA 5.1. (*i*) *A function $g(t)=\{g^n(t)\}$: $[0,T_0]\to c(Z)$ is continuous if and only if each g^n: $[0,T_0]\to Z$ is continuous and the sequence $\{g^n(t)\}$ converges in Z uniformly in $t\in[0,T_0]$.*

(*ii*) *A function $g=\{g^n\}$: $[0,T_0]\to c(Z)$ is strongly measurable in $c(Z)$ if and only if each g^n is strongly measurable in Z and the sequence $\{g^n(t)\}$ converges in Z at each $t\in[0,T_0]$.*

PROOF. (*i*) follows from the Ascoli-Arzela theorem. See [15; p. 85]. Only if part of (*ii*) is trivial. Suppose that g^n ($n \in N$) are strongly measurable in Z and that $\{g^n(t)\} \in c(Z)$ for each $t \in [0, T_0]$. Then $g^\infty = \lim g^n$ is also strongly measurable. For each $N \in N$, we define $\boldsymbol{g}_N(t)$ by

$$\boldsymbol{g}_N(t) = \{g^1(t), \cdots, g^N(t), g^\infty(t), g^\infty(t), \cdots\} .$$

It follows that $\boldsymbol{g}_N$ is strongly measurable in $c(Z)$ and that

$$\|\boldsymbol{g}(t) - \boldsymbol{g}_N(t)\|_{c(Z)} = \sup_{n>N} \|g^n(t) - g^\infty(t)\|_Z \to 0 , \qquad \text{as} \qquad N \to \infty .$$

Therefore, $\boldsymbol{g}$ is strongly measurable in $c(Z)$. Q.E.D.

Let (X, Y) be the pair of Banach spaces satisfying (X) with the isometry S and the open subset $W \subset Y$. We write $\boldsymbol{X} = c(X)$ and $\boldsymbol{Y} = c(Y)$. $\boldsymbol{Y}$ is densely and continuously embedded in $\boldsymbol{X}$. For each $\boldsymbol{y} = \{y^n\} \in \boldsymbol{Y}$, we define $\boldsymbol{S}\boldsymbol{y}$ by $\boldsymbol{S}\boldsymbol{y} = \{Sy^n\}$ $(\in \boldsymbol{X})$. $\boldsymbol{S}$: $\boldsymbol{Y} \to \boldsymbol{X}$ is a linear isometry onto $\boldsymbol{X}$. $\boldsymbol{W}$ denotes the subset of $\boldsymbol{Y}$ consists of all $\boldsymbol{w} = \{w^n\} \in \boldsymbol{Y}$ satisfying $w^n \in W$ for all $n \in N$ and $\lim w^n \in W$. $\boldsymbol{W}$ is the open subset of $\boldsymbol{Y}$. Therefore, the pair $(\boldsymbol{X}, \boldsymbol{Y})$ of the Banach spaces also satisfies (X) with the isometry $\boldsymbol{S}$ and the open subset $\boldsymbol{W}$.

For each $t \in [0, T_0]$ and $\boldsymbol{w} = \{w^n\} \in \boldsymbol{W}$, we define a linear operator $\boldsymbol{A}(t, \boldsymbol{w})$ in $\boldsymbol{X}$ as follows:

> The domain $\mathrm{D}(\boldsymbol{A}(t, \boldsymbol{w}))$ of $\boldsymbol{A}(t, \boldsymbol{w})$ is the set of all $\boldsymbol{x} = \{x^n\} \in \boldsymbol{X}$ satisfying $x^n \in \mathrm{D}(A^n(t, w^n))$ for all $n \in N$ and $\{A^n(t, w^n)x^n\} \in \boldsymbol{X}$, and for each $\boldsymbol{x} = \{x^n\} \in \mathrm{D}(\boldsymbol{A}(t, \boldsymbol{w}))$, we define $\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{x} \equiv \{A^n(t, w^n)x^n\}$.

$\boldsymbol{A}(t, \boldsymbol{w})$ is the linear operator in $\boldsymbol{X}$. We will prove that $\boldsymbol{A}$ satisfies $(A1)$ to $(A4)$ with the same constants α, μ_1, λ_1 and μ_3 as for $\{A^n\}$. By (5.1), for each $\boldsymbol{w} = \{w^n\} \in \boldsymbol{W}$ and $\boldsymbol{y} = \{y^n\} \in \boldsymbol{Y}$, we have

$$\sup_t \|A^n(t, w^n)y^n - A^\infty(t, w^\infty)y^\infty\| \to 0 , \qquad \text{as } n \to \infty , \tag{5.5}$$

where $w^\infty = \lim w^n$ and $y^\infty = \lim y^n$. Therefore, we have

$$\mathrm{D}(\boldsymbol{A}(t, \boldsymbol{w})) \supset \boldsymbol{Y} \text{ and } t \to \boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{y} \text{ is strongly continuous in } \boldsymbol{X},$$

by Lemma 5.1.

Let $t \in [0, T_0]$, $\boldsymbol{w} = \{w^n\} \in \boldsymbol{W}$, $\boldsymbol{x} = \{x^n\} \in \boldsymbol{X}$ and $0 < \alpha h < 1$, and put

$$y^n = [1 + hA^n(t, w^n)]^{-1}x^n \qquad (\in \mathrm{D}(A^n(t, w^n))) .$$

Then, by (5.1), we have

$$\lim_{n\to\infty} [1 + hA^n(t, w^n)]^{-1}x^n = [1 + hA^\infty(t, w^\infty)]^{-1}x^\infty \qquad \text{in } X , \tag{5.6}$$

where $w^\infty = \lim w^n$ and $x^\infty = \lim x^n$. Therefore, we have $\boldsymbol{y} \equiv \{y^n\} \in \boldsymbol{X}$. However, since

$$\lim_{n\to\infty} A^n(t, w^n)y^n = \{x^\infty - [1 + hA^\infty(t, w^\infty)]^{-1}x^\infty\}/h\,,$$

we have $\boldsymbol{y} \in \mathrm{D}(\boldsymbol{A}(t, \boldsymbol{w}))$, and so $\boldsymbol{y}$ is the unique solution of

$$\boldsymbol{y} + h\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{y} = \boldsymbol{x}\,.$$

Therefore, $[1 + h\boldsymbol{A}(t, \boldsymbol{w})]^{-1}$ exists in $\boldsymbol{B}(\boldsymbol{X})$ and satisfies

$$\begin{aligned}\|[1 + h\boldsymbol{A}(t, \boldsymbol{w})]^{-1}\boldsymbol{x}\|_{\boldsymbol{X}} &= \sup_n \|[1 + hA^n(t, w^n)]^{-1}x^n\| \\ &\le (1 - h\alpha)^{-1}\|\boldsymbol{x}\|_{\boldsymbol{X}}\end{aligned}$$

for $\boldsymbol{x} \in \boldsymbol{X}$. Let $t \in [0, T_0]$, $\boldsymbol{w} = \{w^n\} \in \boldsymbol{W}$, $\boldsymbol{z} = \{z^n\} \in \boldsymbol{W}$ and $\boldsymbol{y} = \{y^n\} \in \boldsymbol{Y}$. Then we have

$$\begin{aligned}\|\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{y} - \boldsymbol{A}(t, \boldsymbol{z})\boldsymbol{y}\|_{\boldsymbol{X}} &= \sup_n \|A^n(t, w^n)y^n - A^n(t, z^n)y^n\| \\ &\le \mu_1\|\boldsymbol{w} - \boldsymbol{z}\|_{\boldsymbol{X}}\cdot \|\boldsymbol{y}\|_{\boldsymbol{Y}}\,.\end{aligned}$$

Therefore, we have proved conditions $(A1)$ and $(A2)$ for $\boldsymbol{A}$.

To prove $(A3)$, we put $\boldsymbol{B}(t, \boldsymbol{w})\boldsymbol{x} \equiv \{B^n(t, w^n)x^n\}$ for each $t \in [0, T_0]$, $\boldsymbol{w} = \{w^n\} \in \boldsymbol{W}$ and $\boldsymbol{x} = \{x^n\} \in \boldsymbol{X}$. Then, by $(A4)$, $(A3)$ and (5.2), we have

$$\lim_{n\to\infty} B^n(t, w^n)x^n = B^\infty(t, w^\infty)x^\infty\,,$$

where $w^\infty = \lim w^n$ and $x^\infty = \lim x^n$. Therefore, we have

$$\begin{aligned}&\boldsymbol{B}(t, \boldsymbol{w})\boldsymbol{x} \in \boldsymbol{X}\,, \qquad \boldsymbol{B}(t, \boldsymbol{w}) \in \boldsymbol{B}(\boldsymbol{X}) \qquad \text{and} \\ &t \to \boldsymbol{B}(t, \boldsymbol{w}) \qquad \text{is strongly measurable in } \boldsymbol{B}(\boldsymbol{X})\,.\end{aligned}$$

Furthermore, for each $t \in [0, T_0]$, $\boldsymbol{w} = \{w^n\} \in \boldsymbol{W}$ and $\boldsymbol{z} = \{z^n\} \in \boldsymbol{W}$, we have

$$\begin{aligned}&\|\boldsymbol{B}(t, \boldsymbol{w})\|_{\boldsymbol{X}} \le \sup_n \|B^n(t, w^n)\| \le \lambda_1\,, \qquad \text{and} \\ &\|\boldsymbol{B}(t, \boldsymbol{w}) - \boldsymbol{B}(t, \boldsymbol{z})\|_{\boldsymbol{X}} \le \sup_n \|B^n(t, w^n) - B^n(t, z^n)\| \le \mu_3\|\boldsymbol{w} - \boldsymbol{z}\|_{\boldsymbol{Y}}\,.\end{aligned}$$

This proves condition $(A4)$ for $\boldsymbol{B}$.

Next, we prove

$$\mathrm{D}(\boldsymbol{S}\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{S}^{-1}) = \mathrm{D}(\boldsymbol{A}(t, \boldsymbol{w})) \qquad (\equiv \mathrm{D}(\boldsymbol{A}(t, \boldsymbol{w}) + \boldsymbol{B}(t, \boldsymbol{w})))\,, \qquad \text{and} \tag{5.7}$$

$$\boldsymbol{S}\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{S}^{-1} = \boldsymbol{A}(t, \boldsymbol{w}) + \boldsymbol{B}(t, \boldsymbol{w})\,. \tag{5.8}$$

For each $\boldsymbol{x} = \{x^n\} \in \mathrm{D}(\boldsymbol{A}(t, \boldsymbol{w}))$, we have $\boldsymbol{S}^{-1}\boldsymbol{x} \in \boldsymbol{Y} \subset \mathrm{D}(\boldsymbol{A}(t, \boldsymbol{w}))$ and

$$\begin{aligned}\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{S}^{-1}\boldsymbol{x} &= \{S^{-1}A^n(t, w^n)x^n + S^{-1}B^n(t, w^n)x^n\} \\ &= \boldsymbol{S}^{-1}\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{x} + \boldsymbol{S}^{-1}\boldsymbol{B}(t, \boldsymbol{w})\boldsymbol{x}\,,\end{aligned} \tag{5.9}$$

since $\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{x} \in \boldsymbol{X}$ and $\boldsymbol{B}(t, \boldsymbol{w})\boldsymbol{x} \in \boldsymbol{X}$. Therefore, we have

$$\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{S}^{-1}\boldsymbol{x} \in \boldsymbol{Y}\,.$$

This implies that $\boldsymbol{x} \in \mathrm{D}(\boldsymbol{S}\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{S}^{-1})$, and so we have

$$\mathrm{D}(\boldsymbol{A}(t, \boldsymbol{w})) \subset \mathrm{D}(\boldsymbol{S}\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{S}^{-1}) .$$

Conversely, let $\boldsymbol{x} = \{x^n\} \in \mathrm{D}(\boldsymbol{S}\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{S}^{-1})$. Then we have

$$\begin{aligned} &\boldsymbol{A}(t, \boldsymbol{w})\boldsymbol{S}^{-1}\boldsymbol{x} \in \boldsymbol{Y}, \qquad \text{and} \\ &x^n \in \mathrm{D}(SA^n(t, w^n)S^{-1}) = \mathrm{D}(A^n(t, w^n)) \qquad \text{for} \qquad n \in \boldsymbol{N}. \end{aligned}$$

Therefore, the limit

$$\begin{aligned} y_t &\equiv \lim_{n\to\infty} A^n(t, w^n)S^{-1}x^n \\ &= \lim_{n\to\infty} S^{-1}\{A^n(t, w^n) + B^n(t, w^n)\}x^n \end{aligned}$$

exists in Y for each $t \in [0, T_0]$. On the other hand, since

$$\lim_{n\to\infty} B^n(t, w^n)x^n = B^\infty(t, w^\infty)x^\infty$$

in X, we have

$$\lim_{n\to\infty} A^n(t, w^n)x^n = Sy_t - B^\infty(t, w^\infty)x^\infty \qquad \text{in } X .$$

This implies that $\boldsymbol{x} \in \mathrm{D}(\boldsymbol{A}(t, \boldsymbol{w}))$, and (5.7) is proved. (5.8) follows from (5.9).

For each $t \in [0, T_0]$ and $\boldsymbol{w} = \{w^n\} \in \boldsymbol{W}$, we put $\boldsymbol{f}(t, \boldsymbol{w}) \equiv \{f^n(t, w^n)\}$ and $w^\infty = \lim w^n$. Then since

$$\lim_{n\to\infty} \|f^n(t, w^n) - f^\infty(t, w^\infty)\|_Y = 0$$

by (5.3), we have $\boldsymbol{f}(t, \boldsymbol{w}) \in \boldsymbol{Y}$. Similarly, $t \to \boldsymbol{f}(t, \boldsymbol{w})$ is continuous in $\boldsymbol{X}$ by (5.4). Conditions $(f2)$ and $(f3)$ for $\boldsymbol{f}$ follow immediately.

Now, we have obtained another system $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{S}, \boldsymbol{W}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{f})$ which satisfies (X) through $(f3)$. We are ready to apply our results in Section 4 to the Cauchy problem $(\boldsymbol{CP})$ in $\boldsymbol{X}$, and we obtain the following:

THEOREM 5.2. *Let conditions* $(\boldsymbol{S})$ *and* $(\boldsymbol{C})$ *be satisfied. Then, for each* $\boldsymbol{a} = \{a^n\} \in \boldsymbol{W}$ *with* $a^\infty = \lim a^n$ $(\in W)$, *there is a* $T \in (0, T_0]$ *such that the following hold.*

(*i*) *The solution* u_Δ^n *of* (D^n) *exists in* W *for every partition* Δ *of* $[0, T]$ *and* $n \in \bar{N}$.

(*ii*) *There is a sequence* $\{\Delta(m)\}$ *of partitions of* $[0, T]$ *such that* $\lim |\Delta(m)| = 0$, *and that*

$$\lim_{m\to\infty} \sup_{t, n \in \bar{N}} \|u_{\Delta(m)}^n(t) - u^n(t)\|_Y = 0 \tag{5.10}$$

holds, where each $u^n(t)$ $(n \in \bar{N})$ *is the unique classical solution of* (CP^n). *Moreover, we have*

$$\lim_{n\to\infty} \sup_t \|u^n(t) - u^\infty(t)\|_Y = 0 . \tag{5.11}$$

(*iii*) *In addition to the assumptions above, assume that* (Bf) *is satisfied for every* (CP^n) ($n \in \bar{N}$), *and that* (5.2) *and* (5.3) *hold uniformly on* $[0, T_0]$. *Then,* (5.10) *holds for every sequence* $\{\Delta(m)\}$ *satisfying* $\lim |\Delta(m)| = 0$.

PROOF. Let $\boldsymbol{a} = \{a^n\} \in \boldsymbol{W}$ and $a^\infty = \lim a^n \in W$. Then, by Proposition 1.1, there is a $T \in (0, T_0]$ such that the solution $\boldsymbol{u}_\Delta(t) = \{u_\Delta^n(t)\}$ of

$$(\boldsymbol{D}) \qquad \begin{aligned} &\frac{\boldsymbol{u}_\Delta(t) - \boldsymbol{u}_k}{t - t_k} + \boldsymbol{A}_k \boldsymbol{u}_\Delta(t) = \boldsymbol{f}_k\,, \quad t_k \le t \le t_{k+1}\,, \quad 0 \le k \le N-1\,, \\ &\boldsymbol{u}_\Delta(0) = \boldsymbol{a}\,, \end{aligned}$$

exists in $\boldsymbol{W}$ for every partition $\Delta = \{t_k\}_{k=0}^N$ of $[0, T]$, where $\boldsymbol{u}_k = \boldsymbol{u}_\Delta(t_k)$, $\boldsymbol{A}_k = \boldsymbol{A}(t_k, \boldsymbol{u}_k)$ and $\boldsymbol{f}_k = \boldsymbol{f}(t_k, \boldsymbol{u}_k)$. Each component u_Δ^n ($n \in \boldsymbol{N}$) of $\boldsymbol{u}_\Delta$ is the solution of (D^n). Since $\boldsymbol{u}_\Delta(t)$ is continuous in $\boldsymbol{Y}$,

$$u_\Delta^\infty(t) \equiv \lim_{n\to\infty} u_\Delta^n(t) \tag{5.12}$$

exists in W. The limit is taken with respect to the Y-norm and the convergence holds uniformly on $[0, T]$. Taking the limit of the both sides of (D^n) ($n \in \boldsymbol{N}$), as $n \to \infty$, we see that u_Δ^∞ is the solution of (D^∞). This proves (*i*). Choose smaller $T \in (0, T_0]$ as in Lemma 4.1. Then, by Theorem 4.5, there is a sequence $\{\Delta(m)\}$ of partitions of $[0, T]$ such that $\lim |\Delta(m)| = 0$ and that

$$\lim_{m\to\infty} \sup_t \|\boldsymbol{u}_{\Delta(m)}(t) - \boldsymbol{u}(t)\|_{\boldsymbol{Y}} = \lim_{m\to\infty} \sup_{t, n \in \boldsymbol{N}} \|u_{\Delta(m)}^n(t) - u^n(t)\|_Y = 0\,, \tag{5.13}$$

holds, where $\boldsymbol{u}(t) = \{u^n(t)\}$ is the unique classical solution of ($\boldsymbol{CP}$). By (5.13), each component $u^n(t)$ ($n \in \boldsymbol{N}$) of $\boldsymbol{u}(t)$ is the unique classical solution of (CP^n). Since $\boldsymbol{u}(t)$ is continuous in $\boldsymbol{Y}$, $\{u^n(t)\}_{n \in \boldsymbol{N}}$ converges in the Y-norm uniformly on $[0, T]$. Put $u^\infty \equiv \lim u^n$. Then, taking the limit of the both sides of (CP^n) ($n \in \boldsymbol{N}$), as $n \to \infty$, we see that u^∞ is the unique classical solution of (CP^∞). Therefore, we obtain (5.11). Now, by (5.11) and (5.12), we have

$$\sup_{n \in \boldsymbol{N}} \|u_\Delta^n(t) - u^n(t)\|_Y = \sup_{n \in \bar{\boldsymbol{N}}} \|u_\Delta^n(t) - u^n(t)\|_Y\,.$$

Therefore, (5.10) follows from (5.13), and we have proved (*ii*). (*iii*) follows from (*ii*) of Theorem 4.5 and Lemma 5.1. Q.E.D.

COROLLARY 5.3. *Let conditions* ($\boldsymbol{S}$) *and* ($\boldsymbol{C}$) *be satisfied. Choose* $T \in (0, T_0]$ *as in Theorem* 5.2. *Then, there is a sequence* $\{\Delta(m)\}$ *of partitions of* $[0, T]$ *such that* $\lim |\Delta(m)| = 0$, *and that*

$$\begin{aligned} &\lim_{m,n\to\infty} \sup_t \|u_{\Delta(m)}^n(t) - u^\infty(t)\|_Y \\ &= \lim_{m\to\infty} \{\lim_{n\to\infty} \sup_t \|u_{\Delta(m)}^n(t) - u^\infty(t)\|_Y\} \\ &= \lim_{n\to\infty} \{\lim_{m\to\infty} \sup_t \|u_{\Delta(m)}^n(t) - u^\infty(t)\|_Y\} = 0\,. \end{aligned} \tag{5.14}$$

In addition to the assumptions above, assume that (Bf) is satisfied for every (CP^n) $(n \in \bar{N})$, *and that* (5.2) *and* (5.3) *hold uniformly on* $[0, T_0]$. *Then,* (5.14) *holds for every* $\{\Delta(m)\}$ *satisfying* $\lim |\Delta(m)| = 0$.

Corollary 5.3 follows immediately from (5.10), (5.11) and (5.12).

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