

Semigroups of locally Lipschitzian operators in Banach spaces

Dedicated to Professor Hiroshi Fujita on the occasion of his sixtieth birthday

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Introduction

Let $(X, |\cdot|)$ be a Banach space and D a subset of X . A one-parameter family $S = \{S(t) : t \geq 0\}$ of (possibly nonlinear) operators from D into itself is called a (*nonlinear*) *semigroup on D* if it has the two properties below:

(S1) For $s, t \geq 0$ and $x \in D$, $S(0)x = x$ and $S(s+t)x = S(s)S(t)x$.

(S2) For $x \in D$, $u(\cdot) \equiv S(\cdot)x$ is continuous on $[0, \infty)$ with respect to t .

In order to advance a general theory of nonlinear semigroups, it is necessary to restrict the continuity of the operators $S(t)$. In this paper we employ a lower semi-continuous functional $\varphi : X \rightarrow [0, \infty]$ with $D \subset D(\varphi) = \{x \in X : \varphi(x) < \infty\}$ to subdivide the set D into the “level” sets $D_\alpha = \{x \in D : \varphi(x) \leq \alpha\}$, $\alpha \geq 0$, and impose the following type of Lipschitz condition in a local sense:

(L) For $\alpha \geq 0$ and $\tau \geq 0$ there exists $\omega \equiv \omega(\alpha, \tau) \in \mathbf{R}$ such that

$$|S(t)x - S(t)u| \leq e^{\omega t}|x - u| \quad \text{for } x, u \in D_\alpha \text{ and } t \in [0, \tau].$$

Condition (L) defines a fairly general class of semigroups on D and this class is of our main interest in this paper. Here a semigroup S on D satisfying condition (L) for some lower semi-continuous functional φ is said to belong to the class $\mathfrak{S}(D, \varphi)$ in accordance with a choice of subsets D of X and functionals φ on X .

The objective of this paper is threefold. First, we impose an exponential type of growth condition on semigroups belonging to the class $\mathfrak{S}(D, \varphi)$ in terms of nonnegative functions $\varphi(S(\cdot)x)$, $x \in D$, and investigate basic properties of such semigroups. Semigroups in the class $\mathfrak{S}(D, \varphi)$ arise as families of solution operators to the initial-value problems for differential inclusions of the form

$$(DI) \quad (d/dt)u(t) \in Au(t), \quad t > 0;$$

$$(IC) \quad u(0) = x,$$

where x is an initial-value given in D and A is a possibly multi-valued operator in X . The initial-value problem (DI)–(IC) has been studied by many authors.

Especially, under the assumption that A is quasi-dissipative in X , various types of sufficient conditions on A ensuring the existence of solutions (in a generalized sense) have been given and some of the basic results in this direction are found in the papers by Kōmura [17], Kato [14, 15], Crandall and Liggett [7, 8], Kenmochi and Oharu [16], Takahashi [34], Kobayashi [19, 20], Pierre [30, 31], Walker [35], Martin [23, 24], Pazy [28, 29], Schechter [33] and Goldstein [13]. On the other hand, nonlinear analogues in a Hilbert space of the Hille-Yosida theorem due to Kōmura [18], Crandall and Pazy [6] and Dorroh [12] were extended to the case of "smooth" Banach spaces by Baillon [2] and then Reich [32]. We here show that the results cited above can be extended to the case where the nonlinear operator A in (DI) is locally quasi-dissipative in the sense that

(LQD) $D(A) \subset D$, and for each $\alpha \geq 0$ there exists $\omega \equiv \omega(\alpha) \in \mathbf{R}$ such that

$$[x - u, y - v]_- \leq \omega |x - u|$$

for $x, u \in D(A) \cap D_\alpha$, $y \in Ax$ and $v \in Au$.

Condition (LQD) is proper for the class $\mathfrak{S}(D, \varphi)$ in the sense that under condition (LQD) on A the semigroup consisting of the solution operators of (DI) belongs to the class $\mathfrak{S}(D, \varphi)$ and, conversely, that the infinitesimal generator (if it exists in the ordinary sense) of a semigroup belonging to the class $\mathfrak{S}(D, \varphi)$ satisfies condition (LQD).

Secondly, we discuss the generation of semigroups in the class $\mathfrak{S}(D, \varphi)$ under condition (LQD) and so-called range condition. These conditions together guarantee the existence of the discrete scheme

$$\begin{aligned} \text{(DS)} \quad & (t_k - t_{k-1})^{-1}(x_k - x_{k-1}) - z_k \in Ax_k, \quad k = 1, 2, \dots, \\ & z_k \in X, \quad x_0 \in D, \quad 0 \leq t_0 < t_1 < \dots < t_k < \dots, \end{aligned}$$

so far as the norm of the partition $\Delta = (t_k)$ and the error terms (z_k) are sufficiently small. Hence a modified version of the standard method of discretization in time can be applied under the localized quasi-dissipativity condition (LQD) and the aimed semigroup is obtained through the limits of solutions of the discrete problem (DS) as the norm of Δ and the errors (z_k) tend to zero. It can then be verified that the semigroup provides mild solutions of the problem (DI)–(IC) in the sense of Crandall [10] and Kobayashi, Kobayashi and Oharu [21]. Our results extend those of Chambers and Oharu [5] and Goldstein [13], and it is expected that the generation results can be applied to a broad class of nonlinear partial differential equations. In this connection we notice that in the recent papers by Oharu and Takahashi [25, 26] nonlinear semigroups associated with semilinear evolution equations are discussed from the same point of view.

Thirdly, we investigate the generators and the differentiability of semigroups in the class $\mathfrak{S}(D, \varphi)$ under the additional assumption that X is reflexive and the norm $|\cdot|$ is uniformly Gâteaux differentiable. We shall introduce a notion of generalized infinitesimal generator for semigroups in the class $\mathfrak{S}(D, \varphi)$ and show that such infinitesimal generators have dense domains in D and satisfy condition (LQD) and the range condition. It turns out that in smooth reflexive Banach spaces as mentioned above one can assert the existence of the generalized infinitesimal generator for each semigroup $S \equiv \{S(t)\}$ in the class $\mathfrak{S}(D, \varphi)$ satisfying the exponential growth condition with respect to φ , and that one can discuss the characterization of the set $Lip(S)$ of all elements x in D such that $u(\cdot) \equiv S(\cdot)x$ gives a strong solution of the problem (DI)–(IC). The restrictions on the Banach space X were first proposed by Reich [32] and seem to be optimal to obtain the infinitesimal generators of semigroups belonging to the class $\mathfrak{S}(D, \varphi)$ as far as we employ the techniques developed by Baillon [2], Reich [32], Bruck and Reich [4] and Kobayashi [20]. As treated in Miyadera [25] there is a different method for treating the differentiability of nonlinear contraction semigroups, although we do not go into the approach. We here focus our attention on the study of semigroups in the class $\mathfrak{S}(D, \varphi)$ satisfying the growth condition of exponential type and make an attempt to establish a nonlinear analogue of the Hille-Yosida theorem for such semigroups under the above-mentioned assumptions on X . We will see that it is quite delicate to discuss the definite correspondence between a given semigroup in the class $\mathfrak{S}(D, \varphi)$ and its infinitesimal generator. Consequently, we obtain a (self-contained) general theory for semigroups of locally Lipschitzian operators which includes the theory of quasi-contractive semigroups as a special case.

Section 1 introduces a class of nonlinear operators which are quasi-dissipative in a local sense and then the associated class $\mathfrak{G}(D, \varphi)$ of semigroups locally Lipschitzian operators. In Section 2 two notions of generalized solutions of the initial-value problem for (DI)–(IC) are introduced and their properties are investigated. Section 3 deals with the generation of semigroups in the class $\mathfrak{G}(D, \varphi)$ satisfying a growth condition of exponential type. In Section 4 infinitesimal generators in a generalized sense of semigroups in the class $\mathfrak{G}(D, \varphi)$ are treated in some detail and the question of the differentiability of the semigroups satisfying the exponential growth condition is discussed. Section 5 concerns the range conditions for the generalized infinitesimal generators.

1. A class of nonlinear operators and the associated semigroups

Let X be a real Banach space with norm $|\cdot|$. The dual space of X is denoted by X^* . Given a subset C of X we write \bar{C} for the norm closure of C . The distance from the set C to $x \in X$ is denoted by $d(C, x)$. An

operator A in X means a (possibly multi-valued) operator with domain $D(A)$ and range $R(A)$ in X . In this paper A is identified with its graph $\{(x, y) \in X \times X : x \in D(A), y \in Ax\}$. An operator A is said to be φ -closed if $(x_n, y_n) \in A$, $x_n \rightarrow x$, $y_n \rightarrow y$ in X and $\limsup_{n \rightarrow \infty} \varphi(x_n) < \infty$ imply that $(x, y) \in A$. The identity operator on X is denoted by I .

For $x, y \in X$ we define $[x, y]_\lambda = \lambda^{-1}(|x + \lambda y| - |x|)$ for $\lambda \in \mathbf{R} - \{0\}$,

$$(1.1) \quad [x, y]_+ = \inf_{\lambda > 0} [x, y]_\lambda = \lim_{\lambda \downarrow 0} [x, y]_\lambda \quad \text{and} \quad [x, y]_- = -[x, -y]_+.$$

The functional $[\cdot, \cdot]_+ : X \times X \rightarrow \mathbf{R}$ is upper semi-continuous and has the following properties:

PROPOSITION 1.1 ([9], [22]). For $x, y, z \in X$ and $\alpha \in \mathbf{R}$, we have

$$(1.2) \quad \begin{aligned} [x, \alpha x + y]_+ &= \alpha|x| + [x, y]_+, & [x, |\alpha|y]_+ &= |\alpha|[x, y]_+, \\ [x, y]_- - [x, z]_+ &\leq [x, y - z]_- \leq [x, y]_+ - [x, z]_+, \\ [x, y + z]_+ &\leq [x, y]_+ + [x, z]_+, & |[x, y]_+| &\leq |y|, & [x, x]_\pm &= |x|. \end{aligned}$$

Let $C \subset X$. An operator A in X is said to be *dissipative on C* , if

$$[x - u, y - v]_- \leq 0 \quad \text{for } (x, y), (u, v) \in A \text{ with } x, u \in C.$$

If in particular $C \supset D(A)$, we say simply that A is dissipative. If A is dissipative and satisfies the range condition $R(I - \lambda A) = X$ for $\lambda > 0$, then A is said to be *m-dissipative*. Let $\omega \in \mathbf{R}$. Then the operator $A - \omega I$ is dissipative if and only if $(1 - \lambda\omega)|x - u| \leq |(x - \lambda y) - (u - \lambda v)|$ for $\lambda > 0$ and $(x, y), (u, v) \in A$. Accordingly, if $A - \omega I$ is dissipative and $\lambda\omega < 1$ then the inverse operator $(I - \lambda A)^{-1}$ exists as a Lipschitzian operator which has a Lipschitz constant $(1 - \lambda\omega)^{-1}$ and maps $R(I - \lambda A)$ onto $D(A)$. In what follows, we say that A is *quasi-dissipative on C* if $A - \omega I$ is dissipative on C for some $\omega \geq 0$.

Let D be a subset of X and let $\varphi : X \rightarrow [0, \infty]$ be a lower semi-continuous functional on X such that $D \subset D(\varphi) = \{x \in X : \varphi(x) < \infty\}$. We permit ourselves the common abbreviation, an *l.s.c.* functional on X , in referring to a lower semi-continuous functional on X . For each $\alpha \geq 0$ the level set in D of φ is defined as

$$(1.3) \quad D_\alpha = \{x \in D : \varphi(x) \leq \alpha\}.$$

By Ind_D we denote the indicator function of D . By means of Ind_D the sets D_α , $\alpha \geq 0$, can be characterized as follows.

PROPOSITION 1.2. *The functional $\varphi + \text{Ind}_D$ is l.s.c. on X if and only if the level set D_α is closed in X for each $\alpha \geq 0$. Furthermore, D_α is exactly the level set $\{x \in X : \varphi(x) + \text{Ind}_D(x) \leq \alpha\}$ of $\varphi + \text{Ind}_D$ for each $\alpha \geq 0$.*

Therefore, we may assume without loss of generality that D coincides with the effective domain $D(\varphi)$ and each D_α is the usual level set $\{x \in X : \varphi(x) \leq \alpha\}$ of φ itself. Notice that in this case each D_α is closed in X . Given a pair of numbers $\alpha, \beta \in [0, \infty)$, $\alpha \vee \beta$ and $\alpha \wedge \beta$ denote the maximum and the minimum of the numbers α and β , respectively. Finally, N is the set of all positive integers.

We then introduce a class of nonlinear operators in X that are locally quasi-dissipative with respect to the functional $\varphi : X \rightarrow [0, \infty]$.

DEFINITION 1.1. An operator A in X is said to belong to the class $\mathfrak{G}(D, \varphi)$, if it satisfies the following condition:

(LQD) $D(A) \subset D$ and for each $\alpha \geq 0$ there exists $\omega \equiv \omega(\alpha) \in \mathbf{R}$ such that

$$[x - u, y - v]_- \leq \omega |x - u| \quad \text{for } x, u \in D(A) \cap D_\alpha, y \in Ax \text{ and } v \in Au.$$

Given an operator A belonging to $\mathfrak{G}(D, \varphi)$ we shall impose various conditions on it; in such cases we call it an operator in the class $\mathfrak{G}(D, \varphi)$ for simplicity in description.

As will be seen in Section 3, semigroups generated by operators in the class $\mathfrak{G}(D, \varphi)$ satisfy the local Lipschitz condition (L) as mentioned in Introduction. This leads us to the following

DEFINITION 1.2. Let $\varphi : X \rightarrow [0, \infty]$ be proper and l.s.c. and let $D \equiv D(\varphi)$. A semigroup $S = \{S(t) : t \geq 0\}$ on D is said to belong to the class $\mathfrak{S}(D, \varphi)$, if (L) for each $\alpha \geq 0$ and each $\tau \geq 0$ there exists $\omega \in \mathbf{R}$ such that

$$|S(t)x - S(t)u| \leq e^{\omega t} |x - u|$$

holds for $x, u \in D_\alpha$ and $t \in [0, \tau]$. In case that we consider a semigroup S which belongs to the class $\mathfrak{S}(D, \varphi)$ and satisfies some additional conditions on it, we often call it a semigroup in the class $\mathfrak{S}(D, \varphi)$ for simplicity in description.

The most natural way to attempt to associate the initial-value problem (DI)–(IC) involving an operator A in the class $\mathfrak{G}(D, \varphi)$ is to compute the operator

$$(1.4) \quad A_+ x = \lim_{h \downarrow 0} h^{-1} (S(h)x - x)$$

whose domain $D(A_+)$ is the set of $x \in D$ such that the limit exists in X , and then hope that “solving” (DI)–(IC) with A replaced by an appropriate extension of A_+ will return S . The operator A_+ is usually called the *infinitesimal generator* of S in the theory of operator semigroups. For an arbitrary semigroup S in the class $\mathfrak{S}(D, \varphi)$ in a general Banach space X , the domain $D(A_+)$ may be empty in general as indicated by Crandall and Liggett [8]. Moreover, it is observed by Webb [36] that the operator A_+ need not be large enough to satisfy the range condition and does not necessarily determine the semigroup S even though $D(A_+)$ is dense in D . It is interesting to seek an optimal concept of infinitesimal generator and find conditions on S , its domain D , the functional φ and the space X under consideration which together assure the existence of such an infinitesimal generator. This can be accomplished if φ is convex on X and if the Banach space X is reflexive and smooth in the following sense.

DEFINITION 1.3. The Banach space $(X, |\cdot|)$ is said to have a *Gâteaux differentiable norm* whenever

$$(1.5) \quad \lim_{\lambda \downarrow 0} (|x + \lambda y|^2 + |x - \lambda y|^2 - 2|x|^2)/(2\lambda) = 0$$

holds for $x, y \in X$. If formula (1.5) holds uniformly for bounded x in the sense that for $M > 0$, $y \in X$ and $\varepsilon > 0$ one finds $\delta > 0$ such that

$$(|x + \lambda y|^2 + |x - \lambda y|^2 - 2|x|^2)/(2\lambda) \leq \varepsilon$$

for $\lambda \in (0, \delta]$ and x with $|x| \leq M$, then we say that $(X, |\cdot|)$ has a *uniformly Gâteaux differentiable norm*.

The class of reflexive Banach spaces with uniformly Gâteaux differentiable norms contains an important class of reflexive Banach spaces. See Diestel's book [11, p. 36].

PROPOSITION 1.3. *Any uniformly smooth Banach space has a uniformly Gâteaux differentiable norm.*

In Section 4 and 5 we shall treat infinitesimal generators in a generalized sense and discuss the differentiability of semigroups in the class $\mathfrak{S}(D, \varphi)$ in reflexive Banach spaces with uniformly Gâteaux differentiable norms.

2. Mild solutions and integral solutions

Throughout this section we fix a proper l.s.c. functional $\varphi: X \rightarrow [0, \infty]$ with $D \equiv D(\varphi)$ and define the family of level sets $\{D_\alpha: \alpha > 0\}$ by (1.3). Let A be an operator in the class $\mathfrak{G}(D, \varphi)$ and consider the differential inclusion

$$(DI) \quad (d/dt)u(t) \in Au(t), \quad t > 0.$$

We here introduce two notions of generalized solutions of the differential inclusion (DI) and investigate their properties in some detail. In what follows, τ denotes an arbitrary but fixed positive number.

We begin by recalling the notion of strong solution of (DI).

DEFINITION 2.1. A function $u : [0, \tau] \rightarrow X$ is said to be a *strong solution* of (DI) on $[0, \tau]$, if it is Lipschitz continuous over $[0, \tau]$, differentiable a.e. in $(0, \tau)$, $u(t) \in D(A)$ and the strong derivative $u'(t)$ belongs to the set $Au(t)$ for a.e. $t \in (0, \tau)$.

In case that X is a general Banach space, the inclusion (DI) does not necessarily admit strong solutions even though the initial values lie in $D(A)$. We here adopt a notion of solution which refers directly to the approximation method used to establish the existence of solutions, so-called *method of discretization in time*.

DEFINITION 2.2. Let $\varepsilon > 0$. A piecewise constant function $v : [0, \tau] \rightarrow X$ is said to be an ε -*approximate solution* of (DI) on $[0, \tau]$, if there exists a partition $\{0 = t_0 < t_1 < \dots < t_N\}$ of the interval $[0, t_N]$ and a finite sequence $((x_i, z_i) : i = 1, \dots, N)$ with the three properties (ε.1), (ε.2), (ε.3) below:

$$(\varepsilon.1) \quad v(0) = x_0, v(t) = x_i \quad \text{for } t \in (t_{i-1}, t_i] \cap [0, \tau] \quad \text{and}$$

$$(t_i - t_{i-1})^{-1}(x_i - x_{i-1}) - z_i \in Ax_i, \quad i = 1, \dots, N,$$

$$(\varepsilon.2) \quad t_i - t_{i-1} \leq \varepsilon, i = 1, \dots, N, \quad \text{and} \quad \tau \leq t_N < \tau + \varepsilon,$$

$$(\varepsilon.3) \quad \sum_{i=1}^N (t_i - t_{i-1})|z_i| \leq \varepsilon t_N.$$

DEFINITION 2.3. A continuous function $u : [0, \tau] \rightarrow X$ is said to be a *mild solution* of (DI) on $[0, \tau]$, provided that for each $\varepsilon > 0$ there is an ε -approximate solution v^ε of (DI) on $[0, \tau]$ such that $|u(t) - v^\varepsilon(t)| \leq \varepsilon$ for $t \in [0, \tau]$. If there is a constant $\alpha \in [0, \infty)$ such that $v^\varepsilon(t) \in D_\alpha$ for $\varepsilon > 0$ and $t \in [0, \tau]$, then we say that the mild solution is *confined to D_α* .

Notice that if u is a mild solution on $[0, \tau]$ confined to D_α then $u(t) \in D_\alpha$ for $t \in [0, \tau]$ since D_α is closed in X . A mild solution confined to some D_α is therefore a continuous uniform limit of approximate solutions confined to D_α and this notion represent a considerable generalization of the strong notion. A strong solution confined to some D_α is a mild solution confined to D_α , but the proof is not entirely obvious. The following result is essentially proved in the papers [15] and [20].

PROPOSITION 2.1. *If $u: [0, \tau] \rightarrow X$ is a strong solution of (DI) on $[0, \tau]$, then it is a mild solution of (DI) on $[0, \tau]$. If in addition $u(t) \in D_\alpha$ for $t \in [0, \tau]$ and some $\alpha > 0$, then the mild solution u is a mild solution confined to D_α .*

REMARK. In case that only the values $u(t)$ lie in the level set D_α for $t \in [0, \tau]$, it seems that a mild solution of (DI) on $[0, \tau]$ is not necessarily confined to D_α . However we do not have any examples which illustrate this situation.

We next introduce the notion of integral solution which plays an important role in not only giving a framework of the theory of semigroups of locally Lipschitzian operators which are generated by operators in the class $\mathfrak{G}(D, \varphi)$, but also in establishing the uniqueness of mild solutions.

DEFINITION 2.4. A continuous function $u: [0, \tau] \rightarrow X$ is said to be an *integral solution* (with respect to φ) of (DI) on $[0, \tau]$, if for each $\beta \in [0, \infty)$ there is $\omega(\beta) \in [0, \infty)$ such that the integral inequality

$$(2.1) \quad |u(t) - x| - |u(s) - x| \leq \int_s^t ([u(\xi) - x, y]_+ + \omega(\beta)|u(\xi) - x|) d\xi$$

holds for $s, t \in [0, \tau]$ with $s \leq t$ and $(x, y) \in A$ with $x \in D_\beta$.

The number $\omega(\beta)$ appearing in (2.1) is determined by condition (LQD) and corresponds to the Lipschitz constant stated in condition (L). Notice that (2.1) holds for any number $\omega \in [\omega(\beta), \infty)$.

THEOREM 2.2 (Bénilan [3], Kobayasi-Kobayashi-Oharu [21]). *Let $\alpha \geq 0$ and let $u: [0, \tau] \rightarrow X$ be a mild solution of (DI) on $[0, \tau]$ confined to D_α . Then we have:*

(a) *The mild solution u is an integral solution of (DI) on $[0, \tau]$.*

(b) *If v is an integral solution of (DI) on $[0, \tau]$, then there exists $\omega \equiv \omega(\alpha) \in [0, \infty)$ such that*

$$(2.2) \quad |v(t) - u(t)| \leq e^{\omega t} |v(0) - u(0)| \quad \text{for } t \in [0, \tau].$$

(c) *If v is a mild solution of (DI) on $[0, \tau]$ confined to D_α , then $v(t) \equiv u(t)$ on $[0, \tau]$ provided that $v(0) = u(0)$.*

PROOF. Let $\{u^\varepsilon: \varepsilon > 0\}$ be a family of approximate solutions in the sense of Definition 2.2 and assume that $u^\varepsilon(t) \in D_\alpha$ and $|u^\varepsilon(t) - u(t)| \leq \varepsilon$ for $t \in [0, \tau]$ and $\varepsilon > 0$. For each $\varepsilon > 0$ one finds a partition $\{0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_{N(\varepsilon)}^\varepsilon\}$ of the interval $[0, t_{N(\varepsilon)}^\varepsilon]$ and a finite sequence $((x_i^\varepsilon, z_i^\varepsilon): i = 1, \dots, N(\varepsilon))$ satisfying

$$(2.3) \quad \begin{cases} u^\varepsilon(0) = x_0^\varepsilon, u^\varepsilon(t) = x_i^\varepsilon & \text{for } t \in (t_{i-1}^\varepsilon, t_i^\varepsilon] \cap [0, \tau], \\ y_i^\varepsilon \equiv (t_i^\varepsilon - t_{i-1}^\varepsilon)^{-1}(x_i^\varepsilon - x_{i-1}^\varepsilon) - z_i^\varepsilon \in Ax_i^\varepsilon & \text{for } i = 1, \dots, N(\varepsilon), \\ h_i^\varepsilon \equiv t_i^\varepsilon - t_{i-1}^\varepsilon \leq \varepsilon & \text{for } i = 1, \dots, N(\varepsilon), \tau \leq t_{N(\varepsilon)} < \tau + \varepsilon, \\ \sum_{i=1}^{N(\varepsilon)} h_i^\varepsilon |z_i^\varepsilon| \leq \varepsilon t_{N(\varepsilon)} \leq \varepsilon(\tau + \varepsilon). \end{cases}$$

We first prove (a). Let $\beta \in [0, \infty)$ and set $\gamma = \alpha \vee \beta$. Since $A \in \mathfrak{G}(D, \varphi)$, there is $\omega \equiv \omega(\gamma) \in [0, \infty)$ such that $A - \omega I$ is dissipative on D_γ . Let $(x, y) \in A$ and $x \in D_\beta$. Then $x, x_i^\varepsilon \in D(A) \cap D_\gamma$, and the application of (1.2) implies $[x_i^\varepsilon - x, y_i^\varepsilon]_- - [x_i^\varepsilon - x, y]_+ \leq [x_i^\varepsilon - x, y_i^\varepsilon - y]_- \leq \omega |x_i^\varepsilon - x|$, and so

$$(2.4) \quad [x_i^\varepsilon - x, h_i^\varepsilon y_i^\varepsilon]_- \leq ([x_i^\varepsilon - x, y]_+ + \omega |x_i^\varepsilon - x|) h_i^\varepsilon.$$

The term $h_i^\varepsilon y_i^\varepsilon$ can be written as $(x_i^\varepsilon - x) - (x_{i-1}^\varepsilon - x) - h_i^\varepsilon z_i^\varepsilon$, so that the left side of (2.4) is estimated as

$$\begin{aligned} [x_i^\varepsilon - x, h_i^\varepsilon y_i^\varepsilon]_- &= |x_i^\varepsilon - x| + [x_i^\varepsilon - x, -(x_{i-1}^\varepsilon - x) - h_i^\varepsilon z_i^\varepsilon]_- \\ &\geq |x_i^\varepsilon - x| - |x_{i-1}^\varepsilon - x| - h_i^\varepsilon |z_i^\varepsilon|, \end{aligned}$$

where we have applied Proposition 1.1. Therefore,

$$(2.5) \quad |x_i^\varepsilon - x| - |x_{i-1}^\varepsilon - x| \leq ([x_i^\varepsilon - x, y]_+ + \omega |x_i^\varepsilon - x| + |z_i^\varepsilon|) h_i^\varepsilon$$

for $i = 1, \dots, N(\varepsilon)$. Let $0 \leq j < k \leq N(\varepsilon)$. Adding up both sides of inequalities (2.5) from $i = j + 1$ to $i = k$, we have

$$|x_k^\varepsilon - x| - |x_j^\varepsilon - x| \leq \sum_{i=j+1}^k ([x_i^\varepsilon - x, y]_+ + \omega |x_i^\varepsilon - x| + |z_i^\varepsilon|) h_i^\varepsilon.$$

This together with (2.3) implies

$$(2.6) \quad \begin{aligned} &|u^\varepsilon(t_k^\varepsilon) - x| - |u^\varepsilon(t_j^\varepsilon) - x| \\ &\leq \int_{t_j^\varepsilon}^{t_k^\varepsilon} ([u^\varepsilon(\xi) - x, y]_+ + \omega |u^\varepsilon(\xi) - x|) d\xi + \varepsilon(\tau + \varepsilon). \end{aligned}$$

Let $0 \leq s < t \leq \tau$ and let $t_j^\varepsilon < t_k^\varepsilon, t_j^\varepsilon \rightarrow s$ and $t_k^\varepsilon \rightarrow t$ as $\varepsilon \downarrow 0$ in (2.6). Then, the application of (1.2) and the upper-semicontinuity of the functional $[\cdot, \cdot]_+$ and (2.6) implies the integral inequality (2.1).

Next we demonstrate that (b) is valid. Let $v : [0, \tau] \rightarrow X$ be an integral solution of (DI) on $[0, \tau]$. Then there is $\omega \equiv \omega(\alpha) \in [0, \infty)$ such that

$$(2.7) \quad |v(t) - x| - |v(s) - x| \leq \int_s^t ([v(\xi) - x, y]_+ + \omega |v(\xi) - x|) d\xi$$

for $s, t \in [0, \tau]$ with $s \leq t$ and $(x, y) \in A$ with $x \in D_\alpha$. Let $0 \leq s \leq t \leq \tau$ and set $x = x_i^\varepsilon$ and $y = y_i^\varepsilon$ in (2.7). Then we have

$$(2.8) \quad (|v(t) - x_i^\varepsilon| - |v(s) - x_i^\varepsilon|)h_i^\varepsilon \leq \int_s^t (|v(\xi) - x_i^\varepsilon, h_i^\varepsilon y_i^\varepsilon|_+ + \omega |v(\xi) - x_i^\varepsilon| h_i^\varepsilon) d\xi$$

for $i = 1, \dots, N(\varepsilon)$. Since $h_i^\varepsilon y_i^\varepsilon$ is written as $(x_i^\varepsilon - v(\xi)) - (x_{i-1}^\varepsilon - v(\xi)) - h_i^\varepsilon z_i^\varepsilon$, the application of (1.2) implies

$$\begin{aligned} |v(\xi) - x_i^\varepsilon, h_i^\varepsilon y_i^\varepsilon|_+ &= -|v(\xi) - x_i^\varepsilon| + |v(\xi) - x_i^\varepsilon, -(x_{i-1}^\varepsilon - v(\xi)) - h_i^\varepsilon z_i^\varepsilon|_+ \\ &\leq -|v(\xi) - x_i^\varepsilon| + |v(\xi) - x_{i-1}^\varepsilon| + h_i^\varepsilon |z_i^\varepsilon|. \end{aligned}$$

Combining this with (2.8) gives

$$(2.9) \quad \begin{aligned} &(|v(t) - x_i^\varepsilon| - |v(s) - x_i^\varepsilon|)h_i^\varepsilon \\ &\leq \int_s^t (-|v(\xi) - x_i^\varepsilon| + |v(\xi) - x_{i-1}^\varepsilon| + \omega |v(\xi) - x_i^\varepsilon| h_i^\varepsilon) d\xi \end{aligned}$$

for $i = 1, \dots, N(\varepsilon)$. Let $1 \leq j < k \leq N(\varepsilon)$. Adding up both sides of (2.9) from $i = j + 1$ to $i = k$ and using (2.3), we get

$$(2.10) \quad \begin{aligned} &\int_{t_j^\varepsilon}^{t_k^\varepsilon} (|v(t) - u^\varepsilon(\zeta)| - |v(s) - u^\varepsilon(\zeta)|) d\zeta \\ &\leq \int_s^t \left(-|v(\xi) - u^\varepsilon(t_k^\varepsilon)| + |v(\xi) - u^\varepsilon(t_j^\varepsilon)| \right. \\ &\quad \left. + \varepsilon(\tau + \varepsilon) + \omega \int_{t_j^\varepsilon}^{t_k^\varepsilon} |v(\xi) - u^\varepsilon(\zeta)| d\zeta \right) d\xi. \end{aligned}$$

We now take any pair $\rho, \sigma \in [0, \tau]$ with $\rho \leq \sigma$ and choose two sequences (t_j^ε) and (t_k^ε) so that $t_j^\varepsilon < t_k^\varepsilon$, $t_j^\varepsilon \rightarrow \rho$ and $t_k^\varepsilon \rightarrow \sigma$ in $[0, \tau]$ as $\varepsilon \downarrow 0$. Passing to the limit as $\varepsilon \downarrow 0$ in (2.10), we obtain the integral inequality

$$(2.11) \quad \begin{aligned} &\int_\rho^\sigma (|v(t) - u(\zeta)| - |v(s) - u(\zeta)|) d\zeta + \int_s^t (|v(\xi) - v(\sigma)| - |v(\xi) - u(\rho)|) d\xi \\ &\leq \omega \int_s^t \left(\int_\rho^\sigma |v(\xi) - u(\zeta)| d\zeta \right) d\xi \end{aligned}$$

Let $h \in [0, \tau)$ and define $F_h : [0, \tau - h] \rightarrow [0, \infty)$ by

$$F_h(t) = h^{-2} \int_t^{t+h} \left(\int_t^{t+h} |v(\xi) - u(\zeta)| d\zeta \right) d\xi \quad \text{for } t \in [0, \tau - h].$$

Then (2.11) implies that F_h satisfies the differential inequality $F_h'(t) \leq \omega F_h(t)$ for $t \in (0, \tau - h)$, and hence it follows that $F_h(t) \leq e^{\omega t} F_h(0)$ for $t \in [0, \tau - h]$. Letting $h \downarrow 0$ and using the strong continuity of u and v on $[0, \tau]$, we obtain the desired estimate (2.2).

Finally, assertion (c) follows directly from (a) and (b). This completes the proof of Theorem 2.2. \square

DEFINITION 2.5. Let $u : [0, \infty) \rightarrow X$ be continuous over $[0, \infty)$. We say that u is a *locally φ -bounded global mild solution* of (DI) on $[0, \infty)$, if to each $\tau > 0$ there corresponds $\alpha \in [0, \infty)$ such that the restriction of u to $[0, \tau]$ gives a mild solution of (DI) on $[0, \tau]$ confined to D_α . Further, u is called a *global integral solution* of (DI) if for each $\tau > 0$ the restriction of u to $[0, \tau]$ is an integral solution of (DI) on $[0, \tau]$ in the sense of Definition 2.4.

The next result is an immediate consequence of Theorem 2.2.

COROLLARY 2.3. Let $u : [0, \infty) \rightarrow X$ be a global mild solution of (DI) which is locally φ -bounded on $[0, \infty)$. Let $v : [0, \infty) \rightarrow X$ be a global integral solution of (DI). Then

(a) u is a global integral solution of (DI);

and

(b) for every $\tau \in [0, \infty)$ there is $\omega \in [0, \infty)$ such that

$$|u(t) - v(t)| \leq e^{\omega t} |u(0) - v(0)| \quad \text{for } t \in (0, \tau].$$

Suppose that for each $x \in D$ there is a unique global mild solution $u(\cdot; x)$ of (DI) which is locally φ -bounded on $[0, \infty)$ and satisfies $u(0; x) = x$. Then one can define for each $t \geq 0$ an operator $S(t) : D \rightarrow D$ by

$$(2.12) \quad S(t)x = u(t; x) \quad \text{for } x \in D.$$

To assert that the family $S = \{S(t) : t \geq 0\}$ forms a semigroup belonging to the class $\mathfrak{S}(D, \varphi)$, we need condition (C) below:

(C) For each $\alpha \in [0, \infty)$ and each $\tau \in [0, \infty)$ there is $\beta \in [0, \infty)$ such that for $x \in D_\alpha$ the restriction of the associated global mild solution $u(\cdot; x)$ to $[0, \tau]$ is confined to D_β .

THEOREM 2.4. Let $S = \{S(t) : t \geq 0\}$ be a family of self maps of D defined by (2.12). Then S forms a semigroup on D . Assume further that condition (C) holds. Then the semigroup S belongs to the class $\mathfrak{S}(D, \varphi)$.

PROOF. By the definition of global mild solution of (DI) it is clear that $S(0)x = x$ for $x \in D$ and the X -valued function $S(\cdot)x$ is continuous on $[0, \infty)$. Fix any $x \in D$ and any $s > 0$ and define

$$v(t) = u(t + s; x) \quad \text{for } t \in [0, \infty).$$

Then v is a global mild solution of (DI), $v(0) = u(s, x) \in D$, and v is locally

φ -bounded on $[0, \infty)$. By Corollary 2.3 we have $v(t) = u(t; u(s; x))$, and this means that $S(t + s)x = S(t)S(s)x$. Therefore S forms a semigroup on D . Next, assume condition (C). Let $\alpha \in [0, \infty)$, $\tau \in [0, \infty)$ and let β be a nonnegative number provided by (C). Then the function $S(\cdot)x$ restricted to $[0, \tau]$ gives a mild solution of (DI) on $[0, \tau]$ confined to D_β provided that $x \in D_\alpha$. Let $v: [0, \tau] \rightarrow X$ be any integral solution of (DI) on $[0, \tau]$. Then Theorem 2.2 implies that $|v(t) - S(t)x| \leq e^{\omega t}|v(0) - x|$ for $x \in D_\alpha$, $t \in [0, \tau]$ and some $\omega \in [0, \infty)$. Taking any $y \in D_\alpha$ and setting $v(t) = S(t)y$ we obtain the Lipschitz condition (L). This shows that S belongs to the class $\mathfrak{S}(D, \varphi)$. \square

Given a semigroup $S = \{S(t) : t \geq 0\}$ on D , one can assign to each $x \in D$ a D -valued function $u(\cdot; x)$ by (2.12). However condition (C) does not necessarily hold for the family of functions $\{u(\cdot; x) : x \in D\}$. In the next section we introduce a growth condition of exponential type to define a specific but natural class of semigroups on D for which condition (C) holds.

3. Generation of semigroups of class $\mathfrak{S}(D, \varphi)$

In this section we establish a generation theorem for semigroups in the class $\mathfrak{S}(D, \varphi)$ satisfying a growth condition introduced as below.

Let $a, b \geq 0$ and define the linear function g by

$$(3.1) \quad g(r) = ar + b, \quad r \in [0, \infty).$$

We write $\pi(\cdot; \alpha)$ for the solution of the initial-value problem

$$r'(t) = g(r(t)), \quad t > 0; \quad r(0) = \alpha \in [0, \infty).$$

The solution $\pi(\cdot; \alpha)$ can be explicitly represented as

$$(3.2) \quad \pi(t; \alpha) = \alpha e^{at} + b \int_0^t e^{a(t-s)} ds.$$

We observe that the one-parameter family $\Pi = \{\pi(t; \cdot) : t \geq 0\}$ of the solution operators forms an order-preserving affine semigroup on the real half-line $[0, \infty)$ such that $\pi(t; \alpha) \vee \pi(t; \beta) = \pi(t; \alpha \vee \beta)$ for $t \geq 0$ and $\alpha, \beta \in [0, \infty)$.

Given a semigroup S in the class $\mathfrak{S}(D, \varphi)$ we introduce the following condition:

$$(G) \quad \varphi(S(t)x) \leq \pi(t; \varphi(x)) \quad \text{for } x \in D \text{ and } t \in [0, \infty).$$

We call condition (G) the *exponential growth condition* for S with respect to φ .

A semigroup S on D does not necessarily satisfy the growth condition (G), even if it provides mild solutions of some differential inclusion (DI) via the relation (2.12) and the nonlinear operator A in (DI) belongs to the class

$\mathfrak{G}(D, \varphi)$. In applications to partial differential equations the use of such functionals φ corresponds to *a priori* estimates or energy estimates which assure the global existence of the solutions as well as their asymptotic properties. In case that $a = b = 0$ in the growth condition (G), the functional φ may be called a Lyapunov function for the nonlinear operator A . Appropriate functionals φ are often derived in accordance with the nature of the equation under consideration so that the mild solutions may satisfy a growth condition of the type (G). See also the recent papers [26] and [27]. Quasicontractive semigroups treated for instance in [7, 16, 19, 34] satisfy the exponential growth condition with respect to the *l.s.c.* functionals as mentioned below. Let $A - \omega I$ be dissipative on X and assume that A generates a semigroup $S = \{S(t)\}$ on $D \equiv \overline{D(A)}$ in the sense of [7, 16, 19, 34]. Then $|S(t)x - S(t)y| \leq e^{\omega t}|x - y|$ for $x, y \in D$ and $t \geq 0$ and $|S(t)z - z| \leq \|Az\| \int_0^t e^{\omega s} ds$ for $z \in D(A)$ and $t \geq 0$, where $\|Az\| = \inf \{|v| : v \in Az\}$. Fix any $z \in D(A)$ and define

$$\varphi(z) = \begin{cases} |x - z|, & \text{if } x \in D, \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $|S(t)x - z| \leq |S(t)x - S(t)z| + |S(t)z - z| \leq e^{\omega t}|x - z| + \|Az\| \int_0^t e^{\omega s} ds$, the quasicontractive semigroup S satisfies (G) with $a = \omega$ and $b = \|Az\|$.

In what follows, we are mainly concerned with semigroups in the class $\mathfrak{S}(D, \varphi)$ satisfying the exponential growth condition (G). Let A be an operator in X belonging to the class $\mathfrak{G}(D, \varphi)$. We consider the following condition (R) which we call the *range condition* for the operator A in the sequel.

(R) For $\varepsilon > 0$ and $x \in D$ there exist $\delta \in (0, \varepsilon]$, $x_\delta \in D(A)$ and $z_\delta \in X$ which satisfy $|z_\delta| < \varepsilon$ and the two relations below:

$$\begin{aligned} \delta^{-1}(x_\delta - x) - z_\delta &\in Ax_\delta, \\ \delta^{-1}(\varphi(x_\delta) - \varphi(x)) - \varepsilon &\leq g(\varphi(x_\delta)). \end{aligned}$$

The generation theorem is then stated as follows:

THEOREM 3.1. *Let $A \in \mathfrak{G}(D, \varphi)$ and suppose $D \subset \overline{D(A)}$ and the range condition (R) holds. Then there exists a semigroup $S \equiv \{S(t) : t \geq 0\}$ in the class $\mathfrak{S}(D, \varphi)$ satisfying the growth condition (G) such that for each $x \in D$ the function $u(\cdot) \equiv S(\cdot)x$ gives a unique global mild solution of (DI) and $u(\cdot)$ is locally φ -bounded on $[0, \infty)$.*

Before giving the proof of this theorem we first recall the following result which follows readily from the generation theorems due to Kobayashi [19], Crandall and Evans [9] and Kobayasi, Kobayashi and Oharu [21].

THEOREM 3.2. *Let A be an operator in the class $\mathfrak{G}(D, \varphi)$ satisfying $D \subset \overline{D(A)}$ $\tau > 0$, $\alpha > 0$ and let $x \in D_\alpha$. Suppose that there exists a positive number ε_0 , and that for each $\varepsilon \in (0, \varepsilon_0)$ there is an ε -approximate solution $u^\varepsilon : [0, \tau] \rightarrow X$ such that $u^\varepsilon(t) \in D_\alpha$ for $t \in [0, \tau]$. If $\lim_{\varepsilon \downarrow 0} u^\varepsilon(0) = x$, then there exists a unique mild solution u of (DI) on $[0, \tau]$ confined to D_α and*

$$\lim_{\varepsilon \downarrow 0} (\sup \{|u^\varepsilon(t) - u(t)| : t \in [0, \tau]\}) = 0.$$

For each $\varepsilon > 0$ we write $\pi_\varepsilon(t; \alpha)$ for the solution of the initial-value problem

$$(3.3) \quad r'(t) = g_\varepsilon(r(t)), \quad t > 0; \quad r(0) = \alpha,$$

where g_ε is defined by

$$g_\varepsilon(r) = g(r) + \varepsilon, \quad r \in [0, \infty).$$

It is seen that the solution $\pi_\varepsilon(t; \alpha)$ is represented as

$$(3.4) \quad \pi_\varepsilon(t; \alpha) = \alpha e^{at} + (b + \varepsilon) \int_0^t e^{a(t-s)} ds.$$

We prove Theorem 3.1 after preparing the following lemma which contains fundamental estimates in the generation theory.

LEMMA 3.3. *Let $A \in \mathfrak{G}(D, \varphi)$. Suppose that $D \subset \overline{D(A)}$ and the range condition (R) holds. Let $x_0 \in D$. Then for each $\varepsilon > 0$ there exists a sequence $(h_n, x_n, y_n)_{n=1}^\infty$ in $(0, \varepsilon] \times D(A) \times X$ with the following properties:*

$$(3.5) \quad \sum_{n=1}^\infty h_n = +\infty, \quad y_n \in Ax_n, \quad n \in N,$$

$$(3.6) \quad |x_n - x_{n-1} - h_n y_n| \leq \varepsilon h_n, \quad n \in N,$$

$$(3.7) \quad \varphi(x_n) \leq \pi_\varepsilon(h_n; \varphi(x_{n-1})), \quad n \in N.$$

PROOF. Let $x \in D$ and $\varepsilon > 0$. By the assumptions one finds a sequence $(\delta_k, x_k, y_k)_{k=1}^\infty$ in $(0, \varepsilon) \times D(A) \times X$ satisfying

$$|\delta_k^{-1}(x_k - x) - y_k| \leq \varepsilon, \quad y_k \in Ax_k,$$

$$\delta_k^{-1}(\varphi(x_k) - \varphi(x)) \leq g(\varphi(x_k)) + \varepsilon/2, \quad \text{and} \quad \delta_k \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.$$

The sequence (δ_k) may be chosen so that it is bounded away from 0, although we necessitate choosing a null sequence (δ_k) to get the second estimate of (3.8) below. In view of (3.1) we have $\varphi(x_k) \leq (1 - a\delta_k)^{-1}(\varphi(x) + \delta_k(b + \varepsilon/2))$ for $k \in N$, and so $\limsup_{k \rightarrow \infty} \varphi(x_k) \leq \varphi(x)$. Hence $g(\varphi(x_k)) + \varepsilon/2 \leq g(\varphi(x)) + \varepsilon$ for k sufficiently large. From this it follows that

$$\varphi(x_k) \leq \varphi(x) + \delta_k(g(\varphi(x)) + \varepsilon) \leq \pi_\varepsilon(\delta_k; \varphi(x))$$

for k sufficiently large. Therefore, for each $\varepsilon > 0$ and each $x \in D$ there exist $\delta \in (0, \varepsilon]$, $x_\delta \in D(A)$ and $y_\delta \in Ax_\delta$ such that

$$(3.8) \quad |\delta^{-1}(x_\delta - x) - y_\delta| \leq \varepsilon, \quad \varphi(x_\delta) \leq \pi_\varepsilon(\delta; \varphi(x)).$$

Fix any $\varepsilon > 0$. Given $x \in D$, we write $\delta(x)$ for the supremum of the numbers $\delta \in (0, \varepsilon]$ for which there exist x_δ and $y_\delta \in Ax_\delta$ satisfying (3.8).

Let $x_0 \in D$. Then by induction one can construct a sequence $(h_n, x_n, y_n)_{n=1}^\infty$ in $(0, \varepsilon] \times D(A) \times X$ in such a way that

$$(3.9) \quad \begin{aligned} \delta(x_{n-1})/2 < h_n, \quad |h_n^{-1}(x_n - x_{n-1}) - y_n| \leq \varepsilon, \\ \varphi(x_n) \leq \pi_\varepsilon(h_n; \pi(x_{n-1})) \end{aligned}$$

for $n \in \mathbb{N}$. It is now sufficient to show that $\sum_{n=1}^\infty h_n = +\infty$. To this end, we assume $\sum_{n=1}^\infty h_n \equiv \tau < \infty$ and derive a contradiction. By (3.9) we have

$$(3.10) \quad \begin{aligned} \varphi(x_n) &\leq \pi_\varepsilon(\sum_{j=\ell+1}^n h_j; \varphi(x_\ell)) \\ &\leq \pi_\varepsilon(\sum_{j=1}^n h_j; \varphi_\varepsilon(\tau; \varphi(x_0))) \leq \pi_\varepsilon(\tau; \varphi(x_0)) \end{aligned}$$

for $\ell = 0, 1, \dots$. This shows that $x_n \in D_\alpha$ for $n \in \mathbb{N}$, where $\alpha = \pi_\varepsilon(\tau; \varphi(x_0))$. For the number α there exists $\omega \in [0, \infty)$ such that $A - \omega I$ is dissipative on D_α . Also, $h_n \rightarrow 0$ as $n \rightarrow \infty$ by the hypothesis on the sequence (h_n) . Hence $h_n \omega < 1/2$ for $n \geq N$ and some N sufficiently large. Therefore it follows that $|x_m - x_n|$ is bounded above by

$$\exp(2\omega((t_n - t_\ell) + (t_m - t_\ell)))[(t_n - t_m)|y_\ell| + \varepsilon(t_n - t_\ell) + \varepsilon(t_m - t_\ell)]$$

for $N \leq \ell \leq m \leq n$, where $t_k = \sum_{j=1}^k h_j$. For the detailed proof of this estimate we refer to Kobayashi [19, p. 647], Pierre [30, Paragraph II], [31, p. 194], Kobayasi, Kobayashi and Oharu [21, Lemma 3.4]. It should be noted that the above estimate plays a central role in the basic convergence results such as Theorem 3.2, which state that if an approximate difference scheme of the type (DS) as stated in the Introduction can be solved then their solutions will converge. Since

$$\limsup_{m, n \rightarrow \infty} |x_m - x_n| \leq 2\varepsilon(\tau - t_\ell) \exp(4\omega(\tau - t_\ell)),$$

we see that the sequence $(x_n)_{n=1}^\infty$ is Cauchy in X . Set $x_\infty = \lim_{n \rightarrow \infty} x_n$. Using the first inequality in (3.10) and the lower semicontinuity of φ , we have $x_\infty \in D$ and

$$\varphi(x_\infty) \leq \liminf_{n \rightarrow \infty} \varphi(x_n) \leq \pi_\varepsilon(\sum_{j=\ell+1}^\infty h_j; \varphi(x_\ell))$$

Hence it follows from (3.8) that there exist $\delta \in (0, \varepsilon/2)$, $x_\delta \in D(A)$ and $y_\delta \in Ax_\delta$ such that

$$(3.11) \quad |\delta^{-1}(x_\delta - x_\infty) - y_\delta| \leq \varepsilon/2 \quad \text{and} \quad \varphi(x_\delta) \leq \pi_\varepsilon(\delta; \varphi(x_\infty)).$$

Choose $k \in N$ so that $k \geq N$ (hence $h_k \omega < 1/2$ by the choice of N),

$$(3.12) \quad \begin{aligned} h_{k+1} &< \delta/2, & \sum_{j=k+1}^{\infty} h_j &< \varepsilon/2, \\ \sum_{j=k+1}^{\infty} h_j \cdot |y_\delta| &\geq \delta\varepsilon/4 & \text{and} & \quad |x_k - x_\infty| \geq \delta\varepsilon/4. \end{aligned}$$

The estimates (3.11) and (3.12) together imply $\delta + \sum_{j=k+1}^{\infty} h_j \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$,

$$\begin{aligned} |x_\delta - x_k - (\delta + \sum_{j=k+1}^{\infty} h_j)y_\delta| &\leq |x_\delta - x_\infty - \delta y_\delta| + |x_k - x_\infty| + \sum_{j=k+1}^{\infty} h_j |y_\delta| \\ &\leq \delta\varepsilon/2 + \delta\varepsilon/4 + \delta\varepsilon/4 = \delta\varepsilon \leq (\delta + \sum_{j=k+1}^{\infty} h_j)\varepsilon, \end{aligned}$$

and

$$\begin{aligned} \varphi(x_\delta) &\leq \pi_\varepsilon(\delta; \varphi(x_\infty)) \leq \pi_\varepsilon(\delta; \pi_\varepsilon(\sum_{j=k+1}^{\infty} h_j; \varphi(x_k))) \\ &\leq \pi_\varepsilon(\delta + \sum_{j=k+1}^{\infty} h_j; \varphi(x_k)), \end{aligned}$$

where the last inequality follows from the representation (3.4) of $\pi_\varepsilon(\cdot; \alpha)$. On the other hand, we see from the definition of $\delta(x_k)$ that

$$\delta + \sum_{j=k+1}^{\infty} h_j \leq \delta(x_k).$$

However (3.9) implies that $\delta(x_k)/2 < h_{k+1} < \delta/2$. Hence $\delta(x_k) < \delta$ and we would have $\delta + \sum_{j=k+1}^{\infty} h_j < \delta$. This is a contradiction. Thus it is concluded that $\sum_{n=1}^{\infty} h_n = +\infty$, and the proof of Lemma 3.3 is complete. \square

We are now ready to prove the Generation Theorem.

PROOF OF THEOREM 3.1. Let $x \in D$ and $\varepsilon \in (0, 1)$. By Lemma 3.3 one finds a sequence $(h_i^\varepsilon, x_i^\varepsilon, y_i^\varepsilon)_{i=1}^{\infty}$ in $(0, \varepsilon] \times D(A) \times X$ such that $x_0^\varepsilon = x$,

$$(3.13) \quad \begin{aligned} \sum_{i=1}^{\infty} h_i^\varepsilon &= +\infty, & y_i^\varepsilon &\in Ax_i^\varepsilon \quad \text{for } i \in N, \\ |x_i^\varepsilon - x_{i-1}^\varepsilon - h_i^\varepsilon y_i^\varepsilon| &\leq \varepsilon h_i^\varepsilon & \text{for } i \in N, \\ \varphi(x_i) &\leq \pi_\varepsilon(h_i^\varepsilon; \varphi(x_{i-1}^\varepsilon)) & \text{for } i \in N. \end{aligned}$$

Put $t_0^\varepsilon = 0$ and $t_n^\varepsilon = \sum_{i=1}^n h_i^\varepsilon$ for $n \in N$. We define a function $u^\varepsilon(\cdot): [0, \infty) \rightarrow X$ by putting $u^\varepsilon(0) = x_0^\varepsilon = x$ and

$$u^\varepsilon(t) = x_i^\varepsilon \quad \text{for } t \in (t_{i-1}^\varepsilon, t_i^\varepsilon] \text{ and } i \in N.$$

For each $\tau \in (0, \infty)$ the restriction of u^ε to the interval $[0, \tau]$ gives an ε -approximate solution of (DI) on $[0, \tau]$. Since $u^\varepsilon(0) = x \in D$ and $\varphi(u^\varepsilon(t)) \leq \pi_\varepsilon(\tau + \varepsilon; \varphi(x))$ for $t \in [0, \tau]$ by (3.13), it follows from Theorem 3.2 that $u^\varepsilon(t)$ converges as $\varepsilon \downarrow 0$ uniformly for $t \in [0, \tau]$, and that the limit function on $[0, \tau]$ is a unique mild solution of (DI) on $[0, \tau]$ confined to $D_{\alpha(\tau)}$,

where $\alpha(\tau) = \pi_\varepsilon(\tau + \varepsilon; \varphi(x))$. Since τ was arbitrary in $(0, \infty)$, a function $u(\cdot; x) : [0, \infty) \rightarrow X$ is defined by

$$u(t; x) = \lim_{\varepsilon \downarrow 0} u^\varepsilon(t; x) \quad \text{for } t \in [0, \infty)$$

and it satisfies

$$\varphi(u(t; x)) \leq \liminf_{\varepsilon \downarrow 0} \pi_\varepsilon(t + \varepsilon; \varphi(x)) = \pi(t; \varphi(x)) \quad \text{for } t \in [0, \infty).$$

Therefore u is a global mild solution of (DI) which is locally φ -bounded on $[0, \infty)$ and is uniquely determined by the initial-value x by Theorem 3.2. For each $t \in [0, \infty)$ we define an operator $S(t) : D \rightarrow D$ by (2.12), namely,

$$S(t)x = u(t; x) \quad \text{for } x \in D.$$

Then the one-parameter family $S = \{S(t) : t \geq 0\}$ forms a semigroup in the class $\mathfrak{S}(D, \varphi)$ satisfying the growth condition (G). This completes the proof of Theorem 3.1. \square

REMARK. We have presented a generation theorem for semigroups in the class $\mathfrak{S}(D, \varphi)$ under the range condition (R) and the exponential growth condition (G). As far as the generation of semigroups belonging the class $\mathfrak{S}(D, \varphi)$ is concerned, it is possible to think of more general conditions than (R) and more general growth condition than (G). For semilinear autonomous evolution equations, generation theorems can be obtained under different types of conditions which are called explicit, semi-implicit and implicit substangential conditions. See Oharu and Takahashi [27, Section 5].

4. Infinitesimal generators of semigroups belonging to the class $\mathfrak{S}(D, \varphi)$

This section is devoted to the study of infinitesimal generators of semigroups in the class $\mathfrak{S}(D, \varphi)$. The principal result of this section is established under the assumption that $(X, |\cdot|)$ is reflexive and has a uniformly Gâteaux differentiable norm, D is convex in X , and that φ is convex on X . Let $S \equiv \{S(t) : t \geq 0\}$ belong to the class $\mathfrak{S}(D, \varphi)$ and define for each $h > 0$ an operator $A_h : D \rightarrow X$ by

$$(4.1) \quad A_h x = h^{-1}(S(h)x - x) \quad \text{for } x \in D.$$

We then introduce two notions of "infinitesimal" generators of S .

DEFINITION 4.1. Given a semigroup $S \equiv \{S(t) : t \geq 0\}$ in the class $\mathfrak{S}(D, \varphi)$ the *right infinitesimal generator* A_+ is defined as follows: $v \in D(A_+)$ and $w \in A_+v$ if and only if $v \in D$ and there exist $t \in [0, \infty)$ and $x \in D$ such that $v = S(t)x$ and w equals the right-hand strong derivative $(d^+/dt)S(t)x$. Likewise, the *left*

infinitesimal generator A_- is defined in the following way: $v \in D(A_-)$ and $w \in A_-v$ if and only if $v \in D$ and there exist $t \in (0, \infty)$ and $x \in D$ such that $v = S(t)x$ and w is equal to the left-hand strong derivative $(d^-/dt)S(t)x$.

The domain $D(A_+)$ is the set of all elements $S(t)x$ such that the strong limit as $h \downarrow 0$ of $h^{-1}(S(t+h)x - S(t)x)$ exists, and hence it is the set of elements $x \in D$ such that the strong limit $\lim_{h \downarrow 0} h^{-1}(S(h)x - x)$ exists. The domain $D(A_-)$ is the set of elements $S(t)x$ such that $\lim_{h \downarrow 0} h^{-1}(S(t)x - S(t-h)x)$ exists. The domains $D(A_+)$ and $D(A_-)$ may be empty. G. Webb showed in [35] that in a space of continuous functions with supremum norm there is a semigroup S of nonlinear contractions which is associated with a semilinear evolution equation and has the property that $D(A_-) = \emptyset$ but $D(A_+)$ is dense in the domain of S .

The right infinitesimal generator A_+ is necessarily single-valued and what so called the infinitesimal generator of S in the usual sense, while the left infinitesimal generator A_- is multi-valued in general. Let $v \in D(A_+)$ and let $v = S(t)x = S(s)y$ for some $s, t \in [0, \infty)$ and some $x, y \in D$. Then there exists $\omega \in [0, \infty)$ such that $|S(t+h)x - S(s+h)y| \leq e^{\omega h}|S(t)x - S(s)y| = 0$ for $h \in (0, 1]$. Hence $h^{-1}(S(t+h)x - S(t)x) = h^{-1}(S(s+h)y - S(s)y)$ for $h \in (0, 1)$ and $(d^+/d\xi)S(\xi)x|_{\xi=t} = (d^+/d\xi)S(\xi)y|_{\xi=s}$, where $(d^+/d\xi)S(\xi)y|_{\xi=s}$ denotes the value of the right-hand derivative of $S(\xi)y$ at the point s and so on. This shows that A_+ is necessarily single-valued. If $v \in D(A_-)$ and $v = S(t)x = S(s)y$ for some $s, t \in [0, \infty)$ and some $x, y \in D$, it is possible that the left-hand derivative $(d^+/d\xi)S(\xi)x|_{\xi=t}$ differs from the left-hand derivative $(d^+/d\xi)S(\xi)y|_{\xi=s}$. Accordingly, the left infinitesimal generator A_- should be understood as a multi-valued operator in general.

The situation may be illustrated by the following example:

EXAMPLE. Let $X = \mathbf{R}$ and $D = [0, \infty)$. The space X is regarded as a 1-dimensional Hilbert space. On the closed convex set D we define a semigroup $S \equiv \{S(t) : t \geq 0\}$ by $S(t)x = (x - t) \vee 0$ for $t \geq 0$ and $x \in D$. For each $v \in D$ let $v = S(s)x = S(t)y$ for some $x, y \in D$ and some $s, t \geq 0$. Assume that $0 < x < y$. Then $0 \leq s \leq t$. If $0 \leq s < x$, then $y - t = x - s > 0$ and so $(d^+/d\xi)S(\xi)x|_{\xi=s} = (d^+/d\xi)S(\xi)y|_{\xi=t} = -1$. If $s \geq x$, then $v = 0$ and $t \geq y$. Therefore in this case $(d^+/d\xi)S(\xi)x|_{\xi=s} = (d^+/d\xi)S(\xi)y|_{\xi=t} = 0$. If in particular $x < s < t = y$, then $(d^-/d\xi)S(\xi)y|_{\xi=t} = -1$, while $v = S(\sigma)x = 0$ for $x < \sigma < y$ and $(d^-/d\xi)S(\xi)x|_{\xi=s} = 0$. From this we see that the right and left infinitesimal generators A_+ and A_- of S are the operators defined, respectively, by

$$A_+x = 0 \quad \text{for } x = 0, \quad A_+x = -1 \quad \text{for } x > 0,$$

$$A_-x = \{-1, 0\} \quad \text{for } x = 0 \quad \text{and} \quad A_-x = -1 \quad \text{for } x > 0$$

In this case, $A_+ \subset A_-$ an A_- is a multi-valued dissipative operator in X satisfying the range condition (R). In fact, for $x = 0$ put $x_\lambda = 0$ for $\lambda > 0$. Then $x_\lambda - \lambda A_- x_\lambda = 0 - \lambda \{-1, 0\} \ni 0$. For $x > 0$, let $0 < \lambda < x$ and $x_\lambda = x - \lambda > 0$. Then $x_\lambda - \lambda A_- x_\lambda = x - \lambda + \lambda = x$.

As indicated by Webb's example, it should be noted that both A_+ and A_- need not be large enough to satisfy the range condition and does not necessarily determine the original semigroup S . We then introduce an extended notion of infinitesimal generator.

DEFINITION 4.2. Let f be a positive nondecreasing function on $(0, \infty)$ such that $f(\alpha) > \alpha$ for $\alpha > 0$. For the function f a family $\{A_{f,\alpha} : \alpha > 0\}$ of possibly multi-valued operators in X is defined as follows: For each $\alpha > 0$, $v \in D(A_{f,\alpha})$ and $(v, w) \in A_{f,\alpha}$ if and only if $v \in D_\alpha$ and there is a function $v(\cdot) : (0, \infty) \rightarrow D_{f(\alpha)}$ satisfying

$$(i) \quad \lim_{h \downarrow 0} v(h) = v \quad \text{and} \quad \lim_{h \downarrow 0} A_h v(h) = w \quad \text{in } X,$$

$$(ii) \quad \limsup_{h \downarrow 0} \varphi(v(h)) \leq f(\alpha).$$

REMARK. Let $\{A_{f,\alpha} : \alpha > 0\}$ be a family of operators in X defined for positive nondecreasing function f on $(0, \infty)$ as in Definition 4.2. Then one can replace the function f by any positive nondecreasing function g such that $g \geq f$ on $(0, \infty)$. If we take such a function g in Definition 4.2, it may be possible to extend the family $\{A_{f,\alpha}\}$ to a larger family $\{A_{g,\alpha}\}$ such that $A_{f,\alpha} \subset A_{g,\alpha}$ for $\alpha > 0$. Accordingly, in what follows, we assume that the function f is fixed to the family $\{A_{f,\alpha}\}$.

PROPOSITION 4.1. For $0 < \alpha < \beta$, we have the inclusion $A_{f,\alpha} \subset A_{f,\beta}$.

PROOF. Let $0 < \alpha < \beta$, and $(v, w) \in A_{f,\alpha}$. Since $f(\alpha) \leq f(\beta)$, $v \in D_\beta$ and one finds a function $v(\cdot) : (0, \infty) \rightarrow D_{f(\beta)}$ in such a way that conditions (i) and (ii) with $f(\alpha)$ replaced by $f(\beta)$ are satisfied. In condition (ii) of Definition 4.2 we may replace the number $f(\alpha)$ by $f(\beta)$. Therefore $v \in D(A_{f,\beta})$ and $w \in A_{f,\beta}v$. This means that $A_{f,\alpha} \subset A_{f,\beta}$. \square

The above fact leads us to the following

DEFINITION 4.3. By the *generalized infinitesimal generator* A (with respect to f) of a semigroup $S \equiv \{S(t) : t \geq 0\}$ in the class $\mathfrak{S}(D, \varphi)$ we mean the operator defined by

$$A = \bigcup_{\alpha > 0} A_{f,\alpha},$$

where $\{A_{f,\alpha} : \alpha > 0\}$ is a family of operators defined for a positive nondecreasing

function f on $(0, \infty)$ such that $f(\alpha) > \alpha$ for $\alpha > 0$.

The relation between the generalized infinitesimal generators and the right and left infinitesimal generators may be described as follows:

PROPOSITION 4.2. *Let $S \equiv \{S(t) : t \geq 0\}$ be a semigroup in the class $\mathfrak{S}(D, \varphi)$ satisfying the growth condition (G). Then we have:*

- (a) $D(A_+) \subset D(A)$ and $A_+v \in Av$ for $v \in D(A_+)$.
- (b) For each $v \in D$ the nonnegative function $\varphi(S(\cdot)v)$ is right continuous on $[0, \infty)$. If in addition $\varphi(S(\cdot)v)$ is left-continuous on all of $(0, \infty)$ for $v \in D$, then $A_-v \subset Av$ for $v \in D(A_-)$. Therefore, in this case, $A_+ \cup A_- \subset A$ in the sense of graphs of operators.
- (c) If, in particular, φ is the indicator function Ind_D of D , then

$$A = \liminf_{h \downarrow 0} A_h$$

in the sense of graphs of operators.

PROOF. To see (a), assume that $D(A_+) \neq \emptyset$. Let $v \in D(A_+)$ and put $v(h) = v$ for $h > 0$. Then $v \in D_\alpha$ for some $\alpha > 0$, $\varphi(v(h)) = \varphi(v) \leq \alpha < f(\alpha)$, and $\lim_{h \downarrow 0} A_h v(h) = A_+v$. This shows that $v \in D(A_{f, \alpha})$ and $A_+v \in A_{f, \alpha}v \subset Av$. Next, to prove (b), let $v \in D$. Then $\varphi(v) \leq \liminf_{h \downarrow 0} \varphi(S(h)v) \leq \lim_{h \downarrow 0} \pi(h, \varphi(v)) = \varphi(v)$ by the lower semicontinuity of φ and (3.2). This means that $\lim_{h \downarrow 0} \varphi(S(h)v) = \varphi(v)$, and hence that $\varphi(S(\cdot)v)$ is right continuous on $[0, \infty)$ by the semigroup property of S . Assume then that $D(A_-) \neq \emptyset$. Let $v \in D(A_-) \cap D_\alpha$, $v = S(t)x$ for some $t \in (0, \infty)$ and $x \in D$, and let $w = (d^-/dt)S(t)x$. Since $\varphi(S(\cdot)x)$ is left-continuous at t by assumption, one finds $h_1 \in (0, t)$ such that $\varphi(S(t-h)x) \leq \varphi(S(t)x) + f(\alpha) - \alpha < f(\alpha)$ for $h \in (0, h_1)$. We then put $v(h) = S(t-h)x$ for $h \in (0, h_1)$ and $v(h) = v(h_1)$ for $h \in [h_1, \infty)$. Then $v(h) \rightarrow v$ and $A_h v(h) \rightarrow (d^-/dt)S(t)x \equiv w$ in X as $h \downarrow 0$ and $\varphi(v(h)) \leq f(\alpha)$ for $h > 0$. Hence $v \in D(A_{f, \alpha})$ and $w \in A_{f, \alpha}v \subset Av$. This shows that $A_+ \cup A_- \subset A$ in the sense of graphs of operators. Finally we demonstrate the last assertion (c). If φ is the indicator function of D , then $\varphi(x) = 0$ for $x \in D$. From this it follows that $A_{f, \alpha} = A$ for $\alpha > 0$. This fact together with Definitions 4.2 and 4.3 implies the assertion (c). This completes the proof. \square

We then investigate some of basic properties of the generated infinitesimal generators of semigroups in the class $\mathfrak{S}(D, \varphi)$.

PROPOSITION 4.3. *Let $S = \{S(t)\}$ belong to the class $\mathfrak{S}(D, \varphi)$. Let A be the infinitesimal generator A of S with respect to t . Then A is an operator in the class $\mathfrak{G}(D, \varphi)$.*

PROOF. By definition $D(A) \subset D$. Let $\alpha \in [0, \infty)$, $\beta = f(\alpha)$ and let $\tau > 0$. Then there exists $\omega \equiv \omega(\beta, \tau) \in [0, \infty)$ such that

$$(4.2) \quad |S(t)x - S(t)u| \leq e^{\omega t}|x - u|$$

for $x, u \in D_\beta$ and $t \in [0, \tau]$. Take any pair x, u in $D(A) \cap D_\alpha$ and a pair y, v satisfying $y \in Ax$ and $v \in Au$, respectively. Then one finds two D_β -valued functions $x(\cdot)$ and $u(\cdot)$ on $(0, \infty)$ such that

$$x(h) \rightarrow x, \quad A_h x(h) \rightarrow y, \quad u(h) \rightarrow u \quad \text{and} \quad A_h u(h) \rightarrow v \text{ in } X \text{ as } h \downarrow 0,$$

$$\limsup_{h \downarrow 0} \varphi(x(h)) \leq f(\alpha) \quad \text{and} \quad \limsup_{h \downarrow 0} \varphi(u(h)) \leq f(\alpha).$$

Hence a number $h(\alpha)$ can be chosen in $(0, \tau)$ so that $x(h), u(h) \in D_{f(\alpha)}$ for $h \in (0, h(\alpha))$. Therefore the application of (4.2) implies that for $h \in (0, h(\alpha))$ and $\lambda > 0$

$$\begin{aligned} & |(x(h) - u(h)) - \lambda(A_h x(h) - A_h u(h))| \\ &= |(1 + \lambda/h)(x(h) - u(h)) - (\lambda/h)(S(h)x(h) - S(h)u(h))| \\ &\geq (1 + \lambda/h)|x(h) - u(h)| - (\lambda/h)|S(h)x(h) - S(h)u(h)| \\ &\geq (1 + (\lambda/h)(1 - e^{\omega h}))|x(h) - u(h)|. \end{aligned}$$

Letting $h \downarrow 0$, we have $|(x - u) - \lambda(y - v)| \geq (1 - \omega\lambda)|x - u|$, or

$$\lambda^{-1}(|x - u| - |(x - u) - \lambda(y - v)|) \leq \omega|x - u|.$$

Passing to the limit as $\lambda \downarrow 0$, we obtain $[x - u, y - v]_- \leq \omega|x - u|$. This shows that A satisfies (LQD) and $A \in \mathfrak{G}(D, \varphi)$, thereby completing the proof. \square

In the previous proposition we considered semigroups in the class $\mathfrak{S}(D, \varphi)$. We here show that the growth condition (G) for a semigroup in the class $\mathfrak{S}(D, \varphi)$ restricts the constants ω in the Lipschitz condition (L) (stated in Definition 1.2) in terms of the functions $\pi(\cdot; \alpha)$, $\alpha > 0$.

PROPOSITION 4.4. Let $S = \{S(t)\}$ be a semigroup in the class $\mathfrak{S}(D, \varphi)$ and suppose S satisfies the growth condition (G). Then there is a nondecreasing right-continuous function $\omega : [0, \infty) \rightarrow [0, \infty)$ such that for $t \geq 0$ and $x, u \in D$

$$|S(t)x - S(t)u| \leq |x - u| \exp \left(\int_0^t \omega(\pi(s, \varphi(x) \vee \varphi(u)) ds \right).$$

PROOF. For $t \in [0, \infty)$ and $\alpha \in [0, \infty)$ we write $L(t; \alpha)$ for the number

$$\inf \{ \ell \in [0, \infty) : |S(t)x - S(t)u| \leq |x - u|e^\ell \text{ for } x, u \in D_\alpha \}.$$

Let $\tau > 0$. Then by condition (L) one finds a positive number $\omega(\alpha, \tau)$ such that $L(t; \alpha) \leq t\omega(\alpha, \tau)$ for $t \in [0, \tau]$. Hence for each $\alpha \in [0, \infty)$ we get $\lim_{t \downarrow 0} L(t; \alpha) = 0$ and

$$(4.4) \quad \omega(\alpha) \equiv \liminf_{t \downarrow 0} L(t; \alpha)/t < \infty,$$

so that (4.4) defines a nondecreasing function $\omega(\cdot): [0, \infty) \rightarrow [0, \infty)$. As seen from the argument below, we may assume that $\omega(\cdot)$ is right-continuous on $[0, \infty)$. Let $s, t \in [0, \infty)$ and $x, u \in D_\alpha$. Since $\varphi(S(s)x) \vee \varphi(S(s)u) \leq \pi(s; \alpha)$, we have

$$\begin{aligned} |S(t+s)x - S(t+s)u| &= |S(t)S(s)x - S(t)S(s)u| \\ &\leq |S(s)x - S(s)u| \exp(L(t; \pi(s; \alpha))) \\ &\leq |x - u| \exp(L(s; \alpha) + L(t; \pi(s; \alpha))) \end{aligned}$$

for $x, u \in D_\alpha$. From this and the definition of $L(t+s; \alpha)$ we obtain

$$(4.5) \quad L(t+s; \alpha) \leq L(s; \alpha) + L(t; \pi(s; \alpha))$$

Fix any $t \in (0, \infty)$ and any $\alpha > 0$. For $h \in (0, t)$ we can write $t = nh + r$ for some $n \in N$ and some $r \in [0, h)$. Hence (4.5) and (2.14) together imply

$$\begin{aligned} L(t; \alpha) &= L(nh + r; \alpha) \\ &\leq L(r; \alpha) + \sum_{k=1}^n L(h; \pi((k-1)h + r; \alpha)) \leq L(r; \alpha) + nL(h; \pi(t; \alpha)). \end{aligned}$$

From this it follows that

$$L(t; \alpha)/t \leq L(r; \alpha)/t + nL(h; \pi(t; \alpha))/t = L(r; \alpha)/t + ((t-r)/t)L(h; \pi(t; \alpha))/h.$$

Noting that $r \downarrow 0$ and $L(r; \alpha) \rightarrow 0$ as $h \downarrow 0$, we see that

$$(4.6) \quad L(t; \alpha)/t \leq \omega(\pi(t; \alpha)) \quad \text{or} \quad L(t; \alpha) \leq t\omega(\pi(t; \alpha)).$$

Next, for $n \in N$ we write $h = t/n$ for brevity in notation. Then (4.5), (3.2) and (4.6) together yield

$$\begin{aligned} L(t; \alpha) &= L(nh; \alpha) \leq \sum_{k=0}^{n-1} L(h; \pi(kh; \alpha)) \\ &\leq h \sum_{k=0}^{n-1} \omega(\pi(h; \pi(kh; \alpha))) = h \sum_{k=0}^{n-1} \omega(\pi(kh + h; \alpha)) \\ &\leq \sum_{k=1}^n \int_{kh}^{(k+1)h} \omega(\pi(s; \alpha)) ds = \int_h^{t+h} \omega(\pi(s; \alpha)) ds. \end{aligned}$$

Let $x, u \in D$ and $\alpha = \varphi(x) \vee \varphi(u)$. Then we get

$$|S(t)x - S(t)u| \leq |x - u| \exp\left(\int_h^{t+h} \omega(\pi(s; \alpha)) ds\right)$$

for $h = t/n$. Letting $n \rightarrow \infty$ gives the desired estimate (4.3). Finally, the function ω can be redefined as a right-continuous function if necessary. This concludes the proof of Proposition 4.4. \square

Let $S = \{S(t)\}$ be a semigroup in the class $\mathfrak{S}(D, \varphi)$ satisfying the growth condition (G) and suppose that the generalized infinitesimal generator A of S in the sense of Definition 4.3 has a nonempty domain. Then it is expected that S is a family of solution operators (perhaps in a generalized sense) of the differential inclusion

$$(DE) \quad (d/dt)u(t) \in Au(t), \quad t > 0.$$

Indeed, we have the following result:

THEOREM 4.5. *Let $S = \{S(t)\}$ be a semigroup in the class $\mathfrak{S}(D, \varphi)$ satisfying the growth condition (G) and possessing the generalized infinitesimal generator A . Suppose that $D(A) \neq \emptyset$. Then for each $x \in D$ the function $u(\cdot) \equiv S(\cdot)x$ is a global integral solution of (DI).*

Prior to proving the theorem we observe that the function $u(\cdot) \equiv S(\cdot)x$ becomes a strong solution of (DI) under additional assumptions. Let $x \in D_\alpha$, $\tau > 0$, $\beta = \pi(\tau; \alpha)$ and $\omega = \omega(\beta)$. Then $u(t) \in D_\beta$ for $t \in [0, \tau]$. Suppose then that $A - \omega I$ is maximal dissipative on D_β and $u(\cdot)$ is Lipschitz continuous on $[0, \tau]$. If the function u is weakly right-differentiable at $t \in (0, \tau)$, then we infer from (2.1) and (1.1) that

$$\begin{aligned} [u(t) - x, h^{-1}(u(t+h) - u(t))]_+ &\leq h^{-1}(|u(t+h) - x| - |u(t) - x|) \\ &\leq h^{-1} \int_t^{t+h} ([u(\xi) - x, y]_+ + \omega|u(\xi) - x|) d\xi \end{aligned}$$

for $x \in D(A) \cap D_\alpha$, $y \in Ax$ and $h \in (0, \tau - t)$. Letting $h \downarrow 0$ and applying Proposition 1.1, we have

$$[u(t) - x, D_w^+ u(t) - y]_- \leq \omega|u(t) - x|,$$

where $D_w^+ u(t)$ denotes the weak right-derivative of $u(\cdot)$ at t . The maximal dissipativity of $A - \omega I$ on D_β then implies that $u(t) \in D(A)$ and $D_w^+ u(t) \in Au(t)$. If in particular $u(\cdot)$ is weakly differentiable a.e. on $[0, \tau]$, then the weak derivative $D_w u(\cdot)$ is Bochner integrable over $[0, \tau]$ and hence $u(\cdot)$ becomes a strong solution of (DI).

To prove Theorem 4.5, we need the following lemma.

LEMMA 4.6. *Let $\alpha \in [0, \infty)$, $\tau \in [0, \infty)$, $h \in (0, \tau)$ and let $n \in \mathbb{N}$ satisfy $nh \in [0, \tau]$. Then for any pair $x, u \in D_\alpha$ we have*

$$|S(nh)x - u| - |x - u| \\ \leq h \sum_{k=1}^n ([S(kh)x - u, A_h u]_+ + h^{-1}(\exp(h\omega(\pi(\tau; \alpha)) - 1)|S((k-1)h)x - u|)).$$

PROOF. Let $x, u \in D_\alpha$ and $k \in \{1, \dots, n\}$. Then

$$\varphi(S((k-1)h)x) \leq \pi((k-1)h; \varphi(x)) \leq \pi(\tau - h; \varphi(x)) \leq \pi(\tau - h; \alpha),$$

and so Proposition 4.4 yields

$$|S(kh)x - S(h)u| = |S(h)S((k-1)h)x - S(h)u| \\ \leq |S((k-1)h)x - u| \exp(h\omega(\pi(h; \pi(\tau - h; \alpha))) \\ \leq |S((k-1)h)x - u| \exp(h\omega(\pi(\tau; \alpha))).$$

From this it follows that

$$\begin{aligned} [S(kh)x - u, A_h u]_+ h &\geq [S(kh)x - u, S(h)u - u]_- \\ &\geq |S(kh)x - u| - |S(kh)x - u - (S(h)u - u)| \\ &= |S(kh)x - u| - |S(kh)x - S(h)u| \\ (4.7) \quad &\geq |S(kh)x - u| - |S((k-1)h)x - u| \exp(h\omega(\pi; \alpha)) \\ &= |S(kh)x - u| - |S((k-1)h)x - u| \\ &\quad + (1 - \exp(h\omega(\pi; \alpha)))|S((k-1)h)x - u|. \end{aligned}$$

Adding up both sides of the inequalities (4.7) from $k = 1$ to $k = n$, we obtain the desired estimate. \square

PROOF OF THEOREM 4.5. Let $x \in D$, $\tau > 0$ and $\alpha \in [0, \infty)$. We first observe that $\varphi(S(s)x) \leq \pi(s, \varphi(x)) \leq \pi(\tau; \varphi(x))$ for $s \in [0, \tau]$. Choose β so that $\beta \geq f(\alpha) \vee \pi(\tau; \varphi(x))$. Then $S(s)x \in D_\beta$ for $s \in [0, \tau]$. We now take any pair u, v satisfying $u \in D(A) \cap D_\alpha$ and $v \in Au$. Then, according to Definition 4.3, there exists a D -valued function $u(\cdot)$ on $(0, \infty)$ such that $\varphi(u(h)) \leq \beta$ for $h \in (0, \infty)$, $u(h) \rightarrow u$ and $A_h u(h) \rightarrow v$ as $h \downarrow 0$. Let $h \in (0, \tau]$, $s \in (0, \tau]$, $n \in \mathbb{N}$ and let $nh \in [0, \tau]$. Then by Lemma 4.6 we have

$$(4.8) \quad |S(nh)S(s)x - u(h)| - |S(s)x - u(h)| \\ \leq h \sum_{k=1}^n ([S(kh)S(s)x - u(h), A_h u(h)]_+ \\ + h^{-1}(\exp(h\omega(\pi(\tau; \beta))) - 1)|S((n-1)h)S(s)x - u(h)|).$$

We next take any pair s, t with $0 \leq s < t \leq \tau$ and choose a positive integer valued function $n(h)$ on $(0, \infty)$ so that $n(h)h \rightarrow t - s$ as $h \downarrow 0$. Substituting $n = n(h)$ into (4.8) and passing to the limit as $h \downarrow 0$, we obtain the integral inequality

$$\begin{aligned} |S(t)x - u| - |S(s)x - u| &= |S(t-s)S(s)x - u| - |S(s)x - u| \\ &\leq \int_0^{t-s} ([S(\sigma)S(s)x - u, v]_+ \\ &\quad + \omega(\pi(\tau; \beta))|S(\sigma)S(s)x - u|) d\sigma \\ &= \int_s^t ([S(\sigma)x - u, v]_+ + \omega(\pi(\tau; \beta))|S(\sigma)x - u|) d\sigma. \end{aligned}$$

This shows that $S(\cdot)x$ is a global integral solution of (DI) and concludes the proof of Theorem 4.5. \square

If in Theorem 4.5 the generalized infinitesimal generator A has a sufficiently large domain, then we obtain a result converse to Theorem 3.1.

COROLLARY 4.7. *Let $S = \{S(t)\}$ be a semigroup in the class $\mathfrak{S}(D, \varphi)$ satisfying the growth condition (G) and A the generalized infinitesimal generator of S . If $\overline{D(A)} \supset D$ and A satisfies the range condition (R), then for each $x \in D$ the function $u(\cdot) \equiv S(\cdot)x$ becomes a global mild solution of (DI) satisfying (G).*

PROOF. Under the assumption, Theorem 3.1 can be applied to conclude that there is a semigroup $S_1 \equiv \{S_1(t)\}$ of the class $\mathfrak{S}(D, \varphi)$ such that for $x \in D$ the function $S(\cdot)x$ gives a global mild solution of (DI) confined to D . On the other hand, Theorem 4.5 states that for $x \in D$ the function $u(\cdot) \equiv S(\cdot)x$ is a global integral solution of (DI). Hence, by Theorem 2.2, $S(t)x = S_1(t)x$ for $x \in D$ and $t \in [0, \infty)$. This shows that for each $x \in D$ the function $S(\cdot)x$ is a global mild solution of (DE) confined to D . This completes the proof. \square

The very strong conditions imposed on A in Corollary 4.7 are automatically satisfied if we assume that X is reflexive, the norm $|\cdot|$ is uniformly Gâteaux differentiable, and that φ is convex on X . This is the main result of this section and the assertion is stated as below.

THEOREM 4.8. *Let $(X, |\cdot|)$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm and suppose that φ is convex on X . Let $S = \{S(t)\}$ be a semigroup on D satisfying the growth condition (G). Let A be the generalized infinitesimal generator of S . Then $\overline{D(A)} \supset D$ and A satisfies the range condition of the following form:*

(R₀) To each $x \in D$ there corresponds a positive number $\lambda(x)$ such that for each $\lambda \in (0, \lambda(x)]$ there is $x_\lambda \in D(A)$ satisfying

$$\lambda^{-1}(x_\lambda - x) \in Ax_\lambda \quad \text{and} \quad \lambda^{-1}(\varphi(x_\lambda) - \varphi(x)) \leq g(\varphi(x_\lambda)),$$

where g is the affine function defined by (2.13).

We notice that for an operator A in the class $\mathfrak{G}(D, \varphi)$ the range condition (R₀) is much stronger than (R). In this paper condition (R₀) is called the *strict range condition*. The proof is given after discussing the ranges of the approximate operators A_h which are defined by the formula (4.1) and will play an important role in the next section. Combining Theorem 4.8 with Corollary 4.7, we obtain the following result.

THEOREM 4.9. *Let $(X, |\cdot|)$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm and suppose that φ is convex on X . Let $S = \{S(t)\}$ be a semigroup on D satisfying the growth condition (G). Then the generalized infinitesimal generator A of S in the sense of Definition 4.3 has the domain $D(A)$ with $\overline{D(A)} \supset D$ and satisfies the strict range condition (R₀). Furthermore, for each $x \in D$ the function $u(\cdot) \equiv S(\cdot)x$ gives a global mild solution of (DI) satisfying (G).*

The above result together with Theorem 3.1 implies a nonlinear version of the Hille-Yosida theorem. In order to discuss the differentiability of a semigroup $S = \{S(t)\}$ in the class $\mathfrak{S}(D, \varphi)$, so called Lipschitz domain of S plays an important role.

DEFINITION 4.4. Let $S = \{S(t)\}$ be a semigroup belonging to the class $\mathfrak{S}(D, \varphi)$. The *Lipschitz domain* of S is the set of all elements x in D such that $S(\cdot)x$ is Lipschitz continuous on bounded subintervals of $[0, \infty)$ with respect to t , and we write $Lip(S)$ for the Lipschitz domain.

PROPOSITION 4.10. *Let $S = \{S(t)\}$ be a semigroup in the class $\mathfrak{S}(D, \varphi)$ satisfying the growth condition (G) and $Lip(S)$ the associated Lipschitz domain. Let A_+ , A_- and A be the right-, left, and generalized infinitesimal generator of S in the sense of Definition 4.3, respectively. Put $A_0 = A_+ \cap A_-$. Then we have:*

(a) $Lip(S) = \{x \in D : \liminf_{h \downarrow 0} h^{-1} |S(h)x - x| < \infty\}$ and the Lipschitz domain $Lip(S)$ is invariant under $S(t)$ for $t \geq 0$.

(b) $D(A) \subset \overline{Lip(S)}$. If in particular X has the Radon-Nikodým property, then $Lip(S) \subset D(A_0)$ and for each $x \in Lip(S)$, $(d/dt)S(t)x = A_0 S(t)x$ for a.e. $t \geq 0$.

PROOF. To prove the first half of Assertion (a), let $x \in Lip(S)$. Then there is $A_1 \in [0, \infty)$ such that $|S(h)x - x| \leq A_1 h$ for $h \in [0, 1]$. Hence

$\limsup_{h \downarrow 0} h^{-1} |S(h)x - x| \leq A_1 < \infty$. Conversely, let $x \in D$, $\tau > 0$, and assume that $\liminf_{h \downarrow 0} h^{-1} |S(h)x - x| = A_2 < \infty$. Let $\beta = \pi(\tau; \varphi(x))$ and $\gamma = \omega(\beta)$, where $\omega(\cdot)$ denotes a nondecreasing right-continuous function constructed in Proposition 4.4. Then $S(t)x \in D_\beta$ and $|S(t)x - S(s)x| \leq e^{\gamma(t-s)} |S(t-s)x - x|$ for $0 \leq s < t \leq \tau$. Also, there is a null sequence $(h_i)_{i=1}^\infty$ in $(0, \tau)$ such that

$|S(h_i)x - x| \leq (A_2 + 1)h_i$ for $i = 1, 2, \dots$. Combining these estimates we see that $|S(t)x - S(s)x| \leq e^{\gamma t}(A_2 + 1)|t - s|$ for $s, t \in [0, \tau]$. This shows that $x \in \text{Lip}(S)$. The latter half of Assertion (a) is clear from the definition of $\text{Lip}(S)$. To show the first half of Assertion (b), let $\alpha > 0$, $\tau > 0$ and $(v, w) \in A_{f, \alpha}$. Then there is a function $v(\cdot) : (0, \infty) \rightarrow D_{f(\alpha)}$ such that $\lim_{h \downarrow 0} v(h) = v$, $\lim_{h \downarrow 0} A_h v(h) = w$ and $\limsup_{h \downarrow 0} \varphi(v(h)) \leq f(\alpha)$. Let $\gamma = \omega(\pi(\tau; f(\alpha)))$. Then for $t \in [0, \tau]$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} |S(t)v - v| &\leq |S(t)v(t/n) - v(t/n)| + (e^{\gamma t} + 1)|v(t/n) - v| \\ &\leq \left(\sum_{k=1}^n |S(kt/n)v(t/n) - S((k-1)t/n)v(t/n)|\right) + (e^{\gamma t} + 1)|v(t/n) - v| \\ &\leq te^{\gamma t}|A_{t/n}v(t/n)| + (e^{\gamma t} + 1)|v(t/n) - v|. \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we get $|S(t)v - v| \leq te^{\gamma t}|w|$. This shows that $v \in \text{Lip}(S)$. The proof of the latter half of Assertion (b) is rather elementary. Assume that X has the Radon-Nikodým property, and that $v \in \text{Lip}(S)$. Then $S(\cdot)v$ is Lipschitz continuous on bounded subintervals of $[0, \infty)$, and so it is norm-differentiable a.e. on $[0, \infty)$. Since $(d/dt)S(t)v = A_+ S(t)v = A_- S(t)v$ for a.e. $t \in [0, \infty)$, it follows that $(d/dt)S(t)v = A_0 S(t)v$ and $v \in \overline{D(A_0)}$.

From Theorem 4.9 and Proposition 4.10 we obtain the following result on the differentiability of semigroups in the class $\mathfrak{S}(D, \varphi)$ provided that $(X, |\cdot|)$ is a smooth reflexive space.

COROLLARY 4.11. *Let $(X, |\cdot|)$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm and suppose that φ is convex on X . Let $S = \{S(t)\}$ be a semigroup in the class $\mathfrak{S}(D, \varphi)$ satisfying the growth condition (G). Then the generalized infinitesimal generator A of S is densely defined in D and for each $x \in \text{Lip}(S)$ the X -valued function $S(\cdot)x$ is Lipschitz continuous on bounded subintervals of $[0, \infty)$ and satisfies*

$$(d/dt)S(t)x \in AS(t)x \quad \text{for a.e. } t \geq 0.$$

As shown in Theorem 3.1, an operator A in the class $\mathfrak{G}(D, \varphi)$ satisfying $\overline{D(A)} \supset D$ and the range condition (R) generates a semigroup S of class $\mathfrak{S}(D, \varphi)$ satisfying (G). It is a delicate but deep problem to investigate the relationship

between the operator A and the generalized infinitesimal generator of the semigroups S so obtained. For earlier results in this direction we refer to for instance [5; Section 4] and [16; Section 5]. However it is possible to treat the generalized infinitesimal generators from a different point of view, and we shall discuss this problem in a subsequent paper entitled "Some remarks on semigroups of locally Lipschitzian operators".

5. Range condition for the generalized infinitesimal generators

Here we give the proof of Theorem 4.8 and show that a semigroup in the class $\mathfrak{S}(D, \varphi)$ has a generalized infinitesimal generator satisfying the strict range condition provided that the growth condition (G) holds, φ (and hence D) is convex and that $(X, |\cdot|)$ is a reflexive "smooth" Banach space.

In what follows, we assume without further mention that φ is convex on X , and that $(X, |\cdot|)$ is a reflexive Banach space with uniformly Gâteaux differentiable norm. The main objective here is to prove the following theorem.

THEOREM 5.1. *Let $S = \{S(t)\}$ be a semigroup in the class $\mathfrak{S}(D, \varphi)$ satisfying the growth condition (G). For each $h > 0$ let $A_h: D \rightarrow X$ be the operator defined by (4.1) and let $g_h: [0, \infty) \rightarrow \mathbf{R}$ be defined by*

$$(5.1) \quad g_h(\alpha) = h^{-1}(\pi(h; \alpha) - \alpha) \quad \text{for } \alpha \in [0, \infty).$$

Then for each $x \in D$ there exist $\lambda_0 \equiv \lambda_0(x) \in (0, \infty)$ and $h_0 = h_0(x) \in (0, \infty)$ with the two properties below:

(a) *For each $\lambda \in (0, \lambda_0)$ and each $h \in (0, h_0)$ there is $x_{\lambda, h} \in D$ satisfying*

$$\lambda^{-1}(x_{\lambda, h} - x) = A_h x_{\lambda, h} \quad \text{and} \quad \lambda^{-1}(\varphi(x_{\lambda, h}) - \varphi(x)) \leq g_h(\varphi(x_{\lambda, h})).$$

(b) *The limit $\lim_{h \downarrow 0} x_{\lambda, h} = x_\lambda$ exists and $\lim_{\lambda \downarrow 0} x_\lambda = x$.*

Before proving this theorem we complete the proof of Theorem 4.8 by assuming Theorem 5.1.

PROOF OF THEOREM 4.8. Assume that Theorem 5.1 is already established. Let $x \in D$. Then one finds numbers λ_0 and h_0 in $(0, \infty)$ with the properties (a) and (b) stated in Theorem 5.1. Let f be a positive nondecreasing function satisfying $f(\alpha) > \alpha$ on $(0, \infty)$ and assume that A is the generalized infinitesimal generator of S in the sense of definition 4.3. Fix any $\beta \geq (1 - a\lambda_0)^{-1}(\varphi(x) + b\lambda_0)$, $\lambda \in (0, \lambda_0)$, $h \in (0, h_0)$ and let $x_{\lambda, h}$ be the element in D as mentioned in Assertion (a). Then $\varphi(x_{\lambda, h}) \leq \beta_{\lambda, h}$, where

$$(5.2) \quad \beta_{\lambda, h} = (1 - \lambda h^{-1}(e^{ah} - 1))^{-1} \left(\varphi(x) + \lambda b h^{-1} \int_0^h e^{a(h-s)} ds \right).$$

This fact and Assertion (b) together imply the estimates

$$\varphi(x_\lambda) \leq \liminf_{h \downarrow 0} \varphi(x_{\lambda,h}) \leq \limsup_{h \downarrow 0} \varphi(x_{\lambda,h}) \leq (1 - a\lambda)^{-1}(\varphi(x) + b\lambda)$$

and

$$\varphi(x) \leq \liminf_{\lambda \downarrow 0} \varphi(x_\lambda) \leq \limsup_{\lambda \downarrow 0} \varphi(x_\lambda) \leq \varphi(x).$$

Therefore $\lim_{\lambda \downarrow 0} \varphi(x_\lambda) = \varphi(x)$ and

$$\begin{aligned} & \limsup_{\lambda \downarrow 0} (\limsup_{h \downarrow 0} \varphi(x_{\lambda,h}) - \varphi(x_\lambda)) \\ & \leq \limsup_{\lambda \downarrow 0} (\limsup_{h \downarrow 0} \varphi(x_{\lambda,h})) - \varphi(x) \leq \varphi(x) - \varphi(x) = 0. \end{aligned}$$

This shows that there is a sufficiently small positive number $\lambda(x)$ such that

$$(5.3) \quad \limsup_{h \downarrow 0} \varphi(x_{\lambda,h}) - \varphi(x_\lambda) \leq f(\beta) - \beta \quad \text{for } \lambda \in (0, \lambda(x)).$$

Also, we have $\lim_{h \downarrow 0} x_{\lambda,h} = x_\lambda$ and $\lim_{h \downarrow 0} A_h x_{\lambda,h} = \lim_{h \downarrow 0} \lambda^{-1}(x_{\lambda,h} - x) = \lambda^{-1}(x_\lambda - x)$. Combining these formulae and (5.3), we infer from Definition 4.2 that $x_\lambda \in D(A_{f,\beta})$ and $\lambda^{-1}(x_\lambda - x) \in Ax_\lambda$. Since $\varphi(x_\lambda) \leq (1 - a\lambda)^{-1}(\varphi(x) + b\lambda)$, it follows that $\lambda^{-1}(\varphi(x_\lambda) - \varphi(x)) \leq g(\varphi(x_\lambda))$. This shows that A satisfies the strict range condition (R_0) . Recalling that $x_\lambda \in D(A)$ and $\lim_{\lambda \downarrow 0} x_\lambda = x$, we see that $x \in \overline{D(A)}$. Since x was arbitrary in D , it is concluded that $\overline{D(A)} \supset D$. This completes the proof of Theorem 4.8. \square

REMARK. In the above argument, Assertions (a) and (b) in Theorem 5.1 are essential. That is, Theorem 4.8 is valid without any restrictions on the Banach space $(X, |\cdot|)$ if Theorem 5.1 holds for general Banach spaces. In fact, the first assertion (a) is obtained for any Banach space, although it is not possible to obtain the second assertion (b) via the method employed in this section. It is known that if the semigroup S is associated with a class of semilinear evolution equations of the form.

$$(d/dt)u(t) = Au(t) + Bu(t), \quad t > 0,$$

then Theorem 5.1 is valid for arbitrary Banach spaces. See the recent works of Oharu and Takahashi [26, 27] for the semilinear Hille-Yosida theory in general Banach spaces.

In what follows, we give the proof of Theorem 5.1. Without further mention we put all of the conditions imposed in Theorem 5.1. Fix any $x \in D$, any $\tau \in (0, \infty)$ and take any α with $\varphi(x) < \alpha$. Put $\omega^* = \omega(\pi(\tau; \alpha))$ and take $\lambda^* \in (0, \infty)$ so small that

$$(5.4) \quad \lambda^* \omega^* < 1, \quad \lambda^* \alpha < 1, \quad \lambda^* g(\alpha) < \alpha - \varphi(x).$$

Also, choose $h^* \in (0, \infty)$ in such a way that

$$(5.5) \quad \begin{aligned} \lambda^* h^{-1}(e^{h\omega^*} - 1) < 1, \quad \lambda^* h^{-1}(e^{ha} - 1) < 1, \\ \lambda^* h^{-1}(\pi(h; \alpha) - \alpha) < \alpha - \varphi(x) \quad \text{for } h \in (0, h^*). \end{aligned}$$

Therefore we have

$$(5.6) \quad \begin{aligned} \lambda h^{-1}(e^{h\omega^*} - 1) < 1, \quad \lambda h^{-1}(e^{ha} - 1) < 1, \\ \lambda h^{-1}(\pi(h; \alpha) - \alpha) \geq \alpha - \varphi(x), \quad \text{for } \lambda \in (0, \lambda^*) \text{ and } h \in (0, h^*). \end{aligned}$$

We now take any $\lambda \in (0, \lambda^*)$ and any $h \in (0, h^*)$ and define an operator $K : D \rightarrow X$ by

$$Kz = (\lambda + h)^{-1}hx + (\lambda + h)^{-1}\lambda S(h)z \quad \text{for } z \in D.$$

Since φ is convex on X , we have

$$\begin{aligned} \varphi(Kz) &\leq (\lambda + h)^{-1}h\varphi(x) + (\lambda + h)^{-1}\lambda\varphi(S(h)z) \\ &\leq (\lambda + h)^{-1}h\varphi(x) + (\lambda + h)^{-1}\lambda\pi(h; \varphi(z)) \\ &\leq (\lambda + h)^{-1}h\varphi(x) + (\lambda + h)^{-1}\lambda\pi(h; \alpha) \leq \alpha \end{aligned}$$

for $z \in D_\alpha$, where we have used (5.6) in the last inequality. This means that K maps D_α into itself. To show that K is a strict contraction on D_α , we observe that $(\lambda + h)^{-1}\lambda e^{h\omega^*} < 1$ by (5.6). For any pair $y, z \in D_\alpha$, we have

$$\begin{aligned} |Ky - Kz| &= (\lambda + h)^{-1}\lambda |S(h)y - S(h)z| \\ &\leq (\lambda + h)^{-1}\lambda e^{h\omega(\pi(h; \alpha))} |y - z| \leq (\lambda + h)^{-1}\lambda e^{h\omega^*} |y - z|, \end{aligned}$$

so that K is a strict contraction on D_α . Since D_α is closed in X , the contracting mapping principle implies that there is $x_{\lambda, h} \in D_\alpha$ satisfying $x_{\lambda, h} = Kx_{\lambda, h}$ or $\lambda^{-1}(x_{\lambda, h} - x) = A_h x_{\lambda, h}$. On the other hand, the number $\beta_{\lambda, h}$ defined by (5.2) satisfies the relations

$$(5.7) \quad \begin{aligned} \lambda^{-1}(\beta_{\lambda, h} - \varphi(x)) &= g_h(\beta_{\lambda, h}), \\ \beta_{\lambda, h} &= (\lambda + h)^{-1}h\varphi(x) + (\lambda + h)^{-1}\lambda\pi(h; \beta_{\lambda, h}). \end{aligned}$$

Using (2.14), (5.6) and (5.7), we infer that $\beta_{\lambda, h} \in [0, \alpha]$ and it is a unique fixed point in the interval $[0, \alpha]$ of the mapping $k : [0, \infty) \rightarrow [0, \infty)$ defined by

$$k(\beta) = (\lambda + h)^{-1}h\varphi(x) + (\lambda + h)^{-1}\lambda\pi(h; \beta) \quad \text{for } \beta \in [0, \infty).$$

Now $\varphi(x_{\lambda, h})$ satisfies the inequality

$$\varphi(x_{\lambda, h}) \leq (\lambda + h)^{-1}h\varphi(x) + (\lambda + h)^{-1}\lambda\pi(h; \varphi(x_{\lambda, h})),$$

since $x_{\lambda,h}$ is a fixed point of K and φ is convex on X . Hence $\varphi(x_{\lambda,h}) \leq \beta_{\lambda,h}$ and we have $\lambda^{-1}(\varphi(x_{\lambda,h}) - \varphi(x)) \leq g_h(\varphi(x_{\lambda,h}))$. Thus it is concluded that Assertion (a) of Theorem 5.1 is valid for any number $\lambda_0 \in (0, \lambda^*)$ and any $h_0 \in (0, h^*)$.

It now remains to show that Assertion (b) is obtained for the family of elements $\{x_{\lambda,h} : \lambda \in (0, \lambda_0(x)], h \in (0, h_0(x))\}$ for some numbers $\lambda_0(x) \in (0, \lambda^*)$ and $h_0(x) \in (0, h^*)$. To this end we need the following lemma.

LEMMA 5.2. $\limsup_{\lambda \downarrow 0} (\limsup_{h \downarrow 0} |x_{\lambda,h} - x|) = 0$.

PROOF. We have already seen that $x, x_{\lambda,h} \in D_\alpha$ for $\lambda \in (0, \lambda^*)$ and $h \in (0, h^*)$. Let $h \in (0, h^*)$, $n \in N$ and $nh \in [0, \infty)$. Furthermore let τ and ω^* be the numbers appearing in (5.4). Then Lemma 4.6 yields

$$(5.8) \quad \begin{aligned} & |S(nh)x - x_{\lambda,h}| - |x - x_{\lambda,h}| \\ & \leq h \sum_{k=1}^n ([S(kh)x - x_{\lambda,h}, A_h x_{\lambda,h}]_+ + h^{-1}(e^{h\omega^*} - 1)|S((k-1)h)x - x_{\lambda,h}|). \end{aligned}$$

With regard to this inequality we have the three estimates below:

$$\begin{aligned} [S(kh)x - x_{\lambda,h}, A_h x_{\lambda,h}]_+ & \leq \lambda^{-1}(|S(kh)x - x_{\lambda,h} + \lambda A_h x_{\lambda,h}| - |S(kh)x - x_{\lambda,h}|) \\ & = \lambda^{-1}(|S(kh)x - x| - |S(kh)x - x_{\lambda,h}|) \\ & \leq \lambda^{-1}(2|S(kh)x - x| - |x - x_{\lambda,h}|), \\ |S((k-1)h)x - x_{\lambda,h}| & \leq |S((k-1)h)x - x| + |x - x_{\lambda,h}| \end{aligned}$$

and

$$|S(nh)x - x_{\lambda,h}| - |x - x_{\lambda,h}| \geq -|S(nh)x - x|,$$

where we have used (1.1) in the first estimate. Applying these estimates to (5.8), we obtain

$$\begin{aligned} n(1 - \lambda h^{-1}(e^{h\omega^*} - 1))|x - x_{\lambda,h}| & \leq \lambda h^{-1}|S(nh)x - x| + 2 \sum_{k=1}^n |S(kh)x - x| \\ & \quad + \lambda \sum_{k=1}^n h^{-1}(e^{h\omega^*} - 1)|S((k-1)h)x - x|, \end{aligned}$$

and so

$$\begin{aligned} & (1 - \lambda h^{-1}(e^{h\omega^*} - 1))|x - x_{\lambda,h}| \\ & \leq \lambda(nh)^{-1}|S(nh)x - x| + 2(nh)^{-1} \sum_{k=1}^n |S(kh)x - x|h \\ & \quad + \lambda(nh)^{-1} \sum_{k=1}^n h^{-1}(e^{h\omega^*} - 1)|S((k-1)h)x - x|h. \end{aligned}$$

Taking any $t \in (0, \tau]$ and letting $h \downarrow 0$ and $nh \uparrow t$ in the above estimate, we get the integral inequality

$$\begin{aligned} & (1 - \lambda\omega^*) \limsup_{h \downarrow 0} |x - x_{\lambda, h}| \\ & \leq \lambda t^{-1} |S(t)x - x| + 2t^{-1} \int_0^t |S(s)x - x| ds + \lambda\omega^* t^{-1} \int_0^t |S(s)x - x| ds. \end{aligned}$$

Therefore, passing to the limit as $\lambda \downarrow 0$, we have

$$\limsup_{\lambda \downarrow 0} (\limsup_{h \downarrow 0} |x - x_{\lambda, h}|) \leq 2t^{-1} \int_0^t |S(s)x - x| ds.$$

Consequently, we obtain the desired assertion of the lemma by letting $t \downarrow 0$ in the above inequality. \square

In view of Lemma 5.2, it is sufficient for the proof of (b) to show the following lemma.

LEMMA 5.3. *There is $\lambda(x) \in (0, \lambda^*]$ such that the limit $x_\lambda \equiv \lim_{h \downarrow 0} x_{\lambda, h}$ exists for each $\lambda \in (0, \lambda(x)]$.*

The proof of this lemma is considerably technical, although it requires a new idea based on the so-called asymptotic center and actually this is the central part of the proof of Theorem 5.1.

For $\lambda \in (0, \lambda^*]$ we write $\beta_\lambda = (1 - a\lambda)^{-1}(\varphi(x) + \lambda b)$. Let $\varphi(x) < \alpha$ as before. Then by Lemma 5.2 we have

$$\lim_{\lambda \downarrow 0} (\beta_\lambda + [\limsup_{h \downarrow 0} |x_{\lambda, h} - x|^2 + |\beta_\lambda - \varphi(x)|^2]^{1/2}) = \varphi(x) < \alpha.$$

Hence one can choose $\lambda(x) \in (0, \lambda^*]$ so small that

$$\beta_\lambda + [\limsup_{h \downarrow 0} |x_{\lambda, h} - x|^2 + |\beta_\lambda - \varphi(x)|^2]^{1/2} \leq \alpha \quad \text{for } \lambda \in (0, \lambda(x)].$$

Furthermore, we write $h(x)$ for the number h^* appearing in (5.5). Then the desired assertion of Lemma 5.3 is obtained for the number $\lambda(x)$ and assertion (b) of Theorem 5.1 is valid for the numbers $\lambda_0 = \lambda(x)$ and $h_0 = h(x)$. Therefore the rest of this section is devoted to the proof of Lemma 5.3.

Fix any $\lambda \in (0, \lambda(x)]$ and take any null sequence $(h(n))_{n=1}^\infty$ in $(0, h(x))$. Then, in view of (5.7), we have two bounded sequences $(x_{\lambda, h(n)})$ in D and $(\beta_{\lambda, h(n)})$ in $[0, \alpha]$. We treat Banach limits of these bounded sequences. Fix any functional $L \in (\ell^\infty)^*$ such that given a bounded sequence $(\xi_n)_{n=1}^\infty \in \ell^\infty$ the value of L at (ξ_n) becomes a Banach limit. In order to emphasize the Banach limit, we write the value $\langle (\xi_n), L \rangle$ as $\text{LIM}_{n \rightarrow \infty} \xi_n$ in the following.

We now define a function $\Phi: X \times \mathbf{R} \rightarrow \mathbf{R}$ by

$$\begin{aligned} \Phi((y, \beta)) &= \text{LIM}_{n \rightarrow \infty} (|x_{\lambda, h(n)} - y|^2 + |\beta_{\lambda, h(n)} - \beta|^2) \\ &= \text{LIM}_{n \rightarrow \infty} (|x_{\lambda, h(n)} - y|^2) + |\beta_\lambda - \beta|^2 \end{aligned}$$

for $(y, \beta) \in X \times \mathbf{R}$, where the norm of $X \times \mathbf{R}$ is defined by

$$|(y, \beta)|^2 = |y|^2 + |\beta|^2.$$

The functional Φ is convex and continuous on $X \times \mathbf{R}$. Since the norm of X is uniformly Gâteaux differentiable, it is easily seen that Φ is Gâteaux differentiable on $X \times \mathbf{R}$. Moreover, $\Phi(y, \beta) \rightarrow +\infty$ whenever $|(y, \beta)| \rightarrow +\infty$.

We here think of the use of an analogue of the asymptotic center of bounded sequence in $X \times \mathbf{R}$. Referring to Ekeland [1], Section 5.2, we consider the epigraph of φ which we here denote by

$$E(\varphi) = \{(y, \beta) \in X \times \mathbf{R} : \varphi(y) \leq \beta\}.$$

The set $E(\varphi)$ is closed and convex in $X \times \mathbf{R}$. Also, the growth condition (G) and the Lipschitz condition stated in Remark after Proposition 4.4 can be rewritten, respectively, in the following forms:

- (G') For $(y, \beta) \in E(\varphi)$ and $t \geq 0$, $(S(t)y, \pi(t; \beta)) \in E(\varphi)$.
- (L') For $(y, \beta), (z, \gamma) \in E(\varphi)$ and $t \geq 0$,

$$|S(t)x - S(t)y| \leq |x - y| \exp\left(\int_0^t \omega(\pi(s; \alpha \vee \beta)) ds\right).$$

Since the Banach space $X \times \mathbf{R}$ is reflexive and $\Phi(y, \beta) \rightarrow \infty$ as $|(y, \beta)| \rightarrow \infty$, the functional Φ attains its minimum in $X \times \mathbf{R}$. Namely, there is $(x_{\lambda,0}, \beta_{\lambda,0}) \in E(\varphi)$ satisfying

$$(5.9) \quad \Phi(x_{\lambda,0}, \beta_{\lambda,0}) = \inf \{\Phi(y, \beta) : (y, \beta) \in E(\varphi)\}.$$

Now it is clear that the sequence $(x_{\lambda,h(n)}, \beta_{\lambda,h(n)})$ is bounded in $X \times \mathbf{R}$ and lies in $E(\varphi)$. We then demonstrate that $\Phi(x_{\lambda,0}, \beta_{\lambda,0}) = 0$. If this would be accomplished, then

$$\liminf_{h \rightarrow \infty} (|x_{\lambda,h(n)} - x_{\lambda,0}|^2 + |\beta_{\lambda,h(n)} - \beta_{\lambda,0}|^2) = 0$$

and it would be asserted that there is a subsequence converging to $x_{\lambda,0}$. Therefore, if it would be verified that there is a unique limit point of the sequence $(x_{\lambda,h(n)})$, then Lemma 5.3 would be proved.

Noting that

$$|\beta_{\lambda,0} - \beta_\lambda|^2 \leq \Phi(x_{\lambda,0}, \beta_{\lambda,0}) \leq \Phi(x, \varphi(x)) \leq \limsup_{h \downarrow 0} |x_{\lambda,h} - x|^2 + |\beta_\lambda - \varphi(x)|^2,$$

we have

$$\beta_{\lambda,0} \leq \beta_\lambda + [\limsup_{h \downarrow 0} |x_{\lambda,h} - x|^2 + |\beta_\lambda - \varphi(x)|^2]^{1/2} \leq \alpha$$

and hence $\varphi(x_{\lambda,0}) \leq \beta_{\lambda,0} \leq \alpha$. Let $s \in [0, \tau)$ and choose $h \in (0, h(x)]$ so that $s + h \in [0, \tau]$. We are going to show that for every $z \in D$, $\beta \in [0, \infty)$ and every $\theta > 0$ the inequality

$$\begin{aligned}
& (1 - \lambda(a \vee \omega^*))t^{-1} \int_0^t \Phi(S(s)z, \pi(s; \beta)) ds \\
(5.10) \quad & \leq (2\theta t)^{-1} \int_0^t (\Phi(S(s)z + \theta(S(s)z - x), \pi(s; \beta)) + \theta(\pi(s; \beta) - \varphi(x))) \\
& \quad - \Phi(S(s)z, \pi(s; \beta))) ds + \lambda(2t)^{-1}(\Phi(z, \beta) - \Phi(S(t)z, \pi(t; \beta)))
\end{aligned}$$

holds; from this one can deduce the desired identity $\Phi(x_{\lambda,0}, \beta_{\lambda,0}) = 0$ as mentioned below. Let $z \in D_\alpha$. Then by Assertion (a) we have

$$x_{\lambda,h} = (\lambda + h)^{-1}hx + (\lambda + h)^{-1}\lambda S(h)x_{\lambda,h}$$

and so

$$\begin{aligned}
& |x_{\lambda,h} - [(\lambda + h)^{-1}hx + (\lambda + h)^{-1}\lambda S(s + h)z]| \\
& = (\lambda + h)^{-1}\lambda |S(h)x_{\lambda,h} - S(s + h)z| \leq (\lambda + h)\lambda |x_{\lambda,h} - S(s)z| e^{h\omega^*} \\
& \leq (\lambda + h)^{-1}\lambda |x_{\lambda,h} - S(s)z| + (\lambda + h)^{-1}\lambda(e^{h\omega^*} - 1)|x_{\lambda,h} - S(s)z| \\
& = (1 - (\lambda + h)^{-1}h)|x_{\lambda,h} - S(s)z| + (\lambda + h)^{-1}\lambda(e^{h\omega^*} - 1)|x_{\lambda,h} - S(s)z|.
\end{aligned}$$

The above inequality can be transformed into

$$\begin{aligned}
& ((\lambda + h)^{-1}h - (\lambda + h)^{-1}\lambda(e^{h\omega^*} - 1))|x_{\lambda,h} - S(s)z| \\
& \leq |x_{\lambda,h} - S(s)z| - |z_{\lambda,h} - [(\lambda + h)^{-1}hx + (\lambda + h)^{-1}\lambda S(s + h)z]| \\
& = |x_{\lambda,h} - S(s)z| - |x_{\lambda,h} - S(s + h)z - (\lambda + h)^{-1}h(x - S(s + h)z)|.
\end{aligned}$$

From this we obtain the estimate

$$\begin{aligned}
0 & \leq (1 - \lambda h^{-1}(e^{h\omega^*} - 1))|x_{\lambda,h} - S(s)z| \\
& \leq (\lambda + h)h^{-1}(|x_{\lambda,h} - S(s)z| - |x_{\lambda,h} - S(s + h)z - (\lambda + h)^{-1}h(x - S(s + h)z)|),
\end{aligned}$$

where we have used (5.6) in the first inequality. Multiplying both sides of the above inequality by $|x_{\lambda,h} - S(s)z|$ and using the relation

$$|x_{\lambda,h} - S(s)z| \geq |x_{\lambda,h} - S(s + h)z - (\lambda + h)^{-1}h(x - S(s + h)z)|$$

(which follows from the above inequality), we have

$$\begin{aligned}
& (1 - \lambda h^{-1}(e^{h\omega^*} - 1))|x_{\lambda,h} - S(s)z|^2 \\
& \leq (\lambda + h)h^{-1}(|x_{\lambda,h} - S(s)z|^2 - |x_{\lambda,h} - S(s + h)z - (\lambda + h)^{-1}h(x - S(s + h)z)|^2) \\
& = (\lambda + h)h^{-1}(|x_{\lambda,h} - S(s + h)z|^2 - |x_{\lambda,h} - S(s + h)z \\
& \quad - (\lambda + h)^{-1}h(x - S(s + h)z)|^2).
\end{aligned}$$

Let $t \in (0, \tau)$, $h \in (0, h(x)]$ and let $t + h \in [0, \tau]$. Integrating the above inequality over $[0, t]$ with respect to s , we get

$$\begin{aligned}
 & (1 - \lambda h^{-1}(e^{h\omega^*} - 1))t^{-1} \int_0^t |x_{\lambda, h} - S(s)z|^2 ds \\
 & \leq (\lambda + h)(ht)^{-1} \int_0^t (|x_{\lambda, h} - S(s+h)z|^2 \\
 & \quad - |x_{\lambda, h} - S(s+h)z - (\lambda + h)^{-1}h(x - S(s+h)z)|^2) ds \\
 & \quad + (\lambda + h)(ht)^{-1} \left(\int_0^t (|x_{\lambda, h} - S(s)z|^2 ds - \int_0^t |x_{\lambda, h} - S(s+h)z|^2 ds) \right) \\
 & = (\lambda + h)(ht)^{-1} \int_h^{t+h} (|x_{\lambda, h} - S(s)z|^2 \\
 & \quad - |x_{\lambda, h} - S(s)z - (\lambda + h)^{-1}h(x - S(s)z)|^2) ds \\
 & \quad + (\lambda + h)(ht)^{-1} \left(\int_0^h (|x_{\lambda, h} - S(s)z|^2 ds - \int_t^{t+h} |x_{\lambda, h} - S(s)z|^2 ds) \right).
 \end{aligned}$$

We here recall (1.1) to assert that for any $\theta > 0$ the above inequality can be replaced by the following

$$\begin{aligned}
 & (1 - \lambda h^{-1}(e^{h\omega^*} - 1))t^{-1} \int_0^t |x_{\lambda, h} - S(s)z|^2 ds \\
 (5.11) \quad & \leq (\theta t)^{-1} \int_h^{t+h} (|x_{\lambda, h} - S(s)z + \theta(x - S(s)z)|^2 - |x_{\lambda, h} - S(s)z|^2) ds \\
 & \quad + (\lambda + h)(ht)^{-1} \left(\int_0^h |x_{\lambda, h} - S(s)z|^2 ds - \int_t^{t+h} |x_{\lambda, h} - S(s)z|^2 ds \right).
 \end{aligned}$$

Furthermore, we infer from (5.7) and the same argument as above that for $\beta \in [0, \infty)$, $t \in (0, \tau)$, $h \in (0, h(x)]$ with $t + h \in [0, \tau]$ and $\theta > 0$ the inequality below is valid:

$$\begin{aligned}
 & (1 - \lambda h^{-1}(e^{h\alpha} - 1))t^{-1} \int_0^t |\beta_{\lambda, h} - \pi(s, \beta)|^2 ds \\
 (5.12) \quad & \leq (\theta t)^{-1} \int_h^{t+h} (|\beta_{\lambda, h} - \pi(s; \beta) + \theta(\varphi(x) - \pi(s; \beta))|^2 - |\beta_{\lambda, h} - \pi(s; \beta)|^2) ds \\
 & \quad + (\lambda + h)(ht)^{-1} \left(\int_0^h |\beta_{\lambda, h} - \pi(s; \beta)|^2 ds - \int_t^{t+h} |\beta_{\lambda, h} - \pi(s; \beta)|^2 ds \right).
 \end{aligned}$$

Letting $h = h(n)$ in (5.11) and (5.12) and taking the Banach limits we obtain the desired estimate (5.9) for the function $\Phi(S(\cdot)z, \pi(\cdot; \beta))$. Since $(x_{\lambda,0}, \beta_{\lambda,0}) \in E(\varphi)$ and $\varphi(S(t)x_{\lambda,0}) \leq \pi(t; \beta_{\lambda,0}) \leq \pi(t; \beta_{\lambda,0})$, it follows that $(S(t)x_{\lambda,0}, \pi(t; \beta_{\lambda,0})) \in E(\varphi)$ and

$$\Phi(x_{\lambda,0}, \beta_{\lambda,0}) \leq \Phi(S(t)x_{\lambda,0}, \pi(t; \beta_{\lambda,0})) \quad \text{for } t \in [0, \infty).$$

Therefore, setting $(z, \beta) = (x_{\lambda,0}, \beta_{\lambda,0})$ in (5.10) gives

$$\begin{aligned} & (1 - \lambda(a \vee \omega^*))t^{-1} \int_0^t \Phi(S(s)x_{\lambda,0}, \pi(s; \beta_{\lambda,0})) ds \\ & \leq (\theta t)^{-1} \int_0^t [\Phi(S(s)x_{\lambda,0} + \theta(S(s)x_{\lambda,0} - x), \pi(s; \beta_{\lambda,0}) + \theta(\pi(s; \beta_{\lambda,0}) - \varphi(x))) \\ & \quad - \Phi(S(s)x_{\lambda,0}, \pi(s; \beta_{\lambda,0}))] ds. \end{aligned}$$

Letting $t \downarrow 0$ in the above inequality, we have

$$\begin{aligned} & (1 - \lambda(a \vee \omega^*))\Phi(x_{\lambda,0}, \beta_{\lambda,0}) \\ & \leq \theta^{-1}(\Phi(x_{\lambda,0} + \theta(x_{\lambda,0} - x), \beta_{\lambda,0} + \theta(\beta_{\lambda,0} - \varphi(x)) - \Phi(x_{\lambda,0}, \beta_{\lambda,0})), \end{aligned}$$

Finally, passing to the limit as $\theta \downarrow 0$, we obtain

$$(5.13) \quad (1 - \lambda(a \vee \omega^*))\Phi(x_{\lambda,0}, \beta_{\lambda,0}) \leq \Phi'(x_{\lambda,0}, \beta_{\lambda,0}; x_{\lambda,0} - x, \beta_{\lambda,0} - \varphi(x))$$

where $\Phi'(x, \beta; z, \gamma)$ denotes the Gâteaux derivative at (x, β) in the direction of (z, γ) . Hence the right-hand side of (5.13) is equal to the limit

$$\lim_{\theta \downarrow 0} \theta^{-1}(\Phi(x_{\lambda,0}, \beta_{\lambda,0}) - \Phi(x_{\lambda,0} + \theta(x - x_{\lambda,0}), \beta_{\lambda,0} + \theta(\varphi(x) - \beta_{\lambda,0}))).$$

Since $(x_{\lambda,0}, \beta_{\lambda,0}), (x, \varphi(x)) \in E(\varphi)$ and $E(\varphi)$ is convex, we infer that $(x_{\lambda,0}, \beta_{\lambda,0}) + \theta(x - x_{\lambda,0}, \varphi(x) - \beta_{\lambda,0}) \in E(\varphi)$ for $\theta \in (0, 1)$. Since $(x_{\lambda,0}, \beta_{\lambda,0})$ is the minimum point of the functional Φ in the sense of (5.9), this fact implies that the above limit is nonpositive. From this and (5.13) it follows that $\Phi(x_{\lambda,0}, \beta_{\lambda,0}) = 0$, and that there must exist a subsequence of $(x_{\lambda, h(n)})$ which converges to $x_{\lambda,0}$.

Finally, we demonstrate that the set of limit points of the sequence $(x_{\lambda, h(n)})$ is a singleton set, namely, the element $x_{\lambda,0}$ is a unique limit point of $(x_{\lambda, h(n)})$. Take any pair of null sequences $(h(n))$ and $(h'(n))$ in $(0, h(x))$ and suppose that the sequences $(x_{\lambda, h(n)})$ and $(x_{\lambda, h'(n)})$ converge to some $x_{\lambda,0}$ and $u_{\lambda,0}$, respectively. Since $x_{\lambda, h} \in D_\alpha$ for $h \in (0, h(x))$, we see that both $x_{\lambda,0}$ and $u_{\lambda,0}$ belong to D_α . Letting $h = h(n)$ in (5.10) and passing to the limit as $n \rightarrow \infty$, we have

$$(1 - \lambda\omega^*)t^{-1} \int_0^t |x_{\lambda,0} - S(s)z|^2 ds$$

$$(5.14) \quad \leq (\theta t)^{-1} \int_0^t (|x_{\lambda,0} - S(s)z + \theta(x - S(s)z)|^2 - |x_{\lambda,0} - S(s)z|^2) ds \\ + t^{-1} \lambda (|x_{\lambda,0} - x|^2 - |x_{\lambda,0} - S(t)z|^2).$$

The term $|x_{\lambda,0} - S(t)z|$ on the right-hand side can be written as $|x_{\lambda,0} - z - (\lambda/t)^{-1} \lambda A_t z|$. Hence the application of (1.1) to the second term on the right-hand side of (5.14) implies that the right-hand side is bounded above by

$$(\theta t)^{-1} \int_0^t (|x_{\lambda,0} - S(s)z + \theta(x - S(s)z)|^2 - |x_{\lambda,0} - S(s)z|^2) ds \\ + \theta^{-1} (|x_{\lambda,0} - z + \theta \lambda A_t z|^2 - |x_{\lambda,0} - z|^2).$$

for $\theta > 0$. We then put $t = h'(n)$ and $z = x_{\lambda, h'(n)}$ in the above estimate to get $\lambda A_t z = x_{\lambda, h'(n)} - x$ and

$$(1 - \lambda \omega^*) (h'(n))^{-1} \int_0^{h'(n)} |x_{\lambda,0} - S(s)x_{\lambda, h'(n)}|^2 ds \\ \leq (\theta h'(n))^{-1} \int_0^{h'(n)} (|(x_{\lambda,0} - S(s)x_{\lambda, h'(n)} + \theta(x - S(s)x_{\lambda, h'(n)})|^2 \\ - |x_{\lambda,0} - S(s)x_{\lambda, h'(n)}|^2) ds \\ + \theta^{-1} (|x_{\lambda,0} - x_{\lambda, h'(n)} + \theta(x_{\lambda, h'(n)} - x)|^2 - |x_{\lambda,0} - x_{\lambda, h'(n)}|^2)$$

for $\theta > 0$. Passing to the limit as $n \rightarrow \infty$, we obtain

$$(1 - \lambda \omega^*) |x_{\lambda,0} - u_{\lambda,0}|^2 \\ (5.15) \quad \leq \theta^{-1} (|x_{\lambda,0} - u_{\lambda,0} + \theta(x - u_{\lambda,0})|^2 - |x_{\lambda,0} - u_{\lambda,0}|^2) \\ + \theta^{-1} (|x_{\lambda,0} - u_{\lambda,0} + \theta(x_{\lambda,0} - u)|^2 - |x_{\lambda,0} - u_{\lambda,0}|^2)$$

for $\theta > 0$. We here apply the Gâteaux differentiability of the norm $|\cdot|$. The right-hand side of (5.15) tends to

$$|x_{\lambda,0} - u_{\lambda,0}| ([x_{\lambda,0} - u_{\lambda,0}, x - u_{\lambda,0}]_+ + [x_{\lambda,0} - u_{\lambda,0}, u_{\lambda,0} - x]_+) = 0,$$

and it follows that $(1 - \lambda \omega^*) |x_{\lambda,0} - u_{\lambda,0}| = 0$. This means that any sequence $(x_{\lambda, h(n)})$ converges to $x_{\lambda,0}$ as $h(n) \rightarrow 0$.

For each $x \in D$ we write $\lambda_0(x)$ for the number $\lambda(x) \in (0, \lambda^*]$ obtained in Lemma 5.3 and $h_0(x)$ for the number h^* determined by (5.5). Then both assertions (a) and (b) stated in Theorem 5.1 are thus obtained, and this completes the proof of Theorem 5.1.

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