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Existence theorems for Monge-Ampère equations in $\boldsymbol{R}^N$

Takaŝi KUSANO and Charles A. SWANSON

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1. Introduction

Our aim is to establish the existence of positive radial entire solutions $u(x)$ of nonlinear partial differential equations of the Monge-Ampère type

(1) $$\det(D^2u) = \alpha\Delta u + \lambda f(|x|, u, |Du|), \qquad x \in \boldsymbol{R}^N, \qquad N \geq 3$$

which grow like constant multiples of $|x|^2$ as $|x| \to \infty$, where $\alpha > 0$ and $\lambda \in \boldsymbol{R}$ are constants and $f \in C(\boldsymbol{D}, \boldsymbol{R})$, $\boldsymbol{D} = \bar{\boldsymbol{R}}_+ \times \boldsymbol{R}_+ \times \bar{\boldsymbol{R}}_+$, $\boldsymbol{R}_+ = (0, \infty)$, $\bar{\boldsymbol{R}}_+ = [0, \infty)$. Detailed hypotheses on f are listed in §2. Under modified conditions we also prove (Theorem 3) the existence of radial entire solutions of (1) which are positive in some neighborhood of infinity.

As usual, $|x|$ denotes the Euclidean length of a point $x = (x_1, \ldots, x_N)$ in $\boldsymbol{R}^N$, $D_i = \partial/\partial x_i$, $D_{ij} = D_iD_j$ for $i, j = 1, \ldots, N$, $Du = (D_1u, \ldots, D_Nu)$, $\Delta = \Sigma_{i=1}^N D_{ii}$, and D^2u is the Hessian matrix $(D_{ij}u)$.

An *entire solution* of (1) is defined to be a function $u \in C^2(\boldsymbol{R}^N)$ satisfying (1) at every point $x \in \boldsymbol{R}^N$. We seek radially symmetric entire solutions $u(x) = y(t)$, $t = |x|$, of (1) such that

(2) $$0 < \liminf_{t\to\infty} t^{-2}y(t), \qquad \limsup_{t\to\infty} t^{-2}y(t) < \infty .$$

In particular our results apply to the following special cases of (1):

(3) $$\det(D^2u) = \alpha\Delta u + \lambda p(|x|)u^\gamma, \qquad x \in \boldsymbol{R}^N ;$$

(4) $$\det(D^2u) = \alpha\Delta u + \lambda p(|x|)e^u, \qquad x \in \boldsymbol{R}^N ,$$

where γ is a positive constant and $p \in C(\bar{\boldsymbol{R}}_+, \boldsymbol{R})$. If $p(t) = 0(t^{-2\gamma})$ as $t \to \infty$, Theorem 1 implies that (3) has an infinitude of positive radial entire solutions $u(x) = y(|x|)$ satisfying (2), for all sufficiently small $|\lambda|$. If in addition $\gamma < N$ and $p(t) \geq 0$ on $\bar{\boldsymbol{R}}_+$, Theorem 2 shows that (3) has positive radial entire solutions satisfying (2) for all $\lambda \geq 0$.

If $p(t) = 0[\exp(-2\alpha_N t^2)]$ as $t \to \infty$, where

(5) $$\alpha_N = (\alpha N)^{1/(N-1)}, \qquad N \geq 3 ,$$

Theorem 1 implies that (4) has an infinitude of positive radial entire solutions $u(x) = y(|x|)$ satisfying (2) for sufficiently small $|\lambda|$. Theorem 3 establishes, for arbitrary $\lambda \in \boldsymbol{R}$, the existence of infinitely many radial entire solutions of (4) satisfying (2) which are positive in a neighborhood of infinity.

Theorems 1 and 2 also apply to generalizations of (3) having the form

$$\det(D^2u) = \alpha\Delta u + \lambda p(|x|)u^{\gamma}(1 + |Du|^2)^{\delta}, \qquad x \in \boldsymbol{R}^N, \tag{6}$$

where γ, δ are nonnegative constants with $\gamma + \delta > 0$. The condition $p(t) = 0(t^{-2\gamma-2\delta})$ as $t \to \infty$ implies the existence of positive radial solutions of (6) satisfying (2) if $|\lambda|$ is small enough; and if in addition $\gamma + 2\delta < N$ and $p(t)$ is nonnegative, implies the existence of such solutions for arbitrary $\lambda \geq 0$.

If $\alpha = 0$, $\gamma = 0$, and $\delta = (N+2)/2$ equation (6) arises in differential geometry as the equation for prescribed Gaussian curvature [6, p. 38]. If $\alpha > 0$, (6) is an equation for prescribed generalized Gaussian curvature, as described by Pogorelov [14, Chap. 10–13]. Since the case $\alpha = 0$ was treated in [8, 9] our attention here is directed toward the case $\alpha > 0$, $N \geq 3$. If $N = 2$ sufficient conditions are given in [8] for equation (1) to have infinitely many positive radial entire solutions which are strictly convex in $\boldsymbol{R}^2$ and asymptotic to constant multiples of $|x|$ (if $\alpha = 0$) or $|x|^2$ (if $\alpha > 0$) as $|x| \to \infty$. These results are extended to dimensions $N \geq 3$ by our theorems in §2 (for $\alpha > 0$) together with those in [9] (for $\alpha = 0$).

The significance of Monge-Ampère equations (1) in geometry and analysis have led to many recent investigations [1–7, 10–19], mostly devoted to existence and regularity questions for boundary value problems in *bounded* domains. The results for unbounded domains seem to be limited to those of Popivanov and Kutev [17] for exterior domains and the authors [8, 9] for (1), as described above.

2. Statement of theorems and outline of method

The hypotheses on the function f in (1) will be selected from the following list:

(f_1) $|f(t, u, v)|$ is nondecreasing in u and in v for fixed values of the other variables.

(f_2) $F(k) = \sup_{t \in \bar{\boldsymbol{R}}_+} |f(t, k(1+t^2), 2kt)| < \infty$ for all $k > 0$.

(f_3) $\lim_{k \to \infty} k^{-N}F(k) = 0$.

(f_4) $H(c) = \sup_{t \in \bar{\boldsymbol{R}}_+} |f(t, c + 2\alpha_N t^2, 4\alpha_N t)| < \infty$ for all $c \in \boldsymbol{R}$ where α_N is defined by (5).

(f_5) $\lim_{c \to -\infty} H(c) = 0$.

THEOREM 1. *If $f \in C(\boldsymbol{D}, \boldsymbol{R})$ satisfies* (f_1) *and* (f_2), *then there exists $\lambda_0 > 0$ such that equation* (1) *has an infinitude of positive radial entire solutions $u(x) = y(|x|)$ satisfying* (2) *for all $|\lambda| \leq \lambda_0$.*

THEOREM 2. *If* $f \in C(\boldsymbol{D}, \bar{\boldsymbol{R}}_+)$ *and satisfies* (f_1)–(f_3), *then equation* (1) *has an infinitude of positive radial entire solutions* $u(x) = y(|x|)$ *satisfying* (2) *for all* $\lambda \geq 0$.

THEOREM 3. *If* $f \in C(\bar{\boldsymbol{R}}_+ \times \boldsymbol{R} \times \bar{\boldsymbol{R}}_+, \boldsymbol{R})$ *satisfies* (f_1), (f_4), *and* (f_5), *then equation* (1) *has an infinitude of radial entire solutions which are positive in a neighborhood of infinity and satisfy* (2) *for all real* λ.

To prove these theorems we seek radial entire solutions $u(x) = y(t)$, $t = |x|$, of (1) such that $y(0) = c > 0$ and $y'(t) > 0$. Standard calculations [5] yield the polar forms

$$\det(D^2u) = t^{1-N}(y')^{N-1}y'', \qquad \Delta u = t^{1-N}(t^{N-1}y')', \tag{7}$$

where a prime denotes d/dt. It follows that $u(x)$ is a positive entire solution of (1) if and only if $y(t)$ is a positive $C^2[0, \infty)$-solution of the ordinary differential equation

$$(y')^{N-1}y'' - \alpha(t^{N-1}y')' = \lambda t^{N-1}f(t, y, y'), \qquad t > 0 \tag{8}$$

subject to the initial conditions

$$y(0) = c > 0, \qquad y'(0) = 0. \tag{9}$$

Integration of (8) yields

$$(y'(t))^N - \alpha N t^{N-1}y'(t) = \lambda N \int_0^t s^{N-1}f(s, y(s), y'(s))\, ds, \qquad t > 0. \tag{10}$$

In order to write this integro-differential equation in the more accessible form $y(t) = (\mathscr{F}y)(t)$ (see (19) below), we define

$$z(t) = \alpha_N^{-1}t^{-1}y'(t), \qquad t > 0,$$

where α_N is given by (5), and rewrite (10) in the form

$$[z(t)]^N - z(t) = \lambda N \alpha_N^{-N} t^{-N} \int_0^t s^{N-1}f(s, y(s), y'(s))\, ds, \qquad t > 0. \tag{11}$$

To solve (11) for $z(t)$, we note that the function ϕ defined by $\phi(\zeta) = \zeta^N - \zeta$ is strictly increasing for $\zeta \geq N^{-1/(N-1)}$, and in fact ϕ is a bijective map from $(N^{-1/(N-1)}, \infty)$ onto $(-(N-1)N^{-N/(N-1)}, \infty)$ such that $\phi(1) = 0$. Therefore ϕ has a uniquely defined inverse function Φ from $(-(N-1)N^{-N/(N-1)}, \infty)$ onto $(N^{-1/(N-1)}, \infty)$ with $\Phi(0) = 1$. Moreover, standard inversion theorems show that Φ is analytic, strictly increasing, and concave; in particular

$$\Phi'(\eta) = \frac{1}{N[\Phi(\eta)]^{N-1} - 1} > 0\,, \tag{12}$$

$$\Phi''(\eta) = -\frac{N(N-1)[\Phi(\eta)]^{N-2}}{(N[\Phi(\eta)]^{N-1} - 1)^3} < 0 \tag{13}$$

on dom Φ. It can also be seen easily that

$$\Phi(\eta) \leq 2(1+\eta)^{1/N} \qquad \text{for } \eta \geq 0\,. \tag{14}$$

For the right side of (11) belonging to dom Φ, i.e., exceeding $-(N-1)N^{-N/(N-1)}$, for all $t > 0$, it follows that (11) is equivalent to

$$z(t) = \Phi\left[\lambda N \alpha_N^{-N} t^{-N} \int_0^t s^{N-1} f(s, y(s), y'(s))\, ds\right], \qquad t > 0\,,$$

or

$$y'(t) = \alpha_N t \Phi\left[\lambda N \alpha_N^{-N} t^{-N} \int_0^t s^{N-1} f(s, y(s), y'(s))\, ds\right], \qquad t \geq 0\,. \tag{15}$$

Equation (15) extends to $t = 0$ by continuity since L'Hôpital's rule yields

$$\lim_{t \to 0+} t^{-N} \int_0^t s^{N-1} f(s, y(s), y'(s))\, ds = N^{-1} f(0, c, 0)$$

for any C^1-function y satisfying the initial conditions (9). Integration of (15) leads to the following integro-differential equation, appropriate for the initial value problem (8), (9):

$$y(t) = c + \alpha_N \int_0^t s\Phi\left[\lambda N \alpha_N^{-N} s^{-N} \int_0^s r^{N-1} f(r, y(r), y'(r))\, dr\right] ds\,, \qquad t \geq 0\,. \tag{16}$$

As soon as a positive solution $y \in C^1(\bar{\boldsymbol{R}}_+)$ of (16) has been demonstrated, as will be done in §3, it will follow by differentiation and application of the mapping ϕ that y solves the initial value problem (8), (9), and hence that $u(x) = y(|x|)$ is a positive radial entire solution of (1).

3. Proofs of theorems

To construct a solution $y \in C^1(\bar{\boldsymbol{R}}_+)$ of (16) under the hypotheses of Theorem 1, we choose $\lambda_0 > 0$ such that

$$N\lambda_0 F(2\alpha_N) \leq \alpha_N^N\,, \tag{17}$$

and fix $c \in (0, 2\alpha_N)$ arbitrarily. Let C^1 denote the Fréchet space of all C^1-functions in $\bar{\boldsymbol{R}}_+$, with the topology of uniform convergence of functions and their first derivatives on compact subintervals of $\bar{R}_+$. Consider the closed convex set

$$\mathscr{Y} = \{y \in C^1 : c \le y(t) \le c + 2\alpha_N t^2, 0 \le y'(t) \le 4\alpha_N t, t \ge 0\} \tag{18}$$

and the mapping $\mathscr{F} : \mathscr{Y} \to C^1$ defined by

$$(\mathscr{F}y)(t) = c + \alpha_N \int_0^t s\Phi[w(s)]\, ds\,, \qquad t \ge 0\,, \qquad |\lambda| \le \lambda_0\,, \tag{19}$$

where

$$w(s) = \lambda N \alpha_N^{-N} s^{-N} \int_0^s r^{N-1} f(r, y(r), y'(r))\, dr\,, \qquad y \in \mathscr{Y}. \tag{20}$$

If $y \in \mathscr{Y}$, then for all $s \ge 0$, $|\lambda| \le \lambda_0$,

$$\begin{aligned} |w(s)| &\le |\lambda| N \alpha_N^{-N} s^{-N} \int_0^s r^{N-1} |f(r, 2\alpha_N(1 + r^2), 4\alpha_N r)|\, dr \\ &\le \lambda_0 \alpha_N^{-N} F(2\alpha_N) \le N^{-1} < (N-1) N^{-N/(N-1)}\,, \end{aligned} \tag{21}$$

showing that $\mathscr{F}$ is well-defined on $\mathscr{Y}$. Also, if $y \in \mathscr{Y}$, (12), (14), and (17) yield

$$0 < \alpha_N \Phi[w(s)] \le \alpha_N \Phi(N^{-1}) \le 2\alpha_N(1 + N^{-1})^{1/N} \le 4\alpha_N\,, \qquad s \ge 0\,,$$

and hence

$$c \le (\mathscr{F}y)(t) \le c + 2\alpha_N t^2\,, \qquad t \ge 0\,.$$

Furthermore

$$0 \le (\mathscr{F}y)'(t) \le \alpha_N t \Phi[w(t)] \le 4\alpha_N t\,, \qquad t \ge 0\,, \tag{22}$$

showing that $\mathscr{F}$ maps $\mathscr{Y}$ into itself.

To prove the continuity of $\mathscr{F}$ in the C^1-topology, let $\{y_n\}$ be a sequence in $\mathscr{Y}$ converging to $y \in \mathscr{Y}$ in this topology, and define

$$w_n(t) = \lambda N \alpha_N^{-N} t^{-N} \int_0^t r^{N-1} f(r, y_n(r), y_n'(r))\, dr\,, \qquad t \ge 0\,.$$

Then by (20) and (22), for $|\lambda| \le \lambda_0$, $t \ge 0$,

$$|w_n(t) - w(t)| \le \lambda_0 \alpha_N^{-N} \sup\nolimits_{0 \le r \le t} |f(r, y_n(r), y_n'(r)) - f(r, y(r), y'(r))|$$

and

$$|(\mathscr{F}y_n)'(t) - (\mathscr{F}y)'(t)| = \alpha_N t |\Phi[w_n(t)] - \Phi[w(t)]|\,.$$

The continuity of Φ therefore implies that $(\mathscr{F}y_n)'(t) \to (\mathscr{F}y)'(t)$ as $n \to \infty$ uniformly on every compact subinterval of $\bar{\boldsymbol{R}}_+$. Likewise, from (19), $(\mathscr{F}y_n)(t) \to (\mathscr{F}y)(t)$ uniformly on such subintervals, establishing the continuity of $\mathscr{F}$ in C^1.

To prove that $\mathscr{F}\mathscr{Y}$ has compact closure in C^1 via Ascoli's theorem, we note that $\mathscr{F}y \in C^2(\bar{\boldsymbol{R}}_+)$ for all $y \in \mathscr{Y}$, and

$$(\mathscr{F}y)''(t) = \alpha_N \Phi[w(t)] + \lambda N \alpha_N^{1-N} \Phi'[w(t)]\left[f(t, y(t), y'(t)) - Nt^{-N}\int_0^t r^{N-1} f(r, y(r), y'(r))\, dr\right], \quad t \geq 0 .$$

Then (12), (13), and (21) imply the uniform bound

$$|(\mathscr{F}y)''(t)| \leq \alpha_N \Phi(N^{-1}) + 2\lambda_0 N \alpha_N^{1-N} \Phi'(-N^{-1}), \qquad t \geq 0,$$

from which $\mathscr{F}'\mathscr{Y} = \{(\mathscr{F}y)' : y \in \mathscr{Y}\}$ is locally equicontinuous in $\bar{\boldsymbol{R}}_+$. Similarly $\mathscr{F}\mathscr{Y}$ is locally equicontinuous, and the local uniform boundedness of $\mathscr{F}\mathscr{Y}$ and $\mathscr{F}'\mathscr{Y}$ is easily verified. Hence $\mathscr{F}\mathscr{Y}$ is relatively compact in the C^1-topology by Ascoli's theorem.

We can then apply the Schauder-Tychonov fixed point theorem to conclude that there exists an element $y \in \mathscr{Y}$ such that $\mathscr{F}y = y$, i.e., $y(t)$ satisfies (16), yielding a positive entire solution $u(x) = y(|x|)$ of equation (1) in $\boldsymbol{R}^N$. The fact that $y(t)$ satisfies (2) follows from the inequalities

$$c + \tfrac{1}{2}\alpha_N N^{-1/(N-1)} t^2 \leq y(t) \leq c + 2\alpha_N t^2, \qquad t \geq 0. \tag{23}$$

The right inequality (23) is obvious from (18), and the left inequality is a consequence of the fact

$$\Phi(\eta) \geq N^{-1/(N-1)} \qquad \text{for } \eta \geq -(N-1)N^{-N/(N-1)} .$$

Since any $c \in (0, 2\alpha_N]$ will serve as an initial value $y(0) = c$ in (9), there exists an infinitude of positive radial entire solutions of equation (1). This completes the proof of Theorem 1.

PROOF OF THEOREM 2. For arbitrary (fixed) $\lambda \geq 0$, (f_2) and (f_3) imply the existence of a constant $\beta \geq \alpha_N$ such that

$$\lambda N F(2c) \leq c^N \qquad \text{for all } c \geq \beta . \tag{24}$$

For such a number c, consider the following analogue of (18):

$$\mathscr{Y} = \{y \in C^1 : c \leq y(t) \leq c(1 + 2t^2), 0 \leq y'(t) \leq 4ct, t \geq 0\} . \tag{25}$$

Since f has only nonnegative values by hypothesis, the mapping $\mathscr{F}$ defined by (19) is well-defined on $\mathscr{Y}$. Furthermore, exactly as indicated below (20), if $y \in \mathscr{Y}$, $s \geq 0$, then

$$0 < \alpha_N \Phi[w(s)] \leq \alpha_N \Phi[\lambda \alpha_N^{-N} F(2c)]$$
$$\leq 2\alpha_N[1 + \lambda \alpha_N^{-N} F(2c)]^{1/N} = 2[\alpha_N^N + \lambda F(2c)]^{1/N} \leq 4c$$

in view of (24), implying that $\mathscr{F}$ maps $\mathscr{Y}$ into itself. The remainder of the proof is virtually the same as that for Theorem 1, and will be deleted.

PROOF OF THEOREM 3. For fixed $\lambda \in \boldsymbol{R}$, hypotheses (f_4) and (f_5) show that there exists a number $c_0 \in \boldsymbol{R}$ such that $|\lambda| NH(c) \leq \alpha_N^N$ for all $c \leq c_0$. Almost identical procedure to that used for Theorem 1 then yields a fixed point y of the mapping $\mathscr{F}$ defined by (19) in the set (18). Since c_0 could be negative, the entire solution $u(x) = y(|x|)$ of (1) obtained in this fashion could be negative near $x = 0$, but it is still easy to verify that $u(x)$ grows like a positive constant multiple of $|x|^2$ as $|x| \to \infty$. The details will be left to the reader.

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Department of Mathematics,
Faculty of Science,
Hiroshima University
and
Department of Mathematics,
University of British Columbia,
Vancouver, Canada V6T 1Y4