HIROSHIMA MATH. J.
**20** (1990), 231–258

# Nonoscillatory solutions of neutral differential equations

Yūki NAITO

(Received April 7, 1989)

## 1. Introduction

In this paper we are concerned with neutral differential equations of the form

$$\frac{d^n}{dt^n}[x(t) - h(t)x(\tau(t))] + \sigma p(t)f(x(g(t))) = 0\,, \tag{1.1}$$

where $n \geq 2$, $\sigma = 1$ or $-1$, and the following conditions are always assumed to hold:

(1.2) $\tau(t) \in C[a, \infty)$, $\tau$ is nondecreasing on $[a, \infty)$, $\tau(t) < t$ for $t \geq a$ and $\lim_{t\to\infty} \tau(t) = \infty$;

(1.3) $h(t) \in C[\tau(a), \infty)$, $|h(t)| \leq h < 1$ for $t \geq a$, where $h$ is a constant, and $h(t)h(\tau(t)) \geq 0$ for $t \geq a$;

(1.4) $p(t) \in C[a, \infty)$ and $p(t) > 0$ for $t \geq a$;

(1.5) $f(u) \in C((-\infty, \infty)\backslash\{0\})$ and $f(u)u > 0$ for $u \neq 0$;

(1.6) $g(t) \in C[a, \infty)$ and $\lim_{t\to\infty} g(t) = \infty$.

By a solution of (1.1) we mean a continuous function $x$ which is defined and satisfies (1.1) on $[T_x, \infty)$ for some $T_x \geq a$ (so that $x(t) - h(t)x(\tau(t))$ is $n$-times continuously differentiable on $[T_x, \infty)$). Such a solution is said to be nonoscillatory if it has no zeros on $[T, \infty)$ for some $T \geq T_x$.

Recently there has been an increasing interest in the study of neutral differential equations, and a number of results have been obtained. For typical results we refer in particular to the papers [1–9, 14–18]. In this paper we make an attempt to study in a systematic way the structure of the set of nonoscillatory solutions of equation (1.1). In Section 2 we discuss the relation between two functions $x(t)$ and $x(t) - h(t)x(\tau(t))$. The results obtained in Section 2 will be effectively used in subsequent sections. In Section 3 we classify the nonoscillatory solutions of (1.1) into several classes according to the asymptotic behavior as $t \to \infty$. In Sections 4 and 5 we establish necessary and sufficient conditions for the existence of nonoscillatory solutions of (1.1) with specific asymptotic properties as $t \to \infty$.

If $h(t) \equiv 0$, then equation (1.1) becomes

$$x^{(n)}(t) + \sigma p(t) f(x(g(t))) = 0 . \tag{1.7}$$

Our results extend some of the results for equation (1.7). As a result we see that, concerning the characterization of the existence of nonoscillatory solutions, there is not much difference between equation (1.1) and equation (1.7). Further we see that if $h(t)$ and $\tau(t)/t$ are convergent as $t \to \infty$, then the structure of nonoscillatory solutions of equation (1.1) is similar to that of nonoscillatory solutions of equaton (1.7) or the ordinary differential equation

$$x^{(n)}(t) + \sigma p(t) f(x(t)) = 0 . \tag{1.8}$$

Related results are contained in Jaroš and Kusano [8, 9]. In particular, existence theorems of nonoscillatory solutions of (1.1) have been obtained by Jaroš and Kusano [8, Theorem 1; 9, Theorem 3.1]. However, for the Emden-Fowler type neutral differential equation

$$\frac{d^n}{dt^n}[x(t) - h(t)x(\tau(t))] + \sigma p(t)|x(g(t))|^{\gamma} \operatorname{sgn} x(g(t)) = 0 , \tag{1.9}$$

their theorems cannot be applied to the case of $\gamma < 0$, because they assume that $f$ in (1.1) is a nondecreasing function. In this paper the existence theorems of nonoscillatory solutions of (1.1) are proved by a different method from [8, 9]. Our theorems can be applied to not only the case of $\gamma \geq 0$ but also the case of $\gamma < 0$, provided $h$ and $\tau$ are locally Lipschitz continuous.

## 2. Preliminaries

In this section we study the relation between two continuous functions $x(t)$ and $x(t) - h(t)x(\tau(t))$. As regards $\tau(t)$ and $h(t)$, we assume that conditions (1.2) and (1.3) in Section 1 are satisfied.

Let $T \geq a$. Then we use the notation:

$$T_0(T) = T, \qquad T_i(T) = \sup\{t \geq a; \tau(t) = T_{i-1}(T)\}, \qquad i = 1, 2, \ldots; \tag{2.1}$$

$$\tau^0(t) = t, \qquad \tau^i(t) = \tau(\tau^{i-1}(t)), \qquad i = 1, 2, \ldots . \tag{2.2}$$

Note that $\tau^1(t) = \tau(t)$ and that $\tau^i(t)$ is defined on $[T_i(a), \infty)$, $i = 1, 2, \ldots$. It is easily verified that

$$\tau(T) < T < T_1(T) < \cdots < T_{m-1}(T) < T_m(T) < \cdots,$$

$$\lim_{m \to \infty} T_m(T) = \infty$$

and

$$(2.3)\quad \tau(T) < \tau^{m+1}(t) \le T \qquad \text{for} \quad T_m(T) < t \le T_{m+1}(T)\,, \qquad m = 0, 1, 2, \dots .$$

We define the functions $H_m(t)$ on $[T_{m-1}(a), \infty)$ as follows:

$$(2.4)\qquad H_0(t) = 1\,; \qquad H_m(t) = \prod_{i=0}^{m-1} h(\tau^i(t))\,, \qquad m = 1, 2, \dots .$$

For an $x \in C[\tau(T), \infty)$, we define $Lx \in C[T, \infty)$ by

$$(2.5)\qquad (Lx)(t) = x(t) - h(t)x(\tau(t))\,, \qquad t \ge T\,.$$

LEMMA 2.1. *Let $T \ge a$ and $x \in C[\tau(T), \infty)$. Then*

$$(2.6)\qquad x(t) = \sum_{k=0}^{m} H_k(t)(Lx)(\tau^k(t)) + H_{m+1}(t)x(\tau^{m+1}(t))$$

$$\text{for} \quad t > T_m(T)\,, \qquad m = 0, 1, 2, \dots .$$

PROOF. In view of (2.5) we see that

$$(2.7)\qquad x(t) = (Lx)(t) + h(t)x(\tau(t))\,, \qquad t > T_0(T)\,.$$

Note by (2.1) that $\tau(t) > T_0(T)$ for $t > T_1(T)$. Then equality (2.7) implies that

$$(2.8)\qquad x(t) = (Lx)(t) + h(t)(Lx)(\tau(t)) + h(t)h(\tau(t))x(\tau^2(t))$$

for $t > T_1(T)$. Repeating this argument, we find that (2.6) is satisfied for $t > T_m(T)$.

LEMMA 2.2. *Suppose that $x \in C[\tau(T), \infty)$, $T \ge a$.*

(i) *If $Lx$ is bounded on $[T, \infty)$, then $x$ is also bounded on $[T, \infty)$.*

(ii) *If*

$$(2.9)\qquad \lim_{t\to\infty} (Lx)(t) = 0\,,$$

*then*

$$(2.10)\qquad \lim_{t\to\infty} x(t) = 0\,.$$

PROOF. (i) There are positive constants $c_1$ and $c_2$ such that

$$(2.11)\qquad |(Lx)(t)| \le c_1\,, \qquad t \ge T\,; \qquad |x(t)| \le c_2\,, \qquad \tau(T) \le t \le T\,.$$

Recall that (2.3) holds and notice that $|H_k(t)| \le h^k$ for $t > T_m(T)$, $k = 0, 1, 2, \dots, m + 1$. Then it follows from (2.6) and (2.11) that

$$|x(t)| \le \sum_{k=0}^{m} h^k c_1 + h^{m+1} c_2 \le \frac{c_1}{1-h} + c_2$$

for $T_m(T) < t \le T_{m+1}(T)$, $m = 1, 2, \dots$, which implies that

$$|x(t)| \le \frac{c_1}{1-h} + c_2 \qquad \text{for} \quad t \ge T.$$

Thus $x$ is bounded on $[T, \infty)$.

(ii) In view of (i), $x(t)$ is bounded. Therefore there is a positive constant $c_3$ such that

$$|x(t)| \le c_3, \qquad t \ge \tau(T).$$

Let $\varepsilon > 0$. By condition (2.9) there is a $\tilde{T} \ge T$ such that

$$|(Lx)(t)| < \frac{1-h}{2}\varepsilon, \qquad t \ge \tilde{T}.$$

Since $0 \le h < 1$, there is an integer $m_0$ such that

$$h^{m+1}c_3 < \frac{\varepsilon}{2}, \qquad m = m_0, m_0 + 1, \dots.$$

As in the case of (i), we can obtain the estimate

$$|x(t)| \le \sum_{k=0}^{m} h^k \frac{1-h}{2}\varepsilon + h^{m+1}c_3 < \varepsilon$$

for $t > T_m(\tilde{T})$, $m = m_0, m_0 + 1, \dots$. Therefore we have

$$|x(t)| < \varepsilon \qquad \text{for} \quad t > T_{m_0}(\tilde{T}),$$

which shows that $x(t) \to 0$ as $t \to \infty$.

LEMMA 2.3. *Suppose that* $x \in C[\tau(T), \infty)$, $T \ge a$. *If* $|(Lx)(t)|$ *is not identically zero and is nondecreasing on* $[T, \infty)$, *then there are constants* $h^* > 0$ *and* $T^* \ge T$ *such that*

$$|x(t)| \le h^*|(Lx)(t)| \qquad \textit{for} \quad t \ge T^*. \tag{2.12}$$

PROOF. There are positive constants $T^* \ge T$, $d_1$ and $d_2$ such that

$$|(Lx)(t)| \ge d_1, \quad t \ge T^*; \qquad |x(t)| \le d_2, \quad \tau(T^*) \le t \le T^*. \tag{2.13}$$

On account of the nondecreasing property of $|(Lx)(t)|$, we can see from (2.6) that

$$\begin{aligned} |x(t)| &\le |(Lx)(t)| \sum_{k=0}^{m} h^k + h^{m+1}d_2 \\ &\le \frac{1}{1-h}|(Lx)(t)| + d_2 \end{aligned}$$

for $T_m(T^*) < t \le T_{m+1}(T^*)$, $m = 1, 2, \ldots$. Since the first half of (2.13) implies that $1 \le |(Lx)(t)|/d_1$ for $t \ge T^*$, we obtain

$$|x(t)| \le \left(\frac{1}{1-h} + \frac{d_2}{d_1}\right)|(Lx)(t)|\,, \qquad t \ge T^*\,.$$

This completes the proof of Lemma 2.3.

LEMMA 2.4. *Let $x \in C[\tau(T), \infty)$, $T \ge a$. Suppose that $x$ is of constant sign on $[\tau(T), \infty)$ and $x(t)(Lx)(t) \ge 0$ for $t \ge T$. If either*

(2.14) $\qquad |(Lx)(t)|$ *is nondecreasing on $[T, \infty)$, or*

$$\lim_{t\to\infty} (Lx)(t) = l\,, \qquad 0 < |l| < \infty\,, \tag{2.15}$$

*then there are constants $h_* > 0$ and $T_* \ge T$ such that*

$$|x(t)| \ge h_*|(Lx)(t)| \qquad \textit{for} \quad t \ge T_* \tag{2.16}$$

PROOF. We may assume that $x(t) > 0$ and $(Lx)(t) \ge 0$ for $t \ge T$, since a parallel argument holds if $x(t) < 0$ and $(Lx)(t) \le 0$ for $t \ge T$. We have (2.8) for $t \ge T_1(T)$. From the underlying condition (1.3) and the positivity of $x$ it follows that

$$x(t) \ge (Lx)(t) + h(t)(Lx)(\tau(t)) \ge (Lx)(t) - h(Lx)(\tau(t)) \tag{2.17}$$

for $t \ge T_1(T)$. Suppose first that $Lx$ satisfies (2.14). Then we obtain

$$x(t) \ge (Lx)(t) - h(Lx)(t) = (1-h)(Lx)(t)\,, \qquad t \ge T_1(T)\,.$$

Suppose next that $Lx$ satisfies (2.15). We choose a positive constant $\eta$ such that $h < \eta < 1$. By (2.15), there is a $T_* \ge T_1(T)$ such that

$$\sqrt{\eta}l \le (Lx)(\tau(t)) \le \frac{l}{\sqrt{\eta}}\,, \qquad \sqrt{\eta}l \le (Lx)(t) \le \frac{l}{\sqrt{\eta}} \qquad \text{for} \quad t \ge T_* \tag{2.18}$$

and in particular

$$1 \le \frac{(Lx)(t)}{\sqrt{\eta}l} \qquad \text{for} \quad t \ge T_*\,. \tag{2.19}$$

From (2.18) and (2.19) it follows that

$$(Lx)(\tau(t)) \le \frac{l}{\sqrt{\eta}} \le \frac{l}{\sqrt{\eta}}\frac{(Lx)(t)}{\sqrt{\eta}l} \le \frac{1}{\eta}(Lx)(t)\,, \qquad t \ge T_*\,.$$

From (2.17) we have

$$x(t) \geq (Lx)(t) - \frac{h}{\eta}(Lx)(t) = \left(1 - \frac{h}{\eta}\right)(Lx)(t)\,, \qquad t \geq T_* \,.$$

This completes the proof of Lemma 2.4.

Lemma 2.5. *Let $x \in C[\tau(T), \infty)$, $T \geq a$. Suppose that $x$ is of constant sign on $[\tau(T), \infty)$ and*

$$\lim_{t\to\infty} (Lx)(t) = l\,, \qquad -\infty \leq l \leq \infty\,,$$

*then*

$$0 \leq l \cdot \operatorname{sgn} x(t) \leq \infty\,.$$

Proof. We may suppose with no loss of generality that $x(t) > 0$ for $t \geq \tau(T)$. We claim that $0 \leq l \leq \infty$. Assume to the contrary that $-\infty \leq l < 0$. There is a $\tilde{T} \geq T$ such that

$$(Lx)(t) \equiv x(t) - h(t)x(\tau(t)) < 0\,, \qquad t \geq \tilde{T}\,.$$

We obtain

$$x(t) < h(t)x(\tau(t)) \leq hx(\tau(t))\,, \qquad t \geq \tilde{T}\,.$$

By induction it can be shown that

$$x(t) \leq h^m x(\tau^m(t))\,, \qquad t > T_{m-1}(\tilde{T})\,, \qquad m = 1, 2, \dots\,.$$

Set $\gamma = \max\{x(s) : \tau(\tilde{T}) \leq s \leq \tilde{T}\}$ and recall (2.3) with $T = \tilde{T}$. Then we have

$$x(t) \leq h^m \gamma\,, \qquad T_{m-1}(\tilde{T}) < t \leq T_m(\tilde{T})\,, \qquad m = 1, 2, \dots,$$

which implies that $\lim_{t\to\infty} x(t) = 0$. By (2.5) we have $\lim_{t\to\infty} (Lx)(t) = 0$. However this contradicts the assumption that $\lim_{t\to\infty} (Lx)(t) = l \in [-\infty, 0)$. Thus we conclude that $0 \leq l \leq \infty$.

Remark 2.1. Assume that

$$\begin{cases} x(t) > 0\,, & t \geq \tau(T)\,, \quad \text{and} \\ (Lx)(t) \equiv x(t) - h(t)x(\tau(t)) < 0\,, & t \geq T\,. \end{cases} \tag{2.20}$$

Then in view of the proof of Lemma 2.5 we see that $\lim_{t\to\infty} x(t) = 0$. Notice that (2.20) can occur only when $h(t)$ is positive on $[T, \infty)$.

From Lemmas 2.2, 2.4 and 2.5 we obtain the next lemma.

Lemma 2.6. *Suppose that $x \in C[\tau(T), \infty)$, $T \geq a$. Let $x$ be of constant sign on $[\tau(T), \infty)$.*

(i) *If* $|(Lx)(t)|$ *is nondecreasing on* $[T, \infty)$ *and*

$$\lim_{t\to\infty} |(Lx)(t)| = \infty ,$$

*then*

$$\lim_{t\to\infty} |x(t)| = \infty .$$

(ii) *If*

$$\lim_{t\to\infty} (Lx)(t) = l , \qquad 0 < |l| < \infty ,$$

*then*

$$0 < \liminf_{t\to\infty} |x(t)| \le \limsup_{t\to\infty} |x(t)| < \infty .$$

PROOF. We may suppose with no loss of generality that $x(t) > 0$ for $t \ge \tau(T)$.

(i) From Lemma 2.5 we see that $\lim_{t\to\infty} (Lx)(t) = \infty$. By Lemma 2.4 we have

$$x(t) \ge h_*(Lx)(t) , \qquad t \ge T_* , \tag{2.21}$$

where $h_* > 0$ and $T_* \ge T$ are constants. Then it is clear that $\lim_{t\to\infty} x(t) = \infty$.

(ii) From Lemma 2.5 we obtain $l > 0$; and so $(Lx)(t) > 0$ for all large $t$. By Lemma 2.4 we have (2.21) for some constants $h_* > 0$ and $T_* \ge T$. Then (2.21) gives $\liminf_{t\to\infty} x(t) > 0$. From (i) of Lemma 2.2 we see that $\limsup_{t\to\infty} x(t) < \infty$. The proof of Lemma 2.6 is complete.

LEMMA 2.7. *Let* $x \in C[\tau(T), \infty)$, $T \ge a$, *and* $i$ *be a nonnegative integer.*

(i) *If* $\lim_{t\to\infty} (Lx)(t)/t^i = 0$, *then* $\lim_{t\to\infty} x(t)/t^i = 0$.

(ii) *Suppose in addition that* $x$ *is of constant sign on* $[\tau(T), \infty)$. *If* $|(Lx)(t)|/t^i$ *is nondecreasing on* $[T, \infty)$ *and* $\lim_{t\to\infty} |(Lx)(t)|/t^i = \infty$, *then* $\lim_{t\to\infty} |x(t)|/t^i = \infty$.

(iii) *Suppose in addition that* $x$ *is of constant sign on* $[\tau(T), \infty)$. *If* $\lim_{t\to\infty} (Lx)(t)/t^i$ *exists and is a nonzero finite value, then*

$$0 < \liminf_{t\to\infty} \frac{|x(t)|}{t^i} \le \limsup_{t\to\infty} \frac{|x(t)|}{t^i} < \infty .$$

PROOF. Observe that

$$\frac{(Lx)(t)}{t^i} = \frac{x(t)}{t^i} - h(t)\left[\frac{\tau(t)}{t}\right]^i \frac{x(\tau(t))}{[\tau(t)]^i} , \tag{2.22}$$

and apply (ii) of Lemma 2.2 and Lemma 2.6 with $x(t)$ and $h(t)$ replaced by $x(t)/t^i$ and $h(t)[\tau(t)/t]^i$, respectively.

We can assert that, in (iii) of Lemma 2.7, the limit of $|x(t)|/t^i$ as $t \to \infty$ exists if the following condition is satisfied:

$$\lim_{t\to\infty} h(t)[\tau(t)/t]^i \quad \text{exists and is finite}. \tag{2.23}$$

To see this we first prove the next lemma.

LEMMA 2.8. *Suppose that $x \in C[\tau(T), \infty)$ and that*

$$\lim_{t\to\infty} h(t) = \lambda, \qquad |\lambda| \le h < 1. \tag{2.24}$$

*If*

$$\lim_{t\to\infty} (Lx)(t) = l, \qquad |l| < \infty, \tag{2.25}$$

*then*

$$\lim_{t\to\infty} x(t) = \frac{l}{1-\lambda}. \tag{2.26}$$

PROOF. Set $\hat{x}(t) = x(t) - l(1-\lambda)^{-1}$. We have

$$\begin{aligned}(L\hat{x})(t) &= \hat{x}(t) - h(t)\hat{x}(\tau(t)) \\ &= (Lx)(t) - l + \frac{l}{1-\lambda}(h(t) - \lambda).\end{aligned}$$

From (2.24) and (2.25) it follows that $\lim_{t\to\infty} (L\hat{x})(t) = 0$. In view of (ii) of Lemma 2.2 we have $\lim_{t\to\infty} \hat{x}(t) = 0$, which implies (2.26). The proof of Lemma 2.8 is complete.

LEMMA 2.9. *Let $x \in C[\tau(T), \infty)$, $T \ge a$, and $i$ be a positive integer. Suppose that (2.23) is satisfied. If $\lim_{t\to\infty} (Lx)(t)/t^i \equiv \lim_{t\to\infty} [x(t) - h(t)x(\tau(t))]/t^i$ exists and is a nonzero finite value, then $\lim_{t\to\infty} x(t)/t^i$ exists and is a nonzero finite value.*

PROOF. Note that (2.22) holds, and employ Lemma 2.8 with $x(t)$ and $h(t)$ replaced by $x(t)/t^i$ and $h(t)[\tau(t)/t]^i$.

## 3. Classification of nonoscillatory solutions

In this section we classify nonoscillatory solutions $x$ of (1.1) according to the asymptotic behavior of $(Lx)(t) \equiv x(t) - h(t)x(\tau(t))$ as $t \to \infty$. Some of the results in this section have been obtained by Jaroš and Kusano [9]. However we write the full proofs since a part of the proof is different from [9]. We make use of the following well-known lemma of Kiguradze.

LEMMA 3.1 (Kiguradze [10]). *Let* $n \geq 2$ *and* $\sigma = 1$ *or* $-1$ *and let* $u \in C[T, \infty)$ *satisfy*

$$\sigma u(t)u^{(n)}(t) < 0, \qquad t \geq T.$$

*Then there exist an integer* $j \in \{0, 1, 2, \ldots, n\}$ *and a number* $t_0 \geq T$ *such that* $(-1)^{n-j-1}\sigma = 1$ *and*

$$\begin{cases} u(t)u^{(i)}(t) > 0, & t \geq t_0, \quad 0 \leq i \leq j, \\ (-1)^{i-j}u(t)u^{(i)}(t) > 0, & t \geq t_0, \quad j \leq i \leq n. \end{cases}$$

THEOREM 3.1. *Let* $x$ *be a nonoscillatory solution of* (1.1). *Then one of the following two cases holds*:

(I) *There are an integer* $j$ *with* $0 \leq j \leq n$, $(-1)^{n-j-1}\sigma = 1$ *and a number* $t_0 \geq a$ *such that*

$$x(t)(Lx)(t) > 0, \qquad t \geq t_0, \tag{3.1}$$

$$\begin{cases} (Lx)(t)(Lx)^{(i)}(t) > 0, & t \geq t_0, \quad 0 \leq i \leq j, \\ (-1)^{i-j}(Lx)(t)(Lx)^{(i)}(t) > 0, & t \geq t_0, \quad j \leq i \leq n; \end{cases} \tag{3.2}$$

(II) *There is a number* $t_0 \geq a$ *such that*

$$x(t)(Lx)(t) < 0, \qquad t \geq t_0, \tag{3.3}$$

$$(-1)^i(Lx)(t)(Lx)^{(i)}(t) > 0, \qquad t \geq t_0, \qquad 0 \leq i \leq n \tag{3.4}$$

*and*

$$\lim_{t\to\infty} (Lx)(t) = 0, \qquad \lim_{t\to\infty} x(t) = 0. \tag{3.5}$$

*Furthermore the case* (II) *can hold only when* $(-1)^n\sigma = 1$ *and* $h(t)$ *is eventually positive.*

PROOF. We may assume that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq T_0$ $(\geq a)$. By equation (1.1) we see that $(Lx)^{(n)}(t) = -\sigma p(t)f(x(g(t)))$ is either positive or negative for $t \geq T_0$. Therefore $Lx$ is either decreasing or increasing on $[T_1, \infty)$ for some large $T_1 \geq T_0$. We have the following two possibilities:

$$\text{(I)} \quad (Lx)(t) > 0 \qquad \text{for} \quad t \geq T_2;$$

$$\text{(II)} \quad (Lx)(t) < 0 \qquad \text{for} \quad t \geq T_2,$$

where $T_2$ $(\geq T_1)$ is sufficiently large.

In the case of (I) we have $\sigma(Lx)(t)(Lx)^{(n)}(t) < 0$ for $t \geq T_2$. Applying Lemma 3.1 to the case of $T = T_2$ and $u(t) = (Lx)(t)$, we conclude that there are $j \in \{0, 1, 2, \ldots, n\}$ and $t_0 \geq T_2$ satisfying $(-1)^{n-j-1}\sigma = 1$ and (3.2).

In the case of (II) we have $(-\sigma)(Lx)(t)(Lx)^{(n)}(t) < 0$ for $t \geq T_2$. Lemma 3.1 with $\sigma$, $T$ and $u(t)$ replaced by $-\sigma$, $T_2$ and $(Lx)(t)$, respectively, shows that there are $j \in \{0, 1, 2, \dots, n\}$ and $t_0 \geq T_2$ such that $(-1)^{n-j}\sigma = 1$ and

$$\begin{cases} (Lx)(t)(Lx)^{(i)}(t) > 0\,, & t \geq t_0\,, \quad 0 \leq i \leq j\,, \\ (-1)^{i-j}(Lx)(t)(Lx)^{(i)}(t) > 0\,, & t \geq t_0\,, \quad j \leq i \leq n\,. \end{cases}$$

We claim that $j = 0$. Otherwise, we have $(Lx)(t) < 0$ and $(Lx)'(t) < 0$ for $t \geq t_0$. Then $\lim_{t\to\infty}(Lx)(t) = l$ exists and $l$ satisfies $-\infty \leq l < 0$. On the other hand, Lemma 2.5 implies that $0 \leq l \leq \infty$. This contradiction asserts that the case $0 < j \leq n$ is impossible. Since $j=0$ in the above, we have $(-1)^n\sigma = 1$. Further, since $x(t) > 0$ and $(Lx)(t) < 0$ for $t \geq t_0$, Remark 2.1 implies that $\lim_{t\to\infty} x(t) = 0$ and that the case (II) can occur only when $h(t)$ is eventually positive. The proof of Theorem 3.1 is complete.

Definition 3.1. Let $\mathcal{N}$ denote the set of all nonoscillatory solutions of (1.1). For an integer $j$ with $0 \leq j \leq n$ and $(-1)^{n-j-1}\sigma = 1$, we denote by $\mathcal{N}_j$ the set of all nonoscillatory solutions $x$ of (1.1) which satisfy (3.1) and (3.2). In addition, we denote by $\mathcal{N}_0^-$ the set of all nonoscillatory solutions $x$ of (1.1) which satisfy (3.3)–(3.5).

Theorem 3.1 means that every nonoscillatory solution $x \in \mathcal{N}$ falls into one and only one of the classes $\mathcal{N}_j$ $(0 \leq j \leq n, (-1)^{n-j-1}\sigma = 1)$ and $\mathcal{N}_0^-$. More precisely, $\mathcal{N}$ has the following decomposition:

$$\begin{aligned} &\mathcal{N} = \mathcal{N}_{n-1} \cup \mathcal{N}_{n-3} \cup \cdots \cup \mathcal{N}_1 \cup \mathcal{N}_0^- && \text{for } \sigma = 1 \text{ and } n \text{ is even}\,; \\ &\mathcal{N} = \mathcal{N}_{n-1} \cup \mathcal{N}_{n-3} \cup \cdots \cup \mathcal{N}_2 \cup \mathcal{N}_0 && \text{for } \sigma = 1 \text{ and } n \text{ is odd}\,; \\ &\mathcal{N} = \mathcal{N}_{n} \cup \mathcal{N}_{n-2} \cup \cdots \cup \mathcal{N}_2 \cup \mathcal{N}_0 && \text{for } \sigma = -1 \text{ and } n \text{ is even}\,; \\ &\mathcal{N} = \mathcal{N}_{n} \cup \mathcal{N}_{n-2} \cup \cdots \cup \mathcal{N}_1 \cup \mathcal{N}_0^- && \text{for } \sigma = -1 \text{ and } n \text{ is odd}\,, \end{aligned}$$

where $\mathcal{N}_0^-$ can appear only when $h(t)$ is eventually positive.

Let $x \in \mathcal{N}_j$. Then we see by (3.2) that the asymptotic behavior of $(Lx)(t)$ as $t \to \infty$ is as follows:

(i) If $j = 0$, then either

(i-1) $$\lim_{t\to\infty}(Lx)(t) = \text{const} \neq 0 \quad \text{or}$$

(i-2) $$\lim_{t\to\infty}(Lx)(t) = 0\,.$$

(ii) If $1 \leq j \leq n-1$, then one of the following three cases holds:

(ii-1) $$\lim_{t\to\infty} \frac{(Lx)(t)}{t^j} = \text{const} \neq 0\,;$$

(ii-2) $$\lim_{t\to\infty} \frac{(Lx)(t)}{t^{j-1}} = \text{const} \neq 0\,;$$

(ii-3) $$\lim_{t\to\infty} \frac{(Lx)(t)}{t^j} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{|(Lx)(t)|}{t^{j-1}} = \infty\,.$$

(iii) If $j = n$, then either

(iii-1) $$\lim_{t\to\infty} \frac{(Lx)(t)}{t^{n-1}} = \text{const} \neq 0 \quad \text{or}$$

(iii-2) $$\lim_{t\to\infty} \frac{|(Lx)(t)|}{t^{n-1}} = \infty\,.$$

Notice that the function $|(Lx)(t)|/t^{j-1}$ in (ii-3) is eventually nondecreasing (see Kusano and Natio [13, Lemma, p. 365]). Arguing as in [13], we can prove that $|(Lx)(t)|/t^{n-1}$ in (iii-2) is also eventually nondecreasing. From (i)–(iii) of Lemma 2.7 we find that the asymptotic behavior of $x$ as $t \to \infty$ is as follows:

(i) If $x \in \mathcal{N}_0$, then either

(i-1) $$0 < \liminf_{t\to\infty} |x(t)| \leq \limsup_{t\to\infty} |x(t)| < \infty \quad \text{or}$$

(i-2) $$\lim_{t\to\infty} x(t) = 0\,.$$

(ii) If $x \in \mathcal{N}_j$, $1 \leq j \leq n-1$, then one of the following three cases holds:

(ii-1) $$0 < \liminf_{t\to\infty} \frac{|x(t)|}{t^j} \leq \limsup_{t\to\infty} \frac{|x(t)|}{t^j} < \infty\,;$$

(ii-2) $$0 < \liminf_{t\to\infty} \frac{|x(t)|}{t^{j-1}} \leq \limsup_{t\to\infty} \frac{|x(t)|}{t^{j-1}} < \infty\,;$$

(ii-3) $$\lim_{t\to\infty} \frac{x(t)}{t^j} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{|x(t)|}{t^{j-1}} = \infty\,.$$

(iii) If $x \in \mathcal{N}_n$, then either

(iii-1) $$0 < \liminf_{t\to\infty} \frac{|x(t)|}{t^{n-1}} \leq \limsup_{t\to\infty} \frac{|x(t)|}{t^{n-1}} < \infty \quad \text{or}$$

(iii-2) $$\lim_{t\to\infty} \frac{|x(t)|}{t^{n-1}} = \infty\,.$$

Now consider the case where the next condition holds:

(3.6) $\lim_{t\to\infty} h(t)[\tau(t)/t]^i$ exist and are finite for all $i = 0, 1, 2, \ldots, n-1$.

Condition (3.6) is certainly satisfied if $\lim_{t\to\infty} h(t) = 0$, or if both $\lim_{t\to\infty} h(t)$ and $\lim_{t\to\infty} \tau(t)/t$ exist and are finite. If (3.6) holds, then we can ultilize Lemma 2.8 and Lemma 2.9 instead of (iii) of Lemma 2.7. Then we conclude that, under condition (3.6), the asymptotic behavior of a solution $x$ belonging to $\mathcal{N}_j$ is as follows:

(i) If $j = 0$, then either

(i-1) $$\lim_{t\to\infty} x(t) = \text{const} \neq 0 \quad \text{or}$$

(i-2) $$\lim_{t\to\infty} x(t) = 0\,.$$

(ii) If $1 \leq j \leq n-1$, then one of the following three cases holds:

(ii-1) $$\lim_{t\to\infty} \frac{x(t)}{t^j} = \text{const} \neq 0\,;$$

(ii-2) $$\lim_{t\to\infty} \frac{x(t)}{t^{j-1}} = \text{const} \neq 0\,;$$

(ii-3) $$\lim_{t\to\infty} \frac{x(t)}{t^j} = 0 \qquad \text{and} \qquad \lim_{t\to\infty} \frac{|x(t)|}{t^{j-1}} = \infty\,.$$

(iii) If $j = n$, then either

(iii-1) $$\lim_{t\to\infty} \frac{x(t)}{t^{n-1}} = \text{const} \neq 0 \quad \text{or}$$

(iii-2) $$\lim_{t\to\infty} \frac{|x(t)|}{t^{n-1}} = \infty\,.$$

It is worth while to note that, if (3.6) is satisfied, the structure of the nonoscillatory solutions of the neutral equation (1.1) is exactly the same as that of the nonoscillatory solutions of the non-neutral equation (1.7) or (1.8) with the exception of the $\mathcal{N}_0^-$ for (1.1). For the structure of the nonoscillatory solutions of (1.8), see, for example, [13].

## 4. Nonoscillatory solutions asymptotic to $t^k$

The aim of this section is to find, for each $k = 0, 1, 2, \ldots, n-1$, a necessary and sufficient condition for the existence of a nonoscillatory solution $x$ of (1.1) which behaves like $t^k$ as $t \to \infty$, i.e., a solution $x$ of (1.1) satisfying

$$0 < \liminf_{t\to\infty} \frac{|x(t)|}{t^k} \le \limsup_{t\to\infty} \frac{|x(t)|}{t^k} < \infty .$$

Hereafter, in addition to conditions (1.2)–(1.6), we assume the next conditions (4.1) and (4.2):

(4.1) $\tau$ is locally Lipschitz continuous on $[a, \infty)$;

(4.2) $h$ is locally Lipschitz continuous on $[\tau(a), \infty)$.

First we consider the case of $k = 0$.

THEOREM 4.1. *Assume that* (1.2)–(1.6), (4.1) *and* (4.2) *are satisfied. Then equation* (1.1) *has a nonoscillatory solution $x$ such that*

$$0 < \liminf_{t\to\infty} |x(t)| \le \limsup_{t\to\infty} |x(t)| < \infty \tag{4.3}$$

*if and only if*

$$\int^{\infty} t^{n-1} p(t)\, dt < \infty . \tag{4.4}$$

PROOF. (The "only if" part.) Let $x$ be a nonoscillatory solution of (1.1) having the property (4.3). We may assume that $x(t)$ and $x(g(t))$ are positive on $[T, \infty)$ for some $T \ge a$. We easily find that

$$\lim_{t\to\infty} (Lx)^{(i)}(t) = 0 \qquad \text{for} \quad i = 1, 2, \ldots, n-1 .$$

Therefore, integrating (1.1) repeatedly from $t$ to $\infty$, we have

$$(Lx)^{(i)}(t) = (-1)^{n-i-1}\sigma \int_t^{\infty} \frac{(s-t)^{n-i-1}}{(n-i-1)!} p(s) f(x(g(s)))\, ds , \qquad t \ge T, \tag{4.5}$$

for $i = 1, 2, \ldots, n-1$. Noting that $\lim_{t\to\infty} (Lx)(t)$ exists and is finite and integrating (4.5) with $i = 1$ from $t$ to $\infty$, we obtain

$$(Lx)(t) = (Lx)(\infty) + (-1)^{n-1}\sigma \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} p(s) f(x(g(s)))\, ds , \qquad t \ge T,$$

where $(Lx)(\infty) = \lim_{t\to\infty} (Lx)(t)$. Then we see that

$$\int_T^{\infty} (s-T)^{n-1} p(s) f(x(g(s)))\, ds < \infty . \tag{4.6}$$

In view of (4.3) there are positive constants $c_1$ and $c_2$ such that

$$c_1 \le x(g(t)) \le c_2 \qquad \text{for} \quad t \ge T . \tag{4.7}$$

From (4.6) and (4.7) it follows that

$$f_* \int_T^{\infty} (s-T)^{n-1} p(s)\, ds < \infty ,$$

where $f_* = \min\{f(u): c_1 \le u \le c_2\} > 0$. Thus we get (4.4).

(The "if" part.) Let $c > 0$ be an arbitrary positive number. Put $\mu = (1-h)/3$, where $h$ is a constant appearing in assumption (1.3) and put $f^* = \max\{f(u): \mu c \le u \le c/(3\mu)\}$. Choose $T \ge 1$ so large that

$$\tilde{T} \equiv \min\{\tau(T), \inf_{t \ge T} g(t)\} \ge \max\{a, 0\} \quad \text{and} \tag{4.8}$$

$$\int_T^{\infty} t^{n-1} p(t)\, dt < \frac{(n-1)!\, \mu c}{f^*}. \tag{4.9}$$

For this $T$, let $T_i(T)$, $i = 0, 1, 2, \ldots$, be real numbers defined by (2.1). The solution $x$ of (1.1) satisfying (4.3) will be obtained as a solution $x$ of the integral equation

$$\begin{aligned} x(t) = h(t)x(\tau(t)) &+ (1-\mu)c \\ &+ (-1)^{n-1}\sigma \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} p(s) f(x(g(s)))\, ds, \qquad t \ge T_1(T). \end{aligned} \tag{4.10}$$

Since we are going to get a function $x$ which satisfies (4.10) for $t \ge T_1(T)$, there is no loss of generality in supposing that $h(t)$ satisfies, besides assumption (1.3),

$$h(t) = 0 \qquad \text{for} \qquad \tilde{T} \le t \le T. \tag{4.11}$$

In fact, if $h(t)$ does not satisfy (4.11), then we may replace $h(t)$ in (4.10) by $\tilde{h}(t)$ defined as follows:

$$\tilde{h}(t) = \begin{cases} 0, & \tilde{T} \le t \le T, \\ h(t)(t-T)/(T_1(T)-T), & T \le t \le T_1(T), \\ h(t), & t \ge T_1(T). \end{cases} \tag{4.12}$$

We define the auxiliary function $n(t)$ on $[\tilde{T}, \infty)$ by

$$n(t) = \begin{cases} 1 & \text{if} \quad \tilde{T} \le t \le T, \\ 1 & \text{if} \quad h(t) < 0 \quad \text{and} \quad t > T, \\ \sum_{i=0}^{l} H_i(t) & \text{if} \quad h(t) \ge 0 \quad \text{and} \quad T_{l-1}(T) < t \le T_l(T), \quad l = 1, 2, \ldots, \end{cases} \tag{4.13}$$

where $H_i(t)$, $i = 0, 1, 2, \ldots$, are given by (2.4). Since $n(t) \le \sum_{i=0}^{l} h^i$ for

$T_{l-1}(T) < t \le T_l(T)$, $l = 1, 2, \dots$, we have

$$n(t) \le 1/(1-h) = 1/(3\mu)\,, \qquad t \ge \tilde{T}\,. \tag{4.14}$$

Furthermore, using the condition $h(t)h(\tau(t)) \ge 0$, $t \ge T$, in assumption (1.3), we have

$$n(t) \ge 1\,, \qquad t \ge \tilde{T}\,. \tag{4.15}$$

It is also verified that if $t$ satisfies $h(t) \ge 0$ and $t \ge T$, then

$$h(t)n(\tau(t)) = n(t) - 1\,. \tag{4.16}$$

For the proof of (4.16) we note that $h(t)H_i(\tau(t)) = H_{i+1}(t)$, $i = 0, 1, 2, \dots$, $t \ge T$. If $t$ satisfies $h(t) < 0$ and $t \ge T$, then

$$h(t)n(\tau(t)) = h(t) \ge -h = 3\mu - 1\,. \tag{4.17}$$

Let $L_l^\tau$ and $L_l^h$, $l = 1, 2, \cdots$, be Lipschitz constants for $\tau(t)$ and $h(t)$ on $[T, T_l(T)]$, respectively, i.e.,

$$|\tau(t) - \tau(s)| \le L_l^\tau |t - s| \qquad \text{for} \qquad T \le s, t \le T_l(T)\,, \tag{4.18}$$

$$|h(t) - h(s)| \le L_l^h |t - s| \qquad \text{for} \qquad T \le s, t \le T_l(T)\,. \tag{4.19}$$

We may suppose that $L_l^\tau \ge 1$, $l = 1, 2, \dots$. Define $m(t)$ on $[\tilde{T}, \infty)$ by

$$m(t) = \begin{cases} 0 & \text{for } \tilde{T} \le t \le T\,, \\ L_l^\tau m(\tau(t)) + \dfrac{L_l^h}{3\mu} + (n-1)\mu & \text{for } T_{l-1}(T) < t \le T_l(T)\,,\ l = 1, 2, \dots . \end{cases} \tag{4.20}$$

Observe that $m$ can be inductively determined as follows: If $t \in (T, T_1(T)]$, then, since $\tau(t) \in (\tau(T), T]$, $m(\tau(t))$ is known; and so $m(t)$ is known on $(T, T_1(T)]$. Let $m(t)$ be known on $(T_{l-1}(T), T_l(T)]$ for some $l$, then, since $\tau(t) \in (T_{l-1}(T), T_l(T)]$, $m(\tau(t))$ is known; and so $m(t)$ is known on $(T_l(T), T_{l+1}(T)]$. Thus $m(t)$ is known for all $t \ge \tilde{T}$. We can easily show that $m$ is a nonnegative nondecreasing step function on $[\tilde{T}, \infty)$. Let $C[\tilde{T}, \infty)$ denote the Fréchet space of all continuous functions on $[\tilde{T}, \infty)$ with the topology of uniform convergence on any compact subintervals of $[\tilde{T}, \infty)$. Consider the set $X$ of all $x \in C[\tilde{T}, \infty)$ satisfying

$$\mu c \le x(t) \le cn(t) \qquad \text{for} \quad t \ge \tilde{T}$$

and

$$|x(t_2) - x(t_1)| \le cm(t_2)|t_2 - t_1| \qquad \text{for} \quad t_2 > t_1 \ge \tilde{T}\,.$$

Clearly $X$ is a nonempty, convex and compact subset of $C[\tilde{T}, \infty)$. We define the operator $\mathscr{F}$ on $X$ in the following manner:

$(\mathscr{F}x)(t)$

$$= \begin{cases} h(t)x(\tau(t)) + (1-\mu)c + (-1)^{n-1}\sigma \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} p(s)f(x(g(s)))\,ds & \text{for } t \geq T, \\ (1-\mu)c + (-1)^{n-1}\sigma \displaystyle\int_T^\infty \frac{(s-T)^{n-1}}{(n-1)!} p(s)f(x(g(s)))\,ds & \text{for } \tilde{T} \leq t \leq T. \end{cases}$$

It is easy to see that $\mathscr{F}x$ is well defined on $[\tilde{T}, \infty)$ for each $x \in X$. We seek a fixed point of $\mathscr{F}$ in $X$ with the aid of the Schauder-Tychonoff fixed point theorem.

First we show that $\mathscr{F}$ maps $X$ into $X$. Assume that $x \in X$. Since we suppose that (4.11) holds, $\mathscr{F}x$ is clearly continuous on $[\tilde{T}, \infty)$. We have to verify that

$$\mu c \leq (\mathscr{F}x)(t) \leq cn(t) \quad \text{for} \quad t \geq \tilde{T} \tag{4.21}$$

and

$$|(\mathscr{F}x)(t_2) - (\mathscr{F}x)(t_1)| \leq cm(t_2)|t_2 - t_1| \quad \text{for} \quad t_2 > t_1 \geq \tilde{T}. \tag{4.22}$$

Note by (4.8) and (4.14) that $\mu c \leq x(t) \leq c/(3\mu)$ for $t \geq \tilde{T}$ and so $\mu c \leq x(g(t)) \leq c/(3\mu)$ for $t \geq T$. Let

$$G(t) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} p(s)f(x(g(s)))\,ds \quad \text{for} \quad t \geq T. \tag{4.23}$$

Then, by (4.9), $G$ satisfies

$$|G(t)| \leq \frac{f^*}{(n-1)!} \int_T^\infty s^{n-1}p(s)\,ds \leq \mu c \quad \text{for} \quad t \geq T.$$

Notice that $\mathscr{F}x$ is written as

$$(\mathscr{F}x)(t) = \begin{cases} h(t)x(\tau(t)) + (1-\mu)c + (-1)^{n-1}\sigma G(t) & \text{for } t \geq T, \\ (1-\mu)c + (-1)^{n-1}\sigma G(T) & \text{for } \tilde{T} \leq t \leq T. \end{cases}$$

If $t$ satisfies $t \geq T$ and $h(t) \geq 0$, then, in view of (4.16),

$$\begin{aligned} (\mathscr{F}x)(t) &\leq ch(t)n(\tau(t)) + (1-\mu)c + \mu c \\ &= c[n(t) - 1] + c = cn(t) \end{aligned}$$

and

$$(\mathscr{F}x)(t) \geq (1-\mu)c - \mu c \geq \mu c\,.$$

If $t$ satisfies $t \geq T$ and $h(t) < 0$, then, in view of (4.17),

$$(\mathscr{F}x)(t) \leq (1-\mu)c + \mu c = c = cn(t)$$

and

$$(\mathscr{F}x)(t) \geq ch(t)n(\tau(t)) + (1-\mu)c - \mu c$$
$$\geq c(3\mu - 1) + (1-\mu)c - \mu c = \mu c .$$

Then we get

$$\mu c \leq (\mathscr{F}x)(t) \leq cn(t) \qquad \text{for} \quad t \geq T, \tag{4.24}$$

and in particular $\mu c \leq (\mathscr{F}x)(T) \leq cn(T)$. Since $(\mathscr{F}x)(t) = (\mathscr{F}x)(T)$ and $n(t) = n(T)$ for $\tilde{T} \leq t \leq T$ we have

$$\mu c \leq (\mathscr{F}x)(t) \leq cn(t) \qquad \text{for} \quad \tilde{T} \leq t \leq T. \tag{4.25}$$

Then inequalities (4.24) and (4.25) together yield (4.21).

Let $G$ be the function defined by (4.23). Since

$$|G'(t)| = \int_t^\infty \frac{(s-t)^{n-2}}{(n-2)!} p(s)f(x(g(s)))\, ds$$
$$\leq \frac{f^*}{(n-2)!} \int_T^\infty s^{n-2} p(s)\, ds \leq (n-1)\mu c \qquad \text{for} \quad t \geq T,$$

the mean value theorem gives

$$|G(t_2) - G(t_1)| \leq (n-1)\mu c |t_2 - t_1| \qquad \text{for} \quad t_2 > t_1 \geq T. \tag{4.26}$$

If $\tilde{T} \leq t_1 < t_2 \leq T$, then

$$|(\mathscr{F}x)(t_2) - (\mathscr{F}x)(t_1)| = 0 .$$

If $T \leq t_1 < t_2$ and $T_{l-1}(T) < t_2 \leq T_l(T)$, $l = 1, 2, \ldots$, then, in view of (4.18)–(4.20) and (4.26),

$$|(\mathscr{F}x)(t_2) - (\mathscr{F}x)(t_1)| \leq |h(t_1)||x(\tau(t_2)) - x(\tau(t_1))| + |x(\tau(t_2))||h(t_2) - h(t_1)|$$
$$+ |G(t_2) - G(t_1)|$$
$$\leq cm(\tau(t_2))|\tau(t_2) - \tau(t_1)| + \frac{c}{3\mu}|h(t_2) - h(t_1)|$$
$$+ |G(t_2) - G(t_1)|$$
$$\leq c\left[ L_l^\tau m(\tau(t_2)) + \frac{L_l^h}{3\mu} + (n-1)\mu \right] |t_2 - t_1|$$
$$= cm(t_2)|t_2 - t_1| .$$

Therefore we see that (4.22) is satisfied.

Furthermore it can be shown without difficulty that $\mathscr{F}$ is continuous on $X$. By the Schauder-Tychonoff fixed point theorem there exists an $x \in X$ such that $x = \mathscr{F}x$. This function $x$ satisfies (4.10). It is clear that

$$\lim_{t\to\infty} (Lx)(t) = \lim_{t\to\infty} [x(t) - h(t)x(\tau(t))] = (1 - \mu)c .$$

From (ii) of Lemma 2.6 we see that $x$ satisfies (4.3). The proof of Theorem 4.1 is complete.

The solution $x$ of (1.1) which is obtained in the proof of the "if" part of Theorem 4.1 satisfies $\lim_{t\to\infty} (Lx)(t) = (1 - \mu)c \neq 0$. Therefore by Lemma 2.8 we get the next corollary.

COROLLARY 4.1. *In addition to* (1.2)–(1.6), (4.1) *and* (4.2), *assume that* $\lim_{t\to\infty} h(t)$ *exists and is finite. Then equation* (1.1) *has a nonoscillatory solution* $x$ *such that*

$$\lim_{t\to\infty} x(t) = \text{const} \neq 0$$

*if and only if* (4.4) *holds.*

Next we consider the case of $1 \leq k \leq n - 1$. In this case equation (1.1) is required to be either sublinear or superlinear. Here the sublinearity and superlinearity of (1.1) are defined by the following:

DEFINITION 4.1. Equation (1.1) is called *sublinear* if $f$ in (1.1) satisfies

$$\frac{|f(u_1)|}{|u_1|} \geq \frac{|f(u_2)|}{|u_2|} \quad \text{for} \quad |u_2| > |u_1| , \qquad u_1 u_2 > 0 ;$$

and equation (1.1) is called *superlinear* if $f$ satisfies

$$\frac{|f(u_1)|}{|u_1|} \leq \frac{|f(u_2)|}{|u_2|} \quad \text{for} \quad |u_2| > |u_1| , \qquad u_1 u_2 > 0 .$$

Clearly equation (1.9) is sublinear if $-\infty < \gamma \leq 1$ and is superlinear if $1 \leq \gamma < \infty$.

THEOREM 4.2. *Assume that* (1.2)–(1.6), (4.1) *and* (4.2) *are satisfied. Let* (1.1) *be either sublinear or superlinear and let* $k$ *be an integer with* $1 \leq k \leq n - 1$. *Then equation* (1.1) *has a nonoscillatory solution* $x$ *such that*

$$0 < \liminf_{t\to\infty} \frac{|x(t)|}{t^k} \leq \limsup_{t\to\infty} \frac{|x(t)|}{t^k} < \infty \tag{4.27}$$

*if and only if*

$$\int^{\infty} t^{n-k-1}p(t)|f(c[g(t)]^k)|\,dt < \infty \qquad \text{for some} \quad c \neq 0\,. \tag{4.28}$$

PROOF. (The "only if" part.) Let $x$ be a solution of (1.1) satisfying (4.27). Without loss of generaity we may assume that $x$ is eventually positive. Then we can take a number $T \geq a$ such that $x(t) > 0$, $x(g(t)) > 0$ and $g(t) > 0$ for $t \geq T$. We note that $\lim_{t\to\infty} (Lx)^{(i)}(t) = 0$, $i = k+1,\ k+2,\ \dots,\ n-1$ and $\lim_{t\to\infty} (Lx)^{(k)}(t)$ exists and is a finite value. Integrating (1.1) repeatedly from $t$ to $\infty$, we obtain

$$(Lx)^{(k)}(t) = (Lx)^{(k)}(\infty) + (-1)^{n-k-1}\sigma \int_t^{\infty} \frac{(s-t)^{n-k-1}}{(n-k-1)!} p(s)f(x(g(s)))\,ds$$

for $t \geq T$, where $(Lx)^{(k)}(\infty) = \lim_{t\to\infty} (Lx)^{(k)}(t)$. Thus we have

$$\int_T^{\infty} (s-t)^{n-k-1}p(s)f(x(g(s)))\,ds < \infty\,. \tag{4.29}$$

In view of (4.27) there are positive constants $c_1$ and $c_2$ such that

$$c_1[g(t)]^k \leq x(g(t)) \leq c_2[g(t)]^k \qquad \text{for} \quad t \geq T\,. \tag{4.30}$$

From (4.29) and (4.30) it follows that (4.28) is satisfied for $c = c_2$ if (1.1) is sublinear and for $c = c_1$ if (1.1) is superlinear.

(The "if" part.) Without loss of generality we may assume that $c$ in (4.28) is positive. Let $\mu = (1-h)/3$. Set $c^* = c/\mu$ if (1.1) is sublinear and $c^* = 3\mu c$ if (1.1) is superlinear. Choose $T$ so large that (4.8) and

$$\int_T^{\infty} t^{n-k-1}p(t)f(c[g(t)]^k)\,dt < 3k!(n-k-1)!\mu^2 c \tag{4.31}$$

hold. We shall obtain a solution $x$ of (1.1) satisfying (4.27) as a solution $x$ of the integral equation

$$\begin{aligned} x(t) &= h(t)x(\tau(t)) + (1-\mu)c^* t^k \\ &+ (-1)^{n-k-1}\sigma \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(u-s)^{n-k-1}}{(n-k-1)!} p(u)f(x(g(u)))\,du\,ds\,, \qquad t \geq T_1(T)\,. \end{aligned} \tag{4.32}$$

Arguing as in the proof of Theorem 4.1, we may suppose that (4.11) is satisfied. Let $n(t)$ be the function on $[\tilde{T}, \infty)$ defined by (4.13), where $\tilde{T}$ is a constant in (4.8). Let $L_l^{\tau}$ and $L_l^{h}$ $(l = 1, 2, \dots)$ be the real numbers satisfying (4.18), (4.19) and $L_l^{\tau} \geq 1$, $l = 1, 2, \dots$. We define $m(t)$ on $[\tilde{T}, \infty)$ as follows:

(4.33)

$$m(t) = \begin{cases} kt^{k-1} & \text{for } \tilde{T} \le t \le T, \\ L_l^\tau m(\tau(t)) + \dfrac{L_l^h}{3\mu}[\tau(t)]^k + kt^{k-1} & \text{for } T_{l-1}(T) < t \le T_l(T), \quad l = 1, 2, \dots. \end{cases}$$

Notice that $m(t)$ can be inductively determined on $[\tilde{T}, \infty)$ and that $m(t)$ is a positive nondecreasing piecewise continuous function on $[\tilde{T}, \infty)$. Let $C[\tilde{T}, \infty)$ be the Fréchet space as mentioned in the proof of Theorem 4.1. We denote by $X$ the set of all $x \in C[\tilde{T}, \infty)$ satisfying

$$\mu c^* t^k \le x(t) \le c^* n(t) t^k \qquad \text{for} \quad t \ge \tilde{T},$$

and

$$|x(t_2) - x(t_1)| \le c^* m(t_2)|t_2 - t_1| \qquad \text{for} \quad t_2 > t_1 \ge \tilde{T}.$$

The set $X$ is a nonempty, convex and compact subset of $C[\tilde{T}, \infty)$. We define the operator $\mathscr{F}$ on $X$ in the following manner:

$$(\mathscr{F}x)(t) = \begin{cases} h(t)x(\tau(t)) + (1-\mu)c^*t^k \\ \quad + (-1)^{n-k-1}\sigma \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(u-s)^{n-k-1}}{(n-k-1)!} p(u) f(x(g(u)))\, du\, ds & \text{for } t \ge T, \\ (1-\mu)c^*t^k \qquad \text{for } \tilde{T} \le t \le T. \end{cases}$$

We show that $\mathscr{F}$ maps $X$ into itself. Assume that $x \in X$. By (4.11), $\mathscr{F}x$ belongs to $C[\tilde{T}, \infty)$. Noting (4.14), we find that $\mu c^* t^k \le x(t) \le c^* t^k/(3\mu)$ for $t \ge T$; that is, if (1.1) is sublinear then $ct^k \le x(t) \le ct^k/(3\mu^2)$ for $t \ge T$, and if (1.1) is superlinear then $3\mu^2 ct^k \le x(t) \le ct^k$ for $t \ge T$. Set

$$G(t) = \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(u-s)^{n-k-1}}{(n-k-1)!} p(u) f(x(g(u)))\, du\, ds, \qquad t \ge T. \tag{4.34}$$

Then

$$\begin{aligned} |G(t)| &\le \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} ds \int_T^\infty \frac{(u-T)^{n-k-1}}{(n-k-1)!} p(u) f(x(g(u)))\, du \\ &\le \frac{t^k}{k!(n-k-1)!} \int_T^\infty s^{n-k-1} p(s) f(x(g(s)))\, ds \end{aligned}$$

for $t \ge T$. Therefore, condition (4.31) gives

$$\begin{aligned} |G(t)| &\le \frac{t^k}{k!(n-k-1)!3\mu^2} \int_T^\infty s^{n-k-1} p(s) f(c[g(s)]^k)\, ds \\ &\le ct^k = \mu c^* t^k, \qquad t \ge T, \end{aligned}$$

in the case where (1.1) is sublinear, and

$$|G(t)| \le \frac{t^k}{k!(n-k-1)!} \int_T^\infty s^{n-k-1} p(s) f(c[g(s)]^k)\, ds$$
$$\le 3\mu^2 c t^k = \mu c^* t^k\,, \qquad t \ge T,$$

in the case where (1.1) is superlinear. In either case, we have

$$|G(t)| \le \mu c^* t^k \qquad \text{for} \quad t \ge T. \tag{4.35}$$

Then it can be shown that

$$\mu c^* t^k \le (\mathscr{F}x)(t) \le c^* n(t) t^k \qquad \text{for} \quad t \ge \tilde{T} \tag{4.36}$$

by using (4.35) and the same argument as in the proof of Theorem 4.1.

If $\tilde{T} \le t_1 < t_2 \le T$, then

$$\begin{aligned} |(\mathscr{F}x)(t_2) - (\mathscr{F}x)(t_1)| &= (1-\mu)c^* |t_2^k - t_1^k| \\ &\le (1-\mu)c^* k t_2^{k-1} |t_2 - t_1| \\ &\le c^* m(t_2) |t_2 - t_1|\,, \end{aligned} \tag{4.37}$$

where we have used the mean value theorem for $t^k$. The derivative of $G$ defined by (4.34) is given by the following:

$$G'(t) = \int_t^\infty \frac{(s-t)^{n-2}}{(n-2)!} p(s) f(x(g(s)))\, ds\,, \qquad t \ge T$$

for the case of $k = 1$, and

$$G'(t) = \int_T^t \frac{(t-s)^{k-2}}{(k-2)!} \int_s^\infty \frac{(u-s)^{n-k-1}}{(n-k-1)!} p(u) f(x(g(u)))\, du\, ds\,, \qquad t \ge T$$

for the case of $2 \le k \le n-1$. Therefore we have

$$|G'(t)| \le \frac{1}{(n-2)!} \int_t^\infty s^{n-2} p(s) f(x(g(s)))\, ds$$
$$\le \mu c^*\,, \qquad t \ge T$$

for the case of $k = 1$, and

$$|G'(t)| \le \frac{t^{k-1}}{(k-1)!(n-k-1)!} \int_t^\infty s^{n-k-1} p(s) f(x(g(s)))\, ds$$
$$\le \mu c^* k t^{k-1}\,, \qquad t \ge T$$

for the case of $2 \le k \le n-1$. From the above, we get

$$|G'(t)| \le \mu c^* k t^{k-1} \qquad \text{for} \quad t \ge T \quad \text{and} \quad 1 \le k \le n-1\,.$$

By the mean value theorem we obtain

$$|G(t_2) - G(t_1)| \le \mu c^* k t_2^{k-1} |t_2 - t_1| \quad \text{for} \quad t_2 > t_1 \ge T \quad \text{and} \quad 1 \le k \le n-1 . \tag{4.38}$$

Let $t_2 > t_1 \ge T$ and $T_{l-1}(T) < t_2 \le T_l(T)$, $l = 1, 2, \ldots$. Then we see by (4.18), (4.19), (4.33) and (4.38) that

$$\begin{aligned} &|(\mathscr{F}x)(t_2) - (\mathscr{F}x)(t_1)| \\ &\le |h(t_1)||x(\tau(t_2)) - x(\tau(t_1))| + |x(\tau(t_2))||h(t_2) - h(t_1)| \\ &\quad + (1-\mu)c^*|t_2^k - t_1^k| + |G(t_2) - G(t_1)| \\ &\le c^* m(\tau(t_2))|\tau(t_2) - \tau(t_1)| + \frac{c^*}{3\mu}[\tau(t_2)]^k |h(t_2) - h(t_1)| \\ &\quad + (1-\mu)c^*|t_2^k - t_1^k| + |G(t_2) - G(t_1)| \\ &\le c^* L_l^\tau m(\tau(t_2))|t_2 - t_1| + \frac{c^*}{3\mu}[\tau(t_2)]^k L_l^h |t_2 - t_1| \\ &\quad + (1-\mu)c^* k t_2^{k-1}|t_2 - t_1| + \mu c^* k t_2^{k-1}|t_2 - t_1| \\ &\le c^* \left[ L_l^\tau m(\tau(t_2)) + \frac{L_l^h}{3\mu}[\tau(t_2)]^k + k t_2^{k-1} \right] |t_2 - t_1| \\ &= c^* m(t_2)|t_2 - t_1| . \end{aligned} \tag{4.39}$$

From (4.37) and (4.39) we obtain

$$|(\mathscr{F}x)(t_2) - (\mathscr{F}x)(t_1)| \le c^* m(t_2)|t_2 - t_1| \qquad \text{for} \quad t_2 > t_1 \ge \tilde{T}. \tag{4.40}$$

Then, inequalities (4.36) and (4.40) mean that $\mathscr{F}$ maps $X$ into $X$.

Furthermore it is easily verified that $\mathscr{F}$ is continuous on $X$. By the Schauder-Tychonoff fixed point theorem we can conclude that there exists an $x \in X$ such that $x = \mathscr{F}x$. This $x$ satisfies

$$\begin{aligned} &\frac{d^k}{dt^k}[x(t) - h(t)x(\tau(t))] \\ &\quad = (1-\mu)c^* k! + (-1)^{n-k-1}\sigma \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} p(s) f(x(g(s)))\, ds \end{aligned}$$

for $t \ge T_1(T)$ and is a positive solution of equation (1.1). From the above equality it follows that

$$\lim_{t\to\infty} (Lx)(t)/t^k = \lim_{t\to\infty} [x(t) - h(t)x(\tau(t))]/t^k = (1-\mu)c^* > 0 .$$

Then, by (iii) of Lemma 2.7 we find that $x$ satisfies (4.27). The proof of Theorem 4.2 is complete.

COROLLARY 4.2. *Let* (1.1) *be either sublinear or superlinear, and let $k$ be an integer with $1 \leq k \leq n-1$. In addition to* (1.2)–(1.6), (4.1) *and* (4.2), *assume that $\lim_{t\to\infty} h(t)[\tau(t)/t]^k$ exists and is finite. Then equation* (1.1) *has a nonoscillatory solution $x$ such that*

$$\lim_{t\to\infty} \frac{x(t)}{t^k} = \text{const} \neq 0$$

*if and only if* (4.28) *holds.*

REMARK 4.1. It is easy to verify that if $x$ is a nonoscillatory solution of (1.1) satisfying (4.27), then $x \in \mathcal{N}_k$ for the case of $(-1)^{n-k}\sigma = -1$ and $x \in \mathcal{N}_{k+1}$ for the case of $(-1)^{n-k}\sigma = 1$. This observation is also true in the case of $k = 0$.

EXAMPLE 4.1. Consider the equation

$$\frac{d^n}{dt^n}[x(t) - h\sin t \cdot x(t-2\pi)] + \sigma p(t)|x(t-\tau)|^\gamma \operatorname{sgn} x(t-\tau) = 0, \tag{4.41}$$

where $n \geq 2$, $\sigma = 1$ or $-1$, $p \in C[0, \infty)$, $p(t) > 0$ on $[0, \infty)$, and $h$, $\tau$, $\gamma$ are constant such that $|h| < 1$, $|\tau| < \infty$, $|\gamma| < \infty$. Let $k$ be an integer with $0 \leq k \leq n-1$. Theorems 4.1 and 4.2 show that the condition

$$\int^\infty t^{n-k-1+\gamma k} p(t)\,dt < \infty \tag{4.42}$$

is a necessary and sufficient condition for (4.41) to have a nonoscillatory solution $x$ satisfying

$$0 < \liminf_{t\to\infty} \frac{|x(t)|}{t^k} \leq \limsup_{t\to\infty} \frac{|x(t)|}{t^k} < \infty .$$

EXAMPLE 4.2. Consider the equation

$$\frac{d^n}{dt^n}[x(t) - hx(t-2\pi)] + \sigma p(t)|x(t-\tau)|^\gamma \operatorname{sgn} x(t-\tau) = 0, \tag{4.43}$$

where $n$, $\sigma$, $p$, $h$, $\tau$, $\gamma$ are as in Example 4.1. Then it follows from Corollaries 4.1 and 4.2 that, for an integer $k$ with $0 \leq k \leq n-1$, condition (4.42) is necessary and sufficient for (4.43) to have a nonoscillatory solution $x$ such that

$$\lim_{t\to\infty} \frac{|x(t)|}{t^k} = \text{const} \neq 0 .$$

## 5. Nonoscillatory solutions in $\mathcal{N}_j$, $1 \leq j \leq n-1$

In this section we establish conditions under which equation (1.1) has nonoscillatory solutions of the classes $\mathcal{N}_j$, where $1 \leq j \leq n-1$ and $(-1)^{n-j-1}\sigma = 1$. These results are based upon the following lemmas which are concerned with

$$\{\sigma y^{(n)}(t) + p(t)f(y(g(t)))\} \operatorname{sgn} y(g(t)) \leq 0 . \tag{5.1}$$

Here we assume that $n \geq 2$, $\sigma = 1$ or $-1$, and $p$, $f$ and $g$ satisfy (1.4), (1.5) and (1.6), respectively. We say that a nonoscillatory solution $y$ of (5.1) is of class $\mathcal{N}_j$ if $y$ satisfies

$$\begin{cases} y(t)y^{(i)}(t) > 0 , & 0 \leq i \leq j , \\ (-1)^{i-j}y(t)y^{(i)}(t) > 0 , & j+1 \leq i \leq n , \end{cases}$$

for all sufficiently large $t$. We use the notation

$$g_*(t) = \min\{g(t), t\} .$$

Definition 5.1. Equation (1.1) or inequality (5.1) is called *strictly sublinear* if there is a number $\alpha$ such that $0 < \alpha < 1$ and

$$\frac{|f(u_1)|}{|u_1|^\alpha} \geq \frac{|f(u_2)|}{|u_2|^\alpha} \quad \text{for} \quad |u_1| \leq |u_2| , \qquad u_1 u_2 > 0 .$$

Equation (1.1) or inequality (5.1) is called *strictly superlinear* if there is a number $\beta > 1$ such that

$$\frac{|f(u_1)|}{|u_1|^\beta} \leq \frac{|f(u_2)|}{|u_2|^\beta} \quad \text{for} \quad |u_1| \leq |u_2| , \qquad u_1 u_2 > 0 .$$

Clearly equation (1.9) is strictly sublinear if $-\infty < \gamma < 1$ and is strictly superlinear if $1 < \gamma < \infty$.

Lemma 5.1. *Let* (5.1) *be strictly sublinear and* $1 \leq j \leq n-1$, $(-1)^{n-j-1}\sigma = 1$. *If* (5.1) *has a solution of class* $\mathcal{N}_j$, *then*

$$\int^\infty \left(\frac{g_*(t)}{g(t)}\right)^{\alpha j} t^{n-j-1} p(t) |f(c[g(t)]^j)| \, dt < \infty \qquad \text{for some} \quad c \neq 0 , \tag{5.2}$$

*where* $\alpha$ *is the strict sublinearity constant for* (5.1).

For the proof of Lemma 5.1, see Kitamura [11, Theorem 2]. A close look at the proofs of Theorem 1 of Kitamura [11] and Theorem B of Kitamura and Kusano [12] enables us to obtain the next result.

LEMMA 5.2. *Let* (5.1) *be strictly superlinear and* $1 \le j \le n-1$, $(-1)^{n-j-1}\sigma = 1$. *If* (5.1) *has a solution of class* $\mathcal{N}_j$, *then*

$$\int^{\infty} [g_*(t)]^{n-j} p(t) |f(c[g(t)]^{j-1})| \, dt < \infty \qquad \text{for some} \quad c \neq 0 . \tag{5.3}$$

First we find a necessary condition for the existence of a solution $x$ of (1.1) which belongs to $\mathcal{N}_j$.

THEOREM 5.1. *Let* (1.1) *be strictly sublinear and* $1 \le j \le n-1$, $(-1)^{n-j-1}\sigma = 1$. *If* (1.1) *has a nonoscillatory solution* $x$ *in the class* $\mathcal{N}_j$, *then* (5.2) *holds.*

PROOF. Let $x$ be a solution of (1.1) in the class $\mathcal{N}_j$. Without loss of generality we may assume that $x$ is eventually positive. Then $(Lx)(t)$ is eventually positive and increasing. By Lemmas 2.3 and 2.4 there are $c^* > 0$, $c_* > 0$ and $T \ge a$ such that

$$c_*(Lx)(g(t)) \le x(g(t)) \le c^*(Lx)(g(t)) \qquad \text{for} \quad t \ge T . \tag{5.4}$$

Then from the definition of the strict sublinearity for (1.1) it follows that

$$\begin{aligned} f(x(g(t))) &\ge f(c^*(Lx)(g(t)))\left(\frac{x(g(t))}{c^*(Lx)(g(t))}\right)^{\alpha} \\ &\ge (c_*/c^*)^{\alpha} f(c^*(Lx)(g(t))) \end{aligned} \tag{5.5}$$

for $t \ge T$. From equation (1.1) and (5.5) we obtain

$$\sigma(Lx)^{(n)}(t) + (c_*/c^*)^{\alpha} p(t) f(c^*(Lx)(g(t))) \le 0 , \qquad t \ge T ,$$

and so the inequality

$$\{\sigma y^{(n)}(t) + (c_*/c^*)^{\alpha} p(t) f(c^* y(g(t)))\} \operatorname{sgn} y(g(t)) \le 0$$

has a positive solution $Lx$ of class $\mathcal{N}_j$. Then we conclude by Lemma 5.1 that (5.2) holds. This completes the proof of Theorem 5.1.

THEOREM 5.2. *Let* (1.1) *be strictly superlinear and* $1 \le j \le n-1$, $(-1)^{n-j-1}\sigma = 1$. *If* (1.1) *has a nonoscillatory solution* $x$ *in the class* $\mathcal{N}_j$, *then* (5.3) *holds.*

PROOF. Let $x$ be an eventually positive solution of (1.1) in the class $\mathcal{N}_j$. As in the proof of Theorem 5.1, $Lx$ is eventually positive and (5.4) is satisfied for some $c^* > 0$, $c_* > 0$ and $T \ge a$. By equation (1.1) and (5.4) we have

$$\sigma(Lx)^{(n)}(t) + p(t) f(c_*(Lx)(g(t))) \le 0 , \qquad t \ge T ,$$

which means that the inequality

$$\{\sigma y^{(n)}(t) + p(t)f(c_* y(g(t)))\} \operatorname{sgn} y(g(t)) \le 0$$

has a positive solution $Lx$ of class $\mathcal{N}_j$. Then, by Lemma 5.2 we see that (5.3) holds. The proof of Theorem 5.2 is complete.

THEOREM 5.3. *Let* (1.2)–(1.6), (4.1) *and* (4.2) *be satisfied. Assume that* (1.1) *is strictly sublinear and* $1 \le j \le n-1$, $(-1)^{n-j-1}\sigma = 1$. *Assume in addition that* $g_*(t) = \min\{g(t), t\}$ *satisfies*

$$\liminf_{t\to\infty} \frac{g_*(t)}{g(t)} > 0. \tag{5.6}$$

*Then, a necessary and sufficient condition for* (1.1) *to have a nonoscillatory solution of class* $\mathcal{N}_j$ *is that*

$$\int^{\infty} t^{n-j-1} p(t) |f(c[g(t)]^j)|\, dt < \infty \qquad \textit{for some} \quad c \neq 0. \tag{5.7}$$

PROOF. Note that, under condition (5.6), (5.2) is equivalent to (5.7). Then the necessity part follows from Theorem 5.1, and the sufficient part follows from Theorems 4.1, 4.2 and Remark 4.1.

Likewise, from Theorems 5.2, 4.1, 4.2 and Remark 4.1 we have the following result.

THEOREM 5.4. *Let* (1.2)–(1.6), (4.1) *and* (4.2) *be satisfied. Assume that* (1.1) *is strictly superlinear and* $1 \le j \le n-1$, $(-1)^{n-j-1}\sigma = 1$. *Assume in addition that* $g_*(t) = \min\{g(t), t\}$ *satisfies*

$$\liminf_{t\to\infty} \frac{g_*(t)}{t} > 0. \tag{5.8}$$

*Then, a necessary and sufficient condition for* (1.1) *to have a nonoscillatory solution of class* $\mathcal{N}_j$ *is that*

$$\int^{\infty} t^{n-j} p(t) |f(c[g(t)]^{j-1})|\, dt < \infty \qquad \textit{for some} \quad c \neq 0. \tag{5.9}$$

EXAMPLE 5.1. Let us reconsider equation (4.41). First notice that the case (II) in Theorem 3.1 does not occur (that is, the class $\mathcal{N}_0^-$ for (4.41) is always empty) since the function $h(t) = h \sin t$ takes a nonpositive value on $[T, \infty)$ for all $T$. Let $j$ be an integer satisfying $1 \le j \le n-1$ and $(-1)^{n-j-1}\sigma = 1$. Theorem 5.3 shows that equation (4.41) with $-\infty < \gamma < 1$ has a nonoscillatory solution of class $\mathcal{N}_j$ if and only if

$$\int^{\infty} t^{n-j-1+\gamma j} p(t)\, dt < \infty ,$$

while Theorem 5.4 shows that (4.41) with $1 < \gamma < \infty$ has a nonoscillatory solution of class $\mathcal{N}_j$ if and only if

$$\int^{\infty} t^{n-j+\gamma(j-1)} p(t)\, dt < \infty .$$

Consider the special case that $n$ is even and $\sigma = 1$ in (4.41). We see that if $\gamma < 1$ and the condition

$$\int^{\infty} t^{\gamma(n-1)} p(t)\, dt = \infty \tag{5.10}$$

is satisfied, then all the classes $\mathcal{N}_j$, $j = 1, 3, \ldots, n-1$, for (4.41) are empty. Since $\mathcal{N}_0^-$ is also empty, we can conclude the following: Let $n$ be even, $\sigma = 1$ and $\gamma < 1$. Then equation (4.41) has no nonoscillatory solutions if and only if (5.10) holds. Similarly we have the following result: Let $n$ be even, $\sigma = 1$ and $\gamma > 1$. Then equation (4.41) has no nonoscillatory solutions if and only if

$$\int^{\infty} t^{n-1} p(t)\, dt = \infty .$$

ACKNOWLEDGMENT. The author would like to thank Professor Manabu Naito for his interest in this work and for many helpful suggestions.

## References

[1] D. D. Bainov, A. D. Myshkis and A. I. Zahariev, Asymptotic and oscillatory properties of a class of operator differential inequalities, Ann. Mat. Pura Appl. **143** (1986), 197–205.

[2] K. Gopalsamy, Oscillation and nonoscillation in neutral differential equations with variable parameters, J. Math. Phys. Sci. **21** (1987), 593–611.

[3] S. R. Grace and B. S. Lalli, Oscillations of nonlinear second order neutral delay differential equations, Rad. Mat. **3** (1987), 77–84.

[4] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, Oscillation of second order neutral delay differential equations, Rad. Mat. **1** (1985), 267–274.

[5] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, Oscillation and asymptotic behavior of second order neutral differential equations, Ann. Mat. Pura Appl. **148** (1987), 29–40.

[6] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, Oscillation and asymptotic behavior of higher order neutral equations with variable coefficients, Chinese Ann. Math. Ser. B **9** (1988), 322–338.

[7] J. R. Graef, M. K. Grammatikopoulos and P. W. Spikes, Asymptotic properties of solutions

of nonlinear neutral delay differential equations of the second order, Rad Mat. **4** (1988), 133–149.

[8] J. Jaroš and T. Kusano, Asymptotic behavior of nonoscillatory solutions of nonlinear functional differential equations of neutral type, Funkcial. Ekvac. **32** (1989), 251–263.

[9] J. Jaroš and T. Kusano, Oscillation theory of higher order linear functional differential equations of neutral type, Hiroshima. Math. J. **18** (1988), 509–531.

[10] I. T. Kiguradze, On the oscillation of solutions of the equation $d^m u/dt^m + a(t)|u|^n \operatorname{sign} u = 0$, Mat. Sb. **65** (1964), 172–187. (Russian)

[11] Y. Kitamura, On nonoscillatory solutions of functional differential equations with a general deviating argument, Hiroshima. Math. J. **8** (1978), 49–62.

[12] Y. Kitamura and T. Kusano, An oscillation theorem for a superlinear functional differential equation with general deviating arguments, Bull. Austral. Math. Soc. **18** (1978), 395–402.

[13] T. Kusano and M. Naito, Unbounded nonoscillatory solutions of nonlinear ordinary differential equations of arbitrary order, Hiroshima Math. J. **18** (1988), 361–372.

[14] G. Ladas and Y. G. Sficas, Oscillations of higher-order neutral equations, J. Austral. Math. Soc. Ser. B **27** (1986), 502–511.

[15] J. Ruan, Types and criteria of nonoscillatory solutions for second order linear neutral differential equations, Chinese Ann. Math. Ser. A **8** (1987), 114–124. (Chinese)

[16] J. Ruan, Types and criteria of nonoscillatory solutions of second order NFDE and RFDE with nonpositive coefficients, Chinese Ann. Math. Ser. A **9** (1988), 151–159. (Chinese)

[17] A. I. Zahariev and D. D. Bainov, Oscillating properties of the solutions of a class of neutral type functional differential equations, Bull. Austral. Math. Soc. **22** (1980), 365–372.

[18] A. I. Zahariev and D. D. Bainov, On some oscillation criteria for a class of neutral type functional differential equations, J. Austral. Math. Soc. Ser. B **28** (1986), 229–239.

*Department of Mathematics,*
*Faculty of Science,*
*Hiroshima University*