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Discrete subgroups of convergence type of $U(1, n; C)$

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Introduction

Let C be the field of complex numbers. Let $V = V^{1,n}(C)$ $(n \geqq 1)$ denote the vector space C^{n+1}, together with the unitary structure defined by the Hermitian form

$$\Phi(z^*, w^*) = -\overline{z_0^*}w_0^* + \overline{z_1^*}w_1^* + \cdots + \overline{z_n^*}w_n^*$$

for $z^* = (z_0^*, z_1^*, \ldots, z_n^*)$ and $w^* = (w_0^*, w_1^*, \ldots, w_n^*)$ in V. An automorphism g of V, that is, a linear bijection such that $\Phi(g(z^*), g(w^*)) = \Phi(z^*, w^*)$ for z^*, $w^* \in V$, will be called a *unitary transformation.* We denote the group of all unitary transformations by $U(1, n; C)$. Let $V_0 = \{z^* \in V | \Phi(z^*, z^*) = 0\}$ and $V_- = \{z^* \in V | \Phi(z^*, z^*) < 0\}$. It is clear that V_0 and V_- are invariant under $U(1, n; C)$. Set $V^* = V_- \cup V_0 - \{0\}$. Let $\pi\colon V^* \to \pi(V^*)$ be the projection map defined by $\pi(z_0^*, z_1^*, \ldots, z_n^*) = (z_1^* z_0^{*-1}, z_2^* z_0^{*-1}, \ldots, z_n^* z_0^{*-1})$. Set $H^n(C) = \pi(V_-)$. Let $\overline{H^n(C)}$ denote the closure of $H^n(C)$ in the projective space $\pi(V^*)$. An element g of $U(1, n; C)$ operates in $\pi(V^*)$, leaving $\overline{H^n(C)}$ invariant. Since $H^n(C)$ is identified with the complex unit ball $B^n = B^n(C) = \{z = (z_1, z_2, \ldots, z_n) \in C^n | \|z\|^2 = \Sigma_{k=1}^n |z_k|^2 < 1\}$, we regard a unitary transformation as a transformation operating on B^n. Therefore discrete subgroups of $U(1, n; C)$ are considered to be generalizations of Fuchsian groups.

Our purpose in this paper is to extend results for Fuchsian groups to those for discrete subgroups of $U(1, n; C)$.

Our work is divided into four sections. In Section 1 we consider the Laplace-Beltrami equation. We show in Theorem 1.4 the relation between the type of a discrete subgroup of $U(1, n; C)$ and the existence of a certain automorphic function in B^n. Using this fact, we shall prove in Theorem 1.6 that if G is a discrete subgroup of convergence type, then $\Sigma_{g \in G}(1 - \|g(z)\|)^n$ is uniformly bounded in B^n. In Section 2 we shall discuss the properties of M-harmonic and of M-subharmonic functions. Section 3 is devoted to giving sufficient conditions for a discrete subgroup to be of convergence type. In Section 4 we define a point of approximation and show in Theorem 4.6 that if a

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discrete subgroup G is of convergence type, then the measure of the set of all points of approximation of G is equal to 0. The corresponding results for Fuchsian groups and discrete groups of Möbius transformations in higher dimensions can be found in [1] and [10].

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1. Discrete subgroups of convergence type

Throughout this paper G will always denote a discrete subgroup of $U(1, n; \boldsymbol{C})$. First we recall the definition of a discrete subgroup of convergence type.

DEFINITION 1.1. A discrete subgroup G of $U(1, n; \boldsymbol{C})$ is said to be of *convergence type* if $\Sigma_{g \in G}(1 - \|g(z)\|)^n$ converges for some point $z \in B^n$.

We note that this definition does not depend on the choice of z (see [6; Theorem 5.1]).

For later use we shall quote criteria for a discrete subgroup to be of convergence type from [6] and [7].

THEOREM 1.2 ([6; Theorem 5.3] and [7; Theorem 3.2]). *The following statements are equivalent to one another*:

(a) *G is of convergence type*;

(b) $\Sigma_{g_m \in G}|a_{11}^{(m)}|^{-2n} < \infty$, *where* $g_m = (a_{ij}^{(m)})_{i,j=1,2,\dots,n+1}$;

(c) $\int_0^1 (1 - t)^{n-1} n(t, z)\, dt < \infty$, *where $n(t, z)$ is the number of elements f in G such that $\|f(z)\| < t$ for $z \in B^n$.*

Now we consider the Laplace-Beltrami operator relative to the metric $g_{i\bar{j}}(z) = \delta_{ij}(1 - \|z\|^2)^{-1} + \bar{z}_i z_j (1 - \|z\|^2)^{-2}$ for $z = (z_1, z_2, \dots, z_n) \in B^n$. This operator is given by

$$\tilde{\Delta} = 2(1 - \|z\|^2)\left(\sum_j \frac{\partial^2}{\partial \bar{z}_j \partial z_j} - \sum_{j,k} \bar{z}_j z_k \frac{\partial^2}{\partial \bar{z}_j \partial z_k}\right).$$

We shall show that this operator commutes with the action of all elements in $U(1, n; \boldsymbol{C})$.

PROPOSITION 1.3. *Let* $\tilde{B}^n = \{(w_1, \overline{w_1}, \dots, w_n, \overline{w_n}) | \Sigma_{k=1}^n |w_k|^2 < 1\}$. *If* $u \in C^2(\tilde{B}^n)$, *then* $\tilde{\Delta}(u \circ f)(z) = \tilde{\Delta}(u(w))_{w=f(z)}$ *for any element f of $U(1, n; \boldsymbol{C})$.*

PROOF. We need to prove the following equation

$$(1-\|z\|^2)\{\textstyle\sum_j \bar{D}_j D_j(u\circ f)(z) - \sum_{j,k} \bar{z}_j z_k \bar{D}_j D_k(u\circ f)(z)\}$$
$$= (1-\|w\|^2)\{\textstyle\sum_j \bar{D}_j^* D_j^* u(w) - \sum_{j,k} \bar{w}_j w_k \bar{D}_j^* D_k^* u(w)\}\,,$$

where $D_i = \partial/\partial z_i$, $\bar{D}_i = \partial/\partial \bar{z}_i$, $D_i^* = \partial/\partial w_i$ and $\bar{D}_i^* = \partial/\partial \bar{w}_i$.

Note that

$$\bar{D}_j D_k(u\circ f) = \textstyle\sum_i \{\sum_h (\bar{D}_h^* D_i^* u)(\bar{D}_j \bar{f}_h)(D_k f_i)\}\,.$$

We consider the coefficients of $\bar{D}_h^* D_i^* u$ for $1 \leqq h,\ i \leqq n$. To prove our proposition we have only to show

$$(1-\|z\|^2)\{\textstyle\sum_j (\bar{D}_j \bar{f}_h)(D_j f_i) - \sum_{j,k} \bar{z}_j z_k (\bar{D}_j \bar{f}_h)(D_k f_i)\} = (1-\|w\|^2)(\delta_{hi} - \bar{w}_h w_i)\,.$$

Let $f = (a_{ij})_{i,j=1,2,\dots,n+1}$, $z^* = (1, z_1, z_2, \dots, z_n)$ and $w^* = f(z^*) = (w_0^*, w_1^*, \dots, w_{n+1}^*)$. Nothing that $\Phi(w^*, w^*) = \Phi(z^*, z^*)$, we have $1 - \|z\|^2 = |w_0^*|^2 - |w_1^*|^2 - \cdots - |w_n^*|^2 = |w_0^*|^2(1 - \|w\|^2)$. We see that

$$D_j f_h = w_0^{*-2}\{(D_j w_h^*) w_0^* - (D_j w_0^*) w_h^*\}\,,$$
$$D_j w_h^* = a_{h+1,j+1}\,,$$
$$D_j w_0^* = a_{1,j+1}\,.$$

Using these equalities, we obtain

$$\begin{aligned}\textstyle\sum_{j=1}^n z_j(D_j f_h) &= w_0^{*-2}\{(\textstyle\sum_{j=1}^n a_{h+1,j+1} z_j) w_0^* - (\sum_{j=1}^n a_{1,j+1} z_j) w_h^*\}\\ &= w_0^{*-2}\{(w_h^* - a_{h+1,1}) w_0^* - (w_0^* - a_{11}) w_h^*\}\\ &= w_0^{*-2}(-a_{h+1,1} w_0^* + a_{11} w_h^*)\,.\end{aligned}$$

It follows that

$$\begin{aligned}\textstyle\sum_{j,k} \bar{z}_j z_k (\bar{D}_j \bar{f}_h)(D_k f_i) &= \{\overline{\textstyle\sum_j z_j (D_j f_h)}\}\{\textstyle\sum_k z_k (D_k f_i)\}\\ &= |w_0^*|^{-4}(\overline{w_h^* a_{11} - w_0^* a_{h+1,1}})(w_i^* a_{11} - w_0^* a_{i+1,1})\,.\end{aligned}$$

To compute $\Sigma_{j=1}^n (\bar{D}_j \bar{f}_h)(D_j f_i)$ we use the relations $\Sigma_{j=1}^n \overline{a_{h+1,j+1}} a_{i+1,j+1} = \delta_{hi} + \overline{a_{h+1,1}} a_{i+1,1}$, $\Sigma_{j=1}^n |a_{1,j+1}|^2 = -1 + |a_{11}|^2$, $\Sigma_{j=1}^n \overline{a_{1,j+1}} a_{h+1,j+1} = \overline{a_{11}} a_{h+1,1}$ which follow from the fact $f \in U(1, n; \boldsymbol{C})$. We have

$$\begin{aligned}\textstyle\sum_{j=1}^n (\bar{D}_j \bar{f}_h)(D_j f_i) &= |w_0^*|^{-4}\{|w_0^*|^2 \textstyle\sum_{j=1}^n \overline{a_{h+1,j+1}} a_{i+1,j+1}\\ &\quad + \overline{w_h^*} w_i^* \textstyle\sum_{j=1}^n |a_{1,j+1}|^2 - \overline{w_0^*} w_i^* \sum_{j=1}^n \overline{a_{h+1,j+1}} a_{1,j+1}\\ &\quad - \overline{w_h^*} w_0^* \textstyle\sum_{j=1}^n \overline{a_{1,j+1}} a_{i+1,j+1}\}\\ &= |w_0^*|^{-4}\{|w_0^*|^2(\delta_{hi} + \overline{a_{h+1,1}} a_{i+1,1}) + \overline{w_h^*} w_i^*(-1 + |a_{11}|^2)\\ &\quad - \overline{w_0^*} w_i^* \overline{a_{h+1,1}} a_{11} - \overline{w_h^*} w_0^* \overline{a_{11}} a_{i+1,1}\}\,.\end{aligned}$$

Thus

$$
\begin{aligned}
&(1-\|z\|^2)\{\textstyle\sum_j (\bar{D}_j\bar{f}_h)(D_jf_i) - \sum_{j,k} \bar{z}_k z_j(\bar{D}_k\bar{f}_h)(D_jf_i)\} \\
&= (1-\|z\|^2)|w_0^*|^{-4}(|w_0^*|^2\delta_{hi} - \overline{w_h^*}w_i^*) \\
&= (1-\|w\|^2)(\delta_{hi} - \overline{w_h}w_i)\,.
\end{aligned}
$$

Our proposition is now proved.

Let $u(t)$ be a positive function of t, $0 < t < 1$, such that $\tilde{\Delta}u(\|u\|^2) = 0$, where $u(\|z\|^2)$ is regarded as a function of z. We shall determine u. After a little computation we obtain

$$(1-t)^2 tu''(t) + (1-t)(n-t)u'(t) = 0\,,$$

where $t = \|z\|^2$. If $u'(t) \neq 0$, then this differential equation can be written as

$$u''(t)/u'(t) + n/t + (n-1)/(1-t) = 0$$

or

$$\frac{d}{dt}[\log u'(t) + n\log t - (n-1)\log(1-t)] = 0\,,$$

which gives

$$u'(t)t^n(1-t)^{1-n} = K \text{ (constant)}\,.$$

As a normalized solution, we have

$$
\begin{aligned}
u(t) &= \int_t^1 (1-s)^{n-1}s^{-n}\,ds \\
&= \textstyle\sum_{k=1}^{n-1} (-1)^{n-k-1}k^{-1}(1-t)^k t^{-k} + (-1)^n \log t\,.
\end{aligned}
$$

We shall show the relation between a subgroup of convergence type and the function u.

THEOREM 1.4. *Let u be the function defined as above and let $\{g_0, g_1, \dots\}$ be the complete list of elements of G. Then the following statements* (a) *and* (b) *are equivalent to each other*:

(a) *G is of convergence type*;

(b) *$\Sigma_{g_m \in G} u(\|g_m(z)\|^2)$ converges at some point z in B^n.*

Furthermore, if (b) *is satisfied, then the series in* (b) *is uniformly convergent on every compact subset in $B^n - \bigcup_{m \geq 0} g_m(0)$.*

PROOF. Since

$$\int_t^1 (1-s)^{n-1}\,ds \leqq \int_t^1 (1-s)^{n-1}s^{-n}\,ds$$
$$\leqq \int_t^1 (1-s)^{n-1}t^{-n}\,ds \quad \text{for} \quad t<s<1\,,$$

we have

$$(1/n)(1-\|g_m(z)\|^2)^n \leqq u(\|g_m(z)\|^2)$$
$$\leqq (1/n)\|g_m(z)\|^{-2n}(1-\|g_m(z)\|^2)^n\,.$$

From these inequalities it follows that (a) and (b) are equivalent to each other.

Now we denote the ball $\{\|z\|<r|0<r<1\}$ by D. Since G is discontinuous in B^n, there exist an integer N and a real number $r_1>r$ such that $\|g_m(z)\|>r_1$ for every $z\in D$ and $m>N$. Hence we have

$$(1/n)\|g_m(z)\|^{-2n}(1-\|g_m(z)\|^2)^n \leqq (1/n)r_1^{-2n}(1-\|g_m(z)\|^2)^n\,.$$

Thus the series in (b) is uniformly convergent on every compact subset of $B^n - \bigcup_{m\geqq 0} g_m(0)$.

REMARK 1.5. When G is of convergence type, we put

$$F(z) = \sum_{g_m\in G} u(\|g_m(z)\|^2)\,.$$

Then $F(g(z)) = F(z)$ for every element g in G.

Using the above theorem, we shall show that $\Sigma_{g\in G}(1-\|g(z)\|)^n$ is uniformly bounded in B^n.

THEOREM 1.6. *If G is of convergence type, then $\Sigma_{g\in G}(1-\|g(z)\|)^n \leqq K$ for $z\in B^n$, where K is a constant that does not depend on z.*

To prove Theorem 1.6, we recall

LEMMA 1.7 (cf. [9; Theorem 4.3.2]). *Suppose Ω is an open subset of B^n. Let u be a real-valued continuous function in $\bar{\Omega}$. If $\tilde{\Delta}u=0$ in Ω and $u\leqq 0$ on $\partial\Omega$, then $u\leqq 0$ in Ω.*

From the proof of [6; Theorem 5.1] we obtain

LEMMA 1.8. *If g is an element of $U(1,n;\boldsymbol{C})$, then*

$$1-\|g(z)\| \leqq 4(1-\|z\|^2)^{-1}(1-\|g(0)\|)\,,$$
$$1-\|g(0)\| \leqq 4(1-\|z\|^2)^{-1}(1-\|g(z)\|)$$

for $z\in B^n$.

REMARK 1.9. The latter inequality will be used later.

PROOF OF THEOREM 1.6. Using [3; Proposition 3.2.2] and the fact that G is a countable set, we see that there is a point in B^n which is not fixed by any element of G except the identity. Therefore we can find an element $h=(a_{ij})_{i,j=1,2,\dots,n+1}$ in $U(1, n; C)$ such that the stabilizer $(hGh^{-1})_0$ of the origin 0 consists of only of the identity. Let z be a point in B^n and set $w=h(z)$. Then

$$\begin{aligned}
&\sum_{g\in G}(1-\|g(z)\|)^n \\
&\quad=\sum_{g\in G}(1-\|gh^{-1}(w)\|)^n \\
&\quad=\sum_{g\in G}(1-\|hgh^{-1}(w)\|)^n(1-\|hgh^{-1}(w)\|)^{-n}(1-\|gh^{-1}(w)\|)^n \\
&\quad\leqq\sum_{g\in G}(1-\|hgh^{-1}(w)\|)^n 2^n(1-\|hgh^{-1}(w)\|^2)^{-n}(1-\|gh^{-1}(w)\|^2)^n \\
&\quad=\sum_{g\in G}(1-\|hgh^{-1}(w)\|)^n 2^n|a_{11}+a_{12}Z_1+a_{13}Z_2+\cdots+a_{1,n+1}Z_n|^{2n},
\end{aligned}$$

where $gh^{-1}(w)=(Z_1, Z_2, Z_3,\dots, Z_n)\in B^n$. We note that $|a_{11}+a_{12}Z_1+a_{13}Z_2+\cdots+a_{1,n+1}Z_n|^{2n}$ is bounded in B^n. Hence, if $\Sigma_{g\in G}(1-\|hgh^{-1}(w)\|)^n$ is uniformly bounded in B^n, then so is $\Sigma_{g\in G}(1-\|g(z)\|)^n$. Thus we have only to prove our theorem in the case where the stabilizer $G_0=\{\text{identity}\}$.

We note that

$$\begin{aligned}
\sum_{\|g(z)\|<r}(1-\|g(z)\|)^n &= \int_0^r (1-t)^n\, dn(t, z) \\
&= (1-r)^n n(r, z)-n(0, z) \\
&\quad + n\int_0^r (1-t)^{n-1} n(t, z)\, dt \qquad \text{for } r\in(0, 1).
\end{aligned}$$

By [6, Proposition 4.1], $(1-r)^n n(r, z)$ is bounded. Therefore we need to prove only that $\int_0^r (1-t)^{n-1} n(t, z)\, dt\leqq M$ for any point z in B^n. Let $\{g_0, g_1,\dots\}$ be the complete list of elements of G. Since G is of convergence type, we can define the function $F(z)$ as in Remark 1.5. Set

$$F_i(z)=\sum_{m=0}^{i} u(\|g_m(z)\|^2),$$

where u is the function defined before Theorem 1.4. It is obvious that $F_i(z)\leqq F(z)$ for any point z in B^n.

We use $d(\cdot)$ for the distance which is induced from the metric $g_{i\bar{j}}$. Namely,

$$d(z, w)=\cosh^{-1}\left[|\Phi(z^*, w^*)|\{\Phi(z^*, z^*)\Phi(w^*, w^*)\}^{-1/2}\right],$$

where $z^*\in\pi^{-1}(z)$ and $w^*\in\pi^{-1}(w)$.

Let Ω be an open ball with center at 0 included in the Dirichlet polyhedron $D_0 = \{z \in B^n | d(z, 0) < d(z, g(0))$ for any element g in $G - \{\text{identity}\}\}$ (see [6; p. 181]). Let $g_m(\Omega)$ be denoted by Ω_m. It follows from Proposition 1.3 that the function $F_i(z)$ satisfies $\tilde{\Delta} F_i = 0$ in $B^n - \bigcup_{0 \leqq m \leqq i} \Omega_m$ and $F_i(z) = 0$ on the boundary of B^n. Using Lemma 1.7 and the invariance of $F(z)$ under G, we have

$$0 < F_i(z) \leqq \max_{\zeta \in \bigcup_{0 \leqq m \leqq i} \partial \Omega_m} F_i(\zeta) \leqq \max_{\zeta \in \bigcup_{0 \leqq m \leqq i} \partial \Omega_m} F(\zeta) = \max_{\zeta \in \partial \Omega} F(\zeta)$$

for $z \in B^n - \bigcup_{0 \leqq m \leqq i} \Omega_m$. Hence letting $i \to \infty$, we obtain

$$0 < F(z) \leqq \max_{\zeta \in \partial \Omega} F(\zeta) \qquad \text{for } z \in B^n - \bigcup_{m \geqq 0} \Omega_m .$$

Set $M_1 = \max_{\zeta \in \partial \Omega} F(\zeta)$. It follows that

$$\begin{aligned} \sum_{\|g(z)\| < r} u(\|g(z)\|^2) &= \int_0^r u(t^2)\, dn(t, z) \\ &= [u(t^2) n(t, z)]_0^r + 2 \int_0^r t^{1-2n} (1 - t^2)^{n-1} n(t, z)\, dt \\ &\geqq 2 \int_0^r (1 - t)^{n-1} n(t, z)\, dt . \end{aligned}$$

Therefore it is seen that

$$\int_0^r (1 - t)^{n-1} n(t, z)\, dt \leqq M_1/2 \qquad \text{for } z \in B^n - \bigcup_{m \geqq 0} \Omega_m .$$

Next let z be a point in Ω. Using Lemma 1.8, we have

$$\begin{aligned} n \int_0^r (1 - t)^{n-1} n(t, z)\, dt &\leqq \sum_{g \in G} (1 - \|g(z)\|)^n \\ &\leqq M_2 \sum_{g \in G} (1 - \|g(0)\|)^n \\ &\leqq M_3 , \end{aligned}$$

where M_2 depends only on the radius of Ω. Since the number $n(t, z)$ is invariant under G and $\Sigma_{g \in G} (1 - \|g(z)\|)^n = \Sigma_{g \in G} (1 - \|g(g_m^{-1}(z))\|)^n < M_3$ for any $z \in \Omega_m$, the inequality $\int_0^r (1 - t)^{n-1} n(t, z)\, dt \leqq M_3$ holds for any $z \in \Omega_m$ and hence for any $z \in \bigcup_{m \geqq 0} \Omega_m$. Thus we obtain

$$\int_0^r (1 - t)^{n-1} n(t, z)\, dt \leqq \max(M_1/2, M_3/n) = M \qquad \text{for any } z \in B^n .$$

Our theorem is now proved.

2. M-harmonic functions and M-subharmonic functions

For later use we discuss the properties of M-harmonic and M-subharmonic functions. We need some definitions and notation. We denote the subgroup $\left\{\begin{bmatrix} \alpha & 0 \\ 0 & A \end{bmatrix} \in U(1, n; \boldsymbol{C}) \,\middle|\, |\alpha| = 1, A \in U(n; \boldsymbol{C})\right\}$ of $U(1, n; \boldsymbol{C})$ by $U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$. Let σ be the $U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$-invariant Borel measure on ∂B^n for which $\sigma(\partial B^n) = 1$. Let Ω be a region of B^n. If a real-valued function $f \in C^2(\Omega)$ satisfies $\tilde{\Delta} f = 0$ in Ω, then f is called an *M-harmonic function* in Ω. We have the mean value property as follows.

THEOREM 2.1 (cf. [9; Corollary 2 to Theorem 4.2.4]). *An M-harmonic function f in Ω satisfies*

$$f(a) = \int_{\partial B^n} f(g(r\zeta))\, d\sigma(\zeta)$$

for each $a \in \Omega$ and $r > 0$ such that $g(r\overline{B^n}) \subset \Omega$, where $g \in U(1, n; \boldsymbol{C})$ *with $g(0) = a$.*

Conversely, if a continuous function f in Ω satisfies this mean value property, then f is M-harmonic in Ω.

If a real-valued function f is upper semi-continuous in Ω and satisfies

$$f(a) \leqq \int_{\partial B^n} f(g(r\zeta))\, d\sigma(\zeta)$$

for each $a \in \Omega$ and $r > 0$ as above, instead of the equality in Theorem 2.1, then f is called an *M-subharmonic function* in Ω. In the same manner as in the proof of [5; Chapter I, Theorem 6.3], we have

THEOREM 2.2. *If f is an M-subharmonic function in Ω and there is a constant K such that* $\limsup_{z \to \zeta} f(z) \leqq K (< \infty)$ *for every $\zeta \in \partial\Omega$, then $f(z) \leqq K$ in Ω.*

Next we shall give the definition of K-limit. For $\alpha > 1/2$ and $\zeta \in \partial B^n$, we write $D_\alpha(\zeta)$ for the set of all elements $z \in B^n$ such that

$$|\Phi(z^*, \zeta^*)|\,|\zeta_0^*|^{-1} < \alpha|\Phi(z^*, z^*)|\,|z_0^*|^{-1},$$

where $z^* = (z_0^*, z_1^*, \ldots, z_n^*) \in \pi^{-1}(z)$ and $\zeta^* = (\zeta_0^*, \zeta_1^*, \ldots, \zeta_n^*) \in \pi^{-1}(\zeta)$. It is easy to show that $g(D_\alpha(\zeta)) = D_\alpha(g(\zeta))$ for $g \in U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$. Set $P(z, \zeta) = \{|\zeta_0^*|^2 |\Phi(z^*, z^*)|\,|\Phi(z^*, \zeta^*)|^{-2}\}^n$ and $S(z, \zeta) = \{-\overline{z_0^*}\zeta_0^* \Phi(z^*, \zeta^*)^{-1}\}^n$. We call them the *Poisson kernel* and the *Szegö kernel*, respectively. We note that

$$D_\alpha(\zeta) = \{z \in B^n \,|\, |S(z, \zeta)| P(z, \zeta)^{-1} < \alpha^n\} .$$

DEFINITION 2.3. Suppose $\zeta \in \partial B^n$. Let f be a complex-valued function in B^n. We say that the function f has *K-limit* λ at ζ if $f(z_i) \to \lambda$ as $i \to \infty$ for every $\alpha > 1/2$ and for every sequence $\{z_i\}$ in $D_\alpha(\zeta)$ that converges to ζ. We write $(\text{K-lim}\, f)(\zeta) = \lambda$.

Now we quote a theorem from [9] on the K-limit of the Poisson integral.

THEOREM 2.4 (cf. [9; Theorem 5.4.8]). *If* $f \in L^1(\sigma)$, *then*
$(\text{K-lim} \int_{\partial B^n} f(\zeta)\, P(z, \zeta)\, d\sigma(\zeta))(\xi) = f(\xi)$ *at every Lebesgue point* ξ *of* f.

3. Sufficient conditions for a discrete subgroup to be of convergence type

We shall give sufficient conditions for a discrete subgroup of $U(1, n; \boldsymbol{C})$ to be of convergence type. We begin with preliminaries.

Let G be a discrete subgroup of $U(1, n; \boldsymbol{C})$. Denote the *orbit* $\{g(z) | g \in G\}$ of a point $z \in B^n$ by $G(z)$, and define the *limit set* $L(G)$ of G by $L(G) = \overline{G(z)} \cap \partial B^n$. This set $L(G)$ does not depend on the choice of z (see [3; Lemma 4.3.1]). We observe that $L(G) = \partial B^n$ or $L(G)$ is nowhere dense on ∂B^n (see [8; p. 108]). A discrete subgroup G is said to be of the *first kind* if $L(G) = \partial B^n$, otherwise G is said to be of the *second kind*. We denote the smallest subspace containing $\pi^{-1}(L(G))$ *by* $\langle \pi^{-1}(L(G)) \rangle$, and set $\langle L(G) \rangle = \pi(\langle \pi^{-1}(L(G)) \rangle \cap V_-)$.

Next we shall give the definition of $d^*(z, w)$ for $z, w \in \overline{B^n}$.

DEFINITION 3.1. For z and w in $\overline{B^n}$, we define

$$d^*(z, w) = \{|z_0^*|^{-1} |w_0^*|^{-1} |\Phi(z^*, w^*)|\}^{1/2} ,$$

where $z^* = (z_0^*, z_1^*, \ldots, z_n^*) \in \pi^{-1}(z)$ and $w^* = (w_0^*, w_1^*, \ldots, w_n^*) \in \pi^{-1}(w)$.

It is easy to show that $d^*(z, w)$ does not depend on the choice of z^* and w^*. We shall state some properties of d^*.

PROPOSITION 3.2.

(a) *d^* is invariant under* $U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$.

(b) $d^*(z, w) = d^*(w, z)$ *and* $d^*(z, w) \leqq d^*(z, x) + d^*(x, w)$ *for* $x, z, w \in \overline{B^n}$.

(c) *If g is an element of* $U(1, n; \boldsymbol{C})$, *then*

$$d^*(g(z), g(w)) = \{(1 - \|g(z)\|^2)(1 - \|z\|^2)^{-1}\}^{1/4} \times \{(1 - \|g(w)\|^2)(1 - \|w\|^2)^{-1}\}^{1/4} d^*(z, w)$$

for $z, w \in B^n$.

(d) *d^* is a metric on ∂B^n.*

(e) *Let ζ be a point in ∂B^n and let $S(\zeta, k) = \{\eta \in \partial B^n | d^*(\zeta, \eta) < k\}$. If $g \in U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$, then $g(S(\zeta, k)) = S(g(\zeta), k)$.*

PROOF. (a) It is easy to prove this statement.

(b) The first equality is immediate. We shall show the triangle inequality. By using (a), we may assume that $x = (r, 0, \ldots, 0)$, where $0 \leqq r \leqq 1$. Let $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$. It is easy to see that

$$d^*(z, x)^2 = |1 - rz_1| \quad \text{and} \quad d^*(x, w)^2 = |1 - rw_1| .$$

Setting $\omega = \sum_{j=2}^n \bar{z}_j w_j$, we see that

$$\begin{aligned} |\omega|^2 &\leqq (\textstyle\sum_{j=2}^n |z_j|^2)(\sum_{j=2}^n |w_j|^2) \leqq (1 - |z_1|^2)(1 - |w_1|^2) \\ &\leqq (1 - |rz_1|^2)(1 - |rw_1|^2) \leqq 4|1 - rz_1||1 - rw_1| . \end{aligned}$$

From the above inequality it follows that

$$\begin{aligned} d^*(z, w)^2 &\leqq |1 - \bar{z}_1 w_1 - \omega| \leqq |1 - r\bar{z}_1 + \bar{z}_1(r - w_1) - \omega| \\ &\leqq |1 - rz_1| + |r - w_1| + |\omega| \\ &\leqq |1 - rz_1| + |1 - rw_1| + |\omega| \\ &\leqq |1 - rz_1| + |1 - rw_1| + 2(|1 - rz_1||1 - rw_1|)^{1/2} \\ &= \{d^*(z, x) + d^*(x, w)\}^2 . \end{aligned}$$

Therefore we obtain the triangle inequality.

(c) Let $z^* = (1, z_1, \ldots, z_n)$ and $w^* = (1, w_1, \ldots, w_n)$. We have

$$\begin{aligned} d^*(g(z), g(w))^2 &= |\Phi(g(z)^*, g(w)^*)| \\ &= |g(z^*)_0|^{-1} |g(w^*)_0|^{-1} |\Phi(g(z^*), g(w^*))| \\ &= |g(z^*)_0|^{-1} |g(w^*)_0|^{-1} |\Phi(z^*, w^*)| \\ &= |g(z^*)_0|^{-1} |g(w^*)_0|^{-1} d^*(z, w)^2 . \end{aligned}$$

From the identity $\Phi(g(z^*), g(z^*)) = \Phi(z^*, z^*)$ we derive $|g(z^*)_0|^2(1 - \|g(z)\|^2) = 1 - \|z\|^2$. Hence $|g(z^*)_0|^{-1} = \{(1 - \|g(z)\|^2)(1 - \|z\|^2)^{-1}\}^{1/2}$. Similarly $|g(w^*)_0|^{-1} = \{(1 - \|g(w)\|^2)(1 - \|w\|^2)^{-1}\}^{1/2}$. Substituting these equalities in the above relation, we obtain the required equality.

(d) Let ξ and η be points in ∂B^n. It is obvious that if $\xi = \eta$, then $d^*(\xi, \eta) = 0$. Therefore we have only to prove that if $d^*(\xi, \eta) = 0$, then $\xi = \eta$. Using (a), we may assume that $\xi = (1, 0, \ldots, 0)$. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$. Then we see that

$$d^*(\xi, \eta) = |1 - \eta_1|^{1/2} = 0 .$$

It follows from this equality that $\eta = (1, 0, \ldots, 0)$. Thus $\xi = \eta$.

(e) Let g be an element of $U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$. By definition and (a) we have

$$x \in S(g(\zeta), k) \rightleftarrows d^*(g(\zeta), x) < k \rightleftarrows d^*(\zeta, g^{-1}(x)) < k .$$

Moreover

$$d^*(\zeta, g^{-1}(x)) < k \rightleftarrows g^{-1}(x) \in S(\zeta, x) \rightleftarrows x \in g(S(\zeta, k)) .$$

Therefore $S(g(\zeta), k) = g(S(\zeta, k))$.

Thus our proof is complete.

We shall show that each compact subset of $\overline{B^n} - L(G)$ meets only a finite number of its images under transformations of G. By considering a conjugated group, if necessary, we may assume that the stabilizer G_0 of 0 consists only of the identity. Let D_0 be the Dirichlet polyhedron for G centered at 0. We recall that D_0 is expressed as

$$\{z \in B^n \,|\, |a_{11}^{(m)} + a_{12}^{(m)} z_1 + \cdots + a_{1,n+1}^{(m)} z_n| > 1 \text{ for all } g_m$$
$$= (a_{ij}^{(m)})_{i,j=1,2,\ldots,n+1} \in G - \{\text{identity}\}\}$$

(see [6; p. 181]). Denote the closure of D_0 in $\overline{B^n}$ by $\overline{D_0}$.

Proposition 3.3. *A compact set K in $\overline{B^n} - L(G)$ is covered by a finite number of images of $\overline{D_0}$ under transformations of G.*

To prove Proposition 3.3, we need a lemma.

Lemma 3.4. *Let $g = (a_{ij})_{i,j=1,2,\ldots,n+1}$ be an element of $U(1, n; \boldsymbol{C})$. Let $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ be points in $\overline{B^n}$. If $w = g(z)$, then*

$$|\overline{a_{11}} - \overline{a_{21}} w_1 - \cdots - \overline{a_{n+1,1}} w_n| = |a_{11} + a_{12} z_1 + \cdots + a_{1,n+1} z_n|^{-1} .$$

Proof. We first note that

$$g(0) = (a_{21}/a_{11}, a_{31}/a_{11}, \ldots, a_{n+1}/a_{11}) ,$$
$$|a_{11}|^2 - \textstyle\sum_{i=2}^{n+1} |a_{i1}|^2 = 1 .$$

It follows from these relations that

$$\begin{aligned} &d^*(g(0), g(z))^2/d^*(g(0), g(0)) \\ &= d^*(g(0), w)^2/d^*(g(0), g(0)) \\ &= |1 - (\overline{a_{21}/a_{11}}) w_1 - (\overline{a_{31}/a_{11}}) w_2 - \cdots - (\overline{a_{n+1,1}/a_{11}}) w_n| \\ &\quad \times (1 - \textstyle\sum_{i=2}^{n+1} |a_{i1}/a_{11}|^2)^{-1/2} \\ &= |\overline{a_{11}} - \overline{a_{21}} w_1 - \overline{a_{31}} w_2 - \cdots - \overline{a_{n+1,1}} w_n| . \end{aligned}$$

On the other hand, the proof of (c) in Proposition 3.2 yields that

$$d^*(g(0), g(z))^2/d^*(g(0), g(0))$$
$$= |g(z^*)_0|^{-1} = |a_{11} + a_{12}z_1 + a_{13}z_2 + \cdots + a_{1,n+1}z_n|^{-1} .$$

Thus we obtain our desired equality.

PROOF OF PROPOSITION 3.3. Let $z = (z_1, z_2, \dots, z_n) \in K$ and let $w = (w_1, w_2, \dots, w_n) \in \overline{D_0}$. Assume that $w = g(z)$ for some $g = (a_{ij})_{i,j=1,2,\dots,n+1} \in G$. By Lemma 3.4,

(1) $\quad |\overline{a_{11}} - \overline{a_{21}}w_1 - \cdots - \overline{a_{n+1,1}}w_n|^{-1} = |a_{11} + a_{12}z_1 + \cdots + a_{1,n+1}z_n| \leqq 1 .$

As $K \subset \overline{B^n} - L(G)$, K contains only finitely many points of $G(0)$. Therefore there exist an integer N and $\delta > 0$ such that $m \geqq N$ implies

(2) $$d^*(z, g_m^{-1}(0)) > \delta \qquad \text{for all } z \in K .$$

Let $g_m = (a_{ij}^{(m)})_{i,j=1,2,\dots,n+1}$. Noting that $g_m^{-1}(0) = (-\overline{a_{12}^{(m)}/a_{11}^{(m)}}, -\overline{a_{13}^{(m)}/a_{11}^{(m)}}, \dots, -\overline{a_{1,n+1}^{(m)}/a_{11}^{(m)}})$, we see that (2) is equivalent to

$$|a_{11}^{(m)} + a_{12}^{(m)}z_1 + \cdots + a_{1,n+1}^{(m)}z_n| > \delta^2|a_{11}^{(m)}| \qquad \text{for all } m \geqq N .$$

It follows from [6; Theorem 5.2] and [7; Theorem 3.2] that if $t > n$, then $\Sigma_{g_m \in G}|a_{11}^{(m)}|^{-2t}$ is convergent, so $\Sigma_{g_m \in G}|a_{11}^{(m)} + a_{12}^{(m)}z_1 + \cdots + a_{1,n+1}^{(m)}z_n|^{-2t}$ is uniformly convergent on K. This implies that $\{g \in G \,|\, |a_{11} + a_{12}z_1 + \cdots + a_{1,n+1}z_n| < 1\}$ is a finite set. Denote this set by H. By (1), K is included in $\bigcup_{g \in H} g^{-1}(\overline{D_0})$. Thus our proof is complete.

PROPOSITION 3.5. *If K_1 and K_2 are compact subsets of $\overline{B^n} - L(G)$, then $g(K_1)$ meets K_2 for at most finitely many $g \in G$.*

PROOF. By Proposition 3.3 we may assume that $K_2 \subset \bigcup_{1 \leqq m \leqq h} g_m(\overline{D_0})$. Then $g(K_1)$ can meet K_2 only if $g(K_1)$ meets some $g_m(\overline{D_0})$, $m = 1, 2, \dots, h$, that is, K_1 meets $g^{-1}(g_m(\overline{D_0}))$. Since K_1 is compact, one more application of Proposition 3.3 shows that, for each $m = 1, 2, \dots, h$, K_1 meets $g^{-1}(g_m(\overline{D_0}))$ for only a finite number of g.

Taking $K_1 = K_2$, we have the following corollary.

COROLLARY 3.6. *A compact subset of $\overline{B^n} - L(G)$ meets only a finite number of its images under transformations of G.*

Now we shall consider sufficient conditions for G to be of convergence type.

THEOREM 3.7. *If there is a measurable subset E of ∂B^n with $\sigma(E) > 0$ such that $E \cap g(E) = \emptyset$ for any element g in G except a finite number of elements, then G is of convergence type.*

To prove Theorem 3.7, we need a lemma.

LEMMA 3.8. *Let $g = (a_{ij})_{i,j=1,2,\dots,n+1}$ be an element of $U(1, n; \mathbf{C})$ and let $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$ be a point in ∂B^n. If f is an integrable function in ∂B^n, then*

$$\int_{\partial B^n} f\, d\sigma = \int_{\partial B^n} f(g(\zeta))|a_{11} + a_{12}\zeta_1 + \cdots + a_{1,n+1}\zeta_n|^{-2n}\, d\sigma$$

$$= \int_{\partial B^n} f(g(\zeta))|g(\zeta^*)_0/\zeta_0^*|^{-2n}\, d\sigma\,,$$

where $\zeta^* = (\zeta_0^*, \zeta_1^*, \dots, \zeta_n^*) \in \pi^{-1}(\zeta)$ and $g(\zeta^*) = (g(\zeta^*)_0, g(\zeta^*)_1, \dots, g(\zeta^*)_n)$.

PROOF. Since $g^{-1}(0) = (-\overline{a_{12}/a_{11}}, -\overline{a_{13}/a_{11}}, \dots, -\overline{a_{1,n+1}/a_{11}})$, $P(g^{-1}(0), \zeta) = |a_{11} + a_{12}\zeta_1 + \cdots + a_{1,n+1}\zeta_n|^{-2n}$. Using (5) in [9; p. 45] and $g^{-1}(\partial B^n) = \partial B^n$, we see that

$$\int_{\partial B^n} f\, d\sigma = \int_{\partial B^n} f(g(\zeta))P(g^{-1}(0), \zeta)\, d\sigma$$

$$= \int_{\partial B^n} f(g(\zeta))|a_{11} + a_{12}\zeta_1 + \cdots + a_{1,n+1}\zeta_n|^{-2n}\, d\sigma$$

$$= \int_{\partial B^n} f(g(\zeta))|g(\zeta^*)_0/\zeta_0^*|^{-2n}\, d\sigma\,.$$

PROOF OF THEOREM 3.7. Let $k = \#\{g \in G | E \cap g(E) \neq \emptyset\}$. Put $u(z) = \int_{\partial B^n} \chi_E(\zeta)P(z, \zeta)\, d\sigma(\zeta)$, where $\chi_E(\zeta)$ is the characteristic function of E. Then $u(0) = \int_{\partial B^n} \chi_E(\zeta)P(0, \zeta)\, d\sigma(\zeta) = \int_{\partial B^n} \chi_E(\zeta)\, d\sigma(\zeta) = \sigma(E)$. Using [6; Proposition 5.11, (1) and Lemma 5.12] and Lemma 3.8, we see that

$$u(0) = \int_{\partial B^n} \chi_E(\zeta)P(0, \zeta)\, d\sigma(\zeta)$$

$$= \int_{\partial B^n} \chi_E(\zeta)P(g(0), g(\zeta))|\zeta_0^*/g(\zeta^*)_0|^{2n}\, d\sigma(\zeta)$$

$$\leqq 2^n(1 - \|g(0)\|)^{-n} \int_{\partial B^n} \chi_E(\zeta)|\zeta_0^*/g(\zeta^*)_0|^{2n}\, d\sigma(\zeta)$$

$$= 2^n(1 - \|g(0)\|)^{-n} \int_{\partial B^n} \chi_E(g^{-1}(\zeta))\, d\sigma(\zeta)$$

$$= 2^n(1 - \|g(0)\|)^{-n} \int_{g(E)} d\sigma(\zeta) = 2^n(1 - \|g(0)\|)^{-n}\sigma(g(E))\,.$$

This implies that

$$(1-\|g(0)\|)^n \leqq 2^n\sigma(E)^{-1}\sigma(g(E))$$

for any element g in G. For any point $\zeta \in \bigcup_{g\in G} g(E)$, the number $\#\{g\in G| \zeta\in g(E)\}$ is at most k. Therefore we see that

$$\begin{aligned}\sum_{g\in G}(1-\|g(0)\|)^n &\leqq 2^n\sigma(E)^{-1}\sum_{g\in G}\sigma(g(E)) \leqq 2^n\sigma(E)^{-1}k\sigma(\textstyle\bigcup_{g\in G} g(E))\\ &\leqq 2^n\sigma(E)^{-1}k\sigma(\partial B^n) < \infty\,.\end{aligned}$$

If G is of the second kind, then there exists a spherical cap F with $\bar{F}\subset \partial B^n - L(G)$. Since $\bar{F}$ is compact, the number $\#\{g\in G|\bar{F}\cap g(\bar{F})\neq\emptyset\}$ is finite by Corollary 3.6. Applying Theorem 3.7 to F in place of E yields the following result.

THEOREM 3.9. *If G is of the second kind, then G is of convergence type.*

We shall give an alternative proof of Theorem 3.9.

PROOF OF THEOREM 3.9. Let F be the same spherical cap defined as above. Let $g_m = (a_{ij}^{(m)})_{i,j=1,2,\dots,n+1}$ be an element in G. From the relation $|a_{11}^{(m)}|^2 - \Sigma_{j=2}^{n+1}|a_{1j}^{(m)}|^2 = 1$ we derive

$$\begin{aligned}&|a_{11}^{(m)} + a_{12}^{(m)}\zeta_1 + \cdots + a_{1,n+1}^{(m)}\zeta_n|\\ &\quad\leqq |a_{11}^{(m)}| + |a_{12}^{(m)}\zeta_1 + \cdots + a_{1,n+1}^{(m)}\zeta_n|\\ &\quad\leqq |a_{11}^{(m)}| + (\textstyle\sum_{j=2}^{n+1}|a_{1j}^{(m)}|^2)^{1/2}(\sum_{j=1}^{n}|\zeta_j|^2)^{1/2}\\ &\quad= |a_{11}^{(m)}| + (|a_{11}^{(m)}|^2-1)^{1/2} \leqq 2|a_{11}^{(m)}|\,.\end{aligned}$$

Lemma 3.8 together with this inequality yields

$$\begin{aligned}\infty &> \sum_{g_m\in G}\int_{g_m(F)} d\sigma\\ &= \sum_{g_m\in G}\int_F |a_{11}^{(m)} + a_{12}^{(m)}\zeta_1 + \cdots + a_{1,n+1}^{(m)}\zeta_n|^{-2n}\,d\sigma\\ &\geqq (1/2)^{2n}\sum_{g_m\in G}\int_F |a_{11}^{(m)}|^{-2n}\,d\sigma\\ &= (1/2)^{2n}\sigma(F)\sum_{g_m\in G}|a_{11}^{(m)}|^{-2n}\,.\end{aligned}$$

Hence the series $\Sigma_{g_m\in G}|a_{11}^{(m)}|^{-2n}$ converges. Thus G is of convergence type by Theorem 1.2.

THEOREM 3.10. *If one of the following conditions is satisfied, then G is of convergence type.*

(a) *There exists a non-constant bounded M-harmonic function on B^n/G.*

(b) *There exists a G-invariant measurable subset $E \subset \partial B^n$ with $0 < \sigma(E) < 1$.*

(c) *The orbit $G(p)$ of a point $p \in B^n$ is included in the set $\{z = (z_1, z_2, \ldots, z_{i-1}, b, z_{i+1}, \ldots, z_n) \in B^n\}$ for some $b \in \boldsymbol{C}$.*

(d) *$\langle L(G) \rangle$ is not identical with B^n.*

PROOF. (a) As observed in the beginning of the proof of Theorem 1.6 there exists $h \in U(1, n; \boldsymbol{C})$ such that $g(0) \neq 0$ for any $g \in hGh^{-1} - \{\text{identity}\}$. Since G and hGh^{-1} are of the same type by [6; Theorem 5.9], we may assume $g(0) \neq 0$ for any $g \in G - \{\text{identity}\}$. Let $\{g_0, g_1, \ldots\}$ be the complete list of elements of G. Suppose that there exists a non-constant bounded M-harmonic function on B^n/G. Let p be a point of B^n/G. In the same manner as in the proof of [4; Theorem IV. 3.7], we obtain a positive function H on $B^n/G - \{p\}$ which corresponds to a Green's function for a Riemann surface. The function H has the following properties:

1) H is M-harmonic in $B^n/G - \{p\}$;

2) $H(w) - u(\|w\|^2)$ is M-harmonic in a neighborhood of p, where u is the function defined before Theorem 1.4 and w is a local parameter vanishing at p.

Using the inverse mapping π^{-1}, we can construct a positive G-automorphic function $h(z)$ in B^n with the following properties:

3) $h(z)$ is M-harmonic in $B^n - \bigcup_{m \geqq 0} g_m(0)$;

4) $h(z) - u(\|g_m^{-1}(z)\|^2)$ is M-harmonic in each neighborhood of $g_m(0)$ for $m \geqq 0$.

Let $F_i(z)$ be the function defined in the proof of Theorem 1.6. It follows that $\tilde{\Delta}(F_i(z) - h(z)) = 0$ in $B^n - \bigcup_{m > i} g_m^{-1}(0)$ and $\limsup_{z \to \zeta} (F_i(z) - h(z)) \leqq 0$ for $\zeta \in \partial B^n \cup \bigcup_{m > i} g_m^{-1}(0)$. By Theorem 2.2, $F_i(z) \leqq h(z)$ in B^n. Therefore $\Sigma_{m=0}^{\infty} u(\|g_m(z)\|^2)$ is convergent. From Theorem 1.4 it follows that G is of convergence type.

(b) Assume that there is a G-invariant measurable subset $E \subset \partial B^n$ with $0 < \sigma(E) < 1$ and that G is not of convergence type. Set

$$v(z) = \int_{\partial B^n} \chi_E(\zeta) P(z, \zeta)\, d\sigma(\zeta),$$

where $\chi_E(\zeta)$ is the characteristic function of E and $P(z, \zeta)$ is the Poisson kernel. Then $0 \leqq v(z) \leqq 1$ in B^n.

Let $0 < r < 1$ and let h be an element of $U(1, n; \boldsymbol{C})$. By using Fubini's theorem and [6; Proposition 5.11, (2), (3) and (4)], we see that

$$\begin{aligned}\int_{\partial B^n} v(h(r\zeta))\,d\sigma(\zeta) &= \int_{\partial B^n}\left\{\int_{\partial B^n}\chi_E(\eta)P(h(r\zeta),\eta)\,d\sigma(\eta)\right\}d\sigma(\zeta)\\ &= \int_{\partial B^n}\chi_E(\eta)\left\{\int_{\partial B^n}P(h(r\zeta),\eta)\,d\sigma(\zeta)\right\}d\sigma(\eta)\\ &= \int_{\partial B^n}\chi_E(\eta)\left\{\int_{\partial B^n}P(r\zeta,h^{-1}(\eta))P(h(0),\eta)\,d\sigma(\zeta)\right\}d\sigma(\eta)\\ &= \int_{\partial B^n}\chi_E(\eta)P(h(0),\eta)\left\{\int_{\partial B^n}P(r\zeta,h^{-1}(\eta))\,d\sigma(\zeta)\right\}d\sigma(\eta)\\ &= \int_{\partial B^n}\chi_E(\eta)P(h(0),\eta)\left\{\int_{\partial B^n}P(rh^{-1}(\eta),\zeta)\,d\sigma(\zeta)\right\}d\sigma(\eta)\\ &= \int_{\partial B^n}\chi_E(\eta)P(h(0),\eta)\,d\sigma(\eta) = v(h(0))\,.\end{aligned}$$

By Theorem 2.1, $v(z)$ is M-harmonic in B^n. It follows from Lemma 3.8 that for any element $g \in G$

$$\begin{aligned}v(g(z)) &= \int_{\partial B^n}\chi_E(\zeta)P(g(z),\zeta)\,d\sigma(\zeta)\\ &= \int_{\partial B^n}\chi_E(g(\zeta))P(g(z),g(\zeta))|\zeta_0^*/g(\zeta^*)_0|^{2n}\,d\sigma(\zeta)\\ &= \int_{\partial B^n}\chi_E(g(\zeta))|\zeta_0^*/g(\zeta^*)_0|^{2n}P(z,\zeta)|g(\zeta^*)_0/\zeta_0^*|^{2n}\,d\sigma(\zeta)\\ &= \int_{g^{-1}(E)}P(z,\zeta)\,d\sigma(\zeta) = \int_E P(z,\zeta)\,d\sigma(\zeta) = v(z),\end{aligned}$$

where $\zeta^* = (\zeta_0^*, \zeta_1^*, \dots, \zeta_n^*) \in \pi^{-1}(\zeta)$ and $g(\zeta^*) = (g(\zeta^*)_0, g(\zeta^*)_1, \dots, g(\zeta^*)_n) \in \pi^{-1}(g(\zeta))$. Therefore we can regard $v(z)$ as a bounded M-harmonic function on B^n/G. By (a), $v(z)$ is constant. Moreover we know from Theorem 2.4 and [9; Theorem 5.3.1] that $(\text{K-lim}\, v)(\zeta) = 1$ at a point ζ of E. Hence $v(z) \equiv 1$. Thus we obtain

$$1 = v(0) = \int_E P(0,\zeta)\,d\sigma(\zeta) = \int_E d\sigma(\zeta) = \sigma(E)\,,$$

which is a contradiction. Thus we see that (b) is a sufficient condition for G to be of convergence type.

(c) Noting that $L(G) = \overline{G(p)} \cap \partial B^n \subset \{\zeta = (\zeta_1, \zeta_2, \dots, \zeta_{i-1}, b, \zeta_{i+1}, \dots, \zeta_n) \in \partial B^n\}$, we see that there is an open set $E \subset \partial B^n - L(G)$ with $\sigma(E) > 0$. Hence G

is of the second kind. From Theorem 3.9 it follows that G is of convergence type.

(d) Let $\dim \langle L(G) \rangle = s$. If $s = 0$, then G is of the second kind and hence G is of convergence type by Theorem 3.9. As the limit set $L(G)$ is G-invariant, every element of G leaves $\langle L(G) \rangle$ invariant. If $0 < s < n$, then we may assume that an element of G has the following form:

$$g = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where $A \in U(1, s; \boldsymbol{C})$ and $B \in U(n - s; \boldsymbol{C})$. Therefore the restriction of G to $\langle L(G) \rangle$ may be regarded as a discrete subgroup of $U(1, s; \boldsymbol{C})$. By [6; Theorem 5.2), $\Sigma_{g \in G}(1 - \|g(z)\|)^{s+1} < \infty$ for any $z \in \langle L(G) \rangle$. It follows that $\Sigma_{g \in G}(1 - \|g(z)\|)^{n} < \infty$ and thus G is of convergence type.

Thus our theorem is completely proved.

In the same manner as in (d) we can prove

THEOREM 3.11. *Let Γ be a Fuchsian group keeping $\{z \mid |z| < 1\}$ invariant and let $\{\gamma_0, \gamma_1, \dots\}$ be the complete list of elements of Γ. We consider a correspondence between an element γ_i of Γ and an element g_i of $U(1, n; \boldsymbol{C})$ with $n \geqq 2$ as follows*:

$$\begin{array}{ccc} \Gamma & \longrightarrow & U(1, n; \boldsymbol{C}) \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \gamma_i(z) = \dfrac{a_i z + c_i}{\bar{c}_i z + \bar{a}_i} & \longrightarrow & g_i(z_1, z_2, \dots, z_n) \end{array}$$

$$= \left(\frac{a_i z_1 + c_i}{\bar{c}_i z_1 + \bar{a}_i}, \frac{z_2}{\bar{c}_i z_1 + \bar{a}_i}, \dots, \frac{z_n}{\bar{c}_i z_1 + \bar{a}_i} \right),$$

where $|a_i|^2 - |c_i|^2 = 1$. Denote the group consisting of $g_0, g_1, \dots$ by G. Then G is a discrete subgroup of convergence type in $U(1, n; \boldsymbol{C})$.

4. Set of points of approximation

In this section we shall discuss the measure of the set of points of approximation of G.

We define a point of approximation (cf. [2; p. 261]).

DEFINITION 4.1. Let ζ be a point in the limit set of G. If there exist a sequence $\{g_m\}$ of distinct elements of G and a region $D_\alpha(\zeta)$ defined as in Section 2 such that $g_m(0) \in D_\alpha(\zeta)$ and $g_m(0) \to \zeta$, then the point ζ is called a *point of* *approximation.* We denote the set of all points of approximation of G by $L_D(G)$.

We shall show that the origin 0 is replaced by any point $z \in B^n$.

PROPOSITION 4.2. *Let ζ be a point of approximation of G. Let $\{g_m\}$ be the same sequence of elements of G as in Definition 4.1. For any point z in B^n, there exists a region $D_\beta(\zeta)$ such that $g_m(z) \to \zeta$ in $D_\beta(\zeta)$.*

To prove Proposition 4.2, we need a lemma. By the aid of Lemma 1.8 we obtain

LEMMA 4.3. *If g is an element of $U(1, n; \boldsymbol{C})$, then*

$$(1 - \|g(0)\|^2)\,(1 - \|g(z)\|^2)^{-1} \leqq 8(1 - \|z\|^2)^{-1} \qquad \text{for } z \in B^n.$$

PROOF OF PROPOSITION 4.2. First express $D_\alpha(\zeta)$ as $\{z \in B^n | d^*(z, \zeta)^2 < \alpha d^*(z, z)^2\}$. Use (c) in Proposition 3.2 to yield

$$\begin{aligned}
&d^*(g_m(z), \zeta)/d^*(g_m(z), g_m(z)) \\
&\leqq \{d^*(g_m(z), g_m(0)) + d^*(g_m(0), \zeta)\}/d^*(g_m(z), g_m(z)) \\
&= [\{(1 - \|g_m(z)\|^2)(1 - \|z\|^2)^{-1}(1 - \|g_m(0)\|^2)\}^{1/4} d^*(z, 0) \\
&\quad + d^*(g_m(0), \zeta)](1 - \|g_m(z)\|^2)^{-1/2} \\
&= \{(1 - \|g_m(z)\|^2)^{-1}(1 - \|g_m(0)\|^2)(1 - \|z\|^2)^{-1}\}^{1/4} \\
&\quad + (1 - \|g_m(z)\|^2)^{-1/2} d^*(g_m(0), \zeta)\,.
\end{aligned}$$

Since $g_m(0) \in D_\alpha(\zeta)$,

$$d^*(g_m(0), \zeta) < \alpha^{1/2} d^*(g_m(0), g_m(0)) = \alpha^{1/2}(1 - \|g_m(0)\|^2)^{1/2}\,.$$

It follows from the above inequality and Lemma 4.3 that

$$\begin{aligned}
d^*(g_m(z), \zeta)/d^*(g_m(z), g_m(z)) &\leqq \{(1 - \|g_m(z)\|^2)^{-1}(1 - \|g_m(0)\|^2)\}^{1/4}(1 - \|z\|^2)^{-1/4} \\
&\quad + (1 - \|g_m(z)\|^2)^{-1/2}\alpha^{1/2}(1 - \|g_m(0)\|^2)^{1/2} \\
&\leqq 8^{1/4}(1 - \|z\|^2)^{-1/2} + \{8\alpha(1 - \|z\|^2)^{-1}\}^{1/2} < \beta^{1/2}\,,
\end{aligned}$$

where β is a constant depending only on z. Thus we see that $g_m(z) \to \zeta$ in $D_\beta(\zeta)$.

REMARK 4.4. Let g be an element of $U(1, n; \boldsymbol{C})$ which is not the identity. We shall call g loxodromic if it has exactly two fixed points and they lie on ∂B^n, and g parabolic if it has one fixed point and this lies on ∂B^n.

Every loxodromic fixed point of G is a point of approximation. There is a parabolic fixed point which is not a point of approximation.

PROPOSITION 4.5. *The set $L_D(G)$ is G-invariant.*

PROOF. Let $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ be a point of approximation of G. Then there exists a sequence $\{g_m\}$ of elements of G such that each $g_m(z)$ lies in $D_\alpha(\zeta)$ for some $\alpha > 1/2$ and some point z of B^n. We denote $g_m(z)$ by $W_m = ((W_m)_1, (W_m)_2, \ldots, (W_m)_n)$. Let $g = (a_{ij})_{i,j=1,2,\ldots,n+1}$ be an element of G.

We shall show that $g(\zeta)$ is contained in $L_D(G)$. We have only to prove that there exists a positive number $\beta > 1/2$ such that all $g(W_m)$ lie in $D_\beta(g(\zeta))$. Using that $W_m \in D_\alpha(\zeta)$, we see that

$$\begin{aligned}
&d^*(g(W_m), g(\zeta))^2/d^*(g(W_m), g(W_m))^2\\
&\quad = |\Phi(g(W_m)^*, g(\zeta)^*)||g(\zeta)_0^*|^{-1}|\Phi(g(W_m)^*, g(W_m)^*)|^{-1}|g(W_m)_0^*|\\
&\quad = |\Phi(g(W_m^*), g(\zeta^*))||g(W_m)_0^*||g(W_m^*)_0|^{-1}|g(\zeta)_0^*||g(\zeta^*)_0|^{-1}|g(\zeta)_0^*|^{-1}\\
&\qquad \times |\Phi(g(W_m^*), g(W_m^*))|^{-1}|g(W_m)_0^*|^{-2}|g(W_m^*)_0|^2|g(W_m)_0^*|\\
&\quad = \{|\Phi(W_m^*, \zeta^*)||\zeta_0^*|^{-1}|\Phi(W_m^*, W_m^*)|^{-1}|(W_m^*)_0|\}\{|g(W_m^*)_0||(W_m^*)_0|^{-1}\}\\
&\qquad \times \{|g(\zeta^*)_0|^{-1}|\zeta_0^*|\}\\
&\quad < \alpha\left|a_{11} + \textstyle\sum_{j=2}^{n+1} a_{1j}(W_m)_{j-1}\right|\left|a_{11} + \textstyle\sum_{j=2}^{n+1} a_{1j}\zeta_{j-1}\right|^{-1},
\end{aligned}$$

where $W_m^* = ((W_m)_0^*, (W_m)_1^*, \ldots, (W_m)_n^*) \in \pi^{-1}(W_m)$. It is seen that

$$\begin{aligned}
\left|a_{11} + \textstyle\sum_{j=2}^{n+1} a_{1j}\zeta_{j-1}\right| &\geqq |a_{11}| - \left|\textstyle\sum_{j=2}^{n+1} a_{1j}\zeta_{j-1}\right|\\
&\geqq |a_{11}| - \left(\textstyle\sum_{j=2}^{n+1} |a_{1j}|^2\right)^{1/2}\left(\textstyle\sum_{j=1}^{n} |\zeta_j|^2\right)^{1/2}\\
&= |a_{11}| - (|a_{11}|^2 - 1)^{1/2} > 0\,.
\end{aligned}$$

Therefore we have that $|a_{11} + \Sigma_{j=2}^{n+1} a_{1j}(W_m)_{j-1}||a_{11} + \Sigma_{j=2}^{n+1} a_{1j}\zeta_{j-1}|^{-1}$ is bounded in B^n. This implies that there exists $\beta > 1/2$ such that

$$d^*(g(W_m), g(\zeta))^2/d^*(g(W_m), g(W_m))^2 < \beta\,.$$

Thus $g(\zeta) \in L_D(G)$ and our proof is complete.

THEOREM 4.6. *If G is of convergence type, then $\sigma(L_D(G)) = 0$.*

Before proving our theorem, we prepare a lemma.

LEMMA 4.7 ([9; Proposition 5.1.4]). *Let $I = (1, 0, \ldots, 0)$. If $n = 1$, then $\sigma(S(I, t))/t^{2n}$ decreases from $1/2$ to $1/\pi$ as t decreases from $\sqrt{2}$ to 0. If $n > 1$,* *then* $\sigma(S(I, t))/t^{2n}$ *increases from* 2^{-n} *to a finite limit* $(1/4)\Gamma(n+1)/\Gamma^2(n/2+1)$ *as* t *decreases from* $\sqrt{2}$ *to* 0.

PROOF OF THEOREM 4.6. Given $z \in B^n$ we denote the orbit $G(z)$ of z by $\{a_k\}$. Let $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$ be in $L_D(G)$. Then for some $\alpha > 1/2$ there exists a subsequence $\{a_{k_\nu}\}$ of $\{a_k\}$, $a_k \neq 0$, in $D_\alpha(\zeta)$ such that $a_{k_\nu} \to \zeta$ as $k_\nu \to \infty$. Writing $a_{k_\nu} = ((a_{k_\nu})_1, (a_{k_\nu})_2, \dots, (a_{k_\nu})_n)$, we see that

$$\begin{aligned} d^*(a_{k_\nu}/\|a_{k_\nu}\|, \zeta)^2 &= \|a_{k_\nu}\|^{-1} | - \|a_{k_\nu}\| + \textstyle\sum_{j=1}^n \overline{(a_{k_\nu})_j}\zeta_j | \\ &\leqq \|a_{k_\nu}\|^{-1}\{(1 - \|a_{k_\nu}\|) + |1 - \textstyle\sum_{j=1}^n \overline{(a_{k_\nu})_j}\zeta_j|\} \\ &\leqq \|a_{k_\nu}\|^{-1}\{(1 - \|a_{k_\nu}\|) + \alpha(1 - \|a_{k_\nu}\|)^2\} \\ &\leqq (1 + 2\alpha)\|a_{k_\nu}\|^{-1}(1 - \|a_{k_\nu}\|)\,. \end{aligned}$$

Let

$$S_\alpha(a_k) = \{\eta \in \partial B^n | d^*(a_k/\|a_k\|, \eta) < [(1 + 2\alpha)\|a_k\|^{-1}(1 - \|a_k\|)]^{1/2}\}\,.$$

It is seen that

$$L_D(G) \subset \textstyle\bigcup_{\alpha > 0} (\limsup_{k\to\infty} S_\alpha(a_k))\,.$$

Using Lemma 4.7, we have $\sigma(S_\alpha(a_k)) \leqq M(1 - \|a_k\|)^n$ except for finitely many a_k's, where M is a constant depending only on n and α. Since G is of convergence type, given $\varepsilon > 0$ there exists a positive integer k_0 such that

$$\textstyle\sum_{k>k_0} \sigma(S_\alpha(a_k)) \leqq M \sum_{k>k_0} (1 - \|a_k\|)^n < M\varepsilon\,.$$

Therefore it follows that

$$\sigma(\limsup_{k\to\infty} S_\alpha(a_k)) = 0\,.$$

Noting that $S_\beta(a_k) \supset S_\alpha(a_k)$ for $\beta > \alpha$, we see that $\bigcup_{\alpha>0}(\limsup_{k\to\infty} S_\alpha(a_k))$ can be expressed as a countable union of null sets. Thus $\sigma(L_D(G)) = 0$.

Combining Theorem 4.6 with Proposition 4.5 and (b) in Theorem 3.10, we obtain

THEOREM 4.8. *The measure* $\sigma(L_D(G))$ *is either* 0 *or* 1.

THEOREM 4.9. *If* $\sigma(L_D(G)) > 0$, *then* $L(G) = \partial B^n$.

PROOF. It follows from Theorem 4.8 that $\sigma(L_D(G)) = 1$. By Theorem 4.6, G is not of convergence type. Using Theorem 3.9, we see that G is of the first kind. Thus $L(G) = \partial B^n$.

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