HIROSHIMA MATH. J.
19 (1989), 457–475

Structure of fractional factorial designs derived from two-symbol balanced arrays and their resolution

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(Received September 17, 1988)

1. Introduction

A fractional 2^m factorial (2^m-FF) design derived from a two-symbol orthogonal array (O-array) of strength $2p$ is said to be best in the sense that all factorial effects up to p-factor interactions are estimable uncorrelatedly and these estimates have the same variance under the situation in which all $(p+1)$-factor and higher order interactions are assumed to be negligible. However, such a design can be constructed only for quite restricted number of observations. The concept of an O-array was generalized by Chakravarti [2] to a balanced array (B-array). A 2^m-FF design derived from a two-symbol B-array of strength $2p$ has been investigated by several authors (see [3–15, 17–29, 32–36, 38, 39]).

It is known that there are so many designs having odd $(2p+1)$ resolution in the class of 2^m-FF designs derived from two-symbol B-arrays of strength $2p$ (see [3–12, 17, 18, 20]). Yamamoto, Shirakura and Kuwada [38, 39] introduced the concept of a triangular type multidimensional partially balanced (TMDPB) association scheme among the sets of factorial effects up to p-factor interactions of a 2^m factorial design. The MDPB association scheme was first introduced by Bose and Srivastava [1] as a generalization of the ordinary association scheme. Yamamoto, Shirakura and Kuwada [39] obtained an explicit expression for the characteristic polynomial of the information matrix of a balanced fractional 2^m factorial (2^m-BFF) design of resolution $2p+1$ by utilizing the algebraic structure of the TMDPB association scheme. This includes the results of a 2^m-BFF design of resolution V given by Srivastava and Chopra [27] as a special case. Yamamoto, Shirakura and Kuwada [38] also showed that a 2^m-BFF design of resolution $2p+1$ is equivalent to a design derived from a two-symbol B-array of strength $2p$ provided the information matrix is nonsingular.

It is also known that there are so many designs having even $(2p)$ resolution in the class of 2^m-FF designs derived from two-symbol B-arrays of strength $2p$ (see [19–22]). There are, however, so many designs which have neither odd nor even resolution in the class of those designs (see [13–15, 34–36]). Yamamoto and Hyodo [34, 35] introduced an extended concept of resolution, which includes both odd and even resolution as a special case. Recently, Hyodo and Yamamoto [15] have obtained some algebraic properties of information matrices of 2^m-FF designs derived from two-symbol simple arrays (S-arrays) which belong to a slightly restricted class of two-symbol B-arrays of strength $2p$. In the class of those designs, Yamamoto and Hyodo [34–36] and Hyodo and Yamamoto [13–15] have also obtained some designs having various type resolution, which includes both odd and even, by utilizing the algebraic structure of the information matrix.

In this paper, we shall consider a two-symbol B-array of strength $2p$, m constraints, index set $\{\mu_0^{(2p)}, \mu_1^{(2p)}, \ldots, \mu_{2p}^{(2p)}\}$ and frequency set $\{z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)}\}$ where $z_j^{(m)}$ are the number of row vectors of weight j in the array. Such an array is traditionally denoted as $\mathrm{BA}(N, m, 2, 2p)\{\mu_0^{(2p)}, \mu_1^{(2p)}, \ldots, \mu_{2p}^{(2p)}\}$, where N is the total number of assemblies. We, however, denote it here as $\mathrm{BA}(m, 2p; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$ since the characterization of the information matrix can be explicitly expressed by the frequencies $z_j^{(m)}$. It is well known that its array provides us a two-symbol B-array of strength $u(\leqq 2p)$, m constraints and index set $\{\mu_0^{(u)}, \mu_1^{(u)}, \ldots, \mu_u^{(u)}\}$ where $\mu_i^{(u)} = \sum_{h=0}^{2p} \binom{2p-u}{h-i} \mu_h^{(2p)}$ for $0 \leqq i \leqq u$. The indices $\mu_i^{(u)}$ are completely determined by given $z_j^{(m)}$ as will be seen in Lemma 1. Note that the usual boundary convention for the binomial coefficient $\binom{a}{b}$, i.e., $\binom{a}{b} = 0$ if and only if $b < 0$ or $0 \leqq a < b$, will be used throughout this paper. In Section 3, some algebraic properties of the irreducible matrix representations based on a design derived from a $\mathrm{BA}(m, 2p; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$ will be investigated through the fundamental formula representing a connection between $\mu_i^{(u)}$ and $z_j^{(m)}$ (see [30, 31]). Using their algebraic properties, some class of estimable linear parametric functions as well as resolution of a design derived from a $\mathrm{BA}(m, 2p; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$ will be obtained in Sections 4 and 5.

2. Preliminaries

Consider a 2^m-FF design with m factors $F_1, \ldots, F_m$, each at two levels 0 or 1. Assume that all $(p+1)$-factor and higher order interactions are to be negligible for a fixed integer p satisfying $1 \leqq p \leqq m/2$. The $\nu_p \times 1$ vector of factorial effects is denoted by

(1)
$$\begin{aligned} \underline{\theta}' &= (\theta_\phi; \theta_1, \ldots, \theta_m; \theta_{12}, \ldots, \theta_{m-1m}; \ldots; \theta_{1\ldots p}, \ldots, \theta_{m-p+1\ldots m}) \\ &= (\theta_\phi; \underline{\theta}_1'; \underline{\theta}_2'; \ldots; \underline{\theta}_p'), \end{aligned}$$

where $\nu_p = \sum_{k=0}^{p} \binom{m}{k}$, and θ_ϕ, θ_{t_1} and, in general, $\theta_{t_1 \cdots t_k}$ denote the general mean, the main effect of the factor F_{t_1} and the k-factor interaction of the factors $F_{t_1}, \ldots, F_{t_k}$, respectively. Here $\underline{\theta}_k$ denotes the $\binom{m}{k} \times 1$ vector of k-factor interactions ($k = 0$ and $k = 1$ stand for the general mean, i.e., $\underline{\theta}_0 = \theta_\phi$, and main effects, respectively). Let T be a (0, 1)-array of size $N \times m$ whose rows denote the assemblies under consideration. The linear model based on T is then given by

$$y_T = E_T \underline{\theta} + \underline{e}_T , \tag{2}$$

where $\underline{y}_T$, E_T and $\underline{e}_T$ denote a vector of N observations, the $N \times v_p$ design matrix whose elements are -1 or 1, and an $N \times 1$ error vector with $E[\underline{e}_T] = \underline{0}_N$ and $Cov[\underline{e}_T] = \sigma^2 I_N$, respectively. Here $\underline{0}_N$ and I_N are the $N \times 1$ vector with all zero and the identity matrix of order N, respectively. The normal equation for estimating $\underline{\theta}$ is given by

$$M_T \hat{\underline{\theta}} = E'_T \underline{y}_T , \tag{3}$$

where $M_T = E'_T E_T$ is the information matrix of order v_p.

Among the $p+1$ sets of factorial effects $\{\theta_\phi\}$, $\{\theta_{t_1}\}$, $\{\theta_{t_1 t_2}\}$, $\ldots$, $\{\theta_{t_1 \ldots t_p}\}$, a TMDPB association scheme is defined by introducing a natural relation of association such that $\theta_{t_1 \ldots t_u}$ and $\theta_{t'_1 \ldots t'_v}$ are the a-th associates if and only if

$$|\{t_1, \ldots, t_u\} \cap \{t'_1, \ldots, t'_v\}| = \min(u, v) - a , \tag{4}$$

where $|S|$ and $\min(u, v)$ denote the cardinality of a set S and the minimum of integers u and v, respectively.

It is well known that a TMDPB association algebra $\boldsymbol{R}$ generated by the $(p+1)(p+2)(2p+3)/6$ ordered association matrices $D_a^{(u,v)}$ ($0 \leqq a \leqq \min(u, v)$; $u, v = 0, 1, \ldots, p$) is semi-simple and completely reducible. It is decomposed into $p+1$ two-sided ideals $\boldsymbol{R}_b$ generated by $(p-b+1)^2$ ideal bases $\{D_b^{(u,v)\#} : u, v = 0, 1, \ldots, p\}$ for $b = 0, 1, \ldots, p$. The ideal $\boldsymbol{R}_b$ is isomorphic to the complete $(p-b+1) \times (p-b+1)$ matrix algebra with multiplicity $\binom{m}{b} - \binom{m}{b-1}$ ($= \phi_b$, say). The details of the TMDPB association scheme and its algebra can be seen in Yamamoto, Shirakura and Kuwada [38, 39] and Shirakura [20].

It is shown in Yamamoto, Shirakura and Kuwada [38, 39] that the information matrix M_T of a 2^m-FF design T derived from a two-symbol B-array of strength $2p$, m constraints and index set $\{\mu_0^{(2p)}, \mu_1^{(2p)}, \ldots, \mu_{2p}^{(2p)}\}$ belongs to the TMDPB association algebra $\boldsymbol{R}$ and is given as follows:

$$M_T = \sum_{u=0}^{p} \sum_{v=0}^{p} \sum_{a=0}^{\min(u,v)} \gamma_{|u-v|+2a} D_a^{(u,v)} \in \boldsymbol{R} , \tag{5}$$

or equivalently,

$$M_T = \sum_{b=0}^{p} \sum_{r=0}^{p-b} \sum_{s=0}^{p-b} k_b^{r,s} D_b^{(b+r,b+s)\#} \in \boldsymbol{R} , \tag{6}$$

where

$$\gamma_k = \sum_{h=0}^{2p} \sum_{q=0}^{k} (-1)^q \binom{k}{q} \binom{2p-k}{h-k+q} \mu_h^{(2p)} \qquad \text{for } k = 0, 1, \ldots, 2p , \tag{7}$$

(8) $$k_b^{r,s} = k_b^{s,r} = \sum_{a=0}^{b+r} \gamma_{s-r+2a} z_{ba}^{(b+r,b+s)} \qquad \text{for } 0 \leqq r \leqq s \leqq p-b\,;\ b = 0, 1, \ldots, p$$

and

(9) $$z_{ba}^{(b+r,b+s)} = \sum_{c=0}^{a} (-1)^{a-c} \binom{r}{c} \binom{b+r-c}{b+r-a} \binom{m-2b-r+c}{c} \left\{ \binom{m-2b-r}{s-r} \binom{s}{r} \right\}^{1/2} \Big/ \binom{s-r+c}{c}.$$

The irreducible matrix representation of M_T with respect to each ideal $\boldsymbol{R}_b$ is given by a $(p-b+1) \times (p-b+1)$ symmetric matrix K_b such that

(10) $$K_b = \begin{bmatrix} k_b^{0,0} & k_b^{0,1} & \cdots & k_b^{0,p-b} \\ k_b^{1,0} & k_b^{1,1} & \cdots & k_b^{1,p-b} \\ \vdots & \vdots & \cdots & \vdots \\ k_b^{p-b,0} & k_b^{p-b,1} & \cdots & k_b^{p-b,p-b} \end{bmatrix}.$$

3. Characterization of 2^m-FF designs

The following lemma is due to Yamamoto and Aratani [30, 31]:

LEMMA 1. *Let T be a* BA$(m, 2p; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$. *Then a connection between $\mu_i^{(u)}$ and $z_j^{(m)}$ is given by*

(11) $$\mu_i^{(u)} = \sum_{j=0}^{m} \binom{m-u}{j-i} \left\{ z_j^{(m)} \Big/ \binom{m}{j} \right\} \qquad \textit{for } 0 \leqq i \leqq u \leqq 2p \leqq m\,,$$

where $\mu_0^{(0)} = N$ for convenience.

Applying Lemma 1 into (7), we have the following lemma:

LEMMA 2. *For T being an array of Lemma* 1, *a connection between γ_k and $z_j^{(m)}$ is given by*

(12) $$\gamma_k = \sum_{j=0}^{m} \left\{ \sum_{q=0}^{k} (-1)^q \binom{k}{q} \binom{m-k}{m-j-q} \right\} \left\{ z_j^{(m)} \Big/ \binom{m}{j} \right\} \qquad \textit{for } k = 0, 1, \ldots, 2p\,.$$

THEOREM 3. *The irreducible matrix representation K_b of the information matrix M_T associated with T being an array of Lemma* 1 *with respect to the ideal $\boldsymbol{R}_b$ can be expressed as follows:*

(13) $$K_b = \sum_{j=b}^{m-b} \left\{ z_j^{(m)} \Big/ \binom{m}{j} \right\} \underline{k}_{bj} \underline{k}_{bj}' \qquad \textit{for } b = 0, 1, \ldots, p\,,$$

where $(r+1)$th element of the $(p-b+1)$-dimensional column vector $\underline{k}_{bj}$ is

(14)
$$k_{bj}^r = 2^b \left\{ \sum_{h=0}^{r} (-1)^h \binom{j-b}{r-h} \binom{m-b-j}{h} \right\} \left\{ \binom{m-2b}{j-b} \Big/ \binom{m-2b}{r} \right\}^{1/2} \qquad \textit{for } r = 0, 1, \ldots, p-b\,.$$

PROOF. Substituting (9) into (8), changing the order and region of the summation, and using (12), we have

(15)

$$\begin{aligned} k_b^{r,s} &= \sum_{c=0}^{r}\Big[(-1)^c\tbinom{r}{c}\tbinom{m-2b-r+c}{c}\Big\{\tbinom{m-2b-r}{s-r}\tbinom{s}{r}\Big\}^{1/2}\Big/\tbinom{s-r+c}{c}\Big]\cdot\sum_{a=0}^{b+r}(-1)^a\tbinom{b+r-c}{a-c}\gamma_{2a+s-r} \\ &= \sum_{c=0}^{r}\Big[\tbinom{r}{c}\tbinom{m-2b-r+c}{c}\Big\{\tbinom{m-2b-r}{s-r}\tbinom{s}{r}\Big\}^{1/2}\Big/\tbinom{s-r+c}{c}\Big]\cdot\sum_{a=0}^{b+r-c}(-1)^a\tbinom{b+r-c}{a}\gamma_{2(a+c)+s-r} \\ &= \sum_{c=0}^{r}\Big[\tbinom{r}{c}\tbinom{m-2b-r+c}{c}\Big\{\tbinom{m-2b-r}{s-r}\tbinom{s}{r}\Big\}^{1/2}\Big/\tbinom{s-r+c}{c}\Big] \\ &\quad\cdot\sum_{j=0}^{m}\Big\{\sum_{a=0}^{b+r-c}\sum_{q=0}^{2(a+c)+s-r}(-1)^{a+q}\tbinom{2(a+c)+s-r}{q}\tbinom{m-2(a+c)-s+r}{m-j-q}\tbinom{b+r-c}{a}\Big\} \\ &\quad\cdot\Big\{z_j^{(m)}\Big/\tbinom{m}{j}\Big\}\,. \end{aligned}$$

Putting $x = b + r - c$, $y = s - r + 2c$, $z = m - 2b - s - r$ and $u = m - j$ in Lemma 1 of Hyodo and Yamamoto [15], the following yields

$$\text{(16)}\qquad \begin{aligned} &\sum_{a=0}^{b+r-c}\sum_{q=0}^{2(a+c)+s-r}(-1)^{a+q}\tbinom{2(a+c)+s-r}{q}\tbinom{m-2(a+c)-s+r}{m-j-q}\tbinom{b+r-c}{a} \\ &\quad = 2^{2(b+r-c)}\sum_{h=0}^{s-r+2c}(-1)^h\tbinom{s-r+2c}{h}\tbinom{m-2b-s-r}{j-b-s-c+h}\,. \end{aligned}$$

Since $\binom{m-2b-s-r}{j-b-s-c+h} = 0$ for $j < b$ or $j > m - b$, it follows from (15) and (16) that

$$\text{(17)}\qquad \begin{aligned} k_b^{r,s} &= \sum_{j=b}^{m-b}\Big[\sum_{c=0}^{r}\sum_{h=0}^{s-r+2c}(-1)^h\tbinom{s-r+2c}{h}\tbinom{m-2b-s-r}{j-b-s-c+h}\tbinom{r}{c}\tbinom{m-2b-r+c}{c} \\ &\quad\cdot\Big\{\tbinom{m-2b-r}{s-r}\tbinom{s}{r}\Big\}^{1/2}2^{2(b+r-c)}\Big/\tbinom{s-r+c}{c}\Big]\Big\{z_j^{(m)}\Big/\tbinom{m}{j}\Big\}\,. \end{aligned}$$

The term in [ ] of (17) is identical with $k_{bj}^{r,s}$ in the formula (8) of Hyodo and Yamamoto [15]. Thus we have

$$\text{(18)}\qquad k_b^{r,s} = \sum_{j=b}^{m-b} k_{bj}^{r,s}\Big\{z_j^{(m)}\Big/\tbinom{m}{j}\Big\}\,.$$

This implies (13), since $k_{bj}^{r,s} = k_{bj}^r k_{bj}^s$ holds for $0 \leqq r \leqq s \leqq p - b$, as has been given in Theorem 3 of Hyodo and Yamamoto [15]. This completes the proof.

Note that (14) is identical with the formula in Theorem 3 of Hyodo and Yamamoto [15].

REMARK 1. It is well known that a two-symbol S-array with parameters $(m; \lambda_0, \lambda_1, \dots, \lambda_m)$ belongs to a slightly restricted class of a BA$(m, 2p; z_0^{(m)}, z_1^{(m)}, \dots, z_m^{(m)})$ (see [16, 37]). Since $z_j^{(m)} = \binom{m}{j}\lambda_j$ $(j = 0, 1, \dots, m)$ hold for such an S-array, (13) in Theorem 3 is a generalization of the formula (15) of Hyodo and Yamamoto [15].

REMARK 2. Let T be a BA$(m, 2p; z_0^{(m)}, z_1^{(m)}, \dots, z_m^{(m)})$. Then the information matrix M_T is also given by

$$\text{(19)}\qquad \begin{aligned} M_T &= \sum_{b=0}^{p}\sum_{r=0}^{p-b}\sum_{s=0}^{p-b}\Big[\sum_{j=b}^{m-b}\Big\{z_j^{(m)}\Big/\tbinom{m}{j}\Big\}(k_{bj}^r k_{bj}^s)\Big]D_b^{(b+r,b+s)\#} \\ &= \sum_{j=0}^{m}\Big\{z_j^{(m)}\Big/\tbinom{m}{j}\Big\}M_j \in \boldsymbol{R}\,, \end{aligned}$$

where the $v_p \times v_p$ matrix M_j is

(20) $$M_j = \sum_{b=0}^{\min(j,m-j,p)} \sum_{r=0}^{p-b} \sum_{s=0}^{p-b} (k_{bj}^r k_{bj}^s) D_b^{(b+r,b+s)\#} \in \boldsymbol{R},$$

which is identical with the information matrix of a 2^m-FF design T_j derived from an atomic array of weight j in Hyodo and Yamamoto [15].

By use of Theorem 2 and Lemma 4 of Hyodo and Yamamoto [15], we can obtain the following theorems:

THEOREM 4. *Let T be a* BA$(m, 2p; z_0^{(m)}, z_1^{(m)}, \dots, z_m^{(m)})$. *Then the determinant of the irreducible matrix representation K_b of M_T is represented by*

(21) $$|K_b| = \sum_{b \leqq j_0 < j_1 < \dots < j_{p-b} \leqq m-b} c^*(j_0, j_1, \dots, j_{p-b}) \cdot z_{j_0}^{(m)} z_{j_1}^{(m)} \dots z_{j_{p-b}}^{(m)}, \qquad \text{for } b = 0, 1, \dots, p,$$

where

(22) $$c^*(j_0, j_1, \dots, j_{p-b}) = 2^{(p+b)(p-b+1)} \prod_{s=0}^{p-b} \left[\binom{m-2b}{j_s-b} \Big/ \left\{\binom{m-2b}{s}\binom{m}{j_s}(s!)^2\right\}\right] \cdot \prod_{0 \leqq k < h \leqq p-b} (j_h - j_k)^2 > 0.$$

This theorem implies that the matrix K_b is positive definite if and only if none of the $p - b + 1$ frequencies $z_{j_0}^{(m)}, z_{j_1}^{(m)}, \dots, z_{j_{p-b}}^{(m)}$ is zero for some choice of $\{j_0, j_1, \dots, j_{p-b}\} \subset \{b, b+1, \dots, m-b\}$. Note that $c^*(j_0, j_1, \dots, j_{p-b}) = c^*(m - j_{p-b}, \dots, m - j_1, m - j_0)$.

THEOREM 5. *The rank of K_b based on T being an array of Theorem* 4 *is given by*

(23) $$\text{rank}\,[K_b] = \min\,(w(z_b^{(m)}, z_{b+1}^{(m)}, \dots, z_{m-b}^{(m)}), p - b + 1) \qquad \text{for } b = 0, 1, \dots, p,$$

where $w(\underline{x}')$ denotes the number of nonzero elements of a row vector $\underline{x}'$.

REMARK 3. The matrices K_b have the following properties:

(i) $0 \leqq \text{rank}\,[K_{b+1}] \leqq \text{rank}\,[K_b] \leqq \min\,(\text{rank}\,[K_{b+1}] + 2, p - b + 1)$ for $b = 0, 1, \dots, p - 1$.

(ii) If $\text{rank}\,[K_b] = r$, then the first r rows in K_b are always linearly independent.

4. Estimable linear parametric functions in 2^m-FF designs

Consider a 2^m-FF design T derived from a BA$(m, 2p; z_0^{(m)}, z_1^{(m)}, \dots, z_m^{(m)})$. Let $A_a^{(u,v)}(=(A_a^{(v,u)})')$ $(0 \leqq a \leqq u \leqq v \leqq p)$ be the $\binom{m}{u} \times \binom{m}{v}$ local association matrix of the TMDPB association scheme (see [38]). Further let $A_b^{(u,v)\#}(=(A_b^{(v,u)\#})')$

$(0 \leqq b \leqq u \leqq v \leqq p)$ be the $\binom{m}{u} \times \binom{m}{v}$ matrix which is linearly linked with $A_a^{(u,v)}$ as follows (see [24, 39]):

$$A_a^{(u,v)} = \sum_{b=0}^{u} z_{ba}^{(u,v)} A_b^{(u,v)\#} \qquad \text{for } 0 \leqq a \leqq u \leqq v \leqq p \tag{24}$$

and

$$A_b^{(u,v)\#} = \sum_{a=0}^{u} z_{(u,v)}^{ba} A_a^{(u,v)} \qquad \text{for } 0 \leqq b \leqq u \leqq v \leqq p\,, \tag{25}$$

where

$$z_{(u,v)}^{ba} = \phi_b z_{ba}^{(u,v)} / \{\tbinom{m}{u}\tbinom{u}{a}\tbinom{m-u}{v-u+a}\}\,. \tag{26}$$

The matrices $A_b^{(u,v)\#}$ have the following properties:

$$A_0^{(u,v)\#} = \{\tbinom{m}{u}\tbinom{m}{v}\}^{-1/2} G_{\binom{m}{u} \times \binom{m}{v}}\,, \tag{27}$$

$$\sum_{b=0}^{u} A_b^{(u,u)\#} = I_{\binom{m}{u}}\,, \tag{28}$$

$$A_b^{(u,w)\#} A_c^{(w,v)\#} = \delta_{bc} A_b^{(u,v)\#} \tag{29}$$

and

$$\operatorname{rank}\,[A_b^{(u,v)\#}] = \phi_b\,, \tag{30}$$

where δ_{ab} and $G_{p \times q}$ denote Kronecker's delta and the $p \times q$ matrix with all unity, respectively. It follows from (28) that the vector of u-factor interactions is given by

$$\underline{\theta}_u = \sum_{b=0}^{u} A_b^{(u,u)\#} \underline{\theta}_u\,. \tag{31}$$

Note that (i) every element of the vector of linear parametric functions $A_0^{(u,u)\#}\underline{\theta}_u$ $(0 \leqq u \leqq p)$ represents the average of the effects of u-factor interactions, (ii) the elements of $A_b^{(u,u)\#}\underline{\theta}_u$ $(b \neq 0;\ 1 \leqq u \leqq p)$ represent the contrasts between these effects, (iii) any two contrasts, one belonging to $A_b^{(u,u)\#}\underline{\theta}_u$ and the other to $A_c^{(u,u)\#}\underline{\theta}_u$ $(b \neq c;\ 2 \leqq u \leqq p)$, are orthogonal, and (iv) there are ϕ_b linearly independent functions of $\underline{\theta}_u$ in $A_b^{(u,u)\#}\underline{\theta}_u$ $(0 \leqq b \leqq u;\ 0 \leqq u \leqq p)$. Applying the arguments used in Theorems 8, 9 and 10 of Hyodo and Yamamoto [15], we get the following theorems.

THEOREM 6. *Every estimable linear parametric function of* $\underline{\theta}$ *in* T *being a* BA$(m, 2p; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$ *is given by*

$$\psi = \sum_{b=0}^{p} \sum_{j=b}^{m-b} z_j^{(m)} \underline{x}_{bj}' \{ \sum_{s=b}^{p} (k_{bj}^{s-b} A_b^{(b,s)\#}) A_b^{(s,s)\#} \underline{\theta}_s \} \tag{32}$$
$$\text{for an arbitrary } \underline{x}_{bj} \in R^{\binom{m}{b}};\ b \leqq j \leqq m-b,\ b = 0, 1, \ldots, p\,.$$

There are $\sum_{b=0}^{p} \phi_b \min\,(w(z_b^{(m)}, z_{b+1}^{(m)}, \ldots, z_{m-b}^{(m)}), p - b + 1)$ *linearly independent functions of* $\underline{\theta}$ *in* ψ.

THEOREM 7. *For T being an array of Theorem 6, the vector of linear parametric functions $A_b^{(s,s)\#}\underline{\theta}_s$ is estimable if and only if*

$$\text{rank}\,[K_b^*] = \text{rank}\,[K_b^* : \underline{f}_b^{(s)}]\,, \tag{33}$$

where $K_b^* = [z_b^{(m)}\underline{k}_{bb}, z_{b+1}^{(m)}\underline{k}_{bb+1}, \ldots, z_{m-b}^{(m)}\underline{k}_{bm-b}]$ and $\underline{f}_b^{(s)}$ denotes the $(p-b+1)\times 1$ canonical basis vector whose $(s-b+1)$th element is unity.

REMARK 4. It can be also shown that the vector of linear parametric functions $\sum_{s=b}^{p}(a_s A_b^{(b,s)\#})A_b^{(s,s)\#}\underline{\theta}_s$ is estimable for a given constant vector $\underline{a}_b = (a_b, a_{b+1}, \ldots, a_p)'$ if and only if

$$\text{rank}\,[K_b^*] = \text{rank}\,[K_b^* : \underline{a}_b]\,. \tag{34}$$

THEOREM 8. *For T being an array of Theorem 6, the vector of s-factor interactions $\underline{\theta}_s$ is estimable if and only if*

$$\text{rank}\,[K_b^*] = \text{rank}\,[K_b^* : \underline{f}_b^{(s)}] \qquad \textit{for all } b \in \{0, 1, \ldots, s\}\,. \tag{35}$$

REMARK 5. The vector of h-factor interactions $\underline{\theta}_h$ is not estimable if and only if

$$\text{rank}\,[K_b^*] \neq \text{rank}\,[K_b^* : \underline{f}_b^{(h)}] \qquad \text{for some } b \in \{0, 1, \ldots, h\}\,. \tag{36}$$

To illustrate the usefulness of the results in this section we present an example here.

EXAMPLE 1. Consider a 2^m-FF design T derived from a BA$(m, 2p; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$ for the cases of $m = 2p$, $2p+1$ and $2p+2$ under the assumption that all $(p+1)$-factor and higher order interactions are to be negligible. For $p = 1$, 2 and 3, some class of estimable linear parametric functions in T is given in Tables 1.1, 1.2 and 1.3, respectively. Every estimable linear parametric function in T can be also obtained by linear combinations of each component of the estimable class.

TABLE 1.1. The case $p = 1$, i.e., $\underline{\theta}' = (\theta_\phi; \underline{\theta}_1')$.

m	Conditions on BA$(m, 2; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$	The class of estimable linear parametric functions
2	$z_i^{(2)} > 0\ (i = 0, 2),\ z_1^{(2)} = 0$	$\{\theta_\phi, A_0^{(1,1)\#}\underline{\theta}_1\}$
	$z_1^{(2)} > 0,\ z_i^{(2)} = 0\ (i = 0, 2)$	$\{\theta_\phi, A_1^{(1,1)\#}\underline{\theta}_1\}$
3	$z_i^{(3)} > 0\ (i = 0, 3),\ z_j^{(3)} = 0\ (j = 1, 2)$	$\{\theta_\phi, A_0^{(1,1)\#}\underline{\theta}_1\}$
4	$z_i^{(4)} > 0\ (i = 0, 4),\ z_j^{(4)} = 0\ (j = 1, 2, 3)$	$\{\theta_\phi, A_0^{(1,1)\#}\underline{\theta}_1\}$
	$z_2^{(4)} > 0,\ z_i^{(4)} = 0\ (i = 0, 1, 3, 4)$	$\{\theta_\phi, A_1^{(1,1)\#}\underline{\theta}_1\}$

TABLE 1.2. The case $p = 2$, i.e., $\underline{\theta}' = (\theta_\phi; \underline{\theta}_1'; \underline{\theta}_2')$.

m	Conditions on $BA(m, 4; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$	The class of estimable linear parametric functions
4	$z_i^{(4)} > 0\ (i = 1, 3)$, $z_j^{(4)} = 0\ (j = 0, 2, 4)$	$\{\theta_\phi, \underline{\theta}_1, A_1^{(2,2)\#}\underline{\theta}_2\}$
	$z_i^{(4)} > 0\ (i = 1, 3)$, $z_0^{(4)} + z_4^{(4)} > 0$, $z_2^{(4)} = 0$	$\{\theta_\phi, \underline{\theta}_1, A_0^{(2,2)\#}\underline{\theta}_2, A_1^{(2,2)\#}\underline{\theta}_2\}$
	$z_i^{(4)} > 0\ (i = 0, 2, 4)$, $z_j^{(4)} = 0\ (j = 1, 3)$	$\{\theta_\phi, \underline{\theta}_1, A_0^{(2,2)\#}\underline{\theta}_2, A_2^{(2,2)\#}\underline{\theta}_2\}$
5	$z_i^{(5)} > 0\ (i = 1, 4)$, $z_j^{(5)} = 0\ (j = 0, 2, 3, 5)$	$\{\underline{\theta}_1, 5^{1/2}\theta_\phi + (2^{1/2}A_0^{(0,2)\#})A_0^{(2,2)\#}\underline{\theta}_2, A_1^{(2,2)\#}\underline{\theta}_2\}$
	$z_i^{(5)} > 0\ (i = 2, 3)$, $z_j^{(5)} = 0\ (j = 0, 1, 4, 5)$	$\{\underline{\theta}_1, 5^{1/2}\theta_\phi - (2^{1/2}A_0^{(0,2)\#})A_0^{(2,2)\#}\underline{\theta}_2, A_1^{(2,2)\#}\underline{\theta}_2, A_2^{(2,2)\#}\underline{\theta}_2\}$
	$z_i^{(5)} > 0\ (i = 1, 4)$, $z_0^{(5)} + z_5^{(5)} > 0$, $z_j^{(5)} = 0\ (j = 2, 3)$	$\{\theta_\phi, \underline{\theta}_1, A_0^{(2,2)\#}\underline{\theta}_2, A_1^{(2,2)\#}\underline{\theta}_2\}$
6	$z_i^{(6)} > 0\ (i = 1, 5)$, $z_j^{(6)} = 0\ (j = 0, 2, 3, 4, 6)$	$\{\underline{\theta}_1, 3^{1/2}\theta_\phi + (5^{1/2}A_0^{(0,2)\#})A_0^{(2,2)\#}\underline{\theta}_2, A_1^{(2,2)\#}\underline{\theta}_2\}$
	$z_i^{(6)} > 0\ (i = 2, 4)$, $z_j^{(6)} = 0\ (j = 0, 1, 3, 5, 6)$	$\{\underline{\theta}_1, 15^{1/2}\theta_\phi - (A_0^{(0,2)\#})A_0^{(2,2)\#}\underline{\theta}_2, A_1^{(2,2)\#}\underline{\theta}_2, A_2^{(2,2)\#}\underline{\theta}_2\}$
	$z_i^{(6)} > 0\ (i = 1, 5)$, $z_0^{(6)} + z_6^{(6)} > 0$, $z_j^{(6)} = 0\ (j = 2, 3, 4)$	$\{\theta_\phi, \underline{\theta}_1, A_0^{(2,2)\#}\underline{\theta}_2, A_1^{(2,2)\#}\underline{\theta}_2\}$
	$z_i^{(6)} > 0\ (i = 0, 3, 6)$, $z_j^{(6)} = 0\ (j = 1, 2, 4, 5)$	$\{\theta_\phi, \underline{\theta}_1, A_0^{(2,2)\#}\underline{\theta}_2, A_2^{(2,2)\#}\underline{\theta}_2\}$

TABLE 1.3. The case $p = 3$, i.e., $\underline{\theta}' = (\theta_\phi; \underline{\theta}_1'; \underline{\theta}_2'; \underline{\theta}_3')$.

m	Conditions on $BA(m, 6; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$	The class of estimable linear parametric functions
6	$z_i^{(6)} > 0\ (i = 1, 4, 5)$, $z_j^{(6)} = 0\ (j = 0, 2, 3, 6)$	$\{\underline{\theta}_1, 3 \cdot 5^{1/2}\theta_\phi + (14 \cdot 3^{1/2}A_0^{(0,2)\#})A_0^{(2,2)\#}\underline{\theta}_2 + (9A_0^{(0,3)\#})A_0^{(3,3)\#}\underline{\theta}_3, 3\theta_\phi - (5^{1/2}A_0^{(0,3)\#})A_0^{(3,3)\#}\underline{\theta}_3. A_1^{(3,3)\#}\underline{\theta}_3, A_1^{(2,2)\#}\underline{\theta}_2, A_2^{(2,2)\#}\underline{\theta}_2 + (2^{1/2}A_2^{(2,3)\#})A_2^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(6)} > 0\ (i = 1, 4, 5)$, $z_0^{(6)} + z_6^{(6)} > 0$, $z_j^{(6)} = 0\ (j = 2, 3)$	$\{\theta_\phi, \underline{\theta}_1, A_0^{(2,2)\#}\underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3, A_1^{(2,2)\#}\underline{\theta}_2, A_1^{(3,3)\#}\underline{\theta}_3, A_2^{(2,2)\#}\underline{\theta}_2 + (2^{1/2}A_2^{(2,3)\#})A_2^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(6)} > 0\ (i = 2, 3, 4)$, $z_j^{(6)} = 0\ (j = 0, 1, 5, 6)$	$\{\theta_\phi, \underline{\theta}_2, (5 \cdot 6^{1/2}A_0^{(0,1)\#})A_0^{(1,1)\#}\underline{\theta}_1 - (6 \cdot 5^{1/2}A_0^{(0,3)\#})A_0^{(3,3)\#}\underline{\theta}_3, A_1^{(1,1)\#}\underline{\theta}_1, A_1^{(3,3)\#}\underline{\theta}_3, A_2^{(3,3)\#}\underline{\theta}_3, A_3^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(6)} > 0\ (i = 0, 2, 4, 6)$, $z_j^{(6)} = 0\ (j = 1, 3, 5)$	$\{\theta_\phi, \underline{\theta}_1, \underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3, A_2^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(6)} > 0\ (i = 2, 4, 5)$, $z_0^{(6)} + z_1^{(6)} + z_6^{(6)} > 0$, $z_3^{(6)} = 0$	$\{\theta_\phi, \underline{\theta}_1, \underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3, A_1^{(3,3)\#}\underline{\theta}_3, A_2^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(6)} > 0\ (i = 1, 3, 5)$, $z_j^{(6)} = 0\ (j = 0, 2, 4, 6)$	$\{\theta_\phi, \underline{\theta}_1, \underline{\theta}_2, A_1^{(3,3)\#}\underline{\theta}_3, A_3^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(6)} > 0\ (i = 1, 3, 5)$, $z_0^{(6)} + z_6^{(6)} > 0$, $z_j^{(6)} = 0\ (j = 2, 4)$	$\{\theta_\phi, \underline{\theta}_1, \underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3, A_1^{(3,3)\#}\underline{\theta}_3, A_3^{(3,3)\#}\underline{\theta}_3\}$

TABLE 1.3. (continued)

m	Conditions on BA$(m, 6; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$	The class of estimable linear parametric functions
7	$z_i^{(7)} > 0\ (i = 1, 5, 6),\ z_0^{(7)} + z_7^{(7)} > 0,$ $z_j^{(7)} = 0\ (j = 2, 3, 4)$	$\{\theta_\phi, \underline{\theta}_1, A_0^{(2,2)\#}\underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3\}$ $A_1^{(2,2)\#}\underline{\theta}_2, A_1^{(3,3)\#}\underline{\theta}_3, A_2^{(2,2)\#}\underline{\theta}_2$ $+ (3^{1/2}A_2^{(2,3)\#})A_2^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(7)} > 0\ (i = 0, 1, 4, 7),$ $z_j^{(7)} = 0\ (j = 2, 3, 5, 6)$	$\{\theta_\phi, \underline{\theta}_1, A_0^{(2,2)\#}\underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3,$ $A_1^{(2,2)\#}\underline{\theta}_2 - (2^{1/2}A_1^{(2,3)\#})A_1^{(3,3)\#}\underline{\theta}_3,$ $3^{1/2}A_2^{(2,2)\#}\underline{\theta}_2 + (A_2^{(2,3)\#})A_2^{(3,3)\#}\underline{\theta}_3,$ $A_3^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(7)} > 0\ (i = 1, 4, 6),\ z_0^{(7)} + z_7^{(7)} > 0,$ $z_j^{(7)} = 0\ (j = 2, 3, 5)$	$\{\theta_\phi, \underline{\theta}_1, A_0^{(2,2)\#}\underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3$ $A_1^{(2,2)\#}\underline{\theta}_2, A_1^{(3,3)\#}\underline{\theta}_3, 3^{1/2}A_2^{(2,2)\#}\underline{\theta}_2$ $+ (A_2^{(2,3)\#})A_2^{(3,3)\#}\underline{\theta}_3, A_3^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(7)} > 0\ (i = 0, 2, 5, 7),$ $z_j^{(7)} = 0\ (j = 1, 3, 4, 6)$	$\{\theta_\phi, \underline{\theta}_2, A_0^{(1,1)\#}\underline{\theta}_1, A_0^{(3,3)\#}\underline{\theta}_3,$ $5^{1/2}A_1^{(1,1)\#}\underline{\theta}_1 + 2^{1/2}(A_1^{(1,3)\#})A_1^{(3,3)\#}\underline{\theta}_3,$ $A_2^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(7)} > 0\ (i = 0, 3, 4, 7),$ $z_j^{(7)} = 0\ (j = 1, 2, 5, 6)$	$\{\theta_\phi, \underline{\theta}_2, A_0^{(1,1)\#}\underline{\theta}_1, A_0^{(3,3)\#}\underline{\theta}_3,$ $5^{1/2}A_1^{(1,1)\#}\underline{\theta}_1 - 2^{1/2}(A_1^{(1,3)\#})A_1^{(3,3)\#}\underline{\theta}_3,$ $A_2^{(3,3)\#}\underline{\theta}_3, A_1^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(7)} > 0\ (i = 2, 5, 6),$ $z_0^{(7)} + z_1^{(7)} + z_7^{(7)} > 0,\ z_j^{(7)} = 0\ (j = 3, 4)$	$\{\theta_\phi, \underline{\theta}_1, \underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3, A_1^{(3,3)\#}\underline{\theta}_3, A_2^{(3,3)\#}\underline{\theta}_3\}$
8	$z_i^{(8)} > 0\ (i = 1, 5, 6),$ $z_j^{(8)} = 0\ (j = 0, 2, 3, 4, 7, 8)$	$\{\underline{\theta}_2, \theta_\phi + (2\cdot 2^{1/2}A_0^{(0,1)\#})A_0^{(1,1)\#}\underline{\theta}_1,$ $14\cdot 2^{1/2}\theta_\phi - (7A_0^{(0,1)\#})A_0^{(1,1)\#}\underline{\theta}_1$ $-(9\cdot 7^{1/2}A_0^{(0,3)\#})A_0^{(3,3)\#}\underline{\theta}_3, A_1^{(1,1)\#}\underline{\theta}_1,$ $A_1^{(3,3)\#}\underline{\theta}_3, A_2^{(3,3)\#}\underline{\theta}_3, A_3^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(8)} > 0\ (i = 1, 6, 7),\ z_0^{(8)} + z_8^{(8)} > 0,$ $z_j^{(8)} = 0\ (j = 2, 3, 4, 5)$	$\{\theta_\phi, \underline{\theta}_1, A_0^{(2,2)\#}\underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3,$ $A_1^{(2,2)\#}\underline{\theta}_2, A_1^{(3,3)\#}\underline{\theta}_3, A_2^{(2,2)\#}\underline{\theta}_2$ $+ (2A_2^{(2,3)\#})A_2^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(8)} > 0\ (i = 1, 5, 7),\ z_0^{(8)} + z_8^{(8)} > 0,$ $z_j^{(8)} = 0\ (j = 2, 3, 4, 6)$	$\{\theta_\phi, \underline{\theta}_1, A_0^{(2,2)\#}\underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3,$ $A_1^{(2,2)\#}\underline{\theta}_2, A_1^{(3,3)\#}\underline{\theta}_3, A_2^{(2,2)\#}\underline{\theta}_2$ $+ (A_2^{(2,3)\#})A_2^{(3,3)\#}\underline{\theta}_3, A_3^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(8)} > 0\ (i = 0, 2, 6, 8),$ $z_j^{(8)} = 0\ (j = 1, 3, 4, 5, 7)$	$\{\theta_\phi, \underline{\theta}_2, A_0^{(1,1)\#}\underline{\theta}_1, A_0^{(3,3)\#}\underline{\theta}_3,$ $3^{1/2}A_1^{(1,1)\#}\underline{\theta}_3 + (5^{1/2}A_1^{(1,3)\#})A_1^{(3,3)\#}\underline{\theta}_3,$ $A_2^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(8)} > 0\ (i = 0, 3, 5, 8),$ $z_j^{(8)} = 0\ (j = 1, 2, 4, 6, 7)$	$\{\theta_\phi, \underline{\theta}_2, A_0^{(1,1)\#}\underline{\theta}_1, A_0^{(3,3)\#}\underline{\theta}_3,$ $15^{1/2}A_1^{(1,1)\#}\underline{\theta}_1 - (A_1^{(1,3)\#})A_1^{(3,3)\#}\underline{\theta}_3,$ $A_2^{(3,3)\#}\underline{\theta}_3, A_3^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(8)} > 0\ (i = 1, 4, 7),$ $z_j^{(8)} = 0\ (j = 0, 2, 3, 5, 6, 8)$	$\{\theta_\phi, \underline{\theta}_2, (3A_0^{(0,1)\#})A_0^{(1,1)\#}\underline{\theta}_1$ $+ (7^{1/2}A_0^{(0,3)\#})A_0^{(3,3)\#}\underline{\theta}_3, A_1^{(1,1)\#}\underline{\theta}_1,$ $A_1^{(3,3)\#}\underline{\theta}_3, A_3^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(8)} > 0\ (i = 2, 4, 6),$ $z_j^{(8)} = 0\ (j = 0, 1, 3, 5, 7, 8)$	$\{\theta_\phi, \underline{\theta}_2, (7A_0^{(0,1)\#})A_0^{(1,1)\#}\underline{\theta}_1$ $- (7^{1/2}A_0^{(0,3)\#})A_0^{(3,3)\#}\underline{\theta}_3, A_1^{(1,1)\#}\underline{\theta}_1,$ $A_1^{(3,3)\#}\underline{\theta}_3, A_2^{(3,3)\#}\underline{\theta}_3, A_3^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(8)} > 0\ (i = 3, 4, 5),$ $z_j^{(8)} = 0\ (j = 0, 1, 2, 6, 7, 8)$	$\{\theta_\phi, \underline{\theta}_2, (7A_0^{(0,1)\#})A_0^{(1,1)\#}\underline{\theta}_1$ $+ (3\cdot 7^{1/2}A_0^{(0,3)\#})A_0^{(3,3)\#}\underline{\theta}_3, A_1^{(1,1)\#}\underline{\theta}_1,$ $A_1^{(3,3)\#}\underline{\theta}_3, A_2^{(3,3)\#}\underline{\theta}_3, A_3^{(3,3)\#}\underline{\theta}_3\}$

TABLE 1.3. (continued)

m	Conditions on $BA(m, 6; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$	The class of estimable linear parametric functions
	$z_i^{(8)} > 0$ $(i = 2, 6, 7)$, $z_0^{(8)} + z_1^{(8)} + z_8^{(8)} > 0$, $z_j^{(8)} = 0$ $(j = 3, 4, 5)$	$\{\theta_\phi, \underline{\theta}_1, \underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3, A_1^{(3,3)\#}\underline{\theta}_3, A_2^{(3,3)\#}\underline{\theta}_3\}$
	$z_i^{(8)} > 0$ $(i = 1, 4, 7)$, $z_0^{(8)} + z_8^{(8)} > 0$, $z_j^{(8)} = 0$ $(j = 2, 3, 5, 6)$	$\{\theta_\phi, \underline{\theta}_1, \underline{\theta}_2, A_0^{(3,3)\#}\underline{\theta}_3, A_1^{(3,3)\#}\underline{\theta}_3, A_3^{(3,3)\#}\underline{\theta}_3\}$

5. Resolution of 2^m-FF designs

An extended concept of resolution has been defined by Yamamoto and Hyodo [34, 35] as follows:

DEFINITION 1. Let $P_p = \{0, 1, \ldots, p\}$ and $S \subset P_p$. Then a 2^m-FF design is said to be of resolution $R(S|P_p)$ if

(37) (i) $D_0^{(s,s)}\underline{\theta}$, i.e., a vector of s-factor interactions $\underline{\theta}_s$, is estimable for every $s \in S$

and

(38) (ii) $D_0^{(h,h)}\underline{\theta}$, i.e., a vector of h-factor interactions $\underline{\theta}_h$, is not estimable for every $h \in P_p - S$.

Note that resolution $R(P_p|P_p)$ and $R(P_p - \{p\}|P_p)$ (or $R(P_p - \{0, p\}|P_p)$) correspond, respectively, to resolution $2p + 1$ and $2p$.

DEFINITION 2. A 2^m-FF design of resolution $R(S|P_p)$ is said to be balanced and denoted by 2^m-BFF design of resolution $R(S|P_p)$ if the covariance matrix of the BLUE of $\sum_{s \in S} D_0^{(s,s)}\underline{\theta}$ is invariant under any permutation of m factors.

Now we consider a 2^m-FF design T derived from a $BA(m, 2p; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$. The following theorems, which can be obtained by the arguments similar to Theorems 11 and 12 of Hyodo and Yamamoto [15], are useful for classifying the designs by the structure of resolution.

THEOREM 9. *An array T is a 2^m-BFF design of resolution $R(S|P_p)$ if and only if T satisfies the following conditions*:

(39) (i) $\operatorname{rank}[K_b^*] = \operatorname{rank}[K_b^* : \underline{f}_b^{(s)}]$ *for every* $b \in \{0, 1, \ldots, s\}$ $(s \in S)$

and

(40) (ii) rank $[K_b^*] \neq$ rank $[K_b^* : \underline{f}_b^{(h)}]$ *for some* $b \in \{0, 1, \dots, h\}$ $(h \in P_p - S)$.

THEOREM 10. *An array* T *is a* 2^m*-BFF design of resolution* $R(P_p|P_p)$, *i.e.*, $2p + 1$, *if and only if the vector of p-factor interactions* $\underline{\theta}_p$ *is estimable.*

We now present some examples to illustrate the usefulness of Theorems 9 and 10.

EXAMPLE 2. Let T be a 2^m-FF design derived from a BA$(m, 2p; z_0^{(m)}, z_1^{(m)}, \dots, z_m^{(m)})$ and $m = 2p$, $2p + 1$ and $2p + 2$. For $p = 1$, 2 and 3, all designs under considering possible combination of the ranks of the irreducible matrix representations K_0, K_1, ..., K_p can be classified as in Tables 2.1, 2.2 and 2.3, respectively.

TABLE 2.1. The case $p = 1$, i.e., $P_1 = \{0, 1\}$.

m	Resolution	Conditions on BA$(m, 2; z_0^{(m)}, z_1^{(m)}, \dots, z_m^{(m)})$
2	$R(\{0, 1\}\|P_1)$, i.e., III	$z_g^{(2)}, z_1^{(2)} > 0$ for some $g \in \{0, 2\}$
	$R(\{0\}\|P_1)$, i.e., II	$z_i^{(2)} > 0$ $(i = 0, 2)$, $z_1^{(2)} = 0$; or
		$z_1^{(2)} > 0$, $z_i^{(2)} = 0$ $(i = 0, 2)$
	$R(\phi\|P_1)$	others
3	$R(\{0, 1\}\|P_1)$, i.e., III	$z_g^{(3)}, z_h^{(3)} > 0$ for some $h \in \{1, 2\}$, $g \in \{0, 1, 2, 3\} - \{h\}$
	$R(\{0\}\|P_1)$, i.e., II	$z_i^{(3)} > 0$ $(i = 0, 2)$, $z_j^{(3)} = 0$ $(j = 1, 2)$
	$R(\phi\|P_1)$	others
4	$R(\{0, 1\}\|P_1)$, i.e., III	$z_g^{(4)}, z_h^{(4)} > 0$ for some $h \in \{1, 2, 3\}$,
		$g \in \{0, 1, 2, 3, 4\} - \{h\}$
	$R(\{0\}\|P_1)$, i.e., II	$z_i^{(4)} > 0$ $(i = 0, 4)$, $z_j^{(4)} = 0$ $(j = 1, 2, 3)$; or
		$z_2^{(4)} > 0$, $z_i^{(4)} = 0$ $(i = 0, 1, 3, 4)$
	$R(\phi\|P_1)$	others

TABLE 2.2. The case $p = 2$, i.e., $P_2 = \{0, 1, 2\}$.

m	Resolution	Conditions on BA$(m, 4; z_0^{(m)}, z_1^{(m)}, \dots, z_m^{(m)})$
4	$R(\{0, 1, 2\}\|P_2)$, i.e., V	$z_g^{(4)}, z_h^{(4)}, z_2^{(4)} > 0$ for some $h \in \{1, 3\}$,
		$g \in \{0, 1, 3, 4\} - \{h\}$
	$R(\{0, 1\}\|P_2)$, i.e., IV	$z_i^{(4)} > 0$ $(i = 1, 3)$, $z_j^{(4)} = 0$ $(j = 0, 2, 4)$;
		$z_i^{(4)} > 0$ $(i = 1, 3)$, $z_0^{(4)} + z_4^{(4)} > 0$, $z_2^{(4)} = 0$; or
		$z_i^{(4)} > 0$ $(i = 0, 2, 4)$, $z_j^{(4)} = 0$ $(j = 1, 3)$
	$R(\{0\}\|P_2)$	$z_i^{(4)} > 0$ $(i = 0, 3, 4)$, $z_j^{(4)} = 0$ $(j = 1, 2)$; or
		$z_i^{(4)} > 0$ $(i = 0, 1, 4)$, $z_j^{(4)} = 0$ $(j = 2, 3)$
	$R(\phi\|P_2)$	others

TABLE 2.2. (continued)

m	Resolution	Conditions on BA(m, 4; $z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)}$)
5	$R(\{0, 1, 2\}\|P_2)$, i.e., V	$z_g^{(5)}, z_h^{(5)}, z_i^{(5)} > 0$ for some $i \in \{2, 3\}$, $h \in \{1, 2, 3, 4\} - \{i\}$, $g \in \{0, 1, \ldots, 5\} - \{i, h\}$
	$R(\{0, 1\}\|P_2)$, i.e., IV	$z_i^{(5)} > 0$ $(i = 1, 4)$, $z_0^{(5)} + z_5^{(5)} > 0$, $z_j^{(5)} = 0$ $(j = 2, 3)$
	$R(\{1\}\|P_2)$, i.e., IV	$z_i^{(5)} > 0$ $(i = 1, 4)$, $z_j^{(5)} = 0$ $(j = 0, 2, 3, 5)$; or $z_i^{(5)} > 0$ $(i = 2, 3)$, $z_j^{(5)} = 0$ $(j = 0, 1, 4, 5)$
	$R(\{0\}\|P_2)$	$z_i^{(5)} > 0$ $(i = 0, 4, 5)$, $z_j^{(5)} = 0$ $(j = 1, 2, 3)$; $z_i^{(5)} > 0$ $(i = 0, 1, 5)$, $z_j^{(5)} = 0$ $(j = 2, 3, 4)$; $z_i^{(5)} > 0$ $(i = 0, 3)$, $z_j^{(5)} = 0$ $(j = 1, 2, 4, 5)$; $z_i^{(5)} > 0$ $(i = 2, 5)$, $z_j^{(5)} = 0$ $(j = 0, 1, 3, 4)$; $z_i^{(5)} > 0$ $(i = 0, 3, 5)$, $z_j^{(5)} = 0$ $(j = 1, 2, 4)$; or $z_i^{(5)} > 0$ $(i = 0, 2, 5)$, $z_j^{(5)} = 0$ $(j = 1, 3, 4)$
	$R(\phi\|P_2)$	others
6	$R(\{0, 1, 2\}\|P_2)$, i.e., V	$z_g^{(6)}, z_h^{(6)}, z_i^{(6)} > 0$ for some $i \in \{2, 3, 4\}$, $h \in \{1, 2, 3, 4, 5\} - \{i\}$, $g \in \{0, 1, \ldots, 6\} - \{i, h\}$
	$R(\{0, 1\}\|P_2)$, i.e., IV	$z_i^{(6)} > 0$ $(i = 1, 5)$, $z_0^{(6)} + z_6^{(6)} > 0$, $z_j^{(6)} = 0$ $(j = 2, 3, 4)$; or $z_i^{(6)} > 0$ $(i = 0, 3, 6)$, $z_j^{(6)} = 0$ $(j = 1, 2, 4, 5)$
	$R(\{1\}\|P_2)$, i.e., IV	$z_i^{(6)} > 0$ $(i = 1, 5)$, $z_j^{(6)} = 0$ $(j = 0, 2, 3, 4, 6)$; or $z_i^{(6)} > 0$ $(i = 2, 4)$, $z_j^{(6)} = 0$ $(j = 0, 1, 3, 5, 6)$
	$R(\{0\}\|P_2)$	$z_i^{(6)} > 0$ $(i = 0, 5, 6)$, $z_j^{(6)} = 0$ $(j = 1, 2, 3, 4)$; $z_i^{(6)} > 0$ $(i = 0, 1, 6)$, $z_j^{(6)} = 0$ $(j = 2, 3, 4, 5)$; $z_i^{(6)} > 0$ $(i = 0, 4, 6)$, $z_j^{(6)} = 0$ $(j = 1, 2, 3, 5)$; or $z_i^{(6)} > 0$ $(i = 0, 2, 6)$, $z_j^{(6)} = 0$ $(j = 1, 3, 4, 5)$
	$R(\phi\|P_2)$	others

TABLE 2.3. The case $p = 3$, i.e., $P_3 = \{0, 1, 2, 3\}$.

m	Resolution	Conditions on BA(m, 6; $z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)}$)
6	$R(\{0, 1, 2, 3\}\|P_3\}$, i.e., VII	$z_g^{(6)}, z_h^{(6)}, z_i^{(6)}, z_3^{(6)} > 0$ for some $i \in \{2, 4\}$, $h \in \{1, 2, 4, 5\} - \{i\}$, $g \in \{0, 1, 2, 4, 5, 6\} - \{i, h\}$
	$R(\{0, 1, 2\}\|P_3)$, i.e., VI	$z_i^{(6)} > 0$ $(i = 0, 2, 4, 6)$, $z_j^{(6)} = 0$ $(j = 1, 3, 5)$; $z_i^{(6)} > 0$ $(i = 2, 4, 5)$, $z_0^{(6)} + z_1^{(6)} + z_6^{(6)} > 0$, $z_3^{(6)} = 0$; $z_i^{(6)} > 0$ $(i = 1, 2, 4)$, $z_0^{(6)} + z_5^{(6)} + z_6^{(6)} > 0$, $z_3^{(6)} = 0$; $z_i^{(6)} > 0$ $(i = 1, 3, 5)$, $z_j^{(6)} = 0$ $(j = 0, 2, 4, 6)$; or $z_i^{(6)} > 0$ $(i = 1, 3, 5)$, $z_0^{(6)} + z_6^{(6)} > 0$, $z_j^{(6)} = 0$ $(j = 2, 4)$
	$R(\{0, 2\}\|P_3)$	$z_i^{(6)} > 0$ $(i = 2, 3, 4)$, $z_j^{(6)} = 0$ $(j = 0, 1, 5, 6)$
	$R(\{0, 1\}\|P_3)$	$z_i^{(6)} > 0$ $(i = 1, 4, 5)$, $z_0^{(6)} + z_6^{(6)} > 0$, $z_j^{(6)} = 0$ $(j = 2, 3)$; or $z_i^{(6)} > 0$ $(i = 1, 2, 5)$, $z_0^{(6)} + z_6^{(6)} > 0$, $z_j^{(6)} = 0$ $(j = 3, 4)$
	$R(\{1\}\|P_3)$	$z_i^{(6)} > 0$ $(i = 1, 4, 5)$, $z_j^{(6)} = 0$ $(j = 0, 2, 3, 6)$; or $z_i^{(6)} > 0$ $(i = 1, 2, 5)$, $z_j^{(6)} = 0$ $(j = 0, 3, 4, 6)$
	$R(\{0\}\|P_3)$	$z_i^{(6)} > 0$ $(i = 0, 1, 5, 6)$, $z_j^{(6)} = 0$ $(j = 2, 3, 4)$; $z_i^{(6)} > 0$ $(i = 0, 4, 5, 6)$, $z_j^{(6)} = 0$ $(j = 1, 2, 3)$;

TABLE 2.3. (continued)

m	Resolution	Conditions on $\mathrm{BA}(m, 6; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$
		$z_i^{(6)} > 0$ $(i = 0, 1, 2, 6)$, $z_j^{(6)} = 0$ $(j = 3, 4, 5)$; $z_i^{(6)} > 0$ $(i = 0, 1, 4)$, $z_j^{(6)} = 0$ $(j = 2, 3, 5, 6)$; $z_i^{(6)} > 0$ $(i = 2, 5, 6)$, $z_j^{(6)} = 0$ $(j = 0, 1, 3, 4)$; $z_i^{(6)} > 0$ $(i = 0, 1, 4, 6)$, $z_j^{(6)} = 0$ $(j = 2, 3, 5)$; $z_i^{(6)} > 0$ $(i = 0, 2, 5, 6)$, $z_j^{(6)} = 0$ $(j = 1, 3, 4)$; $z_i^{(6)} > 0$ $(i = 0, 3, 6)$, $z_j^{(6)} = 0$ $(j = 1, 2, 4, 5)$; $z_i^{(6)} > 0$ $(i = 0, 3, 5, 6)$, $z_j^{(6)} = 0$ $(j = 1, 2, 4)$; $z_i^{(6)} > 0$ $(i = 0, 1, 3, 6)$, $z_j^{(6)} = 0$ $(j = 2, 4, 5)$; $z_i^{(6)} > 0$ $(i = 0, 3, 4, 6)$, $z_j^{(6)} = 0$ $(j = 1, 2, 5)$; or $z_i^{(6)} > 0$ $(i = 0, 2, 3, 6)$, $z_j^{(6)} = 0$ $(j = 1, 4, 5)$
	$R(\phi \mid P_3)$	others
7	$R(\{0, 1, 2, 3\} \mid P_3\}$, i.e., VII	$z_g^{(7)}, z_h^{(7)}, z_i^{(7)}, z_j^{(7)} > 0$ for some $j \in \{3, 4\}$, $i \in \{2, 3, 4, 5\} - \{j\}$, $h \in \{1, 2, \ldots, 6\} - \{i, j\}$, $g \in \{0, 1, \ldots, 7\} - \{i, j, h\}$
	$R(\{0, 1, 2\} \mid P_3)$, i.e., VI	$z_i^{(7)} > 0$ $(i = 2, 5, 6)$, $z_0^{(7)} + z_1^{(7)} + z_7^{(7)} > 0$, $z_j^{(7)} = 0$ $(j = 3, 4)$; or $z_i^{(7)} > 0$ $(i = 1, 2, 5)$, $z_0^{(7)} + z_6^{(7)} + z_7^{(7)} > 0$, $z_j^{(7)} = 0$ $(j = 3, 4)$
	$R(\{0, 2\} \mid P_3)$	$z_i^{(7)} > 0$ $(i = 0, 2, 5, 7)$, $z_j^{(7)} = 0$ $(j = 1, 3, 4, 6)$; or $z_i^{(7)} > 0$ $(i = 0, 3, 4, 7)$, $z_j^{(7)} = 0$ $(j = 1, 2, 5, 6)$
	$R(\{0, 1\} \mid P_3)$	$z_i^{(7)} > 0$ $(i = 1, 5, 6)$, $z_0^{(7)} + z_7^{(7)} > 0$, $z_j^{(7)} = 0$ $(j = 2, 3, 4)$; $z_i^{(7)} > 0$ $(i = 1, 2, 6)$, $z_0^{(7)} + z_7^{(7)} > 0$, $z_j^{(7)} = 0$ $(j = 3, 4, 5)$; $z_i^{(7)} > 0$ $(i = 1, 4, 6)$, $z_0^{(7)} + z_7^{(7)} > 0$, $z_j^{(7)} = 0$ $(j = 2, 3, 5)$; $z_i^{(7)} > 0$ $(i = 1, 3, 6)$, $z_0^{(7)} + z_7^{(7)} > 0$, $z_j^{(7)} = 0$ $(j = 2, 4, 5)$; $z_i^{(7)} > 0$ $(i = 0, 1, 4, 7)$, $z_j^{(7)} = 0$ $(j = 2, 3, 5, 6)$; or $z_i^{(7)} > 0$ $(i = 0, 3, 6, 7)$, $z_j^{(7)} = 0$ $(j = 1, 2, 4, 5)$
	$R(\{0\} \mid P_3)$	$z_i^{(7)} > 0$ $(i = 0, 1, 6, 7)$, $z_j^{(7)} = 0$ $(j = 2, 3, 4, 5)$; $z_i^{(7)} > 0$ $(i = 0, 5, 6, 7)$, $z_j^{(7)} = 0$ $(j = 1, 2, 3, 4)$; $z_i^{(7)} > 0$ $(i = 0, 1, 2, 7)$, $z_j^{(7)} = 0$ $(j = 3, 4, 5, 6)$; $z_i^{(7)} > 0$ $(i = 0, 1, 5, 7)$, $z_j^{(7)} = 0$ $(j = 2, 3, 4, 6)$; $z_i^{(7)} > 0$ $(i = 0, 2, 6, 7)$, $z_j^{(7)} = 0$ $(j = 1, 3, 4, 5)$; $z_i^{(7)} > 0$ $(i = 0, 4, 6, 7)$, $z_j^{(7)} = 0$ $(j = 1, 2, 3, 5)$; $z_i^{(7)} > 0$ $(i = 0, 1, 3, 7)$, $z_j^{(7)} = 0$ $(j = 2, 4, 5, 6)$; $z_i^{(7)} > 0$ $(i = 0, 4, 5, 7)$, $z_j^{(7)} = 0$ $(j = 1, 2, 3, 6)$; $z_i^{(7)} > 0$ $(i = 0, 2, 3, 7)$, $z_j^{(7)} = 0$ $(j = 1, 4, 5, 6)$; $z_i^{(7)} > 0$ $(i = 0, 2, 4, 7)$, $z_j^{(7)} = 0$ $(j = 1, 3, 5, 6)$; or $z_i^{(7)} > 0$ $(i = 0, 3, 5, 7)$, $z_j^{(7)} = 0$ $(j = 1, 2, 4, 6)$
	$R(\phi \mid P_3)$	others

TABLE 2.3. (continued)

m	Resolution	Conditions on BA$(m, 6; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$
8	$R(\{0, 1, 2, 3\} \mid P_3)$, i.e., VII	$z_g^{(8)}, z_h^{(8)}, z_i^{(8)}, z_j^{(8)} > 0$ for some $j \in \{3, 4, 5\}$, $i \in \{2, 3, 4, 5, 6\} - \{j\}$, $h \in \{1, 2, \ldots, 7\} - \{i, j\}$, $g \in \{0, 1, \ldots, 8\} - \{i, j, h\}$
	$R(\{0, 1, 2\} \mid P_3)$, i.e., VI	$z_i^{(8)} > 0$ $(i = 2, 6, 7)$, $z_0^{(8)} + z_1^{(8)} + z_8^{(0)} > 0$, $z_j^{(8)} = 0$ $(j = 3, 4, 5)$; $z_i^{(8)} > 0$ $(i = 1, 2, 6)$, $z_0^{(8)} + z_7^{(8)} + z_8^{(8)} > 0$, $z_j^{(8)} = 0$ $(j = 3, 4, 5)$; or $z_i^{(8)} > 0$ $(i = 1, 4, 7)$, $z_0^{(8)} + z_8^{(8)} > 0$, $z_j^{(8)} = 0$ $(j = 2, 3, 5, 6)$
	$R(\{0, 2\} \mid P_3)$	$z_i^{(8)} > 0$ $(i = 0, 2, 6, 8)$, $z_j^{(8)} = 0$ $(j = 1, 3, 4, 5, 7)$; $z_i^{(8)} > 0$ $(i = 0, 3, 5, 8)$, $z_j^{(8)} = 0$ $(j = 1, 2, 4, 6, 7)$; $z_i^{(8)} > 0$ $(i = 1, 4, 7)$, $z_j^{(8)} = 0$ $(j = 0, 2, 3, 5, 6, 8)$; $z_i^{(8)} > 0$ $(i = 2, 4, 6)$, $z_j^{(8)} = 0$ $(j = 0, 1, 3, 5, 7, 8)$; or $z_i^{(8)} > 0$ $(i = 3, 4, 5)$, $z_j^{(8)} = 0$ $(j = 0, 1, 2, 6, 7, 8)$
	$R(\{0, 1\} \mid P_3)$	$z_i^{(8)} > 0$ $(i = 1, 6, 7)$, $z_0^{(8)} + z_8^{(8)} > 0$, $z_j^{(8)} = 0$ $(j = 2, 3, 4, 5)$; $z_i^{(8)} > 0$ $(i = 1, 2, 7)$, $z_0^{(8)} + z_8^{(8)} > 0$, $z_j^{(8)} = 0$ $(j = 3, 4, 5, 6)$; $z_i^{(8)} > 0$ $(i = 1, 5, 7)$, $z_0^{(8)} + z_8^{(8)} > 0$, $z_j^{(8)} = 0$ $(j = 2, 3, 4, 6)$; or $z_i^{(8)} > 0$ $(i = 1, 3, 7)$, $z_0^{(8)} + z_8^{(8)} > 0$, $z_j^{(8)} = 0$ $(j = 2, 4, 5, 6)$
	$R(\{2\} \mid P_3)$	$z_i^{(8)} > 0$ $(i = 1, 5, 6)$, $z_j^{(8)} = 0$ $(j = 0, 2, 3, 4, 7, 8)$; or $z_i^{(8)} > 0$ $(i = 2, 3, 7)$, $z_j^{(8)} = 0$ $(j = 0, 1, 4, 5, 6, 8)$
	$R(\{0\} \mid P_3)$	$z_i^{(8)} > 0$ $(i = 0, 1, 7, 8)$, $z_j^{(8)} = 0$ $(j = 2, 3, 4, 5, 6)$; $z_i^{(8)} > 0$ $(i = 0, 6, 7, 8)$, $z_j^{(8)} = 0$ $(j = 1, 2, 3, 4, 5)$; $z_i^{(8)} > 0$ $(i = 0, 1, 2, 8)$, $z_j^{(8)} = 0$ $(j = 3, 4, 5, 6, 7)$; $z_i^{(8)} > 0$ $(i = 0, 1, 6, 8)$, $z_j^{(8)} = 0$ $(j = 2, 3, 4, 5, 7)$; $z_i^{(8)} > 0$ $(i = 0, 2, 7, 8)$, $z_j^{(8)} = 0$ $(j = 1, 3, 4, 5, 6)$; $z_i^{(8)} > 0$ $(i = 0, 5, 7, 8)$, $z_j^{(8)} = 0$ $(j = 1, 2, 3, 4, 6)$; $z_i^{(8)} > 0$ $(i = 0, 1, 3, 8)$, $z_j^{(8)} = 0$ $(j = 2, 4, 5, 6, 7)$; $z_i^{(8)} > 0$ $(i = 0, 1, 5)$, $z_j^{(8)} = 0$ $(j = 2, 3, 4, 6, 7, 8)$; $z_i^{(8)} > 0$ $(i = 3, 7, 8)$, $z_j^{(8)} = 0$ $(j = 0, 1, 2, 4, 5, 6)$; $z_i^{(8)} > 0$ $(i = 0, 1, 5, 8)$, $z_j^{(8)} = 0$ $(j = 2, 3, 4, 6, 7)$; $z_i^{(8)} > 0$ $(i = 0, 3, 7, 8)$, $z_j^{(8)} = 0$ $(j = 1, 2, 4, 5, 6)$; $z_i^{(8)} > 0$ $(i = 0, 5, 6, 8)$, $z_j^{(8)} = 0$ $(j = 1, 2, 3, 4, 7)$; $z_i^{(8)} > 0$ $(i = 0, 2, 3, 8)$, $z_j^{(8)} = 0$ $(j = 1, 4, 5, 6, 7)$; $z_i^{(8)} > 0$ $(i = 0, 2, 5, 8)$, $z_j^{(8)} = 0$ $(j = 1, 3, 4, 6, 7)$; $z_i^{(8)} > 0$ $(i = 0, 3, 6, 8)$, $z_j^{(8)} = 0$ $(j = 1, 2, 4, 5, 7)$; $z_i^{(8)} > 0$ $(i = 0, 4, 8)$, $z_j^{(8)} = 0$ $(j = 1, 2, 3, 5, 6, 7)$; $z_i^{(8)} > 0$ $(i = 0, 4, 7, 8)$, $z_j^{(8)} = 0$ $(j = 1, 2, 3, 5, 6)$; $z_i^{(8)} > 0$ $(i = 0, 1, 4, 8)$, $z_j^{(8)} = 0$ $(j = 2, 3, 5, 6, 7)$; $z_i^{(8)} > 0$ $(i = 0, 4, 6, 8)$, $z_j^{(8)} = 0$ $(j = 1, 2, 3, 5, 7)$; $z_i^{(8)} > 0$ $(i = 0, 2, 4, 8)$, $z_j^{(8)} = 0$ $(j = 1, 3, 5, 6, 7)$;

TABLE 2.3. (continued)

m	Resolution	Conditions on BA$(m, 6; z_0^{(m)}, z_1^{(m)}, \dots, z_m^{(m)})$
		$z_i^{(8)} > 0$ $(i = 0, 4, 5, 8)$, $z_j^{(8)} = 0$ $(j = 1, 2, 3, 6, 7)$; or
		$z_i^{(8)} > 0$ $(i = 0, 3, 4, 8)$, $z_j^{(8)} = 0$ $(j = 1, 2, 5, 6, 7)$
	$R(\phi \mid P_3)$	others

EXAMPLE 3. For $p = 1$, 2 and 3, the resolution of a 2^m-FF design derived from a BA$(m, 2p; z_0^{(m)}, z_1^{(m)}, \dots, z_m^{(m)})$ can be classified into one of the following possibilities given in Tables 3.1, 3.2 and 3.3, respectively. In these Tables, the symbols $\odot$ and $\times$ stand for the existence and non-existence of a design having specified resolution, respectively. The symbol $*$ indicates the existence of a design having specified resolution for every $m \geqq 2p$.

TABLE 3.1. The case $p = 1$, i.e., $P_1 = \{0, 1\}$.

Resolution	$m \geqq 2$
$*R(\{0, 1\} \mid P_1)$, i.e., III	$\odot$
$*R(\{0\} \mid P_1)$, i.e., II	$\odot$
$*R(\phi \mid P_1)$	$\odot$

TABLE 3.2. The case $p = 2$, i.e., $P_2 = \{0, 1, 2\}$.

Resolution	$m = 4$	$m \geqq 5$
$*R(\{0, 1, 2\} \mid P_2)$, i.e., V	$\odot$	$\odot$
$*R(\{0, 1\} \mid P_2)$, i.e., IV	$\odot$	$\odot$
$*R(\{1\} \mid P_2)$, i.e., IV	$\times$	$\odot$
$*R(\{0\} \mid P_2)$	$\odot$	$\odot$
$*R(\phi \mid P_2)$	$\odot$	$\odot$

TABLE 3.3. The case $p = 3$, i.e., $P_3 = \{0, 1, 2, 3\}$.

Resolution	m: 6	7	8	9	10	11	12	13	14	15
$*R(\{0, 1, 2, 3\} \mid P_3)$, i.e., VII	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$
$*R(\{0, 1, 2\} \mid P_3)$, i.e., VI	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$
$R(\{1, 2\} \mid P_3)$, i.e., VI	$\times$	$\times$	$\times$	$\times$	$\odot$	$\times$	$\times$	$\times$	$\times$	$\times$
$*R(\{0, 2\} \mid P_3)$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$
$*R(\{0, 1\} \mid P_3)$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$
$R(\{2\} \mid P_3)$	$\times$	$\times$	$\odot$	$\times$	$\odot$	$\times$	$\odot$	$\times$	$\odot$	$\times$
$R(\{1\} \mid P_3)$	$\odot$	$\times$	$\times$	$\odot$	$\times$	$\times$	$\times$	$\odot$	$\odot$	$\times$
$*R(\{0\} \mid P_3)$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$
$*R(\phi \mid P_3)$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$

Note that Tables 3.1–3.3 include the results of Hyodo and Yamamoto [15] as a special case.

Acknowledgments

The author would like to express his hearty thanks to Professor Sumiyasu Yamamoto, Okayama University of Science, for his encouragement and his valuable suggestions, and also to Professor Yasunori Fujikoshi and Professor Masahide Kuwada, Hiroshima University, who kindly read over an early draft of this paper with valuable advice and comments. This work was supported in part by both of the Grant of the Ministry of Education, Science and Culture under contract Number 63740126 and the Research Grant of Science University of Tokyo under contract Number 87-1001.

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