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Markov-self-similar sets

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1. Introduction

A theory of non-random self-similar sets has been developed by Moran [11] and Hutchinson [9]. Lately Mauldin-Williams [10], Falconer [5] and Graf [7] investigated random self-similar sets. In this paper we introduce a new concept of *Markov-self-similarity* and investigate deterministic and random Markov-self-similar sets. Takahashi [12] introduced a concept of multi-similarity which is essentially the same concept as Markov-self-semilarity. Markov-self-similarity is a natural extension of self-similarity and Markov-self-similar sets appear as the limit sets of cellular automata [12, 15]. Cellular automata are used to model problems in crystal growth and diffusion and other problems of self-organization. Therefore the patterns appeared in these fields are expected to be Markov-self-similar. On the other hand some Markov-self-similar sets can be constructed as recurrent sets defined by Dekking [3]. (See also Bedford [1, 2].)

A Markov-self-similar set is constructed as follows. First we prepare an N-tuple $(S_{01}, \dots, S_{0N})$ of contraction similarities of $\mathbf{R}^d$ which are initial contractions and used only in the first step. Let F be a non-empty compact subset of $\mathbf{R}^d$, and set

$$A_1 = \bigcup_{k=1}^{N} S_{0k}(F).$$

Next we fix a family of N N-tuples $\{(S_{k1}, \dots, S_{kN})\}_{k=1}^{N}$ of contraction similarities of $\mathbf{R}^d$ which are fundamental contractions and used in the following process repeatedly. We assume that above N N-tuples satisfy the irreducibility condition and the open set condition. (See Section 2.) Set

$$A_2 = \bigcup_{k=1}^{N} S_{0k}(\bigcup_{i=1}^{N} S_{ki}(F)).$$

Note that the contractions S_{ki} are selected depending on the index k of S_{0k}. Set

$$A_3 = \bigcup_{k=1}^{N} S_{0k}(\bigcup_{i=1}^{N} S_{ki}(\bigcup_{j=1}^{N} S_{ij}(F))).$$

We continue this process. Let $K = \lim_{n\to\infty} A_n$ where the limit is taken with respect to the Hausdorff metric. The set K has a Markovian shape structure which is not possessed by a self-similar set constructed in Hutchinson [9].

A random Markov-self-similar set is a probabilistic counterpart of a non-random Markov-self-similar set. The plan of this paper is as follows.

In Section 2 we investigate a Markov-self-similar N-tuple of compact sets which is an extension of a Hutchinson's self-similar set. The fundamental result is as follows: Let $\mathbf{S} = (\underline{S}_1, \ldots, \underline{S}_N)$ be an N-tuple of $\underline{S}_k = (S_{k1}, \ldots, S_{kN})$, $k = 1, \ldots, N$ where S_{ki}, $i = 1, \ldots, N$ are contraction similarities of $\mathbf{R}^d$ which satisfy the open set condition. For a non-negative number β, we define an $N \times N$ non-negative matrix $R(\beta) = [R(\beta)_{kj}]$ by

$$R(\beta)_{kj} = r(S_{kj})^{\beta} \quad k, j = 1, \ldots, N$$

where $r(S_{kj})$ is the contraction ratio of S_{kj}. Let $\lambda(\beta)$ be the maximal eigen value of $R(\beta)$. Let F be a non-empty compact set. Set

$$K_k = \lim_{m \to \infty} \bigcup_{i_1, \ldots, i_m = 1}^{N} S_{ki_1} \circ S_{i_1 i_2} \circ \cdots \circ S_{i_{m-1} i_m}(F)$$

for $k = 1, \ldots, N$ where the limit is taken with respect to the Hausdorff metric. Then

$$\dim_H(K_k) = \alpha$$

and

$$0 < \mathcal{H}^{\alpha}(K_k) < \infty$$

for all $k = 1, \ldots, N$ where α is such that $\lambda(\alpha) = 1$. Furthermore there exists $c > 0$ such that

$$\mathcal{H}^{\alpha}(K_k) = c x_k \qquad \text{for} \quad k = 1, \ldots, N$$

where $(x_1, \ldots, x_N)$ is a positive eigenvector of $R(\alpha)$ corresponding to the maximal eigen value $\lambda(\alpha) = 1$. The N-tuple $(K_1, K_2, \ldots, K_N)$ of compact sets defined above satisfies the conditions:

$$K_k = \bigcup_{i=1}^{N} S_{ki}(K_i) \qquad \text{for} \quad k = 1, \ldots, N,$$

K_k is an α-set and $\mathcal{H}^{\alpha}(S_{ki}(K_i) \cap S_{kj}(K_j)) = 0$ for all $k = 1, \ldots, N$ and $i \neq j$. Such an N-tuple of compact sets is called Markov-self-similar.

In Section 3 we introduce a concept of random Markov-self-similarity and show that the results that correspond to those for the concept of statitical self-similarity obtained in Graf [7] hold. Let $(\mu_1, \ldots, \mu_N)$ be an N-tuple of Borel probability measures on $\mathrm{Con}(X)^N$ where $\mathrm{Con}(X)$ denotes the set of all contractions of a compact set X. Then there exists a unique N-tuple of probability measures $(P_1, \ldots, P_N)$ on $\mathcal{K}(X)$, the set of all non-empty compact sets in X, such that for every Borel set $B \subset \mathcal{K}(X)$,

(i) $P_k(B) = [\mu_k \times \prod_{i=1}^{N} P_i](\{((S_1, \ldots, S_N), (K_1, \ldots, K_N)) \in \mathrm{Con}(X)^N \times \mathcal{K}(X)^N$
$\mid \bigcup_{i=1}^{N} S_i(K_i) \in B\})$

for all $k = 1, \dots, N$. An N-tuple $(P_1, \dots, P_N)$ of probability measures on $\mathscr{K}(X)$ which satisfies (i) is called $(\mu_1, \dots, \mu_N)$-Markov-self-similar. Furthermore the following holds: Let $R(\beta) = [R(\beta)_{ij}]$ be an $N \times N$ matrix defined by

$$R(\beta)_{ij} = \int r(S_j)^\beta \, d\mu_i(S_1, \dots, S_N)$$

where $\beta \geq 0$, and let $\lambda(\beta)$ be the maximal eigen value of non-negative matrix $R(\beta)$. Under some conditions, $\dim_H(K) = \alpha$ for P_k-a.e. $K \in \mathscr{K}(X)$ for all $k = 1, \dots, N$ where α is a positive number such that $\lambda(\alpha) = 1$.

In Section 4 we investigate the Hausdorff-measures of random Markov-self-similar sets. The results are as follows: Suppose that there exists a $\delta > 0$ such that if $R(0)_{ki} > 0$, then $r(S_i) \geq \delta$ for μ_k-a.e. $(S_1, \dots, S_N)$ where $k, i = 1, \dots, N$. Let $(x_1, \dots, x_N)$ be a positive eigenvector of $R(\alpha)$ corresponding to the maximal eigen value 1. Then the following statements are equivalent:

a) $\sum_{i=1}^N r(S_i)^\alpha x_i = x_k$ for μ_k-a.e. $(S_1, \dots, S_N)$ and all $k = 1, \dots, N$.
b) $\mathscr{H}^\alpha(K) > 0$ for P_k-a.e. $K \in \mathscr{K}(X)$ and all $k = 1, \dots, N$.
c) $P_j(\{K \in \mathscr{K}(X) | \mathscr{H}^\alpha(K) > 0\}) > 0$ for some $j \in \{1, \dots, N\}$.

This is an extension of the result given by Graf [7]. Furthermore if $P_j(\{K \in \mathscr{K}(X) | \mathscr{H}^\alpha(K) > 0\}) > 0$ for some $j \in \{1, \dots, N\}$, then there exists $c > 0$ such that

$$\mathscr{H}^\alpha(K) = c x_k \quad \text{for } P_k\text{-a.e. } K \in \mathscr{K}(X) \text{ and all } k = 1, \dots, N.$$

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2. Markov-self-similar sets

Let $Y = (Y, d)$ be a complete metric space. A mapping $S: Y \to Y$ is called a contraction if $d(S(x), S(y)) \leq r d(x, y)$ for all $x, y \in Y$ where $0 < r < 1$, and $r(S) = \inf\{r \geq 0 | d(S(x), S(y)) \leq r d(x, y) \text{ for all } x, y \in Y\}$ is called the contraction ratio of S. By $\mathrm{Con}(Y)$ we denote the set of all contractions of Y. We assume the null contraction ϕ is an element of $\mathrm{Con}(Y)$ where ϕ is such that $\phi(Y) =$ the empty set. Fix a positive integer $N \geq 2$. Let $\mathrm{Con}(Y)^N = \{(S_1, S_2, \dots, S_N) | S_i \in \mathrm{Con}(Y) \text{ for } i = 1, \dots, N, (S_1, S_2, \dots, S_N) \neq (\phi, \phi, \dots, \phi)\}$. Let $\mathscr{K}(Y)$ be the space of all non-empty compact subsets of Y. The topology of $\mathscr{K}(Y)$ is defined by the Hausdorff metric $\rho(A, B) = \sup\{d(a, B), d(A, b) | a \in A, b \in B\}$, $A, B \in \mathscr{K}(Y)$.

Hutchinson [9] proved that for every finite set of contractions $S_1, S_2, \dots, S_N$ of a complete metric space there exists a unique invariant non-empty compact set K, i.e., $K = \bigcup_{i=1}^N S_i(K)$. Furthermore he showed that if S_i are similarities with contraction ratio r_i of $\mathbf{R}^d$ which satisfy the open set condition, the Hausdorff dimension of K equals to α where α is a number such that $\sum_{i=1}^{N} r_i^{\alpha} = 1$. We extend the result as follows.

THEOREM 2.1. *Let* $\mathbf{S} = (\underline{S}_1, \dots, \underline{S}_N)$ *be an* N*-tuple of* $\underline{S}_k = (S_{k1}, \dots, S_{kN}) \in \mathrm{Con}(Y)^N$ $k = 1, \dots, N$. *Then there exists a unique* N*-tuple* $(K_1, \dots, K_N)$ *of non-empty compact sets such that*

$$(1) \qquad K_k = \bigcup_{i=1}^{N} S_{ki}(K_i) \quad \text{for} \quad k = 1, \dots, N.$$

Furthermore for any non-empty compact set F

$$(2) \qquad \lim_{m\to\infty} \bigcup_{i_1 \dots i_m}^{N} S_{ki_1} \circ S_{i_1 i_2} \circ \cdots \circ S_{i_{m-1} i_m}(F) = K_k \quad \textit{for} \quad k = 1, \dots, N$$

where the limit is taken with respect to the Hausdorff metric.

The statement (1) of Theorem 2.1 is a special case of Proposition 3.6 in Section 3, and the statement (2) is proved in the same manner as in Hutchinson [9].

REMARKS (i) Associated with $\mathbf{S} = \{\underline{S}_1, \dots, \underline{S}_N\}$, an operator $T_{\mathbf{S}}: \mathscr{K}(Y)^N \to \mathscr{K}(Y)^N$ is defined by

$$T_{\mathbf{S}}(F_1, \dots, F_N) = \left(\bigcup_{i=1}^{N} S_{1i}(F_i), \dots, \bigcup_{i=1}^{N} S_{Ni}(F_i)\right)$$

for $(F_1, \dots, F_N) \in \mathscr{K}(Y)^N$. Then the equalities (1) imply $T_{\mathbf{S}}(K_1, \dots, K_N) = (K_1, \dots, K_N)$, i.e. $(K_1, \dots, K_N)$ is $T_{\mathbf{S}}$-invariant.

(ii) Let F be a non-empty compact set in Y and $(\underline{S}_1, \dots, \underline{S}_N)$ and $\underline{S}_0$ be such that $\underline{S}_k = (S_{k1}, \dots, S_{kN}) \in \mathrm{Con}(Y)^N$, $k = 0, 1, \dots, N$. Let

$$(2') \qquad K = \lim_{m\to\infty} \bigcup_{i_1, \dots, i_m = 1}^{N} S_{0 i_1} \circ S_{i_1 i_2} \circ \cdots \circ S_{i_{m-1} i_m}(F).$$

Then the set K can be expressed by

$$K = \bigcup_{k=1}^{N} S_{0k}(K_k)$$

where $(K_1, \dots, K_N)$ is the N-tuple of compact sets that satisfy the equalities (1) with respect to $(\underline{S}_1, \dots, \underline{S}_N)$.

Next we give the lower and upper estimates of the Hausdorff measures of compact sets K_k. We introduce some notation.

Let $E \subset Y$, $\delta > 0$ and $\alpha \geq 0$ be arbitrary. Define

$$\mathscr{H}_\delta^\alpha(E) = \inf\left\{\sum_{i=1}^{\infty} |E_i|^\alpha \,\middle|\, E \subset \bigcup_{i=1}^{\infty} E_i,\ |E_i| \leq \delta\right\},$$

and

$$\mathscr{H}^\alpha(E) = \sup_{\delta>0} \mathscr{H}_\delta^\alpha(E)$$

where $|E|$ is the diameter of E. Then $\mathscr{H}^\alpha$ is an outer measure on Y such that all Borel sets are $\mathscr{H}^\alpha$-measurable. $\mathscr{H}^\alpha$ is called the α-dimensional measure. The Hausdorff dimension of E is defined by

$$\begin{aligned}\dim_{\mathrm{H}}(E) &= \sup\{\alpha \geq 0 \mid \mathscr{H}^\alpha(E) > 0\} \\ &= \inf\{\alpha \geq 0 \mid \mathscr{H}^\alpha(E) < \infty\}.\end{aligned}$$

An $\mathscr{H}^\alpha$-measurable set E is called α-set if $0 < \mathscr{H}^\alpha(E) < \infty$.

Let $(\underline{S}_1, \dots, \underline{S}_N)$ be an N-tuple of $\underline{S}_k = (S_{k1}, \dots, S_{kN}) \in \mathrm{Con}(Y)^N$, $k = 1, \dots, N$. For a non-negative number β, we define an $N \times N$ non-negative matrix $R(\beta) = [R(\beta)_{kj}]$ by

$$R(\beta)_{kj} = r(S_{kj})^\beta \qquad k, j = 1, \dots, N$$

where $r(S_{kj})$ is the contraction ratio of S_{kj} and $r(\phi) = 0$ where ϕ is the null contraction. Let $\lambda(\beta)$ be the maximal eigenvalue of $R(\beta)$. Assume that $\lambda(0) > 1$. Then there exists a unique $\alpha > 0$ such that $\lambda(\alpha) = 1$.

PROPOSITION 2.2. *Under the assumption of Theorem* 2.1, *let* $(K_1, \dots, K_N)$ *be the unique N-tuple of non-empty compact sets which satisfies the equalities* (1) *of Theorem* 2.1, *then it holds that*

$$\dim_{\mathrm{H}}(K_k) \leq \alpha \qquad \text{for} \quad k = 1, \dots, N$$

where α *is such that* $\lambda(\alpha) = 1$.

Proposition 2.2 is a special case of Proposition 3.9 in Section 3.

REMARK. If $K = \bigcup_{k=1}^N S_{0k}(K_k)$ for an N-tuple $(S_{01}, \dots, S_{0N})$ of contractions, then $\dim_{\mathrm{H}}(K) \leq \alpha$.

Now we give the definition of Markov-self-similarity. A mappig $S\colon Y \to Y$ is called a similarity if there exists an $r > 0$ such that $d(Sx, Sy) = rd(x, y)$ for all $x, y \in Y$. We define $\mathrm{Sim}(Y)^N$ in the same manner as $\mathrm{Con}(Y)^N$ except that all contractions are contraction similalities.

DEFINITION 2.3. Let $\mathbf{S} = (\underline{S}_1, \dots, \underline{S}_N)$ be an N-tuple of $\underline{S}_k = (S_{k1}, \dots, S_{kN}) \in \mathrm{Sim}(Y)^N$, $k = 1, \dots, N$. An N-tuple $(K_1, \dots, K_N)$ of non-empty compact sets is called *Markov-self-similar* with respect to $\mathbf{S}$ if

$$K_k = \bigcup_{i=1}^N S_{ki}(K_i) \qquad \text{for} \quad k = 1, \dots, N$$

and if for some $\alpha \geq 0$, K_k is an α-set and $\mathscr{H}^\alpha(S_{ki}(K_i) \cap S_{kj}(K_j)) = 0$ for all $k = 1, \dots, N$ and $i \neq j$. A non-empty compact set K is called *Markov-self-similar* with respect to $\mathbf{S}$ if there exist a Markov-self-similar N-tuple $(K_1, \dots, K_N)$ with respect to $\mathbf{S}$ and an N-tuple $(S_1, \dots, S_N)$ of contractions such that $K = \bigcup_{k=1}^N S_k(K_k)$ and $\mathscr{H}^\alpha(S_i(K_i) \cap S_j(K_j)) = 0$.

An $N \times N$ matrix R is called irreducible if for any $i, j \in \{1, \dots, N\}$ there exists a positive integer $m = m(i, j)$ such that $(R^m)_{ij} > 0$. For an irreducible non-negative matrix R, the following Frobenius' Theorem holds:

THEOREM 2.4. (Frobenius). *An irreducible non-negative matrix R has a unique maximal positive eigen value λ for which there correspond positive row and column eigenvectors. Furthermore the inequalities*

$$\lambda z \geq Rz \quad \text{for a vector } z \geq 0 \text{ and } z \neq 0$$

or

$$\lambda z \leq Rz \quad \text{for a vector } z \geq 0 \text{ and } z \neq 0$$

imply that $\lambda z = Rz$ and $z > 0$; and the equality

$$Ry = \eta y \quad \text{for a vector } y \geq 0 \text{ and } y \neq 0$$

impliies that $\eta = \lambda$. Moreover it holds that

$$\lambda = \max_{z \geq 0} \min_{0 \leq i \leq N} (Az)_i / z_i = \min_{z \geq 0} \max_{0 \leq i \leq N} (Az)_i / z_i$$

where $z = (z_1, \dots, z_N)$.

See Gantmacher [6, Ch. 13, §2].

The following theorem states conditions under which an N-tuple of compact sets satisfying (1) in Theorem 2.1 is Markov-self-similar. See Takahashi [12].

THEOREM 2.5. *Let $\mathbf{S} = (\underline{S}_1, \dots, \underline{S}_N)$ be an N-tuple of $\underline{S}_k = (S_{k1}, \dots, S_{kN}) \in \mathrm{Sim}(\mathbf{R}^d)^N$, $k = 1, \dots, N$ which satisfies the following conditions:*

a) *There exists a non-empty open set V for which $S_{ki}(V) \subset V$ and $S_{ki}(V) \cap S_{kj}(V) = \emptyset$ if $i \neq j$ for all $k = 1, \dots, N$.*

b) *The matrix $R(0)$ is irreducible and the maximal eigen value $\lambda(0) > 1$.*

Let $(K_1, \dots, K_N)$ be the unique N-tuple of compact sets that satisfies the condition (1) of Theorem 2.1. Then $(K_1, \dots, K_N)$ is Markov-self-similar with respect to $\mathbf{S}$ for α such that $\lambda(\alpha) = 1$. Furthermore there exists $c > 0$ such that

$$\mathscr{H}^\alpha(K_k) = c x_k \quad k = 1, \dots, N$$

where $(x_1, \dots, x_N)$ is a positive eigenvector of $R(\alpha)$ corresponding to the maximal eigen value 1.

REMARKS (i) If $\underline{S}_k = \underline{S} = (S_1, \dots, S_N)$ for all $k = 1, \dots, N$, the Hausdorff dimension α is obtained as an α for which $\sum_{i=1}^N r(S_i)^\alpha = 1$, because of Theorem 2.4 (Frobenius).

(ii) For $\mathbf{S} = (\underline{S}_1, \dots, \underline{S}_N)$ such that $\underline{S}_k = (S_{k1}, \dots, S_{kN})$ with $r(S_{ki}) = r_k$ for

$i = 1, \dots, N$ and $k = 1, \dots, N$, the Hausdorff dimension α is obtained as an α for which

$$\sum_{k=1}^{N} r_k^\alpha = 1,$$

because $(r_1^\alpha, \dots, r_N^\alpha)$ is a positive eigenvector corresponding to the eigen value 1.

(iii) a) Even if $R(0)$ is reducible, there exists at least one $k \in \{1, \dots, N\}$ such that K_k is an α-set.

b) There exists $\mathbf{S} = \{\underline{S}_1, \dots, \underline{S}_N\}$ for which $R(0)$ is reducible and $\mathscr{H}^\alpha(K_i) = 0$ and $\mathscr{H}^\alpha(K_j) = \infty$ for some $i, j \in \{1, \dots, N\}$.

For the proof of Theorem 2.5 we need a lemma (cf. Falconer [4]).

LEMMA 2.6. *Under the assumptions of Theorem 2.5 there exists an N-tuple $(\mu_1, \dots, \mu_N)$ of Borel probability measures such that, for any measurable set F and $k = 1, \dots, N$,*

(ii) $$\mu_k(F) = \sum_{i=1}^{N} r(S_{ki})^\alpha \mu_i(S_{ki}^{-1}(F))$$

and

$$\mu_k(\mathbf{R}^d) = x_k$$

where $(x_1, \dots, x_N)$ is a positive eigenvector of $R(\alpha)$ corresponding to the maximal eigen value 1. *Furthermore μ_k has the support contained in K_k for $k = 1, \dots, N$.*

PROOF. Choose $y \in K_1$ and write

$$y_{i_1 i_2 \dots i_m} = S_{i_1 i_2} \circ S_{i_2 i_3} \circ \cdots \circ S_{i_{m-1} i_m}(y)$$

for $i_1, \dots, i_m = 1, \dots, N$. Let us write $r(S_{ij})$ by r_{ij}. For $k = 1, \dots, N$ and $m = 1, 2, \dots$, define positive linear functionals $\varphi_m^{(k)}$ on the space $C(K_k)$ of contiuous functions on K_k by

$$\varphi_m^{(k)}(f) = \sum_{i_1 \dots i_m = 1}^{N} (r_{ki_1} r_{i_1 i_2} \cdots r_{i_{m-1} i_m})^\alpha x_{i_m} f(y_{ki_1 \dots i_m}).$$

Note that $y_{ki_1 \dots i_m} \in K_k$ or $y_{ki_1 \dots i_m} = \emptyset$ and that $r(\emptyset) = 0$. Usual arguments show that $\lim_{m \to \infty} \varphi_m^{(k)}$ defines a positive linear functional $\varphi^{(k)}$ on $C(K_k)$. By the Riesz representation theorem, there exists Borel measure μ_k such that

$$\int f \, d\mu_k = \varphi^{(k)} f = \lim_{m \to \infty} \varphi_m^{(k)} f$$

for $f \in \mathrm{C}(K_k)$. Putting $f \equiv 1$, it follows that $\mu_k(\mathbf{R}^d) = x_k$ because

$$\sum_{j=1}^{N} r_i^\alpha x_j = x_i.$$

Since $f \in C(K_k)$, μ_k has the support contained in K_k. For $f \in C(K_k)$,

$$\begin{aligned}\varphi_m^{(k)}(f) &= \textstyle\sum_{i_1=1}^N r_{ki_1}^\alpha \left(\sum_{i_2,\dots i_m=1}^N (r_{i_1 i_2}\cdots r_{i_{m-1}i_m})^\alpha x_{i_m} f(S_{ki_1}(y_{i_1\dots i_m}))\right)\\ &= \textstyle\sum_{i=1}^N r_{ki}^\alpha \varphi_{m-1}^{(i)}(f\circ S_{ki}).\end{aligned}$$

Letting $m\to\infty$ we get

$$\int f\,d\mu_k = \textstyle\sum_{i=1}^N r_{ki}^\alpha \displaystyle\int f\circ S_{ki}\,d\mu_i,$$

so (ii) follows. This completes the proof.

PROOF OF THEOREM 2.5. The proof is similar to that of Theorem 8.6 of Falconer [4]. The upper bound: Iterating (1) we get

$$K_k = \textstyle\bigcup_{i_1,\dots i_m}^N S_{ki_1}\circ S_{i_1 i_2}\circ\cdots\circ S_{i_{m-1}i_m}(K_{i_m}).$$

Using $\sum_{i=1}^N r_{ki}^\alpha x_i = x_k$, we get

$$\begin{aligned}&\textstyle\sum_{i_1,\dots i_m}^N |S_{ki_1}\circ S_{i_1 i_2}\circ\cdots\circ S_{i_{m-1}i_m}(K_{i_m})|^\alpha\\ &\quad= \textstyle\sum_{i_1,\dots,i_m}^N (r_{ki_1}r_{i_1 i_2}\cdots r_{i_{m-1}i_m})^\alpha x_{i_m}|K_{i_m}|^\alpha x_{i_m}^{-1}\\ &\quad\le \frac{x_k}{\min_i x_i}\max_i |K_i|^\alpha < \infty.\end{aligned}$$

As $|S_{ki_1}\circ S_{i_1 i_2}\circ\dots\circ S_{i_{m-1}i_m}(K_{i_m})|^\alpha\to 0$ as $m\to\infty$, we have $\mathscr{H}^\alpha(K_k)<\infty$.

The lower bound: Using similar arguments as in the proof of Theorem 8.6 of Falconer [4] and Lemma 2.6 instead of Lemma 8.4 of Falconer, we can show that

$$\mathscr{H}^\alpha(K_k)\ge x_k(q\max_i x_i)^{-1} > 0$$

where q is a positive finite constant.

Proof of the facts that $\mathscr{H}^\alpha(K_k)=cx_k$ and that $\mathscr{H}^\alpha(S_{ki}(K_k)\cap S_{kj}(K_k))=0$ for $i\neq j$: Using (1) and the fact that S_{ki} are similarities, we get

$$\mathscr{H}^\alpha(K_k)\le \textstyle\sum_{i=1}^N \mathscr{H}^\alpha(S_{ki}(K_i)) = \sum_{i=1}^N r(S_{ki})^\alpha \mathscr{H}^\alpha(K_i)$$

for $k=1,\dots,N$. By Theorem 2.4 (Frobenius) it follows that

(a) $$\mathscr{H}^\alpha(K_k) = \textstyle\sum_{i=1}^N \mathscr{H}^\alpha(S_{ki}(K_i)) = \sum_{i=1}^N r(S_{ki})^\alpha \mathscr{H}^\alpha(K_i)$$

and that there exists $c>0$ such that

$$\mathscr{H}^\alpha(K_k) = cx_k \quad \text{for } k=1,\dots,N$$

where $(x_1,\dots,x_N)$ is a positive eigenvector of $R(\alpha)$ corresponding to the maximal eigen value 1. As $0<\mathscr{H}^\alpha(K_k)<\infty$, (1) and (a) mean that $\mathscr{H}^\alpha(S_{ki}(K_i)\cap S_{kj}(K_j))=0$ for $i\neq j$. This completes the proof.

EXAMPLE 1. Let $Y = [0, 1]$; $N = 2$; $S_{11}(y) = y/3$, $S_{12}(y) = (y + 2)/3$; $S_{21}(y) = y/9$, $S_{22}(y) = (y + 8)/9$ for $0 \leq y \leq 1$. By Remark (ii) of Theorem 2.5, $\alpha \geq 0$ such that $\lambda(\alpha) = 1$ is obtained as an α for which $(1/3)^\alpha + (1/9)^\alpha = 1$, and it follows that $\alpha = (\log(\sqrt{5} + 1) - \log 2)/(\log 3)$. By Theorem 2.5 we have

$$\mathscr{H}^\alpha(K_1) : \mathscr{H}^\alpha(K_2) = (\sqrt{5} - 1) : (3 - \sqrt{5}).$$

EXAMPLE 2. Let $Y = [0, 1]$; $N = 3$; $S_{11}(y) = S_{21}(y) = y/9$, $S_{12}(y) = S_{22}(y) = (y + 4)/9$, $S_{13}(y) = S_{23}(y) = (y + 8)/9$, $S_{31}(y) = y/4$, $S_{32}(y) = (y + 3)/4$, $S_{33} = \phi$ for $0 \leq y \leq 1$. The matrix $R(0) = [r(S_{ki})^0]_{ki}$ is irreducible, $\lambda(1/2) = 1$ and the vector $(1, 1, 1)$ is an eigenvector corresponding to the maximal eigen value 1. Therefore the Hausdorff dimension α equals to $1/2$ and $\mathscr{H}^{1/2}(K_1) : \mathscr{H}^{1/2}(K_2) : \mathscr{H}^{1/2}(K_3) = 1 : 1 : 1$.

3. Random Markov-self-similar sets

Random self-similar sets were investigated by Mauldin-Williams [10], Falconer [5] and Graf [7]. In this section we consider *random* Markov-self-similar sets which are probabilistic counterparts of Markov-self-similar sets defined in Section 2. Our results and techniques were inspired by the work of Graf [7], and all of the results are proved in Appendix.

We introduce the scheme used by Graf [7] with necessary modifications. Let (X, d) be a complete separable metric space whose diameter $|X|$ is *finite*. Fix a positive integer $N \geq 2$. The definition of $\mathrm{Con}(X)^N$ is given in Section 2. Let

$$D = D(N) = \textstyle\bigcup_{m=0}^{\infty} C_m$$

where $C_m = C_m(N) = \{1, 2, \dots, N\}^m$ and $C_0 = \{\emptyset\}$. If $\sigma = (\sigma_1, \dots, \sigma_m) \in D$, then $|\sigma| = m$ is the length of σ (in particular $|\emptyset| = 0$), $\sigma|n = (\sigma_1, \dots, \sigma_n)$ where $n \leq m$ and $t(\sigma) = \sigma_m$. Let $\sigma * \tau = (\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_r)$ for $\tau = (\tau_1, \dots, \tau_r) \in D$.

Our fundamental space is $\Omega = (\mathrm{Con}(X)^N)^D$ equipped with the product topology. The element of $\Omega = (\mathrm{Con}(X)^N)^D$ will be denoted by

$$\mathscr{S} = (\mathscr{S}_\sigma)_{\sigma \in D}$$

where $\mathscr{S}_\sigma = (S_{\sigma*1}, \dots, S_{\sigma*N}) \in \mathrm{Con}(X)^N$.

Let μ and $(\mu_1, \dots, \mu_N)$ be a probability measure and an N-tuple of probability measures on $\mathrm{Con}(X)^N$. As a probabilistic counterpart of (2′) in Section 2 we define a probability measure $\langle \mu \rangle = \langle \mu : \mu_1, \dots, \mu_N \rangle$ on $\Omega = (\mathrm{Con}(X)^N)^D$ as follows: Let $\{B_\sigma | \sigma \in \bigcup_{k=0}^{m} C_k\}$ be a collection of Borel sets in $\mathrm{Con}(X)^N$, i.e. $B_\sigma \in \mathscr{B}(\mathrm{Con}(X)^N)$, then

$$\langle \mu : \mu_1, \dots, \mu_N \rangle (\{\mathscr{S} \in \Omega \mid \mathscr{S}_\sigma \in B_\sigma \text{ for } \sigma \in \textstyle\bigcup_{k=0}^m C_k\})$$
$$= \mu(\mathscr{S}_\emptyset \in B_\emptyset) \textstyle\prod_{\sigma \in \bigcup_{k=1}^m C_k} \mu_{t(\sigma)}(B_\sigma),$$

and Kolmogorov's extension theory determines $\langle \mu : \mu_1, \dots, \mu_N \rangle$ on Ω. Taking $\mu = \mu_k$, we have $\langle \mu_k \rangle = \langle \mu_k : \mu_1, \dots, \mu_N \rangle$ where $k = 1, \dots, N$.

Consider an $N \times N$ matrix $R(\beta) = [R(\beta)_{ij}]$ corresponding to $(\mu_1, \dots, \mu_N)$ defined by

$$R(\beta)_{ij} = \int r(S_j)^\beta d\mu_i(S_1, \dots, S_N)$$

where $\beta \geq 0$ and $0^0 = 0$, and let $\lambda(\beta)$ be the maximal eigen value of non-negative matrix $R(\beta)$. Recall that $r(S)$ is the contraction ratio of a contraction S and that $r(\emptyset) = 0$.

In the following we consider an N-tuple of Borel probability measures $(\mu_1, \dots, \mu_N)$ which satisfies the following conditions (3), (4) and (5):

(3) $R(0)$ is irreducible.

(4) If $R(0)_{ij} > 0$, then $r(S_j) > 0$ for μ_i-a.e. $(S_1, \dots, S_N)$.

(5) $\lambda(0) > 1$.

Furthermore we assume that μ_0 satisfies the following condition (6):

(6) $\sum_{i=1}^N r(S_i) > 0$ μ_0-a.e. $(S_1, \dots, S_N)$.

REMARK. If $R(0)_{ij} = 0$, then $r(S_i) = 0$ for μ_i-a.e. $(S_1, \dots, S_N)$, because $R(0)_{ij} = \int r(S_j)^0 d\mu_i(S_1, \dots, S_N)$.

Recall that $\mathscr{K}(X)$ is the space of all non-empty compact sets of X. In order to construct a probability measure $(\mathscr{K}(X), \mathscr{E}, P_{<\mu_0>})$ from $(\Omega = (\mathrm{Con}(X)^N)^D, \mathscr{B}, \langle \mu_0 \rangle)$, we state necessary results. First the following proposition is obvious by the definition of $\langle \mu_0 \rangle = \langle \mu_0 : \mu_1, \dots, \mu_N \rangle$

PROPOSITION 3.1. *Define* $\varphi : \mathrm{Con}(X)^N \times \Omega^N \to \Omega$ *by*

$$\varphi((S_1, \dots, S_N), (\mathscr{S}^{(1)}, \dots, \mathscr{S}^{(N)})) := \mathscr{S}$$

where

$$\mathscr{S}_\emptyset = (S_1, \dots, S_N) \text{ and } \mathscr{S}_{n*\sigma} = (\mathscr{S}^{(n)})_\sigma \text{ for } \sigma \in D \text{ and } n = 1, \dots, N.$$

Then φ *is Borel measurable and satisfies that for every Borel set* $B \subset \Omega$,

$$[\mu_0 \times \textstyle\prod_{i=1}^N \langle \mu_i \rangle](\varphi^{-1}(B)) = \langle \mu_0 \rangle (B).$$

LEMMA 3.2.

$$\Omega_0 = \{\mathscr{S} \in \Omega \mid \prod_{n=1}^{\infty} r(S_{\sigma|n}) = 0 \text{ for any } \sigma \in C_\infty(N)\}$$

is a Borel set with $\langle \mu_0 \rangle(\Omega_0) = 1$.

By the definition of $\mathrm{Con}(X)^N$, it follows that

$$\bigcap_{m>0} \bigcup_{\sigma \in C_m} \overline{S_{\sigma|1} \circ \cdots \circ S_{\sigma||\sigma|}(X)} \neq \emptyset .$$

PROPOSITION 3.3. *Fix* $\tilde{K} \in \mathscr{K}(X)$ *and define* $\psi: \Omega \to \mathscr{K}(X)$ *by*

$$\psi(\mathscr{S}) = \begin{cases} \bigcap_{m>0} \bigcup_{\sigma \in C_m} \overline{S_{\sigma|1} \circ \cdots \circ S_{\sigma||\sigma|}(X)} & \text{for } \mathscr{S} \in \Omega_0, \\ \tilde{K} & \text{for } \mathscr{S} \notin \Omega_0 . \end{cases}$$

Then ψ *is a Borel measurable map.*

Lemma 3.2 and Proposition 3.3 are proved in Appendix 1.

DEFINITION 3.4. For an N-tuple $(\mu_1, \dots, \mu_N)$ of Borel probability measures and a Borel probability measure μ_0 on $\mathrm{Con}(X)^N$, let $P_{\langle \mu_0 \rangle}$ be the image measure of $\langle \mu_0 \rangle = \langle \mu_0 : \mu_1, \dots, \mu_N \rangle$ with respect to ψ, i.e., for evry Borel set $B \subset \mathscr{K}(X)$,

$$P_{\langle \mu_0 \rangle}(B) = \langle \mu_0 \rangle(\psi^{-1}(B)).$$

REMARK. A $P_{\langle \mu_0 \rangle}$-random set is constructed as follows: Choose an N-tuple $(S_1, \dots, S_N)$ at ramdom with respect to the initial measure μ_0. Let

$$A_1 = \bigcup_{k=1}^{N} S_k(X).$$

Then for $k = 1, \dots, N$, choose an N-tuple $(S_{k1}, \dots, S_{kN})$ with respect to μ_k. Set

$$A_2 = \bigcup_{k=1}^{N} S_k(\bigcup_{i=1}^{N} S_{ki}(X)).$$

Continue this process. The limit set $K = \bigcap_{n \in \mathbf{N}} \bar{A}_n$ is a $P_{\langle \mu_0 \rangle}$-random set. This construction is a stochastic version of that of a Markov-self-similar set in Section 2.

DEFINITION 3.5. Let $(\mu_1, \dots, \mu_N)$ be an N-tuple of Borel probability measures on $\mathrm{Con}(X)^N$. An N-tuple $(P_1, \dots, P_N)$ of probability measures on $\mathscr{K}(X)$ is called $(\mu_1, \dots, \mu_N)$-Markov-self-similar if for every Borel set $B \subset \mathscr{K}(X)$,

$$P_k(B) = [\mu_k \times \prod_{i=1}^{N} P_i](\{((S_1, \dots, S_N), (K_1, \dots, K_N)) \in \mathrm{Con}(X)^N \times \mathscr{K}(X)^N \mid \bigcup_{i=1}^{N} S_i(K_i) \in B\})$$

for all $k = 1, \dots, N$.

PROPOSITION 3.6. *Let* $(\mu_1, \dots, \mu_N)$ *be an* N*-tuple of Borel probability measures on* $\mathrm{Con}(X)^N$. *Then the* N*-tuple* $(P_{\langle \mu_1 \rangle}, P_{\langle \mu_2 \rangle}, \dots, P_{\langle \mu_N \rangle})$ *is the unique*

$(\mu_1, \ldots, \mu_N)$-*Markov-self-similar* N-*tuple of probability measures on* $\mathscr{K}(X)$ *where* $\langle \mu_k \rangle = \langle \mu_k : \mu_1, \ldots, \mu_N \rangle$.

Taking $\mu_k = \delta_{(S_{k1}, \ldots, S_{kN})}$ for $k = 1, \ldots, N$ in Proposition 3.6, we have the statement (1) of Theorem 2.1. Proposition 3.6 is proved in Appendix 2.

The next theorem assures the existance of α such that $P_{\langle \mu_k \rangle}$-a.e. compact set has the Hausdorff dimension α for $k = 1, \ldots, N$.

Theorem 3.7. *Let* $(\mu_1, \ldots, \mu_N)$ *and* μ_0 *be an* N-*tuple of probability measures and a probability measure on* $\mathrm{Con}(X)^N$ *which satisfy the conditions* (3), (4), (5) *and* (6). *Suppose that, for* $k = 1, \ldots, N$, μ_k-*a.e.* $(S_1, \ldots, S_N) \in \mathrm{Con}(X)^N$ *and every* $i = 1, \ldots, N$ *such that* $R(0)_{ki} > 0$, *there exists a* $c > 0$ *with* $d(S_i x, S_i y) \geq c d(x, y)$ *for all* $x, y \in X$. *Then there exists an* $\alpha \geq 0$ *such that*

$$\dim_{\mathrm{H}}(K) = \alpha$$

for $P_{\langle \mu_0 \rangle}$-*a.e.* $K \in \mathscr{K}(X)$. *Especially it holds that* $\dim_{\mathrm{H}}(K) = \alpha$ *for* $P_{\langle \mu_k \rangle}$-*a.e.* $K \in \mathscr{K}(X)$.

Theorem 3.7 is proved in Appendix 3 and the following 0–1 law is used in the proof.

Proposition 3.8. *Assume that an* N-*tuple* $(\mu_1, \ldots, \mu_N)$ *of Borel probability measures on* $\mathrm{Con}(X)^N$ *satisfies the conditions* (3) *and* (5). *Let* B *be a Borel set in* $\Omega = (\mathrm{Con}(X)^N)^D$. *If*

$$\langle \mu_k \rangle (B) = \prod_{i: R(0)_{ki} > 0} \langle \mu_i \rangle (B)$$

for all $k = 1, \ldots, N$, *then*

$$\langle \mu_k \rangle (B) = 0 \qquad \textit{for all} \quad k = 1, \ldots, N,$$

or

$$\langle \mu_k \rangle (B) = 1 \qquad \textit{for all} \quad k = 1, \ldots, N.$$

Proof. Assume that $\langle \mu_j \rangle (B) = 0$ for some $j \in \{1, \ldots, N\}$. Using the irreducibility of $R(0)$ we deduce that $\langle \mu_k \rangle (B) = 0$ for all $k = 1, \ldots, N$. Now assume that $\langle \mu_k \rangle (B) \neq 0$. Note that

$$\prod_{k=1}^{N} \langle \mu_k \rangle (B) = \prod_{k=1}^{N} \prod_{i: R(0)_{ki} \neq 0} \langle \mu_i \rangle (B)$$

and that

$$\sum_{k=1}^{N} \# \{ i \,|\, R(0)_{ki} \neq 0 \} > N$$

because $\lambda(0) > 1$. Therefore there exists a $j \in \{1, \ldots, N\}$ such that

$$\langle \mu_j \rangle (B) = 1.$$

Using the irreducibility of $R(0)$ we duduce that

$$\langle\mu_k\rangle(B)=1 \qquad \text{for all} \quad k=1,\dots,N.$$

REMARK. Under the assumptions of Proposition 3.8, the statement in Proposition 3.8 is true for $(P_{\langle\mu_1\rangle},\dots,P_{\langle\mu_N\rangle})$: Let B be a Borel set in $\mathscr{K}(X)$. If

$$P_{\langle\mu_k\rangle}(B)=\prod_{i:R(0)_{ki}\neq 0} P_{\langle\mu_i\rangle}(B)$$

for all $k=1,\dots,N$, then

$$P_{\langle\mu_k\rangle}(B)=0 \qquad \text{for all} \quad k=1,\dots,N,$$

or

$$P_{\langle\mu_k\rangle}(B)=1 \qquad \text{for all} \quad k=1,\dots,N.$$

An upper bound for the Hausdorff dimension of $P_{\langle\mu_0\rangle}$-random sets is given by the following proposition which is an extension of the result obtained by Mauldin-Williams [10], Falconer [5] and Graf [7].

PROPOSITION 3.9. *Let* $(\mu_1,\dots,\mu_N)$ *and* μ_0 *be an* N*-tuple of probability measures and a probability measure on* $\mathrm{Con}(X)^N$ *which satisfy the condition* (5). *Let* α *be such that* $\lambda(\alpha)=1$. *Then*

$$E_{P_{\langle\mu_0\rangle}}(\mathscr{H}^\alpha(K))<\infty.$$

In particular

$$\mathscr{H}^\alpha(K)<\infty \qquad \text{for } P_{\langle\mu_0\rangle}\text{-a.e. } K\in\mathscr{K}(X)$$

and

$$\dim_{\mathrm{H}}(K)\leq\alpha \qquad \text{for } P_{\langle\mu_0\rangle}\text{-a.e. } K\in\mathscr{K}(X).$$

Especially we have the corresponding statements for $P_{\langle\mu_k\rangle}$*-a.e.* K.

REMARK. The uniqueness of α for which $\lambda(\alpha)=1$ follows from the fact that $\lambda(\beta)$ is continuous and strictly decreasing with respect to β.

The proof of Proposition 3.9. is given in Appendix 4. In the proof we use the following martingale convergence theorem (Theorem 3.10). Let Γ be a subset in D, and define $f^{(k)}_{\Gamma,\beta}\colon(\Omega,\mathscr{B},\langle\mu_k\rangle)\to\mathbf{R}_+$ by

$$f^{(k)}_{\Gamma,\beta}(\mathscr{S})=\sum_{\sigma\in\Gamma}\left[\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\beta\right]x_{t(\sigma)}$$

and

$$f^{(k)}_{\{\emptyset\},\beta}(\mathscr{S})=x_k,$$

for $k=1,\dots,N$ where $(x_1,\dots,x_N)$ is an positive eigenvector of $R(\alpha)$

corresponding to the maximal eigen value 1. We abbreviate $f^{(k)}_{C_m,\beta}$ by $f^{(k)}_{m,\beta}$.

THEOREM 3.10. *Let* $(\mu_1, \dots, \mu_N)$ *be an N-tuple of probability measures on* $\mathrm{Con}(X)^N$ *which satisfies the conditions* (3) *and* (5). *Let* α *be the unique value such that* $\lambda(\alpha) = 1$. *For* $m \in \mathbf{N}$ *let* $\mathscr{B}_m$ *be the* σ*-field of all Borel subsets in* $\Omega = (\mathrm{Con}(X)^N)^D$ *depending only on coodinates from* $D_m = \bigcup_{k \le m} C_m$. *Then for every* $p \in \mathbf{N}$ *and* $k = 1, \dots, N$, $(f^{(k)}_{m,\alpha})_{m\in\mathbf{N}}$ *is an* L^p*-bounded martingale with respect to* $(\mathscr{B}_m)_{m\in\mathbf{N}}$ *which converges* $\langle\mu_k\rangle$*-a.e. and in* $L^p(\Omega, \langle\mu_k\rangle)$ *to a function* $f^{(k)}$. *Furthermore if the condition* (4) *holds, then* $f^{(k)} > 0$ *for* $\langle\mu_k\rangle$*-a.e. and* $k = 1, \dots, N$.

Theorem 3.10 is proved in Appendix 4.

The following theorem gives conditions which assure that, for $P_{\langle\mu_k\rangle}$-a.e. compact sets, the Hausdorff dimension is equal to α.

THEOREM 3.11. *Let* $X \subset \mathbf{R}^d$ *be a compact set with the non-empty interior* $\mathring{X}$. *Let* $(\mu_1, \dots, \mu_N)$ *and* μ_0 *be an N-tuple of probability measures and a probability measure on* $\mathrm{Con}(X)^N$ *which satisfy the conditions* (3), (4), (5) *and* (6). *Suppose that, for* μ_k*-a.e.* $(S_1, \dots, S_N) \in \mathrm{Con}(X)^N$ *and* $k = 1, \dots, N$, *the followng conditions are satisfied.*

a) *For all* $i = 1, \dots, N$, S_i *is a contraction similarity or the null contraction* ϕ.

b) $(S_1, \dots, S_N)$ *satisfies the following open set condition*: $S_i(\mathring{X}) \cap S_j(\mathring{X}) = \emptyset$ *if* $i \ne j$.

Let $\alpha \ge 0$ *be such that* $\lambda(\alpha) = 1$. *Then* $\dim_{\mathrm{H}}(K) = \alpha$ *for* $P_{\langle\mu_0\rangle}$*-a.e.* $K \in \mathscr{K}(X)$. *Especially* $\dim_{\mathrm{H}}(K) = \alpha$ *for* $P_{\langle\mu_k\rangle}$*-a.e.* $K \in \mathscr{K}(X)$ *and* $k = 1, \dots, N$.

Theorem 3.11 is proved in Appendix 5.

EXAMPLE. Let $X = [0, 1]$ and $N = 2$. Let T_1, T_2 and T_3 be similalities which map $[0, 1]$ to $[0, 1/3]$, $[1/3, 2/3]$ and $[2/3, 1]$ respectively, and $\tilde{T}_1$, $\tilde{T}_2$, $\tilde{T}_3$ and $\tilde{T}_4$ be similalities which map $[0, 1]$ to $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$ and $[3/4, 1]$ respectively. Let

$$\mu_1 = 3^{-1}\{\varepsilon_{(T_1,T_2)} + \varepsilon_{(T_2,T_3)} + \varepsilon_{(T_1,T_3)}\}$$

and

$$\mu_2 = 6^{-1} \sum_{1 \le i < j \le 4} \varepsilon_{(\tilde{T}_i,\tilde{T}_j)}\,.$$

Then (μ_1, μ_2) is a pair of probability measures on $\mathrm{Con}(X)^2$, and it satisfies the conditions (3), (4) and (5). By Theorem 3.11,

$$\dim_{\mathrm{H}}(K) = \alpha \text{ for } P_{\langle\mu_k\rangle}\text{-a.e. } K \in \mathscr{K}([0, 1]) \text{ and } k = 1, 2$$

where α is such that $(1/3)^\alpha + (1/4)^\alpha = 1$.

4. Hausdorff measures of random Markov-self-similar sets

First we state a theorem which corresponds to Theorem 7.8 of Graf [7].

THEOREM 4.1. *Let the assumptions of Theorem* 3.11 *be satisfied. Suppose that there exists a* $\delta > 0$ *such that if* $R(0)_{ki} > 0$, *then* $r(S_i) \geq \delta$ *for* μ_k*-a.e.* $(S_1, \dots, S_N)$, $k = 1, \dots, N$. *Let* $(x_1, \dots, x_N)$ *be a positive eigenvector of* $R(\alpha)$ *corresponding to the maximal eigen value* 1. *Then the following statements are equivalent*:

a) $\sum_{i=1}^N r(S_i)^\alpha x_i = x_k$ *for* μ_k*-a.e.* $(S_1, \dots, S_N)$ *and all* $k \in \{1, \dots, N\}$.
b) $\mathscr{H}^\alpha(K) > 0$ *for* $P_{\langle \mu_k \rangle}$*-a.e.* $K \in \mathscr{K}(X)$ *and all* $k \in \{1, \dots, N\}$.
c) $P_{\langle \mu_j \rangle}(\{K \in \mathscr{K}(X) | \mathscr{H}^\alpha(K) > 0\}) > 0$ *for some* $j \in \{1, \dots, N\}$.

Theorem 4.1 is proved in Appendix 6.

The following theorem gives an information about the α-dimensional Haudorff measure $\mathscr{H}^\alpha(K)$ for $P_{\langle \mu_k \rangle}$-a.e. $K \in \mathscr{K}(X)$ for Markov-self-similar $(P_{\langle \mu_N \rangle}, \dots, P_{\langle \mu_N \rangle})$. See [13] and [14].

THEOREM 4.2. *Let the assumptions and the condition c) of Theorem* 4.1 *be satisfied. Then there exists a* $c > 0$ *such that*

$$\mathscr{H}^\alpha(K) = cx_k$$

for $P_{\langle \mu_k \rangle}$*-a.e.* $K \in \mathscr{K}(X)$ *and all* $k \in \{1, \dots, N\}$.

For the proof of Theorem 4.2 we show the following lemma:

LEMMA 4.3. Assume that $0 < E_{\langle \mu_k \rangle}(\mathscr{H}^\alpha(K(\mathscr{S}))) < \infty$ for $k = 1, \dots, N$ and that

$$\sum_{i=1}^N r(S_i)^\alpha x_i = x_k$$

for μ_k*-a.e.* $(S_1, \dots, S_N)$ *and* $k = 1, \dots, N$. *Then it holds that*

$$\mathscr{H}^\alpha(K(\mathscr{S})) = \sum_{i=1}^N r(S_i(\mathscr{S}))^\alpha \mathscr{H}^\alpha(K(\mathscr{S}^{(i)}))$$

for $\langle \mu_k \rangle$*-a.e.* $\mathscr{S}$ *and* $k = 1, \dots, N$. *Furthermore there exists a* $c > 0$ *such that*

$$E_{\langle \mu_k \rangle}(\mathscr{H}^\alpha(K(\mathscr{S}))) = cx_k \quad \textit{for} \quad k = 1, \dots, N.$$

PROOF. Since

$$K(\mathscr{S}) \subset \bigcup_{i=1}^N S_i(K(\mathscr{S}^{(i)}))$$

and S_i are similarities, it follows that

$$\mathscr{H}^\alpha(K(\mathscr{S})) \leq \sum_{i=1}^N r(S_i(\mathscr{S}))^\alpha \mathscr{H}^\alpha(K(\mathscr{S}^{(i)})).$$

Integrating the both sides with respect to $\langle\mu_k\rangle$ and using Proposition 3.1,

$$E_{\langle\mu_k\rangle}[\mathscr{H}^\alpha(K(\mathscr{S}))] \le \sum_{i=1}^N R(\alpha)_{ki} E_{\langle\mu_i\rangle}[\mathscr{H}^\alpha(K(\mathscr{S})]$$

for $k = 1, \dots, N$. Since $0 < E_{\langle\mu_k\rangle}(\mathscr{H}^\alpha(K(\mathscr{S}))) < \infty$, we deduce, by Theorem 2.4 (Frobenius), that there exists a $c > 0$ such that

$$E_{\langle\mu_k\rangle}[\mathscr{H}^\alpha(K(\mathscr{S}))] = cx_k \qquad \text{for} \quad k = 1, \dots, N$$

$$E_{\langle\mu_k\rangle}[\mathscr{H}^\alpha(K(\mathscr{S}))] = \sum_{i=1}^N R(\alpha)_{ki} E_{\langle\mu_i\rangle}[\mathscr{H}^\alpha(K(\mathscr{S})]$$

for $k = 1, \dots, N$. Therefore

$$\mathscr{H}^\alpha(K(\mathscr{S})) = \sum_{i=1}^N r(S_i(\mathscr{S}))^\alpha \mathscr{H}^\alpha(K(\mathscr{S}^{(i)}))$$

for $\langle\mu_k\rangle$-a.e. $\mathscr{S}$ and $k = 1, \dots, N$. This completes the proof.

Proof of Theorem 4.2. Proposition 3.9 and Theorem 4.1 assure the assumptions of Lemma 4.3. Iterating Lemma 4.3, we have

$$\mathscr{H}^\alpha(K(\mathscr{S})) = \sum_{i_1=1}^N r(S_{i_1}(\mathscr{S}))^\alpha \sum_{i_2=1}^N r(S_{i_2}(\mathscr{S}^{(i_1)}))^\alpha \sum \cdots$$
$$\sum_{i_m=1}^N r(S_{i_m}(\mathscr{S}^{(i_1)\dots(i_{m-1})}))^\alpha \mathscr{H}^\alpha(K(\mathscr{S}^{(i_1)\dots(i_m)}))$$

for $\langle\mu_k\rangle$-a.e. $\mathscr{S}$ and $k = 1, \dots, N$ where $\mathscr{S}^{(i_1)(i_2)} = (\mathscr{S}^{(i_1)})^{(i_2)}$ and so on. Consider $E_{\langle\mu_k\rangle}[\mathscr{H}^\alpha(K(\mathscr{S}))|\mathscr{B}_{m-1}]$ where $\mathscr{B}_{m-1}$ are the σ-field of all Borel subsets in $\Omega = (\mathrm{Con}(X)^N)^D$ depending only on coordinates from $\bigcup_{i \le m-1} C_i$. Using Proposition 3.1 we have

$$E_{\langle\mu_k\rangle}[\mathscr{H}^\alpha(K(\mathscr{S}))|\mathscr{B}_{m-1}] =$$
$$\sum_{i_1=1}^N r(S_{i_1}(\mathscr{S}))^\alpha \sum_{i_2=1}^N r(S_{i_2}(\mathscr{S}^{(i_1)}))^\alpha \cdots$$
$$\sum_{i_m=1}^N r(S_{i_m}(\mathscr{S}^{(i_1)\dots(i_{m-1})}))^\alpha E_{\langle\mu_{i_m}\rangle}[\mathscr{H}^\alpha(K(\mathscr{S}))].$$

Since $\sum_{i=1}^N r(S_i)^\alpha x_i = x_k$ and $E_{\langle\mu_k\rangle}(\mathscr{H}^\alpha(K(\mathscr{S}))) = cx_k$, it follows that

$$E_{\langle\mu_k\rangle}[\mathscr{H}^\alpha(K(\mathscr{S}))|\mathscr{B}_{m-1}] = c\,x_k.$$

As m is arbitrary, we have

$$\mathscr{H}^\alpha(K) = cx_k \qquad \text{for } P_{\langle\mu_k\rangle}\text{-a.e. } K \in \mathscr{K}(X) \text{ and } k = 1, \dots, N.$$

Remark. In the case of $\mathscr{H}^\alpha(K) = 0$ for a.e. K, the exact Hausdorff dimension of K was investigated by Graf, Mauldin and Williams [8].

Example. Consider the example stated at the end of Section 3. Theorem 4.2 implies that

$$\mathscr{H}^\alpha(K) = c(1/3)^\alpha \qquad \text{for } P_{\langle\mu_1\rangle}\text{-a.e. } K \in \mathscr{K}(X)$$

and

$$\mathscr{H}^{\alpha}(K) = c(1/4)^{\alpha} \quad \text{for } P_{\langle\mu_2\rangle}\text{-a.e. } K \in \mathscr{K}(X)$$

for some $c > 0$.

APPENDIX

1. Proof of Lemma 3.2 and Proposition 3.3

PROOF OF LEMMA 3.2 (cf. the proof of Lemma 3.2 of Graf [7]). The result that Ω_0 is a Borel set is proved in Lemma 3.2 of Graf [7]. We show that $\langle\mu_0\rangle(\Omega_0) = 1$. By Proposition 3.1, it suffices to prove that $\langle\mu_k\rangle(\Omega_0) = 1$ for $k = 1, \dots, N$. For $a > 0$ set

$$B_a = \{\mathscr{S} \in \Omega \mid \text{there exists } \sigma \in \{1, \dots, N\}^{\mathbf{N}} \text{ such that } \textstyle\prod_{n=0}^{\infty} r(S_{\sigma|n}) \geq a\},$$

then the fact that B_a is Borel measurable is also proved in Lemma 3.2 of Graf [7].

Define $p_k\colon (0, 1) \to [0, 1]$ by $p_k(a) = \langle\mu_k\rangle(B_a)$ for $k = 1, \dots, N$. It follows that from Proposition 3.1 that, for every $a \in (0, 1)$, we have

$$\begin{aligned}
\text{(a1)} \quad p_k(a) &= [\mu_k \times \textstyle\prod_{i=1}^{N} \langle\mu_i\rangle](\{((S_1, \dots, S_N), (\mathscr{S}^{(1)}, \dots, \mathscr{S}^{(N)})) \mid \text{there exist} \\
&\qquad j \in \{1, \dots, N\} \text{ and } \sigma \in \{1, \dots, N\}^{\mathbf{N}} \text{ such that } r(S_j) \textstyle\prod_{n=0}^{\infty} r(\mathscr{S}^{(j)}_{\sigma|n}) \geq a\}) \\
&\leq \textstyle\sum_{j=1}^{N} [\mu_k \times \prod_{i=1}^{N} \langle\mu_i\rangle](\{((S_1, \dots, S_N), (\mathscr{S}^{(1)}, \dots, \mathscr{S}^{(N)})) \mid \text{there exists} \\
&\qquad \sigma \in \{1, \dots, N\}^{\mathbf{N}} \text{ such that } r(S_j) \textstyle\prod_{n=0}^{\infty} r(S^{(j)}_{\sigma|n}) \geq a\}) \\
&\leq \textstyle\sum_{j=1}^{N} \mu_k(\{(S_1, \dots, S_N) \mid r(S_j) \geq a\}) p_j(a).
\end{aligned}$$

Since $r(S) < 1$ there exists a $b \in (0, 1)$ such that

$$\mu_j(\{(S_1, \dots, S_N) \mid \max_{1 \leq i \leq N} r(S_i) \geq b\}) < 1/N$$

for all $j \in \{1, \dots, N\}$. If there exists a k such that $p_k(b) > 0$, let k_1 be such that $p_{k_1}(b) = \max_k p_k(b) > 0$. Then it follows from (a1) that $p_{k_1}(b) < p_{k_1}(b)$. This contradiction implies that $p_k(b) = 0$ for all $k = 1, \dots, N$.

Let $\eta_k = \inf\{a \in (0, 1) \mid p_k(a) = 0\}$ for $k = 1, \dots, N$, and $\eta = \max_{1 \leq k \leq N} \eta_k < 1$. Assume $\eta > 0$. Then there is an $a > \eta$ with $ab < \eta$. We deduce as before

$$\begin{aligned}
p_k(ab) &\leq \textstyle\sum_{j=1}^{N} [\mu_k \times \prod_{i=1}^{N} \langle\mu_i\rangle](\{((S_1, \dots, S_N), (\mathscr{S}^{(1)}, \dots, \mathscr{S}^{(N)})) \mid \text{there exists} \\
&\qquad \sigma \in \{1, \dots, N\}^{\mathbf{N}} \text{ such that } r(S_j) \textstyle\prod_{n=0}^{\infty} r(S^{(j)}_{\sigma|n}) \geq ab\}).
\end{aligned}$$

Since $a > \eta$ we have $p_j(a) = 0$ for $j = 1, \dots, N$, and so

$$\prod_{n=0}^{\infty} r(S^{(j)}_{\sigma|n}) \leq a \text{ for } \langle \mu_j \rangle\text{-a.e. } \mathscr{S}^{(j)} \text{ and } j = 1, \dots, N.$$

This leads to

$$p_k(ab) \leq \sum_{j=1}^{N} \mu_k(\{((S_1, \dots, S_N) | r(S_j) \geq b\}) p_j(ab)$$

for $k = 1, \dots, N$. Assume that there exists a k such that $p_k(ab) > 0$. As before this leads to a contradction, so $p_k(ab) = 0$ for all $k = 1, \dots, N$. This contradicts $ab < \eta$ and the definition of η. Thus $\eta = 0$ and p_k vanishes identically for $k = 1, \dots, N$. This completes the proof.

PROOF OF PROPOSITION 3.3. The proof of Theorem 3.7 of Graf [7] using Lemma 3.2 instead of Lemma 3.2 of Graf [7] implies Proposition 3.3.

2. Proof of Proposition 3.6. (cf. the proof of Theorem 4.5 of Graf [7])

First we give a definition.

DEFINITION. Let $(\mu_1, \dots, \mu_N)$ be an N-tuple of probability measures on $\mathrm{Con}(X)^N$. For $k = 1, \dots, N$, define $T_k = T_k^{(\mu_1, \dots, \mu_N)}: P(\mathscr{K}(X))^N \to P(\mathscr{K}(X))$ by

$$[T_k(Q_1, \dots, Q_N)](B) = [\mu_k \times \prod_{i=1}^{N} Q_i] (\{((S_1, \dots, S_N), (K_1, \dots, K_N)) | \bigcup_{1 \leq j \leq N} S_j(K_j) \in B\})$$

where $P(\mathscr{K}(X))$ is the set of all Borel probability measures on $\mathscr{K}(X)$.

REMARK. An N-tuple $(P_1, \dots, P_N)$ of probability measures on $\mathscr{K}(X)$ is $(\mu_1, \dots, \mu_N)$-Markov-self-similar if and only if

$$P_k = T_k^{(\mu_1, \dots, \mu_N)}(P_1, \dots, P_N)$$

for all $k = 1, \dots, N$.

PROOF OF PROPOSITION 3.6. The proof of Theorem 4.5 of Graf [7] assures that

$$T_k(P_{\langle \mu_1 \rangle}, \dots, P_{\langle \mu_N \rangle}) = P_{\langle \mu_k \rangle}$$

for $k = 1, \dots, N$.

Define $T: P(\mathscr{K}(X))^N \to P(\mathscr{K}(X))^N$ by

$$T(Q_1, \dots, Q_N) = (T_1(Q_1, \dots, Q_N), \dots, T_N(Q_1, \dots, Q_N))$$

for $(Q_1, \dots, Q_N) \in P(\mathscr{K}(X))^N$. Let $A \subset \mathscr{K}(X)$ be a closed set. Using induction on n, we have

$$(T^n(Q_1, \dots, Q_N))_k(A)$$
$$= [\langle\mu_k\rangle \times (\textstyle\prod_{i=1}^N Q_i)^D](\{(\mathscr{S}, (K_{\sigma*1}, \dots, K_{\sigma*N})_{\sigma\in D}) \in \Omega \times (\mathscr{K}(X)^N)^D$$
$$\mid \textstyle\bigcup_{\sigma\in C_{n-1}} \bigcup_{i=1}^N S_{\sigma|1} \circ \cdots \circ S_{\sigma|n-1} \circ S_{\sigma*i}(K_{\sigma*i}) \in A\})$$

Hence we obtain

$$\lim_{n\to\infty}\sup (T^n(Q_1, \dots, Q_N))_k(A)$$
$$= \inf_m \sup_{n\geq m} [\langle\mu_k\rangle \times (\textstyle\prod_{i=1}^N Q_i)^D](\{(\mathscr{S}, (K_{\sigma*1}, \dots, K_{\sigma*N})_{\sigma\in D}) \in \Omega \times$$
$$(\mathscr{K}(X)^N)^D \mid \textstyle\bigcup_{\sigma\in C_{n-1}} \bigcup_{i=1}^N S_{\sigma|1} \circ \cdots \circ S_{\sigma|n-1} \circ S_{\sigma*i}(K_{\sigma*i}) \in A\})$$
$$\leq [\langle\mu_k\rangle \times (\textstyle\prod_{i=1}^N Q_i)^D](\bigcap_m \bigcup_{n\geq m} \{(\mathscr{S}, (K_{\sigma*1}, \dots, K_{\sigma*N})_{\sigma\in D}) \in \Omega \times$$
$$(\mathscr{K}(X)^N)^D \mid \textstyle\bigcup_{\sigma\in C_{n-1}} \bigcup_{i=1}^N S_{\sigma|1} \circ \cdots \circ S_{\sigma|n-1} \circ S_{\sigma*i}(K_{\sigma*i}) \in A\})$$
$$\leq [\langle\mu_k\rangle \times (\textstyle\prod_{i=1}^N Q_i)^D](\{(\mathscr{S}, (K_{\sigma*1}, \dots, K_{\sigma*N})_{\sigma\in D}) \in \Omega \times (\mathscr{K}(X)^N)^D \mid$$
$$\lim_{n\to\infty} \textstyle\bigcup_{\sigma\in C_{n-1}} \bigcup_{i=1}^N S_{\sigma|1} \circ \cdots \circ S_{\sigma|n-1} \circ S_{\sigma*i}(K_{\sigma*i}) \in A\}).$$

By Theorem 2.2 of Graf [7] and the definition of ψ, the last expression equals to

$$[\langle\mu_k\rangle \times (\textstyle\prod_{i=1}^N Q_i)^D](\{(\mathscr{S}, (K_{\sigma*1}, \dots, K_{\sigma*N})_{\sigma\in D}) \in \Omega \times (\mathscr{K}(X)^N)^D \mid \psi(\mathscr{S}) \in A\})$$
$$= \langle\mu_k\rangle(\psi^{-1}(A)).$$

Therefore it holds that

$$\lim_{n\to\infty}\sup (T^n(Q_1, \dots, Q_N))_k(A) \leq P_{\langle\mu_k\rangle}(A).$$

Since this is true for an arbitrary closed set A of $\mathscr{K}(X)$, $\{(T^n(Q_1, \dots, Q_N))_k\}_{n\in\mathbf{N}}$ converges to $P_{\langle\mu_k\rangle}$ in the weak topology. This implie the uniqueness of the $(\mu_1, \dots, \mu_N)$-Markov-self-similar probability measure.

3. Proof of Theorem 3.7

First we show the following 0–1 law (cf. Theorem 7.2 of Graf [7]):

LEMMA A. *For a given* $\beta \geq 0$, it holds that

(a) $P_{\langle\mu_k\rangle}(\{K \in \mathscr{K}(X) \mid \mathscr{H}^\beta(K) = 0\}) = 0$ for all $k = 1, \dots, N$, or $= 1$ for all $k = 1, \dots, N$,

and that

(b) $P_{\langle\mu_k\rangle}(\{K \in \mathscr{K}(X) \mid \mathscr{H}^\beta(K) = \infty\}) = 0$ for all $k = 1, \dots, N$, or $= 1$ for all $k = 1, \dots, N$.

PROOF. By Proposition 3.6 we have

$$P_{\langle\mu_k\rangle}(\{K\in\mathscr{K}(X)|\mathscr{H}^\beta(K)=0\})=[\mu_k\times\prod_{i=1}^N P_{\langle\mu_i\rangle}](\{((S_1,\dots,S_N),(K_1,\dots,K_N))|\mathscr{H}^\beta(\bigcup_{j=1}^N S_j(K_j))=0\})$$
$$=[\mu_k\times\prod_{i=1}^N P_{\langle\mu_i\rangle}](\{((S_1,\dots,S_N),(K_1,\dots,K_N))|\mathscr{H}^\beta(S_j(K_j))=0 \quad \text{for} \quad j=1,\dots,N\})$$
$$=\prod_{i:R(0)_{ki}\neq 0} P_{\langle\mu_i\rangle}(\{K|\mathscr{H}^\beta(K_i))=0\}).$$

By the remark of Proposition 3.8 we have (a). The fact (b) follows in the same way because

$$P_{\langle\mu_i\rangle}(\{K|\mathscr{H}^\beta(K_i))=\infty\})=1-P_{\langle\mu_i\rangle}(\{K|\mathscr{H}^\beta(K_i))<\infty\}).$$

PROOF OF THEOREM 3.7. It is easy to prove the theorem using (a) and (b). See the proof of Corollary 7.3 of Graf [7].

4. Proof of Proposition 3.9 and Theorem 3.10

First we prove Theorem 3.10 (cf. the proof of Theorem 6.3 of Graf [7]).

PROOF OF THEOREM 3.10. Since

$$E_{\langle\mu_k\rangle}[f^{(k)}_{q+1,\alpha}|\mathscr{B}_q]=E_{\langle\mu_k\rangle}[\textstyle\sum_{\tau\in C_{q+1}}\prod_{n=1}^{q+1} r(S_{\tau|n})^\alpha x_{t(\tau)}|\mathscr{B}_q]$$
$$=\textstyle\sum_{\sigma\in C_q}\prod_{n=1}^{q} r(S_{\sigma|n})^\alpha E_{\langle\mu_{t(\sigma)}\rangle}[\sum_{i=1}^N r(S_i)^\alpha x_i]$$
$$=\textstyle\sum_{\sigma\in C_q}\prod_{n=1}^{q} r(S_{\sigma|n})^\alpha x_{t(\sigma)}=f^{(k)}_{q,\alpha}$$

$\langle\mu_k\rangle$-a.e. $\mathscr{S}$ for $q\geq 1$ and

$$E_{\langle\mu_k\rangle}[f^{(k)}_{1,\alpha}|\mathscr{B}_0]=\textstyle\sum_{i=1}^N R(\alpha)_{ki}x_i=x_k=f^{(k)}_{0,\alpha},$$

$(f^{(k)}_{q,\alpha})_{q\in\mathbf{N}}$ is a martingale with respect to $(\mathscr{B}_q)_{q\in\mathbf{N}}$.

By induction on $p\in\mathbf{N}$ we prove $(f^{(k)}_{q,\alpha})_{q\in\mathbf{N}}$ is L^p-bounded. Since $f^{(k)}_{q,\alpha}\geq 0$ and $(f^{(k)}_{q,\alpha})_{q\in\mathbf{N}}$ is a martingale, it is L^1-bounded. Now assume that $p>1$ and that for $m<p$, $(f^{(k)}_{q,\alpha})_{q\in\mathbf{N}}$ is L^m-bounded for all $k=1,\dots,N$. Let

$$M=\sup\{\|f^{(k)}_{q,\alpha}\|_m|q\in\mathbf{N},\ m<p,\ k=1,\dots,N\}<\infty,$$
$$L=\max\{\|f^{(k)}_{0,\alpha}\|_p^p/x_k|k=1,\dots,N\}<\infty,$$
$$C=\max\left\{\frac{1}{x_k}\int(\textstyle\sum_{i=1}^N r(S_i)^\alpha)^p d\mu_k(S_1,\dots,S_N)|k=1,\dots,N\right\}$$
$$\leq N^p/\min_{1\leq k\leq N}x_k$$

and

$$\delta = \max\left\{\int \sum_{i=1}^{N} r(S_i)^{p\alpha} \frac{x_i}{x_k} d\mu_k(S_1, \dots, S_N) | k = 1, \dots, N\right\}.$$

Note that $\delta < 1$ by Theorem 2.4 (Frobenius) because the maximal eigen value of $R(p\alpha)$ is smaller than one. We show by induction on q that

$$\text{(a2)} \qquad \| f_{q,\alpha}^{(k)} \|_p^p \le x_k(\delta^q L + M^p \cdot C \cdot \sum_{i=0}^{q-1} \delta^i).$$

For $q = 0$ it is obvious. Assume that (a2) holds for $q = 1, \dots, n$. For $q = n + 1$, we have

$$\| f_{n+1,\alpha}^{(k)} \|_p^p = \int (f_{n+1,\alpha}^{(k)})^p d\langle\mu_k\rangle$$

$$= \iint \{\sum_{i=1}^{N} r(S_i)^\alpha f_{n,\alpha}^{(i)}(\mathscr{S}^{(i)})\}^p \prod_{i=1}^{N} d\langle\mu_i\rangle(\mathscr{S}^{(i)})\, d\mu_k(S_1, \dots, S_N)$$

$$= \sum_{\nu_1 + \dots + \nu_N = p} \frac{p!}{\nu_1! \cdots \nu_N!} \int r(S_1)^{\nu_1\alpha} \cdots r(S_N)^{\nu_N\alpha} \| f_{n,\alpha}^{(1)} \|_{\nu_1}^{\nu_1} \cdots \| f_{n,\alpha}^{(N)} \|_{\nu_N}^{\nu_N}$$

$$d\mu_k(S_1, \dots, S_N)$$

$$= \int (r(S_1)^{p\alpha} \| f_{n,\alpha}^{(1)} \|_p^p + \cdots + r(S_N)^{p\alpha} \| f_{n,\alpha}^{(N)} \|_p^p) d\mu_k(S_1, \dots, S_N) + \sum_{\substack{\nu_1 + \dots + \nu_N = p \\ \nu_1, \dots, \nu_N < p}}$$

$$\frac{P!}{\nu_1! \cdots \nu_N!} \int r(S_1)^{\nu_1\alpha} \cdots r(S_N)^{\nu_N\alpha} \| f_{n,\alpha}^{(1)} \|_{\nu_1}^{\nu_1} \cdots \| f_{n,\alpha}^{(N)} \|_{\nu_N}^{\nu_N} d\mu_k(S_1, \dots, S_N)$$

$$\le \int \sum_{i=1}^{N} r(S_i)^{p\alpha} x_i(\delta^n L + M^p \cdot C \cdot \sum_{i=0}^{n-1} \delta^i)\, d\mu_k(S_1, \dots, S_N) +$$

$$M^p \sum_{\nu_1 + \cdots \nu_N = p} \frac{p!}{\nu_1! \cdots \nu_N!} \int r(S_1)^{\nu_1\alpha} \cdots r(S_N)^{\nu_N\alpha} d\mu_k(S_1, \dots, S_N)$$

$$\le x_k \delta(\delta^n L + M^p \cdot C \cdot \sum_{i=0}^{n-1} \delta^i) + M^p \int (\sum_{i=1}^{N} r(S_i)^\alpha)^p d\mu_k(S_1, \dots, S_N)$$

$$= x_k(\delta^{n+1} L + M^p C \sum_{i=0}^{n} \delta^i).$$

Since $\delta < 1$, we deduce that $(f_{q,\alpha}^{(k)})_{q\in\mathbf{N}}$ is *L^p-bounded.*

We show that $f^{(k)} > 0$ for $\langle\mu_k\rangle$-a.e. and $k = 1, \dots, N$ if the condition (4) holds. Using Proposition 3.1 and Lemma 6.4 of Graf [7], we deduce

$$\langle\mu_k\rangle(\{\mathscr{S} | f^{(k)}(\mathscr{S}) = 0\}) = [\mu_k \times \prod_{i=1}^{N} \langle\mu_i\rangle](\{((S_1, \dots, S_N), (\mathscr{S}^{(1)}, \dots, \mathscr{S}^{(N)}))|$$

$$\sum_{i=1}^{N} r(S_i)^\alpha f^{(i)}(\mathscr{S}^{(i)}) = 0\})$$

$$= \prod_{i:R(0)_{ki}\neq 0} \langle\mu_i\rangle(\{\mathscr{S}\,|\,f^{(i)}(\mathscr{S})=0\}).$$

By Proposition 3.8 and the fact that $E_{\langle\mu_k\rangle}[f^{(k)}] = x_k > 0$, we deduce that $\langle\mu_i\rangle(\{\mathscr{S}\,|\,f^{(i)}(\mathscr{S})>0\}) = 1$ for all $i = 1,\dots, N$. This completes the proof.

A subset $\Gamma \subset D$ is called a minimal covering if for each $\eta\in\{1,\dots,N\}^{\mathbf{N}}$ there exists a unique $\sigma\in\Gamma$ such that $\eta|j=\sigma$ for some $j\in\mathbf{N}$. Let $\mathrm{Min} = \{\Gamma\subset D\,|\,\Gamma$ is a minimal covering$\}$. For $\Gamma_1, \Gamma_2 \subset D$, we write $\Gamma_1 < \Gamma_2$ if for every $\sigma_1\in\Gamma_1$ there exists $\sigma_2\in\Gamma_2$ such that $\sigma_2|j=\sigma_1$ for some $j\in\mathbf{N}$.

Corollary of Theorem 3.10 (cf. Corollary 6.5 of Graf [7]). *Let the assumptions of Theorem* 3.10 *be satisfied. Then*

$$E_{\langle\mu_k\rangle}[\sup_{\Gamma_0\in\mathrm{Min}} \inf\{f^{(k)}_{\Gamma,\alpha}\,|\,\Gamma\in\mathrm{Min},\ \Gamma>\Gamma_0\}] < \infty$$

for $k=1,\dots,N$. *In particular*

$$\sup_{\Gamma_0\in\mathrm{Min}} \inf\{f^{(k)}_{\Gamma,\alpha}\,|\,\Gamma\in\mathrm{Min},\ \Gamma>\Gamma_0\} < \infty$$

for $\langle\mu_k\rangle$*-a.e.* $\mathscr{S}\in\Omega$ *and* $k=1,\dots,N$.

Proof. For $\langle\mu_k\rangle$-a.e. $\mathscr{S}$ we have

$$\sup_{\Gamma_0\in\mathrm{Min}} \inf\{f^{(k)}_{\Gamma,\alpha}\,|\,\Gamma\in\mathrm{Min},\ \Gamma>\Gamma_0\} \le \sup_{q_0\in\mathbf{N}} \inf_{q\ge q_0} f^{(k)}_{q,\alpha}(\mathscr{S}) = f^{(k)}(\mathscr{S}).$$

Since $\int f^{(k)} d\langle\mu_k\rangle < \infty$ by Theorem 3.10 the corollary is proved.

For the proof of Proposition 3.9 we state a result in Graf [7].

Theorem 2.4 of Graf [7]. *Let* $\mathscr{S}\in\Omega_0$ *be given. Then, for every* $\beta>0$,

$$\mathscr{H}^{\beta}(K(\mathscr{S})) \le |X|^{\beta} \sup_{\Gamma_0\in\mathrm{Min}} \inf\{\textstyle\sum_{\sigma\in\Gamma}\prod_{n=1} r(S_{\sigma|n})^{\beta}\,|\,\Gamma\in\mathrm{Min},\ \Gamma>\Gamma_0\}.$$

Proof of Proposition 3.9 (cf. the proof of Theorem 7.4 of Graf [7]). We show that $E_{P_{<\mu_k>}}[\mathscr{H}^{\alpha}(K)] < \infty$ for $k=1,\dots,N$. Let $\psi:\Omega \to \mathscr{K}(X)$ be as defined in Proposition 3.3. Since $P_{\langle\mu_k\rangle} = \langle\mu_k\rangle\circ\psi^{-1}$, it is enough to show that $E_{\langle\mu_k\rangle}[\mathscr{H}^{\alpha}(\psi(\mathscr{S}))] < \infty$ for $k=1,\dots,N$. By Lemma 3.2 and Theorem 2.4 of Graf [7] it holds that

$$\begin{aligned}\mathscr{H}^{\alpha}(\psi(\mathscr{S})) &\le |X|^{\alpha} \sup_{\Gamma_0\in\mathrm{Min}} \inf\{\textstyle\sum_{\sigma\in\Gamma}^{|\sigma|}\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^{\alpha}\,|\,\Gamma\in\mathrm{Min},\ \Gamma>\Gamma_0\}\\ &\le |X|^{\alpha} \sup_{\Gamma_0\in\mathrm{Min}} \inf\{f^{(k)}_{\Gamma,\alpha}/\min_{1\le i\le N} x_i\,|\,\Gamma\in\mathrm{Min},\ \Gamma>\Gamma_0\}\end{aligned}$$

for $\langle\mu_k\rangle$-a.e. $\mathscr{S}$. By the last corollary, the expectation of this last expression with respect to $\langle\mu_k\rangle$ is finite. This completes the proof.

5. Proof of Theorem 3.11

For the proof of Theorem 3.11 we need a lemma, Lemma D, which is a modification of Theorem 6.8 of Graf [7]. To show Lemma D we state necessary results. For $\mathscr{S}\in(\mathrm{Con}(X)^N)^D$ and $\sigma\in D$, let $\mathscr{S}^\sigma\in(\mathrm{Con}(X)^N)^D$ defined by $(\mathscr{S}^\sigma)_\tau=\mathscr{S}_{\sigma*\tau}$ for $\tau\in D$.

LEMMA B (cf. Lemma 6.6. of Graf [7]). *Let* $(\mu_1,\dots,\mu_N)$ *satisfy the conditions* (3), (4) *and* (5). *Let* α *be such that* $\lambda(\alpha)=1$. *For* $\beta<\alpha$, $\langle\mu_k\rangle$*-a.e.* $\mathscr{S}\in\Omega$ *and* $k=1,\dots,N$, *there exists an* $m\in\mathbf{N}$ *such that, for every* $\sigma\in D$ *with* $|\sigma|\ge m$,

$$\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\alpha f^{(t(\sigma))}(\mathscr{S}^\sigma)\le\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\beta .$$

PROOF. Let $\sigma\in D$ and $p\in\mathbf{N}$ be arbitrary. Using Chebyshev's inequality, we have

$$\begin{aligned}&\langle\mu_k\rangle(\{\mathscr{S}\mid\textstyle\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^{\alpha-\beta}f^{(t(\sigma))}(\mathscr{S}^\sigma)>1\})\\&\quad\le\int\textstyle\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^{p(\alpha-\beta)}d\langle\mu_k\rangle(\mathscr{S})\displaystyle\int\{f^{(t(\sigma))}(\mathscr{S})\}^p d\langle\mu_{t(\sigma)}\rangle(\mathscr{S}).\end{aligned}$$

Therefore

$$\begin{aligned}&\langle\mu_k\rangle(\{\mathscr{S}\mid\text{ there exists a }\sigma\in C_q\text{ such that }\textstyle\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^{\alpha-\beta}f^{(t(\sigma))}(\mathscr{S}^\sigma)>1\})\\&\quad\le\int\textstyle\sum_{\sigma\in C_q}\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^{p(\alpha-\beta)}d\langle\mu_k\rangle(\mathscr{S})\max_{1\le i\le N}\displaystyle\int\{f(\mathscr{S})^{(i)}\}^p d\langle\mu_i\rangle(\mathscr{S}).\end{aligned}$$

Let $p\in\mathbf{N}$ such that $p(\alpha-\beta)>\alpha$. Then we have $\lambda(p(\alpha-\beta))<1$. Let

$$c=\max_{1\le i\le N}\sum_{j=1}^N R(p(\alpha-\beta))_{ij}\frac{x_j}{x_i}$$

where $(x_1,\dots,x_N)$ is a positive eigenvector of $R(\alpha)$ corresponding to the maximal eigen value 1. By Frobenius' theorem we have that $c<1$. Since $\int(f^{(i)})^p d\langle\mu_i\rangle<\infty$ for $i=1,\dots,N$ by Theorem 3.10 and

$$\int\sum_{\sigma\in C_q}\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^{p(\alpha-\beta)}d\langle\mu_k\rangle(\mathscr{S})\le x_k c^q/(\min_{1\le i\le N}x_i),$$

we deduce

$$\begin{aligned}&\sum_{q=1}^\infty\langle\mu_k\rangle(\{\mathscr{S}\mid\text{ there exists a }\sigma\in C_q\text{ such that}\\&\qquad\qquad\textstyle\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^{\alpha-\beta}f^{(t(\sigma))}(\mathscr{S}^\sigma)>1\})<\infty.\end{aligned}$$

By the Borel-Cantelli lemma we have

$$\begin{aligned}&\langle\mu_k\rangle(\textstyle\bigcap_{m\in\mathbf{N}}\bigcup_{q\ge m}\{\mathscr{S}\mid\text{ there exists a }\sigma\in C_q\text{ such that}\\&\qquad\qquad\textstyle\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^{\alpha-\beta}f^{(t(\sigma))}(\mathscr{S}^\sigma)>1\})=0.\end{aligned}$$

This completes the proof.

Lemma C. (cf. Theorem 6.7 of Graf [7]). *Let* $(\mu_1, \dots, \mu_N)$ *satisfy the conditions* (3), (4) *and* (5). *Let* α *be such that* $\lambda(\alpha) = 1$. *For* $\beta < \alpha$, $\langle\mu_k\rangle$*-a.e.* $\mathscr{S} \in \Omega$ *and* $k = 1, \dots, N$,

$$\sup_{\Gamma_0 \in \mathrm{Min}} \inf\{f^{(k)}_{\Gamma,\beta}(\mathscr{S}) \mid \Gamma \in \mathrm{Min},\ \Gamma > \Gamma_0\}) \geq f^{(k)}(\mathscr{S}).$$

Proof. By Lemma B and Lemma 6.4 of Graf [7] we deduce the result. See the proof of Theorem 6.7 of Graf [7].

Lemma D. *Let* $(\mu_1, \dots, \mu_N)$ *satisfy the conditions* (3), (4) *and* (5). *Let* $\beta < \alpha$ *where* $\lambda(\alpha) = 1$. *Then, for* $\langle\mu_k\rangle$*-a.e.* $\mathscr{S}$ *and* $k = 1, \dots, N$,

$$\sup_{\Gamma_0 \in \mathrm{Min}} \inf\{\textstyle\sum_{\sigma \in \Gamma} r(\mathscr{S}_\sigma)^d \prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\beta \mid \Gamma \in \mathrm{Min},\ \Gamma > \Gamma_0\}) > 0.$$

Proof. Since $\lambda(\beta) > 1$, there exists an $\eta > 0$ such that, for $A_k = \{(S_1, \dots, S_N) \mid r(S_i) \geq \eta$ for $i = 1, \dots, N$ with $R(0)_{ki} > 0\}$ $(k = 1, \dots, N)$, the maximal eigen value of a matrix $T = [t_{ki}]$ is greater than 1 where

$$t_{ki} = \int_{A_k} r(S_i)^\beta d\mu_k(S_1, \dots, S_N).$$

Define $r_\eta(S)\colon \mathrm{Con}(X) \to [0, 1)$ by

$$r_\eta(S) = \begin{cases} 0, & r(S) < \eta \\ r(S), & r(S) \geq \eta. \end{cases}$$

Let $f^{(k)}_\eta(\mathscr{S}) = \lim_{m \to \infty} \sum_{\sigma \in C_m} \prod_{n=1}^{|\sigma|} r_\eta(S_{\sigma|n})^\alpha x_{t(\sigma)}$ defined on $(\Omega, \mathscr{B}, \langle\mu_k\rangle)$ for $k = 1, \dots, N$. For $\mathscr{S} \in \Omega$ we have

$$\begin{aligned} &\sup_{\Gamma_0} \inf_{\Gamma > \Gamma_0} \textstyle\sum_{\sigma \in \Gamma} r(S_\sigma)^d \prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\beta \\ &\quad \geq \sup_{\Gamma_0} \inf_{\Gamma > \Gamma_0} \textstyle\sum_{\sigma \in \Gamma} r_\eta(S_\sigma)^d \prod_{n=1}^{|\sigma|} r_\eta(S_{\sigma|n})^\beta \\ &\quad \geq \eta^d \sup_{\Gamma_0} \inf_{\Gamma > \Gamma_0} \textstyle\sum_{\sigma \in \Gamma} \prod_{n=1}^{|\sigma|} r_\eta(S_{\sigma|n})^\beta \\ &\quad \geq \eta^d f^{(k)}_\eta(\mathscr{S}) / \max_{1 \leq i \leq N} x_i \qquad \text{for } \langle\mu_k\rangle\text{-a.e. } \mathscr{S}. \end{aligned}$$

The last inequality follows from Lemma C. Since by Theorem 3.10 $\int f^{(k)}_\eta(\mathscr{S}) d\langle\mu_k\rangle > 0$, we deduce that

$$\sup_{\Gamma_0} \inf_{\Gamma > \Gamma_0} \textstyle\sum_{\sigma \in \Gamma} r(S_\sigma)^d \prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\beta > 0 \tag{a3}$$

with positive probability.

We show that the left-hand side in (a3) is either 0 with probability 1 or > 0 with probability 1. By Proposition 3.1 we have

$$\begin{aligned}
p_k &:= \langle\mu_k\rangle(\{\mathscr{S} \mid \sup_{\Gamma_0}\inf_{\Gamma>\Gamma_0}\textstyle\sum_{\sigma\in\Gamma} r(S_\sigma)^d \prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\beta = 0\}) \\
&= \mu_k \times \textstyle\prod_{i=1}^N \langle\mu_i\rangle(\{((S_1,\dots,S_N),\ (\mathscr{S}^{(1)},\dots,\mathscr{S}^{(N)}))\mid \\
&\qquad \textstyle\sum_{i=1}^N r(S_i)^\beta \sup_{\Gamma_1}\ \inf_{\Gamma>\Gamma_1}\sum_{\sigma\in\Gamma} r(S_\sigma^{(i)})^d \prod_{n=1}^{|\sigma|} r(S_{\sigma|n}^{(i)})^\beta = 0\}) \\
&= \textstyle\prod_{i:R(0)_{ki}\neq 0}^N \langle\mu_i\rangle(\{\mathscr{S} \mid \sup_{\Gamma_1}\inf_{\Gamma>\Gamma_1}\sum_{\sigma\in\Gamma} r(S_\sigma)^d \prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\beta = 0\}),
\end{aligned}$$

because $r(S_j) > 0$ for $j \in \{1,\dots,N\}$ such that $R(0)_{kj} > 0$. By Proposition 3.8 and (a3) we deduce that

$$p_i = 0 \qquad \text{for} \quad i = 1,\dots,N.$$

This completes the proof.

Proof of Theorem 3.11. By Proposition 3.9 we have

$$\dim_{\mathrm{H}}(K) \le \alpha$$

for $P_{\langle\mu_k\rangle}$ a.e. $K \in \mathscr{K}(X)$ and $k = 1,\dots,N$. The converse inequality is shown in the same way as in the proof of Theorem 7.6 of Graf [7] using Theorem 2.5 of Graf [7] and lemma D.

6. Proof of Theorem 4.1

Our fundamental lemma is as follows:

Lemma E (cf. Lemma 6.10 of Graf [7]). *Let $(\mu_1,\dots,\mu_N)$ be an N-tuple of probability measures on* $\mathrm{Con}(X)^N$ *which satisfies the conditions* (3), (4) *and* (5) *in Section* 3. *Let $\alpha > 0$ be such that $\lambda(\alpha) = 1$. For $n \in \mathbf{N}$ define $h_n : \Omega \to \mathbf{R}_+$ by*

$$h_n(\mathscr{S}) = \inf\{f_{\Gamma,\alpha}(\mathscr{S}) \mid \Gamma \in \mathrm{Min},\ \Gamma \neq \{\emptyset\},\ |\Gamma| \le n\}$$

where

$$f_{\Gamma,\alpha}(\mathscr{S}) = \textstyle\sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\alpha x_{t(\sigma)}$$

and $|\Gamma| = \max\{|\sigma| : \sigma \in \Gamma\}$. (Note that for all $k = 1,\dots,N$, $f_{\Gamma,\alpha}(\mathscr{S}) = f_{\Gamma,\alpha}^{(k)}(\mathscr{S})$ for $\Gamma \neq \emptyset$.) *Then $(h_n)_{n\in\mathbf{N}}$ are non-increasing sequences of Borel measurable functions which satisfy the following properties*:

(i) $h_{n+1}(\mathscr{S}) = \sum_{i=1}^N r(S_i)^\alpha \min(x_i, h_n(\mathscr{S}^{(i)}))$ *for all $n \in \mathbf{N}$ and $\mathscr{S} \in \Omega$.*

(ii) $h := \inf_{n\in\mathbf{N}} h_n = \inf_{\Gamma\in\mathrm{Min}\setminus\{\emptyset\}} f_{\Gamma,\alpha}$.

(iii) *If the condition $\langle\mu_j\rangle(\{h > 0\}) > 0$ for some $j \in \{1,\dots,N\}$ holds, then $\sum_{i=1}^N r(S_i)^\alpha x_i = x_k$ for μ_k-a.e. $(S_1,\dots,S_N)$ and all $k = 1,\dots,N$.*

PROOF. We only show (iii) since (i) and (ii) is trivial. It follows from (i) and (ii) that

$$h(\mathscr{S}) = \textstyle\sum_{i=1}^{N} r(S_i)^\alpha \min(x_i, h(\mathscr{S}^{(i)})) \tag{a4}$$

for all $\mathscr{S} \in \Omega$. Let $(y_1, \dots, y_N)$ be a positive vector such that $(y_1, \dots, y_N) = (y_1, \dots, y_N)R(\alpha)$ and $\sum_{k=1}^{N} y_k = 1$. Integrating the both sides of (a4) with respect to $\sum_{k=1}^{N} y_k \langle \mu_k \rangle$, we have by Proposition 3.1 that

$$\begin{aligned}
\textstyle\sum_{k=1}^{N} y_k \int h(\mathscr{S}) d\langle \mu_k \rangle &= \textstyle\sum_{k=1}^{N} y_k \int\int \sum_{i=1}^{N} r(S_i)^\alpha \min(x_i, h(\mathscr{S}^{(i)})) d\langle \mu_i \rangle (\mathscr{S}^{(i)}) \\
&\qquad\qquad d\mu_k(S_1, \dots, S_N) \\
&= \textstyle\sum_{i=1}^{N} \int \min(x_i, h(\mathscr{S})) d\langle \mu_i \rangle \sum_{k=1}^{N} y_k \int r(S_i)^\alpha \\
&\qquad\qquad d\mu_k\ (S_1, \dots, S_N) \\
&= \textstyle\sum_{i=1}^{N} \int \min(x_i, h(\mathscr{S})) y_i d\langle \mu_i \rangle.
\end{aligned}$$

Since $y_k > 0$, we deduce that

$$h(\mathscr{S}) \le x_k \quad \text{for } \langle \mu_k \rangle\text{-a.e. } \mathscr{S} \text{ and } k = 1, \dots, N.$$

Therefore (a4) implies that

$$h(\mathscr{S}) = \textstyle\sum_{i=1}^{N} r(S_i)^\alpha h(\mathscr{S}^{(i)}) \tag{a5}$$

for $\langle \mu_k \rangle$-a.e. $\mathscr{S}$ and $k = 1, \dots, N$. Let η_k be the essential supremum of $h(\mathscr{S})$ with respect to $\langle \mu_k \rangle$ for $k = 1, \dots, N$. Using (a5) and Proposition 3.1 we obtain that

$$\eta_k \ge \textstyle\sum_{i=1}^{N} r(S_i)^\alpha \eta_i$$

for $\langle \mu_k \rangle$-a.e. $\mathscr{S}$ and $k = 1, \dots, N$. Integrating the both sides with respect to $\langle \mu_k \rangle$, we have

$$\eta_k \ge \textstyle\sum_{i=1}^{N} R(\alpha)_{ki} \eta_i \qquad \text{for} \quad k = 1, \dots, N$$

where $(\eta_1, \dots, \eta_N)$ is non-negative nonzero vector by our assumption (iii). By Theorem 2.4 (Frobenius),

$$\eta_k = \textstyle\sum_{i=1}^{N} R(\alpha)_{ki} \eta_i \qquad \text{for} \quad k = 1, \dots, N$$

and $(\eta_1, \dots, \eta_N)$ is positive eigenvector of $R(\alpha)$ corresponding to the maximal eigen value. This implies that

$$\eta_k = \sum_{i=1}^N r(S_i)^\alpha \eta_i \quad \text{for } \langle\mu_k\rangle\text{-a.e. } \mathscr{S} \text{ and } k = 1, \dots, N.$$

Since $\eta_1 : \cdots : \eta_N = x_1 : \cdots : x_N$, we have

$$x_k = \sum_{i=1}^N r(S_i)^\alpha x_i \quad \text{for } \mu_k\text{-a.e. } (S_1, \dots, S_N) \text{ and } k = 1, \dots, N.$$

This completes the proof.

Using Lemma E and the similar arguments to the proof of Theorem 6.11 of Graf [7], we have the following proposition.

PROPOSITION F. *Assume the condition of Lemma E are satisfied. Let $\alpha > 0$ be such that $\lambda(\alpha) = 1$ and $(x_1, \dots, x_N)$ be a positive eigenvector of $R(\alpha)$ corresponding to the maximal eigen value 1. Then the following conditions are equivalent:*

a) *For all $k \in \{1, \dots, N\}$, $\sum_{i=1}^N r(S_i)^\alpha x_i = x_k$ for μ_k-a.e. $(S_1, \dots, S_N)$.*
b) *For all $k \in \{1, \dots, N\}$, $\sup_{\Gamma_0 \in \mathrm{Min}} \inf\{f^{(k)}_{\Gamma,\alpha}(\mathscr{S}) \mid \Gamma \in \mathrm{Min}, \Gamma > \Gamma_0\} > 0$ for $\langle\mu_k\rangle$-a.e. $\mathscr{S}$.*
c) *$\langle\mu_j : \mu_1, \dots, \mu_N\rangle(\{\mathscr{S} \mid \sup_{\Gamma_0 \in \mathrm{Min}} \inf\{f^{(j)}_{\Gamma,\alpha}(\mathscr{S}) \mid \Gamma \in \mathrm{Min}, \Gamma > \Gamma_0\} > 0\}) > 0$ for some $j \in \{1, \dots, N\}$.*

PROOF. (a) $\to$ (b): Under the assumption (a), it holds that $f^{(k)}_{\Gamma,\alpha}(\mathscr{S}) = x_k$ for $\langle\mu_k\rangle$-a.e. $\mathscr{S}$. This measn

$$\sup_{\Gamma_0} \inf_{\Gamma > \Gamma_0} f^{(k)}_{\Gamma,\alpha}(\mathscr{S}) = x_k > 0 \ \langle\mu_k\rangle\text{-a.e. } \mathscr{S}.$$

(b) $\to$ (c) is trivial.

(c) $\to$ (a): Fix $\Gamma_0 \in \mathrm{Min}$ for $\Gamma \in \mathrm{Min}$ with $\Gamma > \Gamma_0$ and $\sigma \in \Gamma_0$, let $\Gamma_\sigma = \{\tau \in D \mid \sigma * \tau \in \Gamma\}$, then $\Gamma_\sigma \in \mathrm{Min}$. It holds that

$$\begin{aligned}
\text{(a6)} \quad \inf_{\Gamma > \Gamma_0} f^{(j)}_{\Gamma,\alpha}(\mathscr{S}) &= \inf_{\Gamma > \Gamma_0} \sum_{\sigma \in \Gamma_0} \Big[\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\alpha \sum_{\tau \in \Gamma_\sigma} \prod_{m=1}^{|\tau|} r(S_{\sigma*(\tau|m)})^\alpha x_{t(\sigma*\tau)}\Big] \\
&= \sum_{\sigma \in \Gamma_0} \Big[\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\alpha \inf_{\Gamma > \Gamma_\sigma} \sum_{\tau \in \Gamma_\sigma} \prod_{m=1}^{|\tau|} r(S_{\sigma*(\tau|m)})^\alpha x_{t(\sigma*\tau)}\Big] \\
&= \sum_{\sigma \in \Gamma_0} \Big[\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\alpha \min(x_{t(\sigma)}, \inf_{\Gamma \in \mathrm{Min} \setminus \{\emptyset\}} f_{\Gamma,\alpha}(\mathscr{S}^\sigma))\Big] \\
&= \sum_{\sigma \in \Gamma_0} \Big[\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\alpha \min(x_{t(\sigma)}, h(\mathscr{S}^\sigma)\Big].
\end{aligned}$$

By (c), there exists a Borel set $B \subset \Omega$ with $\langle\mu_j\rangle(B) > 0$ such that, for any $\mathscr{S} \in B$, there is a Γ_0 with $\inf_{\Gamma > \Gamma_0} f^{(j)}_{\Gamma,\alpha}(\mathscr{S}) > 0$. By (a6), it holds that for any $\mathscr{S} \in B$, there exist $\Gamma_0 \in \mathrm{Min}$ and a $\sigma \in \Gamma_0$ such that $\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\alpha > 0$ and $h(\mathscr{S}^\sigma) > 0$. For $\sigma \in D$, let $\Omega(\sigma) = \{\mathscr{S} \mid \prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\alpha > 0 \text{ and } h(\mathscr{S}^\sigma) > 0\}$. Note that $\langle\mu_j\rangle(\bigcup_{\sigma \in D} \Omega(\sigma)) > 0$, because $B \subset \bigcup_{\sigma \in D} \Omega(\sigma)$. Hence there exists a $\sigma \in D$ such that $\langle\mu_j\rangle(\Omega(\sigma)) > 0$. Since $\langle\mu_j\rangle(\Omega(\sigma)) = \langle\mu_j\rangle(\{\prod_{n=1}^{|\sigma|} r(S_{\sigma|n})^\alpha > 0\}) \langle\mu_{t(\sigma)}\rangle(\{\mathscr{S} \mid h(\mathscr{S}) > 0\}) > 0$, it holds that $\langle\mu_{t(\sigma)}\rangle(\{\mathscr{S} \mid h(\mathscr{S}) > 0\}) > 0$. Therefore Lemma E implies the condition (a).

PROOF OF THEOREM 4.1. (a) $\rightarrow$ (b). By Theorem 2.5 of Graf [7] and Lemma 3.2, there exists a $c>0$ such that

$$c\,|X|^{\alpha}\sup_{\Gamma_0}\inf_{\Gamma>\Gamma_0}\sum_{\sigma\in\Gamma}r(S_\sigma)^d\prod_{n=1}^{|\sigma|}r(S_{\sigma|n})^{\alpha}\leq\mathscr{H}^{\alpha}(\psi(\mathscr{S}))$$

for $\langle\mu_k\rangle$-a.e. $\mathscr{S}$. Using the assumptions of Theorem 4.1 we have

$$c\,\delta^{\alpha}\,|X|^{\alpha}\sup_{\Gamma_0}\inf_{\Gamma>\Gamma_0}\sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|}r(S_{\sigma|n})^{\alpha}\leq\mathscr{H}^{\alpha}(\psi(\mathscr{S}))$$

for $\langle\mu_k\rangle$-a.e.$\mathscr{S}$. Proposition F yields $\mathscr{H}^{\alpha}(\psi(S))>0$ for $\langle\mu_k\rangle$-a.e.$\mathscr{S}$ and by the definition of $P_{\langle\mu_k\rangle}$, we have (b).

(b) $\rightarrow$ (c) is trivial.

(c) $\rightarrow$ (a). By Theorem 2.4 of Graf [7] and Lemma 3.2 in Section 3 it follows that

$$\mathscr{H}^{\alpha}(\psi(\mathscr{S}))\leq|X|^{\alpha}\sup_{\Gamma_0}\inf_{\Gamma>\Gamma_0}\sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|}r(S_{\sigma|n})^{\alpha}$$

for $\langle\mu_k\rangle$-a.e. $\mathscr{S}$ and $k=1,\dots,N$. Assume that $\mu_j((S_1,\dots,S_N)|\sum_{i=1}^{N}r(S_i)^{\alpha}x_i\neq x_j)>0$ for some $j\in\{1,\dots,N\}$. Then Proposition F implies $\sup_{\Gamma_0}\inf_{\Gamma>\Gamma_0}f_{\Gamma,\alpha}^{(k)}(\mathscr{S}):=\sup_{\Gamma_0}\inf_{\Gamma>\Gamma_0}\sum_{\sigma\in\Gamma}\prod_{n=1}^{|\sigma|}r(S_{\sigma|n})^{\alpha}x_{t(\sigma)}=0$ for $\langle\mu_k\rangle$-a.e. $\mathscr{S}$ and $k=1,\dots,N$. It follows that $\mathscr{H}^{\alpha}(\psi(\mathscr{S}))=0$ for $\langle\mu_k\rangle$-a.e. $\mathscr{S}$ and $k=1,\dots,N$. By the definition of $P_{\langle\mu_k\rangle}$ we have $\mathscr{H}^{\alpha}(K)=0$ for $P_{\langle\mu_k\rangle}$-a.e. $K\in\mathscr{K}(X)$ and all $k=1,\dots,N$. This completes the proof.

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