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Chiral models and the Einstein-Maxwell field equations

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1. Introduction

The main objective in this paper is to provide a geometric picture of solutions of a (1 + 1)-dimensional reduction for the (1 + 3)-dimensional principal chiral model taking values in an arbitrary linear algebraic group.

Let G be a closed subgroup of the group scheme GL_N and assume that G is defined over $\boldsymbol{R}$. The equations of motion for the $SO(1, 2)$-invariant chiral model on flat Minkowski space can be written

$$d(t * d\sigma \cdot \sigma^{-1}) = 0 \tag{1.1}$$

for $\sigma \in G(\boldsymbol{C}[[t, z]])$. Here t, z are real variables, d is exterior differentiation, and $*$ is the Hodge operator with respect to the Lorentz metric $(dt)^2 - (dz)^2$.

Let λ be a real parameter. Let $\mathscr{A}$ denote an algebra $\{a = \Sigma_{n \in Z} a_n \lambda^n \in \boldsymbol{C}[[t, z, \lambda, \lambda^{-1}]];\ \mathrm{ord}\, a_n \geq n\}$, where $\mathrm{ord}\, \varphi = \sup \{k \in \boldsymbol{Z};\ \varphi \in (\boldsymbol{C}[[t, z]]t + \boldsymbol{C}[[t, z]]z)^k\}$. Set $\mathscr{A}^{\pm} = \mathscr{A} \cap \boldsymbol{C}[[t, z, \lambda^{\pm 1}]]$, $\mathscr{P}_G = G(\mathscr{A}^+)$ and $\mathscr{N}_G = \{g \in G(\mathscr{A}^-);\ g(t, z, \infty) = 1\}$. Then $G(\mathscr{A}) = \mathscr{N}_G \mathscr{P}_G$ (Lemma 2.3 and K. Takasaki [6, (3.17)]). This decomposition is used for solving (1.1).

THEOREM 1.1. *There exist* $w \in \mathscr{N}_G$ *and* $p \in \mathscr{P}_G$ *such that* $w^{-1}p = \gamma(z + \lambda t^2/2 + 1/2\lambda)$ *for each* $\gamma \in G(\boldsymbol{C}[[z]])$. *Furthermore, if we set* $\sigma = p(t, z, 0)$, *then* σ *is a unique solution of* (1.1) *with* $\sigma(0, z) = \gamma(z)$.

We give a proof of the theorem in §2 and derive an explicit formula for the solution σ with $\sigma(0, z) \in G(\boldsymbol{C}[z])$. Also we consider a transformation group for solutions of (1.1). As an application, we show in §3 a variant of the Geroch conjecture [3], that is to say, a real form $\mathscr{SU}(1, 2)$ of $SL_3(\boldsymbol{C}[[z]])$ acts transitively on the space of plane wave solutions of the Einstein-Maxwell field equations.

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2. The chiral models

To start with, we consider a manifest invariance of (1.1). We note that $d(t * d\tau^{-1} \cdot \tau) = -\mathrm{Ad}\, \tau^{-1}(d(t * d\tau \cdot \tau^{-1}))$ for any $\tau \in G(\boldsymbol{C}[[t, z]])$. The following result is obvious.

LEMMA 2.1. *Let $\theta: G \to G'$ be a homomorphism or an antihomomorphism between linear algebraic groups G and G'. If $\sigma \in G(\boldsymbol{C}[[t, z]])$ satisfies* (1.1), *then $\theta(\sigma)$ does also.*

We shall prove the solvability of (1.1) using this invariance.

PROPOSITION 2.2. *There exists a unique solution $\sigma \in G(\boldsymbol{C}[[t, z]])$ of* (1.1) *with $\sigma(0, z) = \gamma(z)$ for each $\gamma \in G(\boldsymbol{C}[[z]])$.*

PROOF. We rewrite (1.1) as follows:

$$(2.1) \qquad (t\partial_t)^2\sigma = t^2\partial_z^2\sigma + t\partial_t\sigma\cdot\sigma^{-1}t\partial_t\sigma - t\partial_z\sigma\cdot\sigma^{-1}t\partial_z\sigma\,.$$

We set $\varphi[n] = \partial_t^n\varphi(0, z)/n!$ for $\varphi \in \mathfrak{gl}_N(\boldsymbol{C}[[t, z]])$. By (2.1), $\sigma[0]$ determines $\sigma[n]$ for $n > 0$. The proposition is now valid if $G = GL_N$.

Let ρ be a polynomial representation of GL_N on V such that $G = \{g \in GL_N; v_0\rho(g) \in \boldsymbol{C}v_0\}$ with $v_0 \in V$. Let $\sigma \in GL_N(\boldsymbol{C}[[t, z]])$ satisfy $d(t * d\sigma\cdot\sigma^{-1}) = 0$ and $\sigma(0, z) = \gamma(z)$. Then (2.1) combined with Lemma 2.1 implies that

$$\begin{aligned} n^2 v_0\tau[n] = v_0\partial_z^2\tau[n-2] &+ \textstyle\sum_{0<p,q,r<n,\,p+q+r=n}(v_0 p\tau[p]\tau^{-1}[q]r\tau[r] \\ &- v_0\partial_z\tau[p-1]\cdot\tau^{-1}[q]\partial_z\tau[r-1]) \end{aligned}$$

for $\tau = \rho(\sigma)$ and $\rho(\sigma)^{-1}$. Hence $v_0\rho(\sigma)^{\pm 1}[n] \in \boldsymbol{C}[[z]]v_0$. This means that $\sigma \in G(\boldsymbol{C}[[t, z]])$. $\square$

We now consider a linearization of (1.1) (cf. K. Nagatomo [5]). Let α_1 and $\alpha_2 \in \mathfrak{g}(\boldsymbol{C}[[t, z]])$. If $\alpha_1 = \partial_t\sigma\cdot\sigma^{-1}$ and $\alpha_2 = \partial_z\sigma\cdot\sigma^{-1}$ with $\sigma \in G(\boldsymbol{C}[[t, z]])$, then

$$(2.2) \qquad \partial_z\alpha_1 - \partial_t\alpha_2 + [\alpha_1, \alpha_2] = 0\,.$$

Moreover, if σ satisfies (1.1), then

$$(2.3) \qquad \partial_t(t\alpha_1) - \partial_z(t\alpha_2) = 0\,.$$

Conversely, if $(\alpha_1, \alpha_2) \in \mathfrak{g}(\boldsymbol{C}[[t, z]]) \times \mathfrak{g}(\boldsymbol{C}[[t, z]])$ is a solution of (2.2), then there exists a unique $\sigma \in G(\boldsymbol{C}[[t, z]])$ satisfying $\partial_t\sigma = \alpha_1\sigma$, $\partial_z\sigma = \alpha_2\sigma$ and $\sigma(0, 0) = \beta$ for each $\beta \in G(\boldsymbol{C})$. Therefore (1.1) is equivalent to the system (2.2-3).

Here we introduce two vector fields:

$$D_1 = \partial_t - \lambda t\partial_z \qquad \text{and} \qquad D_2 = \partial_z - \lambda t\partial_t + 2\lambda^2\partial_\lambda\,.$$

If α_1 and $\alpha_2 \in \mathfrak{g}(\boldsymbol{C}[[t, z]])$ satisfy

$$(2.4) \qquad D_i w = \alpha_i w\,, \qquad i = 1, 2 \quad \text{with } w \in G(\mathscr{A})\,,$$

then (α_1, α_2) is a solution of (2.2-3), since $[D_1, D_2] = -\lambda D_1$ and $D_1 D_2 w - D_2 D_1 w = \{\partial_t \alpha_2 - \partial_z \alpha_1 + [\alpha_2, \alpha_1] - \lambda(t\partial_z \alpha_2 - t\partial_t \alpha_1)\} w$.

In the remainder of this section, we study the space of solutions of (1.1). Our approach is based on a theory of transformation. We begin with a slight extension of the Birkhoff decomposition theorem due to K. Takasaki.

LEMMA 2.3. *The map $\mathcal{N}_G \times \mathcal{P}_G \to G(\mathcal{A})$ given by $(h, q) \to hq^{-1}$ is bijective.*

PROOF. If $G = GL_N$, the lemma is nothing but [6, (3.17)]. Let ρ, V and v_0 be as in the proof of Proposition 2.2. Let χ be a rational character of G such that $\chi(g)v_0 = v_0\rho(g)$ for every $g \in G$. Also, without loss of generality, we may assume that χ is extended to a polynomial mapping on $\mathfrak{gl}_N$.

Let $h \in \mathcal{N}_{GL_N}$ and $q \in \mathcal{P}_{GL_N}$. Suppose that $g := hq^{-1} \in G(\mathcal{A})$. We set $c = \chi(g)$. Then $c \in GL_1(\mathcal{A})$. Therefore there exist $a \in \mathcal{N}_{GL_1}$ and $b \in \mathcal{P}_{GL_1}$ such that $a^{-1}b = c$. Then $av_0\rho(h) = bv_0\rho(q) \in \boldsymbol{C}[[t, z]]v_0$. This implies that h and $q \in G(\mathcal{A})$. $\square$

PROOF OF THEOREM 1.1. Set $g = \gamma(z + \lambda t^2/2 + 1/2\lambda)$ for $\gamma \in G(\boldsymbol{C}[[z]])$. Then $g = \exp(\lambda t^2 \partial_z/2)\gamma(z + 1/2\lambda) \in G(\mathcal{A})$. Lemma 2.3 implies that $g = w^{-1}p$ with $w \in \mathcal{N}_G$ and $p \in \mathcal{P}_G$. Then $\gamma(z + 1/2\lambda) = w(0, z, \lambda)^{-1}p(0, z, \lambda)$. Furthermore $p(0, z, \lambda) = \gamma(z)$ by the uniqueness of the Birkhoff decomposition.

Also $D_i g = 0$. Hence $D_i w \cdot w^{-1} = D_i p \cdot p^{-1} \in \mathfrak{g}(\boldsymbol{C}[[t, z]])$. Thus $D_i p(t, z, 0) = \partial_i p(t, z, 0) = \alpha_i p(t, z, 0)$, where $\partial_1 = \partial_t$, $\partial_2 = \partial_z$ and $\alpha_i = D_i p \cdot p^{-1}$. In view of the linearization, we see that $\sigma := p(t, z, 0)$ is a solution of (1.1) with $\sigma(0, z) = \gamma(z)$. $\square$

EXAMPLE 2.4. Let $\gamma \in G(\boldsymbol{C}[z])$ with $\deg \gamma = m$. Let $\Sigma_{|n| \le m} h_n \lambda^n = \gamma(z + \lambda t^2/2 + 1/2\lambda)$. We set $a_{ij} = h_{i-j}$, $b_{ij} = h_{i-j-m-1}$ and $c_{ij} = h_{i+m+1-j} \in \mathfrak{gl}_N(\boldsymbol{C}[t, z])$. Let $A = (a_{ij})_{0 \le i,j \le m}$, $B = (b_{ij})_{0 \le i,j \le m}$ and $C = (c_{ij})_{0 \le i,j \le m} \in \mathfrak{gl}_{N(m+1)}(\boldsymbol{C}[t, z])$. We define inductively $A_0 = A$ and $A_i = A - CA_{i-1}^{-1}B$ for $i > 0$. Set $B_i = BA_i^{-1}$ and $C_i = CA_i^{-1}$. Let $E_0 = (1_N, 0, \cdots, 0) \in \bigoplus^m \mathfrak{gl}_N(\boldsymbol{C})$ and ${}^tE_0$ is the transpose of E_0. Then

$$\sigma := E_0 A^{-1}(1 + \textstyle\sum_{k>0} B_1 \cdots B_k C_{k-1} \cdots C_0)\,{}^tE_0$$

is a solution of (1.1) with $\sigma(0, z) = \gamma(z)$.

In fact, if $\gamma(z + \lambda t^2/2 + 1/2\lambda) = w^{-1}p$ with $w \in \mathcal{N}_G$ and $p = \Sigma_{n \ge 0} p_n \lambda^n \in \mathcal{P}_G$, then

$$(p_0, p_1, \cdots)(a_{ij})_{0 \le i,j < \infty} = (1_N, 0, \cdots)\,, \tag{2.5}$$

and it is easy to solve the linear algebraic equation (2.5) since the matrix $(a_{ij})_{0 \le i,j < \infty}$ has the blocks of tridiagonal form

$$\begin{bmatrix} A & B & & & & & & \\ C & A & B & & & & & \\ & C & A & B & & & & \\ & & C & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & \cdot \\ & & & & & & \cdot & \cdot \end{bmatrix}.$$

DEFINITION 2.5. Let $\mathscr{S}(G)$ denote the space of solutions of (1.1). In view of Theorem 1.1, a pair $(w, p) \in \mathscr{N}_G \times \mathscr{P}_G$ is called a potential for $\sigma \in \mathscr{S}(G)$ if $D_i w \cdot w^{-1} = D_i p \cdot p^{-1}$ and if $p(t, z, 0) = \sigma$.

For $g \in G(\mathscr{A})$ and $(w, p) \in \mathscr{N}_G \times \mathscr{P}_G$, set $g \cdot (w, p) = (v, qp)$ with $v \in \mathscr{N}_G$, $q \in \mathscr{P}_G$ satisfying $gw^{-1} = v^{-1}g$. The map $G(\mathscr{A}) \times \mathscr{N}_G \times \mathscr{P}_G \to \mathscr{N}_G \times \mathscr{P}_G$ defined by $(g, w, p) \to g \cdot (w, p)$ is an action of $G(\mathscr{A})$ on $\mathscr{N}_G \times \mathscr{P}_G$. If (w, p) is a potential for $\sigma \in \mathscr{S}(G)$ and if $g \in G(\mathscr{A})$ satisfies $D_i g = 0$, then $D_i v \cdot v^{-1} = qD_i w \cdot w^{-1} q^{-1} + D_i q \cdot q^{-1}$. This implies that (v, qp) is also a potential for a certain $\tau \in \mathscr{S}(G)$. Set $g \cdot \sigma = \tau$. Since $(g \cdot \sigma)(0, z) = g(z)\sigma(0, z)$, Proposition 2.2 implies that the map $\{g \in G(\mathscr{A}); D_i g = 0\} \times \mathscr{S}(G) \to \mathscr{S}(G)$ given by $(g, \sigma) \to g \cdot \sigma$ is a transitive action of the group. Thus we can define an action of $G(\boldsymbol{C}[[z]])$ on $\mathscr{S}(G)$ via the following isomorphism.

PROPOSITION 2.6. *We set* $\iota(\gamma) = \gamma(z + \lambda t^2/2 + 1/2\lambda)$ *for* $\gamma \in G(\boldsymbol{C}[[z]])$. *Then* ι *is an isomorphism*: $G(\boldsymbol{C}[[z]]) \to \{g \in G(\mathscr{A}); D_i g = 0\}$.

PROOF. We change the variables $t = t$, $z = x - \lambda t^2/2$ and $\lambda = \lambda$. Then $\partial_z = \partial_x$, $\partial_t = \partial_t + \lambda t \partial_x$ and $\partial_\lambda = \partial_\lambda + t^2 \partial_x/2$. Hence $D_1 = \partial_t$ and $D_2 = \partial_x - \lambda t \partial_t + 2\lambda^2 \partial_\lambda$ with respect to the new variables.

Let $\psi = \Sigma_{n \in \boldsymbol{Z}} \psi_n \lambda^n \in \mathfrak{gl}_N(\mathscr{A})$ satisfy $D_i \psi = 0$. We set $\varphi(t, x, \lambda) = \psi(t, x - \lambda t^2/2, \lambda) = \exp(-\lambda t^2 \partial_z/2)\psi(t, x, \lambda)$. Then φ is independent of t, since $D_1 \varphi = \partial_t \varphi = 0$. Since $D_2 \varphi = (\partial_x + 2\lambda^2 \partial_\lambda)\varphi = 0$, we have $\partial_x \varphi_n + 2(n-1)\varphi_{n-1} = 0$ in the expansion $\varphi = \Sigma_{n \in \boldsymbol{Z}} \varphi_n \lambda^n$. Hence $\varphi_n = 0$ and $\varphi_{-n} = \partial_x^n \varphi_0 / 2^n n!$ for $n > 0$, since $\varphi = \varphi(0, x, \lambda) = \psi(0, x, \lambda)$. Thus $\varphi = \varphi_0(x + 1/2\lambda)$ and $\psi = \varphi(t, z + \lambda t^2/2, \lambda) = \varphi_0(z + \lambda t^2/2 + 1/2\lambda)$. $\square$

COROLLARY 2.7. *We have a unique potential for a solution of* (1.1).

PROOF. Let $(w, p) \in \mathscr{N}_G \times \mathscr{P}_G$ be a potential for $\sigma \in \mathscr{S}(G)$. Set $g = w^{-1}p$. Then $D_i g = 0$. From Proposition 2.6, it follows that $g = \gamma(z + \lambda t^2/2 + 1/2\lambda)$

with $\gamma \in G(\boldsymbol{C}[[z]])$. Since $w(0, z, \lambda)\gamma(z + 1/2\lambda) = p(0, z, \lambda)$, we see that $\gamma(z) = p(0, z, \lambda) = \sigma(0, z)$. The corollary now follows from the uniqueness of the Birkhoff decomposition. $\square$

3. The Einstein-Maxwell fields

In this section, we study a $(1 + 1)$-dimensional reduction for the Einstein-Maxwell field equations. Those equations are expressed in terms of potentials due to F. J. Ernst $(u, v) \in \boldsymbol{C}^2[[t, z]]$ as follows ([2]):

$$(3.1)\qquad d(t * d(u, v)) = f^{-1}(du - \bar{v}\, dv)t * d(u, v)\,, \qquad 2f = u + \bar{u} - |v|^2 > 0\,.$$

Moreover, following M. Gürses & B. C. Xanthopoulos [4], we shall identify (3.1) with a subclass of the chiral model (1.1) taking values in $SU(2, 1)$. Let

$$(3.2)\qquad \sigma = f^{-1}\begin{bmatrix} 1 & i(f - \bar{u}) & \bar{v} \\ i(\bar{u} - f) & |u|^2 & iu\bar{v} \\ v & -i\bar{u}v & f + |v|^2 \end{bmatrix}.$$

Then, by a direct calculation, we can check that (3.1) is equivalent to (1.1). Hence we identify the space $\mathscr{M}$ of solutions of (3.1) with a subspace of $\mathscr{S}(SL_3)$. Let $J = \begin{bmatrix} & i & \\ -i & & \\ & & 1 \end{bmatrix}$. Let $\mathscr{S}\mathscr{U}(2, 1) = \{g \in SL_3(\boldsymbol{C}[[z]]);\, gJ^\dagger g = J\}$ and $\mathscr{U}(2) = \{g \in \mathscr{S}\mathscr{U}(2, 1);\, g^\dagger g = 1\}$, where † denotes the Hermitian conjugation. We set $g \circ \sigma = {}^\dagger(g \cdot {}^\dagger(g \cdot \sigma))$ for $g \in SL_3(\boldsymbol{C}[[z]])$ and $\sigma \in \mathscr{S}(SL_3)$, where $\cdot$ denotes the action defined in §2. This new action makes $\mathscr{M}$ into a homogeneous space of $\mathscr{S}\mathscr{U}(2, 1)$, that is,

THEOREM 3.1. *Set $v(g) = g \circ 1$ for $g \in \mathscr{S}\mathscr{U}(2, 1)$. Then v induces a bijection: $\mathscr{S}\mathscr{U}(2, 1)/\mathscr{U}(2) \to \mathscr{M}$.*

PROOF. We set

$$n(b, c) = \begin{bmatrix} 1 & 0 & 0 \\ b + i|c|^2/2 & 1 & i\bar{c} \\ c & 0 & 1 \end{bmatrix}$$

for $b \in \boldsymbol{R}$ and $c \in \boldsymbol{C}$. Let $N = \{n(b, c);\, b \in \boldsymbol{R}, c \in \boldsymbol{C}\}$ and $A = \{\operatorname{diag}(a^{-1}, a, 1);\, a > 0\}$. Then we have an Iwasawa decomposition $SU(2, 1) = NAU(2)$. We set $u = a^2 + |c|^2/2 - ib$, $v = c$ and $s = n(b, c)\operatorname{diag}(a^{-1}, a, 1)$ for $a > 0$, $b \in \boldsymbol{R}$ and $c \in \boldsymbol{C}$. Then we see that $s^\dagger s$ is of the same form as σ in (3.2). This implies that $v(\mathscr{S}\mathscr{U}(2, 1)) = \mathscr{M}$, since $v(g) = g^\dagger g$ on $t = 0$. $\square$

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