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# Asymptotic behavior and domain-dependency of solutions to a class of reaction-diffusion systems with large diffusion coefficients

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## §1. Introduction

A lot of reaction-diffusion equation models have been recently used to study pattern formation in population ecology, morphogenesis, neurobiology, chemical reactor theory and in other fields. These are usually described by the following weakly-coupled parabolic systems:

$$(1.1a) \qquad u_t = D\,\Delta u + f(u)\,, \qquad (t, x) \in (0, \infty) \times \Omega\,,$$

where $\Omega$ is a bounded domain in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$, $u = {}^t(u_1, u_2, \dots, u_m)$, $D = \operatorname{diag}(d_1, d_2, \dots, d_m)$ with diffusion coefficients $d_i > 0$ $(i = 1, 2, \dots, m)$, $\Delta$ is the Laplacian and $f$ is a smooth mapping of $\boldsymbol{R}^m$ into itself.

One of the familiar boundary conditions for the system (1.1a) is the homogeneous Neumann boundary condition:

$$(1.1b) \qquad \frac{\partial u}{\partial n} = 0, \qquad (t, x) \in (0, \infty) \times \partial\Omega\,,$$

where $\partial/\partial n$ denotes the outer normal derivative on $\partial\Omega$.

One of the important topics for (1.1) is the problem as to whether or not stable spatially inhomogeneous equilibria or periodic solutions exist from pattern formation point of view.

It is shown by Chafee [3] that any stable equilibrium solution of the scalar reaction-diffusion equations of (1.1) ($m = 1$) in one dimensional interval is *constant*. Later, along this line, there have been a lot of papers including Matano [16] and Casten and Holland [2], in which the same result is valid when $\Omega$ is a bounded, *convex* domain in $\boldsymbol{R}^n$. We should note that this conclusion for scalar versions holds for any nonlinearity of $f$.

Kishimoto and Weinberger [15] generalized the above result to the system (1.1a) satisfying $(\partial/\partial u_j)f_i > 0$ for $i \neq j$, which is called the $m$-cooperating system. Namely, when $\Omega$ is any bounded convex domain, there are no stable inhomogeneous equilibrium solutions of (1.1). On the other hand, for the system (1.1) with $m = 2$ satisfying $(\partial/\partial u_j)f_i < 0$ for $i \neq j$, which is called the competitive system, Matano and Mimura [17] have proved that there exists a bounded *nonconvex* domain $\Omega \subset \boldsymbol{R}^2$ for which the system has *stable* spatially inhomogeneous equilibrium solutions when suitable additional conditions are imposed on $f$ (see also Matano [16] for the scalar equations). We note that when $\Omega$ is convex, the system has no stable spatially inhomogeneous equilibrium solutions (see Kishimoto and Weinberger [15]).

These results indicate that the stability of spatially inhomogeneous equilibrium solutions depend on the shape of domain. In fact, Hale and Vegas [10] discussed this problem by appropriately parametrizing a family of nonconvex domains. Following them, there are a lot of papers on scalar equations (for instance, Vegas [21], Keyfitz and Kuiper [14], Dancer [6], Jimbo [12], [13]) to discuss the changes of solutions by varying the domain.

Recently, Morita [20] has studied (1.1) in the system version, when $D$ is arbitrarily fixed large and $\Omega$ is a nonconvex domain of dumb-bell shape with very narrow handle. His assertion is that there exists a finite dimensional Lipschitz continuous invariant manifold together with its attractivity and the reduced form of ordinary differential equations on the invariant manifold.

On the other hand, Conway, Hoff and Smoller [5] considered the problem (1.1) for arbitrarily fixed $\Omega$. By assuming the existence of invariant region for (1.1) they conclude that if all of the diffusion coefficients are very large, any solution of (1.1) tends to be spatially homogeneous as $t \to +\infty$ and that the asymptotic behavior of the solution of (1.1) is qualitatively determined by the following ODE:

$$\frac{du}{dt} = f(u)\,. \tag{1.2}$$

However, we note that if all of the diffusion coefficients are not large, there exist stable spatially inhomogeneous steady states of (1.1) with $m \geq 2$ for suitable $f(u)$ even if $\Omega$ is convex (see Mimura, Nishiura and Yamaguti [18]).

These two results indicate that the existence and stability of spatially inhomogeneous equilibrium solutions of (1.1) depend on not only the shape of domain but also the diffusion coefficients.

In this paper, we study the dependency of these two effects on solutions of the problem (1.1). To do it, we introduce one parameter $\varepsilon$ into the system (1.1a) in a way that $\Omega$ is a dumb-bell shape domain $\Omega_\varepsilon$ used in [10] and [20] (see Figure 1) and that $D$ takes $D = \varepsilon^{-\theta}\bar{D}$ $(\theta > 0)$ with $\bar{D} = \operatorname{diag}(\bar{d}_1, \ldots, \bar{d}_m)$ $(\bar{d}_i > 0, i = 1, 2, \ldots, m)$. If $\varepsilon$ is sufficiently small, the situation is as follows: $\Omega_\varepsilon$ is a dumb-bell shape domain such that it closes to $\Omega_0$ which is the union of two disjoint convex domains and all the diffusion coefficients are very large with the rate $\theta > 0$. Our aim is to construct a finite dimensional Lipschitz continuous invariant manifold and derive the reduced form of ODE on the invariant manifold from (1.1). We note that the case when $\theta = 0$ was already discussed by Morita [20].

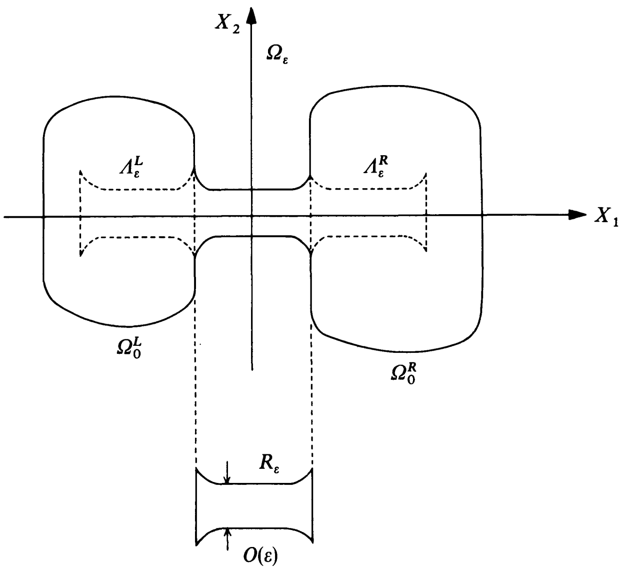


FIGURE 1

In Section 2 we show some results obtained by Hale and Vegas [10] and Vegas [21] and Morita [20]. In Section 3 we construct an invariant manifold of finite dimensions and show its global attractivity. In Section 4 we derive the ordinary differential equations on the manifold. The discussion on the asymptotic behavior of solutions of (1.1) with an application to population dynamics will be reported in a forthcoming paper [19].

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## §2. Preliminaries

We consider the following reaction-diffusion equations with two parameters $\varepsilon > 0$ and $\theta > 0$:

$$(2.1) \qquad \begin{cases} u_t = \dfrac{1}{\varepsilon^\theta} D\, \Delta u + f(u)\,, & (t, x) \in (0, \infty) \times \Omega_\varepsilon\,, \\[2ex] \dfrac{\partial u}{\partial n} = 0\,, & (t, x) \in (0, \infty) \times \partial\Omega_\varepsilon\,. \end{cases}$$

Here $\Omega_\varepsilon \subset \boldsymbol{R}^2$ are the $\varepsilon$-family of nonconvex domains symmetric with respect to $x_1$ axis with smooth boundary $\partial\Omega_\varepsilon$ for $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0 > 0$), which consists of three disjoint unions $\Omega_\varepsilon = \Omega_0^L \cup \Omega_0^R \cup R_\varepsilon$, where $\Omega_0^L$, $\Omega_0^R$ are two disjoint convex domains and $R_\varepsilon$ is a handle satisfying $|R_\varepsilon| \to 0$ as $\varepsilon \to 0$ with respect to the Lebesgue measure $|\cdot|$ in $\boldsymbol{R}^2$. (see Figure 1).

We assume without loss of generality that $f(0) = 0$. We also assume that there exists $K^* > 0$ such that

(H) (2.1) admits an invariant region $V_K = \{u \in \boldsymbol{R}^m | 0 \le u_i \le K, 1 \le i \le m\}$ for any $K > K^*$.

In fact, the existence of such an invariant region for competition-diffusion system is studied in [4]. Now we fix sufficiently large $K$ ($>K^*$) and simply denote $V_K$ by $V$, because we only consider solutions of (2.1) in $V$.

Let $Z_\varepsilon^k = (H^k(\Omega_\varepsilon))^m$ be the $m$-product space of $H^k(\Omega_\varepsilon)$ for $k \ge 0$ and $\varepsilon \in (0, \varepsilon_0]$ with the norm $\|u\|_{Z_\varepsilon^k} = (\Sigma_{i=1}^m \|u_i\|_{H^k(\Omega_\varepsilon)}^2)^{1/2}$. In particular, we write $Z_\varepsilon^0 = (L^2(\Omega_\varepsilon))^m$ as $Z_\varepsilon$ simply. The inner-product in $Z_\varepsilon$ is denoted by $(u, v)_{Z_\varepsilon} = \Sigma_{i=1}^m \int_{\Omega_\varepsilon} u_i(x) v_i(x)\, dx$. $A_\varepsilon$ and $\exp\{-A_\varepsilon t\}$ denote a closed operator $-\varepsilon^{-\theta} D\Delta$ in $Z_\varepsilon$ with the domain $\mathscr{D}(A_\varepsilon) = \{u \in Z_\varepsilon^2 | \partial u/\partial n = 0 \text{ on } \partial\Omega_\varepsilon\}$ and the $C_0$-semigroup in $Z_\varepsilon$ generated by $-A_\varepsilon$, respectively.

Hereafter we modify $f$ to $\tilde{f}$ by multiplying a suitable $C^\infty$-cut-off function such that

(i) $\tilde{f}(u) = f(u)$ for $u \in V$ and $\tilde{f}(u) = 0$ for $u \in \bar{V} = \{u \in \boldsymbol{R}^m | -1 \le u_i \le K + 1, (i = 1, \dots, m)\}$;

(ii) there exists $\rho > 0$ such that $|\tilde{f}(u)|$, $|\tilde{f}_u(u)| \le \rho$ for $u \in \boldsymbol{R}^m$, where $\tilde{f}_u(u)$ means the first derivative of $\tilde{f}$;

(iii) $|\tilde{f}(u^{(1)}) - \tilde{f}(u^{(2)})| \le \rho \max\{|u^{(1)} - u^{(2)}|, 4K\}$ for $u^{(1)}$, $u^{(2)} \in \boldsymbol{R}^m$ and it holds for $\tilde{f}_u(u)$ in place of $\tilde{f}(u)$.

Using the above notation, we may write (2.1) with conditions (i) $\sim$ (iii) as an abstract form

$$(2.2)_\varepsilon \qquad \begin{cases} \dfrac{du}{dt} = -A_\varepsilon u + \tilde{f}(u)\,, \quad t > 0\,, \\ \\ u(0) = u_0 \in U_{K'}\,, \end{cases}$$

where $U_{K'} = \{u \in Z_\varepsilon^1 \cap (L^\infty(\Omega_\varepsilon))^m | u \in V \text{ and } \|u\|_{Z_\varepsilon^1} \le K'\}$. Here we note that the existence and uniqueness of solutions of $(2.2)_\varepsilon$ $u(t) \in \mathbf{C}^1([0, \infty), Z_\varepsilon^1) \cap \mathscr{D}(A_\varepsilon)$ is proved in a standard manner (c.f. D. Henry [11]).

In this paper, we are concerned with $(2.2)_\varepsilon$ and simply write $\tilde{f}$ as $f$ through the rest of this paper.

We first give some results with respect to the family of the domains $\Omega_\varepsilon$ which is proposed by Hale and Vegas [10]. Let $\lambda_\varepsilon^{(k)}$, $\omega_\varepsilon^{(k)}$ be the $k$-th eigenvalue of $-\Delta$ in $\Omega_\varepsilon$ with Neumann boundary condition and the corresponding normalized eigenfunction, respectively. It is well known that $0 = \lambda_\varepsilon^{(1)} < \lambda_\varepsilon^{(2)} \leq \lambda_\varepsilon^{(3)} \leq \cdots$.

PROPOSITION 2.1 (Hale and Vegas [10], [21]).

(1) *$\lambda_\varepsilon^{(2)}$ is continuous in $\varepsilon$ and there exists $\gamma_1 > 0$ such that $\lambda_\varepsilon^{(2)} \leq \gamma_1 \varepsilon$ for small $\varepsilon > 0$.*

(2) *There exists $\gamma_3 > 0$ such that $\lambda_\varepsilon^{(3)} \geq \gamma_3$ for small $\varepsilon > 0$.*

(3) *$\|\omega_\varepsilon^{(2)}\|_{L^2(R_\varepsilon)} = O(\varepsilon^{1/2})$, $\|\omega_\varepsilon^{(2)} - \omega_0^{(2)}\|_{C^2(\bar{\Omega}_0)} = O(\varepsilon^{1/2})$. Here $\Omega_0 = \Omega_0^L \cup \Omega_0^R$ and*

$$\omega_0^{(2)} = \begin{cases} -\left\{\dfrac{\alpha}{(1-\alpha)|\Omega_0|}\right\}^{1/2} \equiv \omega_0^L & \text{in } \Omega_0^L \\ \left\{\dfrac{1-\alpha}{\alpha|\Omega_0|}\right\}^{1/2} \equiv \omega_0^R & \text{in } \Omega_0^R \end{cases}$$

*with $\alpha = |\Omega_0^R|/|\Omega_0|$.*

Now we consider the asymptotic behavior of solutions of $(2.2)_\varepsilon$. Let $Q^\varepsilon$ be the projection from $Z_\varepsilon$ into $(\operatorname{span}\{\omega_\varepsilon^{(1)}, \omega_\varepsilon^{(2)}\})^m$ and $P^\varepsilon = Id - Q^\varepsilon$, where $Id$ is the identity on $Z_\varepsilon$. For $u = (u_1, \ldots, u_m) \in Z_\varepsilon$ and $\omega \in L^2(\Omega_\varepsilon)$, $\langle u, \omega\rangle_{Z_\varepsilon}$ means ${}^t((u_1, \omega)_{L^2(\Omega_\varepsilon)}, \ldots, (u_m, \omega)_{L^2(\Omega_\varepsilon)})$ and for $Y = (y_1, y_2) \in \boldsymbol{R}^{2m}$, $\Psi_\varepsilon = (\omega_\varepsilon^{(1)}, \omega_\varepsilon^{(2)}) \in (L^2(\Omega_\varepsilon))^2$, $Y \cdot \Psi_\varepsilon \in Z_\varepsilon$ does $y_1\omega_\varepsilon^{(1)} + y_2\omega_\varepsilon^{(2)}$. Using the above notations, we note that $Q^\varepsilon$ is represented by $Q^\varepsilon u = Y \cdot \Psi_\varepsilon$, where $Y = (y_1, y_2)$ and $y_i = \langle u, \omega_\varepsilon^{(i)}\rangle_{Z_\varepsilon}$ $(i = 1, 2)$.

PROPOSITION 2.2. *For $t \geq 0$, the followings hold:*

(1) $\|\exp\{-A_\varepsilon t\}P^\varepsilon\varphi\|_{Z_\varepsilon} \leq \exp\left\{-\dfrac{\gamma_3}{\varepsilon^\theta} d_* t\right\} \|\varphi\|_{Z_\varepsilon}$ *for* $\varphi \in Z_\varepsilon$;

(2) $\|\exp\{-A_\varepsilon t\}P^\varepsilon\varphi\|_{Z_\varepsilon^1} \leq \alpha_0 \varepsilon^{\theta/2} t^{-1/2} \exp\left\{-\dfrac{\gamma_3}{\varepsilon^\theta} d_* t\right\} \|\varphi\|_{Z_\varepsilon}$ *for* $\varphi \in Z_\varepsilon$ *and some* $\alpha_0 > 0$;

(3) $\|\exp\{-A_\varepsilon t\}P^\varepsilon\varphi\|_{Z_\varepsilon^1} \leq \exp\left\{-\dfrac{\gamma_3}{\varepsilon^\theta} d_* t\right\} \|P^3\varphi\|_{Z_\varepsilon^1}$ *for* $\varphi \in Z_\varepsilon^1$, *where* $d_* = \min\{d_1, \ldots, d_m\}$.

LEMMA 2.3. *There exists $K_1 = K_1(K', K) > 0$ such that if $u_0 \in U_{K'}$, then $u(t, \cdot) \in U_{K_1}$ for all $t \geq 0$, where $u(t, \cdot)$ is the solution of $(2.2)_\varepsilon$.*

PROOF. Since $V$ is a positively invariant region of $(2.2)_\varepsilon$, we have

$$|y_i(t)| \leq \|u(t)\|_{Z_\varepsilon} \cdot \|\omega_\varepsilon^{(i)}\|_{L^2(\Omega_\varepsilon)} \leq K(m|\Omega_{\varepsilon_0}|)^{1/2} \qquad (i = 1, 2) \tag{2.4}$$

by Schwartz's inequality, where $y_i(t) = \langle u(t), \omega_\varepsilon^{(i)} \rangle_{Z_\varepsilon}$. The solution of $(2.2)_\varepsilon$ can be represented by the following integral equation:

$$(2.5)_\varepsilon \qquad u(t) = \exp\{-A_\varepsilon t\} u_0 + \int_0^t \exp\{-A_\varepsilon(t-s)\} f(u(s))\, ds \qquad \text{for } t \geq 0$$

Operating $P^\varepsilon$ on both sides of $(2.5)_\varepsilon$, we have from (2) and (3) of Proposition 2.2,

$$\begin{aligned}
\|P^\varepsilon u(t)\|_{Z_\varepsilon^1} &\leq \|\exp\{-A_\varepsilon t\} P^\varepsilon u_0\|_{Z_\varepsilon^1} + \int_0^t \|\exp\{-A_\varepsilon(t-s)\} P^\varepsilon f(u(s))\|_{Z_\varepsilon^1}\, ds \\
&\leq \exp\{-(\gamma_3/\varepsilon^\theta)\, d_* t\} \|P^\varepsilon u_0\|_{Z_\varepsilon^1} + \alpha_0 \varepsilon^{\theta/2} \int_0^t \frac{\exp\{-(\gamma_3/\varepsilon^\theta)\, d_*(t-s)\}}{(t-s)^{1/2}} \\
&\quad \times \|f(u(s))\|_{Z_\varepsilon}\, ds \\
&\leq [3K(m|\Omega_{\varepsilon_0}|)^{1/2} + K'] + \alpha_0 \rho (m|\Omega_{\varepsilon_0}|)^{1/2} \varepsilon^{\theta/2} \int_0^\infty \frac{\exp\{-(\gamma_3/\varepsilon^\theta)\, d_* s\}}{s^{1/2}}\, ds \\
&\leq [3K + \alpha_0 \rho (\pi d_*/\gamma_3)^{1/2} \varepsilon_0^\theta](m|\Omega_{\varepsilon_0}|)^{1/2} + K'.
\end{aligned}$$

Thus it follows from the estimate of (2.4) that

$$\begin{aligned}
\|u(t)\|_{Z_\varepsilon^1} &\leq \|Q^\varepsilon u(t)\|_{Z_\varepsilon^1} + \|P^\varepsilon u(t)\|_{Z_\varepsilon^1} \\
&\leq |y_1(t)|\, \|\omega_\varepsilon^{(1)}\|_{H^1(\Omega_\varepsilon)} + |y_2(t)|\, \|\omega_\varepsilon^{(2)}\|_{H^1(\Omega_\varepsilon)} + \|P^\varepsilon u(t)\|_{Z_\varepsilon^1} \\
&\leq [6K + \alpha_0 \rho (\pi d_*/\gamma_3)^{1/2} \varepsilon_0^\theta] \times (m|\Omega_{\varepsilon_0}|)^{1/2} + K' \equiv K_1. \quad \text{Q.E.D.}
\end{aligned}$$

Hereafter, we fix sufficiently large constant $K'$ as well as $K$ and do not write explicitly the dependency on constants $K$, $K'$ and $K_1$. For example, we denote $U_{K'}$ simply by $U$.

The following result can be found in Vegas [21], which is useful in obtaining some estimates of asymptotic behavior of solutions of $(2.2)_\varepsilon$.

LEMMA 2.4 (Vegas [21]). *Let $g_\varepsilon \in Z_\varepsilon$ and $g_0 \in (\mathrm{span}\{\omega_0^{(1)}, \omega_0^{(2)}\})^m$, where $\omega_0^{(1)} = |\Omega_0|^{-1/2}$ and $\omega_0^{(2)}$ has been already defined in Proposition 2.1. Then*

$$\|P^\varepsilon g_\varepsilon\|_{Z_\varepsilon} \leq M_1 \varepsilon^{1/2} \|g_\varepsilon\|_{Z_\varepsilon} + 2\|g_\varepsilon - g_0\|_{Z_0} + \|g_\varepsilon\|_{(L^2(R_\varepsilon))^m},$$

*where $Z_0 = (L^2(\Omega_0))^m$ with the norm $\|\cdot\|_{Z_0}$ and $M_1$ is a positive constant independent of small $\varepsilon > 0$.*

Now we can write $(2.2)_\varepsilon$ as follows:

$$(2.6\text{a})_\varepsilon \qquad \begin{cases} \dfrac{dy_1}{dt} = \langle f(Y\cdot \Psi_\varepsilon + \bar{u}), \omega_\varepsilon^{(1)} \rangle_{Z_\varepsilon}\,, \\[2mm] \dfrac{dy_2}{dt} = -\dfrac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} Dy_2 + \langle f(Y\cdot \Psi_\varepsilon + \bar{u}), \omega_\varepsilon^{(2)} \rangle_{Z_\varepsilon}\,, \\[2mm] y_1(0) = y_1^0\,, \qquad y_2(0) = y_2^0\,, \end{cases}$$

$$(2.6\text{b})_\varepsilon \qquad \begin{cases} \dfrac{d\bar{u}}{dt} = -A_\varepsilon \bar{u} + P^\varepsilon f(Y\cdot \Psi_\varepsilon + \bar{u})\,, \\[2mm] \bar{u}(0) = P^\varepsilon u_0\,, \end{cases}$$

where $y_i(t) = \langle u(t), \omega_\varepsilon^{(i)} \rangle_{Z_\varepsilon}$, $y_i^0 = \langle u_0, \omega_\varepsilon^{(i)} \rangle_{Z_\varepsilon}$ $(i = 1, 2)$ and $Y(t) = (y_1(t), y_2(t))$, $\bar{u}(t) = P^\varepsilon u(t)$. Note that $Y(t)\cdot \Psi_\varepsilon = Q^\varepsilon u(t)$.

Next lemma is proved in the similar manner to Morita [20].

LEMMA 2.5. *There exist $\varepsilon_1 > 0$, $t_0 > 0$ and $c > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$, $\|P^\varepsilon u(t)\|_{Z_\varepsilon^1} \le c\varepsilon^{(\theta+1)/2}$ for any $t \ge t_0$ and any $u_0 \in U$, where $u(t)$ is the solution of $(2.2)_\varepsilon$ or $(2.6)_\varepsilon$.*

PROOF. From the variational characterization we have

$$(2.7) \qquad (\gamma_3 d_*/\varepsilon^\theta)\|\bar{u}\|_{Z_\varepsilon}^2 \le \|A_\varepsilon^{1/2}\bar{u}\|_{Z_\varepsilon}^2 \qquad \text{for } \bar{u} \in P^\varepsilon Z_\varepsilon\,,$$

where $A_\varepsilon^{1/2}\bar{u}$ means $(A_\varepsilon|_{P^\varepsilon Z_\varepsilon})^{1/2}\bar{u}$.

From the equation $(2.6\text{b})_\varepsilon$ and the estimate (2.7), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|A_\varepsilon^{1/2}\bar{u}(t)\|_{Z_\varepsilon}^2 &= \left(A_\varepsilon \bar{u}(t), \frac{d\bar{u}(t)}{dt}\right)_{Z_\varepsilon} \\ &= -\|A_\varepsilon \bar{u}(t)\|_{Z_\varepsilon}^2 + (A_\varepsilon \bar{u}(t), P^\varepsilon f(Y\cdot \Psi_\varepsilon + \bar{u}))_{Z_\varepsilon} \\ &\le -\|A_\varepsilon \bar{u}(t)\|_{Z_\varepsilon}^2 + \|A_\varepsilon \bar{u}(t)\|_{Z_\varepsilon}\cdot \|P^\varepsilon f(Y\cdot \Psi_\varepsilon + \bar{u})\|_{Z_\varepsilon} \\ &\le -\tfrac{1}{2}\|A_\varepsilon \bar{u}(t)\|_{Z_\varepsilon}^2 + \tfrac{1}{2}\|P^\varepsilon f(Y\cdot \Psi_\varepsilon + \bar{u})\|_{Z_\varepsilon}^2 \\ &\le -(\gamma_3 d_*/2\varepsilon^\theta)\|A_\varepsilon^{1/2}\bar{u}(t)\|_{Z_\varepsilon}^2 + \tfrac{1}{2}\|P^\varepsilon f(Y\cdot \Psi_\varepsilon + \bar{u})\|_{Z_\varepsilon}^2\,, \end{aligned}$$

so that

$$\begin{aligned} &\frac{d}{dt}\|A_\varepsilon^{1/2}\bar{u}(t)\|_{Z_\varepsilon}^2 + (\gamma_3 d_*/\varepsilon^\theta)\|A_\varepsilon^{1/2}\bar{u}(t)\|_{Z_\varepsilon}^2 \le \|P^\varepsilon f(Y\cdot \Psi_\varepsilon + \bar{u})\|_{Z_\varepsilon}^2 \\ &\le \left\{\|P^\varepsilon f(Y\cdot \Psi_\varepsilon)\|_{Z_\varepsilon} + \int_0^1 \|P^\varepsilon f_u(Y\cdot \Psi_\varepsilon + \tau\bar{u})\bar{u}\|_{Z_\varepsilon}\, d\tau\right\}^2 \\ &\le 2\{\|P^\varepsilon f(Y\cdot \Psi_\varepsilon)\|_{Z_\varepsilon}^2 + (\rho^2\varepsilon^\theta/\gamma_3 d_*)\|A_\varepsilon^{1/2}\bar{u}(t)\|_{Z_\varepsilon}^2\}\,. \end{aligned}$$

So it follows from Proposition 2.1 and Lemma 2.4 that

$$\begin{aligned}\|P^{\varepsilon}f(Y\cdot\Psi_{\varepsilon})\|_{Z_{\varepsilon}} &\leq M_1\varepsilon^{1/2}\|f(Y\cdot\Psi_{\varepsilon})\|_{Z_{\varepsilon}}+2\|f(Y\cdot\Psi_{\varepsilon})-f(Y\cdot\Psi_0)\|_{Z_0}\\ &\quad+\|f(Y\cdot\Psi_{\varepsilon})\|_{(L^2(R_{\varepsilon}))^m}\\ &\leq c'\varepsilon^{1/2}\end{aligned}$$

for some positive constant $c'$ independent of $\varepsilon$, where $\Psi_0=(\omega_0^{(1)},\omega_0^{(2)})$ and $Y\cdot\Psi_0=y_1\omega_0^{(1)}+y_2\omega_0^{(2)}$ for $Y=(y_1,y_2)\in\boldsymbol{R}^{2m}$. Thus we have

$$\frac{d}{dt}\|A_{\varepsilon}^{1/2}\bar{u}(t)\|_{Z_{\varepsilon}}^2+(\gamma_3 d_*/\varepsilon^{\theta}-2\rho^2\varepsilon^{\theta}/\gamma_3 d_*)\|A_{\varepsilon}^{1/2}\bar{u}(t)\|_{Z_{\varepsilon}}^2\leq 2c'^2\varepsilon$$

and therefore

$$\begin{aligned}\|A_{\varepsilon}^{1/2}\bar{u}(t)\|_{Z_{\varepsilon}}^2 &\leq \exp\{-(\gamma_3 d_*/\varepsilon^{\theta}-2\rho^2\varepsilon^{\theta}/\gamma_3 d_*)t\}\|A_{\varepsilon}^{1/2}\bar{u}_0\|_{Z_{\varepsilon}}^2+\bar{c}\varepsilon\\ &\leq c'\varepsilon\end{aligned}\tag{2.8}$$

for some positive constant $\bar{c}$ and $c'$ and any $t\geq t_0$ where $t_0=\sup_{\varepsilon\leq\varepsilon_1}\left\{-\dfrac{\theta+1}{\delta_{\varepsilon}}\cdot\log\varepsilon\right\}$, because $\delta_{\varepsilon}=\gamma_3 d_*/\varepsilon^{\theta}-2\rho^2\varepsilon^{\theta}/\gamma_3 d_*>0$ for sufficiently small $\varepsilon\in(0,\varepsilon_1)$ and

$$\|A_{\varepsilon}^{1/2}\bar{u}\|_{Z_{\varepsilon}}^2\leq\frac{d^*}{\varepsilon^{\theta}}\|\nabla\bar{u}\|_{Z_{\varepsilon}}^2,\tag{2.9}$$

where $d^*=\max\{d_1,\dots,d_m\}$. On the other hand,

$$\|A_{\varepsilon}^{1/2}\bar{u}(t)\|_{Z_{\varepsilon}}^2=(A_{\varepsilon}\bar{u}(t),\bar{u}(t))_{Z_{\varepsilon}}\geq\frac{d_*}{\varepsilon^{\theta}}(-\Delta\bar{u}(t),\bar{u}(t))_{Z_{\varepsilon}}=\frac{d_*}{\varepsilon^{\theta}}\|\nabla\bar{u}(t)\|_{Z_{\varepsilon}}^2.\tag{2.10}$$

So we have from (2.7) and (2.10)

$$\|\bar{u}(t)\|_{Z_{\varepsilon}^1}\leq c_1\varepsilon^{\theta/2}\|A_{\varepsilon}^{1/2}\bar{u}(t)\|_{Z_{\varepsilon}}\tag{2.11}$$

for some $c_1>0$. The proof of this Lemma is complete with help of (2.8) and (2.11). Q.E.D.

## §3. Existence and attractivity of an invariant manifold

In this section, by using the standard centre manifold theory (for instance, see J. Carr [1]) we construct a global invariant manifold. The main argument owes to Morita [20].

We define a function set $V^{\varepsilon}$:

$$\begin{aligned}V^{\varepsilon}=\{h\in\mathbf{C}(\boldsymbol{R}^{2m};P^{\varepsilon}Z_{\varepsilon}^1)\,|\,|||h|||_{\varepsilon,\infty}\leq\beta_0\varepsilon^{(2\theta+1)/2},\ |||h|||_{\varepsilon,L}\leq\beta_1\varepsilon^{(2\theta+1)/2},\\ h(Y)\equiv 0\text{ for }Y=(y_1,0)\in\boldsymbol{R}^{2m}\},\end{aligned}\tag{3.1}$$

where

$$|||h|||_{\varepsilon,\infty} = \sup_{Y \in R^{2m}} \|h(Y)\|_{Z_\varepsilon^1},$$

$$|||h|||_{\varepsilon,L} = \sup_{\substack{Y_1 \neq Y_2 \\ Y_1, Y_2 \in R^{2m}}} \frac{\|h(Y_1) - h(Y_2)\|_{Z_\varepsilon^1}}{|Y_1 - Y_2|}$$

and $\beta_0$, $\beta_1$ are positive constants determined later. It is easy to see that $V^\varepsilon$ is complete.

We define an operator $\mathscr{R}^\varepsilon$ on $V^\varepsilon$:

$$(3.2) \qquad (\mathscr{R}^\varepsilon h)(Y_0) = \int_{-\infty}^{0} \exp\{A_\varepsilon s\} P^\varepsilon f(Y \cdot \Psi_\varepsilon + h(Y))\, ds$$

for $h \in V^\varepsilon$ and $Y_0 \in \boldsymbol{R}^{2m}$, where $Y = Y(t; Y_0, h) = (y_1(t; Y_0, h), y_2(t; Y_0, h))$ is the solution of the following equation:

$$(3.3) \qquad \begin{cases} \dfrac{dy_1}{dt} = \langle f(Y \cdot \Psi_\varepsilon + h(Y)), \omega_\varepsilon^{(1)} \rangle_{Z_\varepsilon} \\ \dfrac{dy_2}{dt} = -\dfrac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} D y_2 + \langle f(Y \cdot \Psi_\varepsilon + h(Y)), \omega_\varepsilon^{(2)} \rangle_{Z_\varepsilon} \\ Y(0) = Y_0 = (y_1^0, y_2^0) \end{cases}$$

If $h_\varepsilon^*$ is a fixed point of $\mathscr{R}^\varepsilon$, it is obvious that $Y^*(t) \cdot \Psi_\varepsilon + h_\varepsilon^*(Y^*(t))$ is a solution of $(2.5)_\varepsilon$, where $Y^*(t) = Y(t; Y_0, h^*)$ for any $Y_0 \in \boldsymbol{R}^{2m}$. That is, $\mathscr{M}(h_\varepsilon^*) \equiv \{u = Y \cdot \Psi_\varepsilon + h_\varepsilon^*(Y) \mid Y \in \boldsymbol{R}^{2m}\}$ gives a global invariant manifold of $(2.6)_\varepsilon$.

The following two propositions can be easily proved by Proposition 2.1 and Gronwall's inequality.

PROPOSITION 3.1. *There exists $c_0 > 0$ such that*

$$\|Y \cdot \Psi_\varepsilon - Y \cdot \Psi_0\|_{Z_0} \leq c_0 \varepsilon^{1/2} |Y|$$

*and*

$$\|Y \cdot \Psi_\varepsilon\|_{(L^2(R_\varepsilon))^m} \leq c_0 \varepsilon^{1/2} |Y| \qquad \text{for } Y \in \boldsymbol{R}^{2m}.$$

PROPOSITION 3.2. (1) *Let $Y(t) = Y(t; Y_0, h)$ and $\bar{Y}(t) = Y(t; \bar{Y}_0, h)$ for $Y_0, \bar{Y}_0 \in \boldsymbol{R}^{2m}$. Then*

$$|Y(t) - \bar{Y}(t)| \leq |Y_0 - \bar{Y}_0| \exp\left\{-\left(\rho + \frac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} d^* + \beta_1 \rho \varepsilon^{(2\theta+1)/2}\right) t\right\}$$

*for $t \leq 0$.*

(2) *Let* $Y^{(i)}(t) = Y(t; Y_0, h^{(i)})$ *for any* $Y_0 \in \boldsymbol{R}^{2m}$ *and* $h^{(i)} \in V^\varepsilon$ $(i = 1, 2)$. *Then*

$$|Y^{(1)}(t) - Y^{(2)}(t)| \le \rho |||h^{(1)} - h^{(2)}|||_{\varepsilon,\infty} \exp\left\{-\left(\rho + \frac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} d^* + \beta_1 \rho \varepsilon^{(2\theta+1)/2}\right)t\right\}$$

*for* $t \le 0$.

LEMMA 3.3. *There exist* $\beta_0$, $\beta_1 > 0$, *and* $\varepsilon_2 > 0$ $(\varepsilon_2 < \varepsilon_1)$ *such that for any* $\varepsilon \in (0, \varepsilon_2)$, $\mathscr{R}^\varepsilon$ *is a contraction on* $V^\varepsilon$ *with respect to* $|||\cdot|||_{\varepsilon,\infty}$.

PROOF. We first show that $\mathscr{R}^\varepsilon$ maps $V^\varepsilon$ into itself for appropriate $\beta_0$, $\beta_1$. For any $h \in V^\varepsilon$ it follows from Lemma 2.4 that

$$\|(\mathscr{R}^\varepsilon h)(Y_0)\|_{Z_\varepsilon^1} \le \int_{-\infty}^0 \alpha_0 \varepsilon^{\theta/2}(-s)^{-1/2} \exp\left\{\left(\frac{\gamma_3}{\varepsilon^\theta}\right) d_* s\right\} \|P^\varepsilon f(Y\cdot\Psi_\varepsilon + h(Y))\|_{Z_\varepsilon} ds$$

and

$$\begin{aligned}
\|P^\varepsilon f(Y\cdot\Psi_\varepsilon + h(Y))\|_{Z_\varepsilon} &= \|P^\varepsilon f(Y\cdot\Psi_\varepsilon) + \int_0^1 P^\varepsilon f_u(Y\cdot\Psi_\varepsilon + \tau h(Y)) \circ h(Y)\, d\tau\|_{Z_\varepsilon} \\
&\le \|P^\varepsilon f(Y\cdot\Psi_\varepsilon)\|_{Z_\varepsilon} + \rho\beta_0\varepsilon^{(2\theta+1)} \\
&\le M_1 \varepsilon^{1/2} \|f(Y\cdot\Psi_\varepsilon)\|_{Z_\varepsilon} + 2\|f(Y\cdot\Psi_\varepsilon) - f(Y\cdot\Psi_0)\|_{Z_0} \\
&\quad + \|f(Y\cdot\Psi_\varepsilon)\|_{(L^2(R_\varepsilon))^m} + \rho\beta_0\varepsilon^{(2\theta+1)/2} \\
&\le (c_1 + \rho\beta_0\varepsilon^\theta)\varepsilon^{1/2}
\end{aligned}$$

for some positive constant $c_1 > 0$. So we have

$$\begin{aligned}
\|(\mathscr{R}^\varepsilon h)(Y_0)\|_{Z_\varepsilon^1} &\le \alpha_0(c_1 + \rho\beta_0\varepsilon^\theta)\varepsilon^{(\theta+1)/2}\left(\frac{\pi\varepsilon^\theta}{\gamma_3 d_*}\right)^{1/2} \\
&= \alpha_0(c_1 + \rho\beta_0\varepsilon^\theta)\left(\frac{\pi}{\gamma_3 d_*}\right)^{1/2} \varepsilon^{(2\theta+1)/2} \le \beta_0 \varepsilon^{(2\theta+1)/2},
\end{aligned}$$

where $\varepsilon_2$ and $\beta_0$ are constants satisfying

$$\left\{\begin{aligned}
&1 - \varepsilon_2^\theta \alpha_0 \rho \left(\frac{\pi}{\gamma_3 d_*}\right)^{1/2} > 0, \\
&\beta_0 \ge \frac{\alpha_0 c_1 (\pi/\gamma_3 d_*)^{1/2}}{1 - \varepsilon_2^\theta \alpha_0 \rho \left(\dfrac{\pi}{\gamma_3 d_*}\right)^{1/2}}.
\end{aligned}\right. \tag{3.4}$$

Similarly, defining $Y(t) = Y(t; Y_0, h)$ and $\bar{Y}(t) = Y(t; \bar{Y}_0, h)$ for different initial values $Y_0$ and $\bar{Y}_0$, we obtain by Proposition 3.1 and Lemma 2.4

$$\begin{aligned}\|(\mathscr{R}^\varepsilon h)(Y_0) - (\mathscr{R}^\varepsilon h)(\bar{Y}_0)\|_{Z^1_\varepsilon} &\le \int_{-\infty}^0 \alpha_0 \varepsilon^{\theta/2}(-s)^{-1/2} \exp\{(\gamma_3/\varepsilon^\theta)\, d_* s\} \\ &\quad \times \|P^\varepsilon f(Y\cdot\Psi_\varepsilon + h(Y)) - P^\varepsilon f(\bar{Y}\cdot\Psi_\varepsilon + h(\bar{Y}))\|_{Z_\varepsilon}\, ds \\ &\le \int_{-\infty}^0 \alpha_0 \varepsilon^{\theta/2}(-s)^{-1/2} \exp\{(\gamma_3/\varepsilon^\theta)\, d_* s\} \\ &\quad \times \|P^\varepsilon\{f(Y\cdot\Psi_\varepsilon + h(Y)) - f(\bar{Y}\cdot\Psi_\varepsilon + h(Y))\|\}_{Z_\varepsilon} \\ &\quad + \rho\|h(Y) - h(\bar{Y})\|_{Z^1_\varepsilon}]\, ds\end{aligned}$$

and

$$\begin{aligned}&\|P^\varepsilon\{f(Y\cdot\Psi_\varepsilon + h(Y)) - f(\bar{Y}\cdot\Psi_\varepsilon + h(Y))\}\|_{Z_\varepsilon} \\ &= \left\| P^\varepsilon \int_0^1 f_u(Y\cdot\Psi_\varepsilon + h(Y) + \tau((\bar{Y} - Y)\cdot\Psi_\varepsilon) \circ ((\bar{Y} - Y)\cdot\Psi_\varepsilon)\, d\tau \right\|_{Z_\varepsilon} \\ &\le (M_1 + 2c_0)\rho\varepsilon^{1/2}|Y - \bar{Y}| + 2\left\| \int_0^1 \{f_u(Y\cdot\Psi_\varepsilon + h(Y) + \tau((\bar{Y} - Y)\cdot\Psi_\varepsilon) \right. \\ &\qquad \left. \circ ((\bar{Y} - Y)\cdot\Psi_\varepsilon) - f_u(Y\cdot\Psi_0 + \tau((\bar{Y} - Y)\cdot\Psi_0) \circ ((\bar{Y} - Y)\cdot\Psi_0)\}\, d\tau \right\|_{Z_0} \\ &\le (c_2 + 2\rho\beta_0\varepsilon^\theta)\varepsilon^{1/2}|Y - \bar{Y}|\end{aligned} \tag{3.5}$$

for some positive constant $c_2 > 0$. Then it follows that

$$\begin{aligned}&\|(\mathscr{R}^\varepsilon h)(Y_0) - (\mathscr{R}^\varepsilon h)(\bar{Y}_0)\|_{Z^1_\varepsilon} \\ &\le \int_{-\infty}^0 \alpha_0 \varepsilon^{\theta/2}(-s)^{-1/2} \exp\{(\gamma_3/\varepsilon^\theta)\, d_* s\}\cdot(c_2 + 2\rho\beta_0\varepsilon^\theta + \rho\beta_1\varepsilon^\theta)\varepsilon^{1/2}|Y - \bar{Y}|\, ds \\ &\le |Y^0 - \bar{Y}^0|\alpha_0(c_2 + 2\rho\beta_0\varepsilon^\theta + \rho\beta_1\varepsilon^\theta) \\ &\quad \times (\pi/[\gamma_3 d_* - (\rho\varepsilon^\theta + \lambda_\varepsilon^{(2)} d^* + \beta_1\rho\varepsilon^{(4\theta+1)/2})])^{1/2}\varepsilon^{(2\theta+1)/2}\end{aligned}$$

Let $\beta_0$, $\beta_1$ and $\varepsilon_2$ be the constants satisfying

$$\begin{cases} \bar{\gamma} = \gamma_3 d_* - (\rho\varepsilon_2^\theta + \gamma_1\varepsilon_2 d^* + \beta_1\rho\varepsilon_2^{(4\theta+1)/2}) > 0\,, \\ \beta_1 \ge \dfrac{\alpha_0(c_2 + 2\rho\beta_0\varepsilon_2^\theta)(\pi/\bar{\gamma})^{1/2}}{1 - \rho\varepsilon_2^\theta(\pi/\bar{\gamma})^{1/2}}\,. \end{cases} \tag{3.6}$$

Then we have

$$|||(\mathscr{R}^\varepsilon h)|||_{\varepsilon,L} \le \beta_1\varepsilon^{(2\theta+1)/2}\,.$$

Since $y_2(t;\, Y_0, h) \equiv 0$ for $Y_0 = (y_1^0, 0)$ holds from the uniqueness of solution of

(3.3), we have $(\mathscr{R}^\varepsilon h)(Y_0) \equiv 0$ for $Y_0 = (y_1^0, 0)$, which implies that $\mathscr{R}^\varepsilon$ maps $V^\varepsilon$ into itself.

In the rest we show that $\mathscr{R}^\varepsilon$ is a contraction mapping on $V^\varepsilon$. Defining $Y^{(i)}(t) = Y(t;\, Y_0, h^{(i)})$ for any $h^{(i)} \in V^\varepsilon$ $(i = 1, 2)$, we have

$$\begin{aligned}
&\|(\mathscr{R}^\varepsilon h^{(2)})(Y_0) - (\mathscr{R}^\varepsilon h^{(1)})(Y_0)\|_{Z_\varepsilon^1} \\
&\le \int_{-\infty}^{0} \alpha_0 \varepsilon^{\theta/2}(-s)^{-1/2} \exp\{(\gamma_3/\varepsilon^\theta)\, d_* s\} [\|P^\varepsilon\{f(Y^{(1)}(s)\cdot \Psi_\varepsilon + h^{(1)}(Y^{(1)}(s))) \\
&\qquad - f(Y^{(2)}(s)\cdot \Psi_\varepsilon + h^{(1)}(Y^{(1)}(s)))\}\|_{Z_\varepsilon} + \rho\, \|h^{(2)}(Y^{(2)}(s)) - h^{(1)}(Y^{(1)}(s))\|_{Z_\varepsilon^1}]\, ds
\end{aligned}$$

and similarly to (3.5),

$$\begin{aligned}
&\|P^\varepsilon\{f(Y^{(1)}(s)\cdot \Psi_\varepsilon + h^{(1)}(Y^{(1)}(s))) - f(Y^{(2)}(s)\cdot \Psi_\varepsilon + h^{(1)}(Y^{(1)}(s)))\}\|_{Z_\varepsilon} \\
&\quad \le (c_2 + 2\rho\beta_0\varepsilon^\theta)\varepsilon^{1/2}|\, Y^{(2)}(s) - Y^{(1)}(s)|\,.
\end{aligned}$$

So,

$$\begin{aligned}
&\|(\mathscr{R}^\varepsilon h^{(2)})(Y_0) - (\mathscr{R}^\varepsilon h^{(1)})(Y_0)\|_{Z_\varepsilon^1} \\
&\le \left[\alpha_0(c_2 + 2\rho\beta_0\varepsilon^\theta + \rho\beta_1\varepsilon^\theta)\left(\frac{\pi}{\bar{\gamma}}\right)^{1/2} \varepsilon^{(2\theta+1)/2} + \alpha_0\rho\left(\frac{\pi}{\gamma_3 d_*}\right)^{1/2} \varepsilon^\theta\right] |||h^{(2)} - h^{(1)}|||_{\varepsilon,\infty}\,.
\end{aligned}$$

Taking $\beta_0$, $\beta_1$ and sufficiently small $\varepsilon_2$ such that

$$\kappa = \alpha_0(c_2 + 2\rho\beta_0\varepsilon_2^\theta + \rho\beta_1\varepsilon_2^\theta)\left(\frac{\pi}{\bar{\gamma}}\right)^{1/2} \varepsilon_2^{(2\theta+1)/2} + \alpha_0\rho\left(\frac{\pi}{\gamma_3 d_*}\right)^{1/2} \varepsilon_2^\theta < 1\,, \tag{3.7}$$

we have

$$|||(\mathscr{R}^\varepsilon h^{(2)}) - (\mathscr{R}^\varepsilon h^{(1)})|||_{\varepsilon,\infty} \le \kappa\, |||h_\varepsilon^{(2)} - h_\varepsilon^{(1)}|||_{\varepsilon,\infty}$$

for any $\varepsilon \in (0, \varepsilon_2]$, as required. Q.E.D.

From this Lemma, we know that $\mathscr{R}^\varepsilon$ has an unique fixed point on $V^\varepsilon$ and we express it by $h_\varepsilon^*$.

THEOREM 3.4. *There exists $\varepsilon_2 > 0$ such that for any $\varepsilon \in (0, \varepsilon_2)$, there exists a 2m-dimensional Lipschitz continuous manifold represented by $\mathscr{M}_\varepsilon = \{Y\cdot \Psi_\varepsilon + h_\varepsilon^*(Y) \,|\, Y \in \boldsymbol{R}^{2m}\} \cap U$, which is invariant under the semiflow $S(t)u_0 = u(t; u_0)$, where $u(t; u_0)$ is the solution of $(2.2)_\varepsilon$ with $u_0 \in U$. $h_\varepsilon^*$ satisfies $|||h_\varepsilon^*|||_{\varepsilon,\infty} = O(\varepsilon^{(2\theta+1)/2})$, $|||h_\varepsilon^*|||_{\varepsilon,L} = O(\varepsilon^{(2\theta+1)/2})$ and $h_\varepsilon^*(y_1, 0) \equiv 0$. Also, there exist $\varepsilon_3 > 0$ $(\varepsilon_3 < \varepsilon_2)$ and $N > 0$ such that for any $\varepsilon \in (0, \varepsilon_3)$, there is a $\nu = \nu(\varepsilon) > 0$ so that for any solution $u(t) \in U$ of $(2.2)_\varepsilon$, there exists $\tilde{Y}_0 \in \boldsymbol{R}^{2m}$ satisfying*

$$\|u(t) - (\tilde{Y}(t)\cdot \Psi_\varepsilon + h^*_\varepsilon(\tilde{Y}(t)))\|_{Z^1_\varepsilon} \le Ne^{-\nu t} \qquad \textit{for } t \ge 0\,,$$

*where* $\tilde{Y}(t) = Y(t;\, \tilde{Y}_0, h^*)$.

PROOF. By Lemma 3.3, the proof of the former half is obvious. We now prove the attractivity of the manifold. Writing $u(t) = Y(t)\cdot \Psi_\varepsilon + \bar{u}(t) = y_1(t)\omega_\varepsilon^{(1)} + y_2(t)\omega_\varepsilon^{(2)} + \bar{u}(t)$, we see

$$(3.8)\qquad \begin{cases} \dfrac{dy_1}{dt} = \langle f(Y\cdot \Psi_\varepsilon + \bar{u}),\, \omega_\varepsilon^{(1)}\rangle_{Z_\varepsilon}\,, \\[2mm] \dfrac{dy_2}{dt} = -\dfrac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} D y_2 + \langle f(Y\cdot \Psi_\varepsilon + \bar{u}),\, \omega_\varepsilon^{(2)}\rangle_{Z_\varepsilon}\,, \\[2mm] Y_0 = Y(0)\,, \\[2mm] \bar{u}(t) = \exp\{-A_\varepsilon t\}\bar{u}_0 + \displaystyle\int_0^t \exp\{-A_\varepsilon(t-s)\} P^\varepsilon f(Y\cdot \Psi_\varepsilon + \bar{u})\, ds\,. \end{cases}$$

Here, we note that the solution of $(2.2)_\varepsilon$ on the manifold $\mathcal{M}_\varepsilon$ is represented by $\tilde{Y}(t)\cdot \Psi_\varepsilon + h^*(\tilde{Y}(t))$, where $\tilde{Y}(t) = Y(t;\, \tilde{Y}_0, h^*)$ for $\tilde{Y}_0 \in \boldsymbol{R}^{2m}$. Defining $\tilde{Y}(t) - Y(t)$ and $h^*_\varepsilon(\zeta + Y) - \bar{u}(t)$ by $\zeta = (\zeta_1, \zeta_2) = \tilde{Y}(t) - Y(t)$ and $H^*(t;\, \zeta)$, respectively, we have

$$(3.9)\qquad \begin{cases} \dfrac{d\zeta_1}{dt} = \langle f((\zeta + Y)\cdot \Psi_\varepsilon + h^*_\varepsilon(\zeta + Y)) - f(Y\cdot \Psi_\varepsilon + \bar{u}),\, \omega_\varepsilon^{(1)}\rangle_{Z_\varepsilon}\,, \\[2mm] \dfrac{d\zeta_2}{dt} = -\dfrac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} D\zeta_2 + \langle f((\zeta + Y)\cdot \Psi_\varepsilon + h^*_\varepsilon(\zeta + Y)) - f(Y\cdot \Psi_\varepsilon + \bar{u}),\, \omega_\varepsilon^{(2)}\rangle_{Z_\varepsilon}\,, \\[2mm] \zeta(0) = \tilde{Y}_0 - Y_0\,, \end{cases}$$

and

$$(3.10)\qquad \begin{aligned} H^*(t;\, \zeta) &= \exp\{-A_\varepsilon t\} H^*(0;\, \tilde{Y}_0 - Y_0) + \int_0^t \exp\{-A_\varepsilon(t-s)\} \\ &\quad \times P^\varepsilon\{f((\zeta + Y)\cdot \Psi_\varepsilon + h^*_\varepsilon(\zeta + Y)) - f(Y\cdot \Psi_\varepsilon + \bar{u})\}\, ds\,. \end{aligned}$$

We consider the existence of $\zeta(t)$ instead of $\tilde{Y}(t)$. By Lemma 2.5, we can assume $\|\bar{u}_0\|_{Z^1_\varepsilon} \le c\varepsilon^{(\theta+1)/2}$. We shall show that there exists a solution $\zeta(t)$ of (3.9) such that $|\zeta(t)| \le N_1 e^{-\nu t}$ for some positive constants $N_1$ and $\nu$.

We define a set $\Phi_q^\varepsilon$ by

$$\Phi_q^\varepsilon = \{\zeta = (\zeta_1, \zeta_2) \in \mathbf{C}([0, \infty);\, \boldsymbol{R}^{2m}) \mid \|\zeta\|_{\nu,\infty} = \sup_{t\ge 0}(|\zeta(t)|e^{\nu t}) \le q\varepsilon^{(\theta+1)/2}\}\,,$$

where $\nu$ and $q$ are constants to be determined later. We also define an operator $\varphi^\varepsilon$ on $\Phi_q^\varepsilon$ such that $\varphi^\varepsilon(\zeta)(t) = ((\varphi_1^\varepsilon\zeta)(t), (\varphi_2^\varepsilon\zeta)(t)) \in \mathbf{C}([0, \infty);\, \boldsymbol{R}^{2m})$ for

$\zeta(t) = (\zeta_1(t), \zeta_2(t)) \in \Phi_q^\varepsilon$, where

$$\text{(3.11)} \quad \begin{cases} (\varphi_1^\varepsilon\zeta)(t) = -\displaystyle\int_t^\infty \langle f((\zeta + Y)\cdot \Psi_\varepsilon + h^*(\zeta + Y)) - f(Y\cdot\Psi_\varepsilon + \bar{u}), \omega_\varepsilon^{(1)}\rangle_{Z_\varepsilon}\, ds \\ (\varphi_2^\varepsilon\zeta)(t) = -\displaystyle\int_t^\infty \left\{ -\frac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} D\zeta_2 + \langle f((\zeta + Y)\cdot\Psi_\varepsilon \right. \\ \qquad \left. + h_\varepsilon^*(\zeta + Y)) - f(Y\cdot\Psi_\varepsilon + \bar{u}), \omega_\varepsilon^{(2)}\rangle_{Z_\varepsilon} \right\} ds\,. \end{cases}$$

It is obvious that the fixed point of $\varphi^\varepsilon$ is a solution of (3.9). So that it sufficies to show that $\varphi^\varepsilon$ is a contraction on $\Phi_q^\varepsilon$ for appropriate $q > 0$.

Similarly to the procedure of (3.5), we have

$$\text{(3.12)} \quad \begin{aligned} \|H^*(t;\zeta)\|_{Z_\varepsilon^1} \le{}& (\beta_0\varepsilon^{(2\theta+1)/2} + c\varepsilon^{(\theta+1)/2}) \exp\left\{-\left(\frac{\gamma_3}{\varepsilon^\theta}\right) d_* t\right\} \\ &+ \int_0^t \alpha_0\varepsilon^{\theta/2}(t-s)^{-1/2} \exp\left\{-\left(\frac{\gamma_3}{\varepsilon^\theta}\right) d_*(t-s)\cdot[c_3\varepsilon^{1/2}|\zeta| \right. \\ &+ \rho\|H^*(s;\zeta)\|_{Z_\varepsilon^1}]\, ds \end{aligned}$$

for some positive constant $c_3 > 0$. Choosing $\nu$ and $\varepsilon_3$ satisfying

$$\text{(3.13)} \quad \nu < \frac{\gamma_3 d_*}{\varepsilon^\theta}$$

for $0 < \varepsilon \le \varepsilon_3$, we have

$$\begin{aligned} e^{\nu t}\|H^*(t;\zeta)\|_{Z_\varepsilon^1} \le{}& (\beta_0\varepsilon^{\theta/2} + c)\varepsilon^{(\theta+1)/2} + \alpha_0 c_3 q(\pi/[\gamma_3 d_* - \nu\varepsilon^\theta])^{1/2}\varepsilon^{(3\theta+2)/2} \\ &+ \alpha_0\rho\varepsilon^{\theta/2}\int_0^t (t-s)^{-1/2}\exp\left\{-\left(\frac{\gamma_3 d_*}{\varepsilon^\theta} - \nu\right)(t-s)\right\} \\ &\times e^{\nu s}\|H^*(s;\zeta\|_{Z_\varepsilon^1}\, ds\,. \end{aligned}$$

Let $\chi = \sup_{t\ge 0}\{\|H^*(t;\zeta)\|_{Z_\varepsilon^1} e^{\nu t}\}$. If $\varepsilon_3$ satisfies

$$\text{(3.14)} \quad 1 - \alpha_0\rho(\pi/[\gamma_3 d_* - \nu\varepsilon_3^\theta])^{1/2} > 0\,,$$

we have

$$\chi \le \chi_1(q)\varepsilon^{(\theta+1)/2}\,,$$

where

$$\chi_1(q) = \frac{[\beta_0\varepsilon^{\theta/2} + c + \alpha_0 c_3 q(\pi/[\gamma_3 d_* - \nu\varepsilon^\theta])^{1/2}\varepsilon^{(2\theta+1)/2}]\varepsilon^{(\theta+1)/2}}{1 - \alpha_0\rho(\pi/[\gamma_3 d_* - \nu\varepsilon^\theta])^{1/2}\varepsilon^\theta}\,,$$

which implies

$$\|H^*(t;\zeta)\|_{Z_\varepsilon^1} \le \chi_1(q)\varepsilon^{(\theta+1)/2}e^{-\nu t} \qquad \text{for } t \ge 0\,.$$

Let $\varepsilon_3$ be sufficiently small such that there exist constants $\nu = \nu(\varepsilon)$, $q = q(\varepsilon)$ satisfying (3.13) and

$$\text{(3.15)} \qquad \nu > \frac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} d^* + \rho\,, \qquad q > \frac{\rho\chi_1(q)}{\nu - (d^*\lambda_\varepsilon^{(2)}/\varepsilon^\theta + \rho)}$$

for any $\varepsilon \in (0, \varepsilon_3]$. In fact, we can take such constants $\nu(\varepsilon)$ and $q(\varepsilon)$ because $\lambda_\varepsilon^{(2)} \le \gamma_1\varepsilon$ and the coefficient of $q$ in the right side of the second inequality of (3.15) is less than 1 if $\varepsilon_3$ is sufficiently small. Thus, it follows that

$$\begin{aligned} |(\varphi^\varepsilon\zeta)(t)| &\le \int_t^\infty \left\{\left(d^*\frac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} + \rho\right)|\zeta| + \rho\|H^*(s;\zeta)\|_{Z_\varepsilon^1}\right\} ds \\ &\le \int_t^\infty \left[\left(d^*\frac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} + \rho\right) q\varepsilon^{(\theta+1)/2} + \rho\chi_1(q)\varepsilon^{(\theta+1)/2}\right] e^{-\nu s}\, ds \\ &= \frac{(d^*\lambda_\varepsilon^{(2)}/\varepsilon^\theta + \rho)q + \rho\chi_1(q)}{\nu}\varepsilon^{(\theta+1)/2}e^{-\nu t} \le q\varepsilon^{(\theta+1)/2}e^{-\nu t} \end{aligned}$$

for $\zeta \in \Phi_q^\varepsilon$. So we find that $\varphi^\varepsilon$ maps $\Phi_q^\varepsilon$ into itself.

We shall show that $\varphi^\varepsilon$ is contractive on $\Phi_q^\varepsilon$. Letting $\zeta^{(1)}$, $\zeta^{(2)} \in \Phi_q^\varepsilon$ and calculating (3.11) similarly to (3.5), we have

$$\begin{aligned} |(\varphi^\varepsilon\zeta^{(2)})(t) - (\varphi^\varepsilon\zeta^{(1)})(t)| &\le \int_t^\infty \left(\frac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} d^* + c_4\varepsilon^{(\theta+1)/2}\right)|\zeta^{(2)} - \zeta^{(1)}|\, ds \\ &\le \int_t^\infty \left(\frac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta} d^* + c_4\varepsilon^{(\theta+1)/2}\right) e^{-\nu s}\|\zeta^{(2)} - \zeta^{(1)}\|_{\nu,\infty}\, ds \end{aligned}$$

for some positive constant $c_4$, which implies

$$\|\varphi^\varepsilon\zeta^{(2)} - \varphi^\varepsilon\zeta^{(1)}\|_{\nu,\infty} \le \frac{d^*\lambda_\varepsilon^{(2)}/\varepsilon^\theta + c_4\varepsilon^{(\theta+1)/2}}{\nu}\|\zeta^{(2)} - \zeta^{(1)}\|_{\nu,\infty}\,.$$

Therefore if $\varepsilon_3$ satisfies

$$\text{(3.16)} \qquad c_4\varepsilon_3^{(\theta+1)/2} < \rho\,,$$

$\varphi^\varepsilon$ is a contraction and the proof of Theorem 3.4 is complete. Q.E.D.

REMARK 3.1. By this Theorem, we know that $|\tilde{Y}(t)| \le K_1$ for $t \ge t_0$, where $t_0$ is determined by Lemma 2.5.

REMARK 3.2. We note that $h_\varepsilon^*$ is continuous in sufficiently small $\varepsilon > 0$, weakly in $Z_\varepsilon^1$ and strongly in $Z_\varepsilon$ in the sense of [10], by combining Lemma 3.5 and Corollary 3.9 in Hale and Vegas [10] and the smooth dependency of the fixed point of contraction mapping on parameters.

## §4. ODEs on the invariant manifold

In this section we rewrite the ordinary differential equation (3.3) on the manifold in another form and consider the dependency of $\theta$ on the asymptotic behavior of solutions of $(2.2)_\varepsilon$.

We have known that $\tilde{Y}(t) = Y(t; \tilde{Y}_0, h^*)$ for $\tilde{Y}_0 \in \boldsymbol{R}^{2m}$ is the solution of $(2.2)_\varepsilon$ on the invariant manifold. Let

$$v = \tilde{Y}J^{-1}\,, \tag{4.1}$$

where

$$J = |\Omega_\varepsilon|^{1/2}\begin{pmatrix} 1-\alpha & -(\alpha(1-\alpha))^{1/2} \\ \alpha & (\alpha(1-\alpha))^{1/2} \end{pmatrix}.$$

Then we find that $v = (v_1, v_2)$ $(v_i \in \boldsymbol{R}^m)$ satisfies the following equation:

$$\begin{cases} \dfrac{dv_1}{dt} = \dfrac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta}\alpha D(v_2 - v_1) + \left\langle f(v_1\phi_\varepsilon^{(1)} + v_2\phi_\varepsilon^{(2)} + \bar{h}_\varepsilon^*(v)), \dfrac{\phi_\varepsilon^{(1)}}{(1-\alpha)|\Omega_\varepsilon|}\right\rangle_{Z_\varepsilon}, \\ \dfrac{dv_2}{dt} = \dfrac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta}(1-\alpha)D(v_1 - v_2) + \left\langle f(v_1\phi_\varepsilon^{(1)} + v_2\phi_\varepsilon^{(2)} + \bar{h}_\varepsilon^*(v)), \dfrac{\phi_\varepsilon^{(2)}}{\alpha|\Omega_\varepsilon|}\right\rangle_{Z_\varepsilon}, \end{cases} \tag{4.2}$$

where

$$\begin{cases} \phi_\varepsilon^{(1)} = |\Omega_\varepsilon|^{1/2}((1-\alpha)\omega_\varepsilon^{(1)} - (\alpha(1-\alpha))^{1/2}\omega_\varepsilon^{(2)})\,, \\ \phi_\varepsilon^{(2)} = |\Omega_\varepsilon|^{1/2}(\alpha\omega_\varepsilon^{(1)} + (\alpha(1-\alpha))^{1/2}\omega_\varepsilon^{(2)})\,, \end{cases} \tag{4.3}$$

and $\bar{h}_\varepsilon^*(v) = h_\varepsilon^*(|\Omega_\varepsilon|^{1/2}((1-\alpha)v_1 + \alpha v_2),\ |\Omega_\varepsilon|^{1/2}(\alpha(1-\alpha))^{1/2}(-v_1 + v_2))$. Here we note that

$$\phi_0^{(1)} = \begin{cases} 1 & \text{in } \Omega_0^L \\ 0 & \text{in } \Omega_0^R \end{cases}, \qquad \phi_0^{(2)} = \begin{cases} 0 & \text{in } \Omega_0^L \\ 1 & \text{in } \Omega_0^R \end{cases} \tag{4.4}$$

and $\bar{h}^*(v) \equiv 0$ when $v_1 = v_2$. Therefore we obtain

$$f(v_1) = \left\langle f(v_1\phi_0^{(1)} + v_2\phi_0^{(2)}), \frac{\phi_0^{(1)}}{(1-\alpha)|\Omega_0|}\right\rangle_{Z_0},$$

$$f(v_2) = \left\langle f(v_1\phi_0^{(1)} + v_2\phi_0^{(2)}), \frac{\phi_0^{(2)}}{\alpha|\Omega_0|}\right\rangle_{Z_0}.$$

Let

$$G_1^\varepsilon(v) = \left\langle f(v_1\phi_\varepsilon^{(1)} + v_2\phi_\varepsilon^{(2)} + \bar{h}_\varepsilon^*(v)), \frac{\phi_\varepsilon^{(1)}}{(1-\alpha)|\Omega_\varepsilon|}\right\rangle_{Z_\varepsilon} - f(v_1),$$

and

$$G_2^\varepsilon(v) = \left\langle f(v_1\phi_\varepsilon^{(1)} + v_2\phi_\varepsilon^{(2)} + \bar{h}_\varepsilon^*(v)), \frac{\phi_\varepsilon^{(2)}}{\alpha|\Omega_\varepsilon|}\right\rangle_{Z_\varepsilon} - f(v_2).$$

Here we define the following norms:

$$|G|_{K,\infty} = \sup\{|G(v)|;\, |v| \le K\},$$

$$|G|_{K,L} = \sup\left\{\frac{|G(v^{(1)}) - G(v^{(2)})|}{|v^{(1)} - v^{(2)}|};\, v^{(1)} \ne v^{(2)},\, v^{(j)} \in \boldsymbol{R}^{2m} \text{ and } |v^{(j)}| \le K\ (j = 1, 2)\right\}$$

for $G\colon \boldsymbol{R}^{2m} \to \boldsymbol{R}^m$. Then we have the following estimates by Proposition 2.1 and Lemma 2.4:

$$(4.5) \qquad |G_i^\varepsilon|_{K_2,\infty} = O(\varepsilon^{1/2}), \qquad |G_i^\varepsilon|_{K_2,L} = O(\varepsilon^{1/2}) \qquad (i = 1, 2),$$

where $K_2$ is the bound of $v$ ((4.1)) induced by the bound $K_1$ of $\tilde{Y}$. So (4.1) can be written as follows:

$$(4.6)_\varepsilon \qquad \begin{cases} \dfrac{dv_1}{dt} = f(v_1) + \dfrac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta}\alpha D(v_2 - v_1) + G_1^\varepsilon(v), \\[2ex] \dfrac{dv_2}{dt} = f(v_2) + \dfrac{\lambda_\varepsilon^{(2)}}{\varepsilon^\theta}(1-\alpha) D(v_1 - v_2) + G_2^\varepsilon(v). \end{cases}$$

First, we state the convergence of $\lambda_\varepsilon^{(2)}/\varepsilon$ as $\varepsilon \downarrow 0$, which is the key theorem in analysing $(4.6)_\varepsilon$. To do it, $R_\varepsilon$ is assumed to be $R_\varepsilon = \{(x_1, x_2) \,|\, |x_1| \le 1, |x_2| < \varepsilon\beta(x_1)\}$, where the function $\beta(x_1)$ is positive and even from $[-1, 1]$ into $\boldsymbol{R}$ with $\beta(-1) = \beta(1) > \beta(0) > 0$, and $\beta \in \mathbf{C}^\infty(-1, 1)$ increasing in $[0, 1]$ and $d^k\beta(x_1)/dx_1^k \to \infty$ as $x_1 \uparrow 1$ for $k \ge 1$.

THEOREM 4.1.

$$\lim_{\varepsilon\to 0} \frac{\lambda_\varepsilon^{(2)}}{\varepsilon} = \frac{\tau}{\alpha(1-\alpha)},$$

*where*

$$\tau = 2\left\{|\Omega_0| \int_{-1}^{1} \frac{1}{\beta(x_1)}\, dx_1\right\}^{-1}.$$

The proof will be based on the following inequality which is given by Hale and Vegas [10] under some hypothesis on $\Omega_0$.

LEMMA 4.2. *There exists a constant $c > 0$ independent of $\varepsilon$ such that*

$$\|u\|^2_{H^1(R_\varepsilon)} \le c(\|\Delta u\|^2_{L^2(R_\varepsilon)} + \|u\|_{H^1(\Lambda_\varepsilon)})$$

*for any $u \in \mathbf{C}^2(\Omega_\varepsilon)$ satisfying $\partial u/\partial n = 0$ on $\partial\Omega_\varepsilon$. Here, $\Lambda_\varepsilon = \Lambda_\varepsilon^L \cup \Lambda_\varepsilon^R$ and $\Lambda_\varepsilon^L \subset \Omega_0^L$ ($\Lambda_\varepsilon^R \subset \Omega_0^R$) is the symmetric region of $R_\varepsilon$ with respect to $x_1 = -1$ ($x_1 = 1$).* (See Figure 1.)

PROOF OF THEOREM 4.1. We will show that

$$\lim_{\varepsilon\to 0} \frac{\lambda_\varepsilon^{(2)}}{\varepsilon} = 2\int_{-1}^{1} \beta(x_1)\{\phi'(x_1)\}^2\, dx_1\,, \tag{4.7}$$

where $\phi(x_1)$ is the solution of the boundary value problem:

$$\begin{cases} (\beta\phi')' = 0\,, & |x_1| < 1\,, \\ \phi(-1) = \omega_0^L\,, \\ \phi(1) = \omega_0^R\,. \end{cases} \tag{4.8}$$

From the variational characterization, we know that

$$\lambda_\varepsilon^{(2)} = \inf\left\{ \frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla w|^2\, dx_1\, dx_2}{\displaystyle\int_{\Omega_\varepsilon} w^2\, dx_1\, dx_2} \;\middle|\; w \in H^1(\Omega_\varepsilon),\, w \ne 0,\, \int_{\Omega_\varepsilon} w\, dx_1\, dx_2 = 0 \right\}.$$

Defining $\tilde\psi(x_1, x_2)$ by

$$\tilde\psi(x_1, x_2) = \begin{cases} \phi(x_1) & \text{in } R_\varepsilon \\ \omega_0^L & \text{in } \Omega_0^L \\ \omega_0^R & \text{in } \Omega_0^R \end{cases}$$

and $\psi(x_1, x_2) = \tilde\psi(x_1, x_2) - \dfrac{1}{|\Omega_\varepsilon|}\displaystyle\int_{\Omega_\varepsilon} \tilde\psi(x_1, x_2)\, dx_1\, dx_2$, then we know that $\psi \in H^1(\Omega_\varepsilon)$ and $\displaystyle\int_{\Omega_\varepsilon} \psi\, dx_1\, dx_2 = 0$. So we have

$$\frac{\lambda_\varepsilon^{(2)}}{\varepsilon} \le \frac{1}{\varepsilon} \frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla\psi|^2\, dx_1\, dx_2}{\displaystyle\int_{\Omega_\varepsilon} \psi^2\, dx_1\, dx_2} = \frac{2\displaystyle\int_{-1}^{1} \beta(\phi')^2\, dx_1}{\left(1 + 2\varepsilon\displaystyle\int_{-1}^{1} \beta\phi^2\, dx_1 - \frac{1}{|\Omega_\varepsilon|}4\varepsilon^2\left\{\int_{-1}^{1} \beta\phi\, dx_1\right\}^2\right)}\,,$$

which implies that

$$\limsup_{\varepsilon\to 0} \frac{\lambda_\varepsilon^{(2)}}{\varepsilon} \le 2\int_{-1}^{1} \beta(\phi')^2\, dx_1\,. \tag{4.9}$$

It remains to show

$$\liminf_{\varepsilon\to 0}\frac{\lambda_\varepsilon^{(2)}}{\varepsilon}\geq 2\int_{-1}^{1}\beta(\phi')^2\,dx_1\,. \tag{4.10}$$

To this end, let $V_\varepsilon(x_1, z)\equiv\omega_\varepsilon^{(2)}(x_1,\varepsilon z)$, where $(x_1,z)\in R_1=\{(x_1,z)\,|\,|x_1|\leq 1, |z|<\beta(x_1)\}$. Then by Lemma 4.2 we have

$$\begin{aligned}\int_{R_1}\left[V_\varepsilon^2+\left(\frac{\partial V_\varepsilon}{\partial x_1}\right)^2+\frac{1}{\varepsilon^2}\left(\frac{\partial V_\varepsilon}{\partial z}\right)^2\right]dx_1\,dz\\ \leq c\lambda_\varepsilon^{(2)}\int_{R_1}V_\varepsilon^2\,dx_1\,dz\\ +\frac{c}{\varepsilon}\int_{\Lambda_\varepsilon}[(\omega_\varepsilon^{(2)})^2+|\nabla\omega_\varepsilon^{(2)}|^2]\,dx_1\,dx_2\,.\end{aligned} \tag{4.11}$$

In view of this together with Proposition 2.1 (3) we know that $\{V_\varepsilon\}$ is uniformly bounded in $H^1(R_1)$. So we can find a subsequence $\{\varepsilon_n\}$ with $\varepsilon_n\downarrow 0$, and a function $V_0\in H^1(R_1)$ such that $V_{\varepsilon_n}\to V_0$ weakly in $H^1(R_1)$ and strongly in $H^{3/4}(R_1)\hookrightarrow L^2(R_1)$ by Sobolev imbedding theorem, and

$$\liminf_{\varepsilon\to 0}\int_{R_1}\left(\frac{\partial V_\varepsilon}{\partial x_1}\right)^2dx_1\,dz=\lim_{n\to\infty}\int_{R_1}\left(\frac{\partial V_{\varepsilon_n}}{\partial x_1}\right)^2dx_1\,dz\,.$$

The estimate (4.11) implies that $V_0$ is independent of $z$. Let

$$X=\{w\in H^1(-1,1)\,|\,w(-1)=\omega_0^L,\ w(1)=\omega_0^R\}\,.$$

Now we show that $V_0\in X$. In fact, since $V_0\in H^1(R_1)$, we know immediately that $V_0\in H^1(-1,1)$. To find the boundary value of $V_0$, we recall Proposition 2.1 (3) which implies that

$$V_{\varepsilon_n}\to\begin{cases}\omega_0^L\,, & x_1=-1\\ \omega_0^R\,, & x_1=1\end{cases}$$

uniformly on $\partial R_1\cap\partial\Omega_0$. However from the continuity of the boundary trace mapping $H^{3/4}(R_1)\to H^{1/4}(\partial R_1)$, it follows that

$$V_{\varepsilon_n}|_{\partial R_1}\to V_0|_{\partial R_1}\qquad\text{in } H^{1/4}(\partial R_1)\,.$$

Thus, $V_0(-1)=\omega_0^L$ and $V_0(1)=\omega_0^R$, and this proves that $V_0\in X$. Observe that

$$\begin{aligned}\lambda_\varepsilon^{(2)}&=\int_{\Omega_\varepsilon}|\nabla\omega_\varepsilon^{(2)}|^2\,dx_1\,dx_2\geq\int_{R_\varepsilon}|\nabla\omega_\varepsilon^{(2)}|^2\,dx_1\,dx_2\\ &=\int_{R_1}\left[\varepsilon\left(\frac{\partial V_\varepsilon}{\partial x_1}\right)^2+\frac{1}{\varepsilon}\left(\frac{\partial V_\varepsilon}{\partial z}\right)^2\right]dx_1\,dz\geq\int_{R_1}\varepsilon\left(\frac{\partial V_\varepsilon}{\partial x_1}\right)^2dx_1\,dz\end{aligned}$$

and that $\phi$ is the minimizer of the functional $J: X \to \boldsymbol{R}$ defined by

$$J(w) = \int_{-1}^{1} \beta(x_1)(w'(x_1))^2\, dx_1 \qquad \text{for } w \in X\,.$$

It follows that

$$\begin{aligned}\liminf_{\varepsilon\to 0} \frac{\lambda_\varepsilon^{(2)}}{\varepsilon} &\geq \liminf_{\varepsilon\to 0} \int_{R_1} \left(\frac{\partial V_\varepsilon}{\partial x_1}\right)^2 dx_1\, dz \\ &= \lim_{n\to\infty} \int_{R_1} \left(\frac{\partial V_{\varepsilon_n}}{\partial x_1}\right)^2 dx_1\, dz \geq \int_{R_1} \left(\frac{\partial V_0}{\partial x_1}\right)^2 dx_1\, dz \\ &\geq 2\int_{-1}^{1} \beta(V_0')^2\, dx_1 \geq 2\int_{-1}^{1} \beta(\phi')^2\, dx_1\,.\end{aligned}$$

This gives (4.10), which along with (4.9) implies (4.7) and completes the proof of Theorem 4.1. Q.E.D.

Keeping Theorem 4.1 in mind, we define the reduced equations of $(4.6)_\varepsilon$ in the limit $\varepsilon \downarrow 0$:

(i) $(\theta > 1)$

$$(4.12)_1 \qquad \begin{cases} \dfrac{d\tilde{v}}{dt} = f(\tilde{v}) \\ v_1 = v_2 = \tilde{v}\,; \end{cases}$$

(ii) $(\theta = 1)$

$$(4.12)_2 \qquad \begin{cases} \dfrac{dv_1}{dt} = f(v_1) + \dfrac{\tau}{1-\alpha} D(v_2 - v_1) \\ \dfrac{dv_2}{dt} = f(v_2) + \dfrac{\tau}{\alpha} D(v_1 - v_2)\,; \end{cases}$$

(iii) $(0 < \theta < 1)$

$$(4.12)_3 \qquad \begin{cases} \dfrac{dv_1}{dt} = f(v_1) \\ \dfrac{dv_2}{dt} = f(v_2)\,. \end{cases}$$

We now discuss the relation between $(4.6)_\varepsilon$ with sufficiently small $\varepsilon$ and its reduced equations $(\varepsilon \downarrow 0)$, (4.12).

By using Theorem 4.1, the following result can be easily verified.

LEMMA 4.3. *Let $\theta > 1$. For any $\sigma > 0$ there exist $\varepsilon^* > 0$ and $\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$, there exists $\bar{v}(t)$ which satisfies*

$$|v_i(t) - \bar{v}(t)| \le \delta e^{-\sigma t} \qquad (i = 1, 2) \text{ for } t \ge 0\,,$$

*and $d\bar{v}/dt = f(\bar{v}) + g(t)$ with $|g(t)| \le c_1 \exp\{-\sigma_1 t\}$ for $t \ge 0$ and some $c_1, \sigma_1 > 0$, where $(v_1(t), v_2(t))$ is a bounded solution of $(4.6)_\varepsilon$.*

We thus find that the reduced equation $(4.12)_1$ plays a role of the limiting equation of $(4.6)_\varepsilon$ when $\theta > 1$.

Combining Theorem 3.4, (4.1) and Lemma 4.3, we get the following theorem corresponding to the result of Conway, Hoff and Smoller [5].

THEOREM 4.4. *Let $\theta > 1$. There exist $\varepsilon_4 > 0$ ($\varepsilon_4 < \varepsilon_3$) and $M > 0$ so that for any $\varepsilon \in (0, \varepsilon_4)$ there exists $\kappa = \kappa(\varepsilon) = O(\varepsilon^{1-\theta}) > 0$ such that if $u(t)$ is any solution of $(2.2)_\varepsilon$, then*

$$\|u(t) - \bar{v}(t)\|_{Z_0} \le M e^{-\kappa t}$$

*holds for $t \ge 0$ and some $\bar{v}(t)$ which satisfies $d\bar{v}/dt = f(\bar{v}) + g(t)$ with $|g(t)| \le M_1 \exp\{-\kappa_1 t\}$ for $t \ge 0$ and some $M_1$ and $\kappa_1 > 0$.*

The above theorem indicates that when $\theta > 1$ and $\varepsilon$ is sufficiently small, there are no stable spatially inhomogeneous solution of $(2.2)_\varepsilon$.

We next consider the cases when $0 < \theta < 1$ and $\theta = 1$. Since $(4.6)_\varepsilon$ is represented as

$$(4.13)_\varepsilon \qquad \begin{cases} \dfrac{dv_1}{dt} = f(v_1) + o(1) \\[2ex] \dfrac{dv_2}{dt} = f(v_2) + o(1) \end{cases}$$

for $0 < \theta < 1$ and

$$(4.14)_\varepsilon \qquad \begin{cases} \dfrac{dv_1}{dt} = f(v_1) + \dfrac{\tau}{1-\alpha} D(v_2 - v_1) + o(1) \\[2ex] \dfrac{dv_2}{dt} = f(v_2) + \dfrac{\tau}{\alpha} D(v_1 - v_2) + o(1) \end{cases}$$

for $\theta = 1$ as $\varepsilon \downarrow 0$, general theories of ODEs state that the orbits of $(4.12)_3$ and $(4.12)_2$ which approaches the asymptotically stable attractor is close to that of $(4.13)_\varepsilon$ and $(4.14)_\varepsilon$ uniformly in $t$, respectively. So, generically we can say that $(4.12)_3$ and $(4.12)_2$ is the limiting equations of $(4.13)_\varepsilon$ and $(4.14)_\varepsilon$ as $\varepsilon \downarrow 0$, respectively. Thus, we have arrived at the reduced equations of $(4.6)_\varepsilon$ in the limit $\varepsilon \downarrow 0$.

The next problem is the study of the transient as well as asymptotic behaviors of solutions of $(2.2)_\varepsilon$ by solving (4.12). The particularly interesting case is for $\theta = 1$, because it includes three parameters $\tau$, $\alpha$ and $D$. So, the dynamics of solutions of $(4.6)_\varepsilon$ or $(2.2)_\varepsilon$ generally depend on these parameters. From the global bifurcation view point, this will be discussed in a forthcoming paper [19].

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