HIROSHIMA MATH. J.
22 (1992), 129–176

Classification of weighing matrices of small orders

Hiroyuki OHMORI

(Received December 26, 1990)

Summary

The classification problem of weighing matrices of orders not exceeding 14 has been completed by Chan et al. [2] and Ohmori [17, 18]. In this paper, we first consider a construction problem of weighing matrices of order $8a-2$ and weight $4a$ for $a \geq 2$. A general solution for the intersection pattern condition, which is necessary to construct such weighing matrices, is given. Furthermore, the complete classification of weighing matrices for the case $a = 2$ is made.

1. Introduction

A weighing matrix W of order n and weight k is an $n \times n$ matrix with elements $+1$, -1 and 0 such that $WW^t = kI_n$, $k \leq n$, where I_n is the identity matrix of order n and W^t denotes the transpose of W. We refer to such a matrix as a $W(n, k)$. A $W(n, n)$ is called a Hadamard matrix of order n. It is known that the order of a Hadamard matrix is 2 or a multiple of 4. In fact, the concept of weighing matrices was introduced by Taussky [24] as a generalization of Hadamard matrices. However, in the area of design theory, weighing matrices appear naturally as the "coefficient" matrices of an orthogonal design (see Geramita and Seberry [4]) and has applications for weighing designs (for example, see Chakrabarti [1], Federer [3], Raghavarao [22]). Furthermore, weighing matrices have been studied in order to find optimal solutions to the so-called weighing design problem of weighing objects whose weights are small relative to the weights of moving parts of the balance being used. It was shown by Raghavarao [21, 22] that if the variance of the errors in the weights obtained by individual weighing is σ^2 in the usual weighing design set up, then using a $W(n, k)$ as a design of an experiment to weigh n objects will give the variance σ^2/k. Indeed, in the class of all such weighing designs for $n \equiv 0 \pmod 4$, a Hadamard matrix is optimal. Furthermore, in the class of all weighing designs for $n \equiv 2 \pmod 4$, a symmetric conference matrix (that is a kind of $W(n, n-1)$) is optimal. Weighing matrices also have applications in the area of coding theory. A linear code is an l-dimensional subspace of the m-dimensional space over Galois field $GF(q)$. The weight of a vector is defined by the number of non-zero elements of the vector. The minimum weight of a code, denoted by d, is the weight of the non-zero vector having the smallest value of weight in the code. It is quite useful to know the value of minimum weight d since a code of such d can correct $\left[\frac{d-1}{2}\right]$ errors. Thus, given m and l, it is worthwhile to obtain a code having d as large as possible. There are many investigations for linear codes constructed by using $W(n, k)$'s over $GF(3)$, for example, see [16], [19], [20], [23]. Thus, the problem of classifying weighing matrices is important in the area of discrete mathematics and statistics.

Two weighing (Hadamard) matrices are said to be equivalent if one can be transformed into the other by using the following operations: (i) multiply any row or column by -1; (ii) interchange two rows or two columns. If a $W(n, k)$ is equivalent to its transpose, the matrix is said to be self-dual. It is known that the complete classification of Hadamard matrices whose orders are less than or equal to 24 has been completed (see Hall [5, 6, 7], Ito et al. [9], Kimura [11], Wallis [27]). Furthermore, it has been shown (Kimura [10, 12], Kimura and Ohmori [14, 15], Tonchev [25, 26]) that there are at least 486 inequivalent Hadamard matrices of order 28. On the other hand, the problem of classifying weighing matrices started recently. Chan, Rodger and Seberry [2] classified the inequivalent weighing matrices of any order with weights less than 6. For $1 \leq k \leq n \leq 13$, all $W(n, k)$'s have been classified by Chan et al. [2] and Ohmori [17, 18]. As a next step of investigation, it is appropriate to consider the classification problem of weighing matrices of order 14. Geramita and Seberry [4] proved that if $n \equiv 2 \pmod 4$ then for a $W(n, k)$ to exist, $k \leq n-1$ and k is the sum of two squares. Thus it is now sufficient to consider only the cases of $k = 1, 2, 4, 5, 8, 9, 10, 13$ for the classification problem of $W(14, k)$'s. For the cases of $k \leq 5$ and $k = 13$, it has been completed by Chan et al. [2]. The available construction of $W(n, k)$'s is fully based on the intersection pattern condition (IPC) which consists of two linear equations with non-negative integral variables, because it allows us to get considerable information about the structure of a weighing matrix.

In this paper, we shall deal with the classification problem of $W(8a-2, 4a)$'s, where a is an integer greater than or equal to 2. In Section 2, we present a general solution for IPC. It is essential for the problem of constructing weighing matrices to determine whether there are weighing matrices having the "inner structure" associated with solutions of IPC or not. In fact, for some solutions of IPC, it is shown in Section 2 that there is no weighing matrix having the "inner structure" associated with them. In Section 3, we deal with the case $a = 2$. A set of $W(14, 8)$'s which contains all inequivalent weighing matrices of order 14 and weight 8 is provided. Furthermore, all $W(14, 8)$'s are classified into matrices of some types by solutions of IPC. The set of these matrices is obtained by first constructing all inequivalent admissible and feasible matrices belonging to each of types, secondly extending feasible matrices to weighing matrices with the aid of a personal computer or through the trial and error method, and thirdly removing equivalent weighing matrices by using automorphism groups of feasible matrices. These matrices are also classified into some classes by using the **C**- or **T**-distribution associated with each weighing matrix. Two tables are also presented in Section 3. **T**-distributions are listed in Table 1. They are helpful to classify weighing matrices. All weighing matrices $W(14, 8)$'s constructed in Section 2 are given in Table 2. They are divided into representative matrices of inequivalent classes and others. In conclusion, $W(14, 8)$'s will be classified into 65 inequivalent classes, and the result is useful for further classification of all inequivalent $W(14n, 8k)$'s by combining a $W(n, k)$ and $W(14, 8)$'s, and of all inequivalent $W(m, 8)$'s, where $m > 14$.

2. General solution for IPC with parameters $8a - 2$ and $4a$

Let $\mathbf{x}$ and $\mathbf{y}$ be row (column) vectors of the same size, and $\mathbf{x} * \mathbf{y}$ denote the Hadamard product, i.e. elementwise product. In this case, $|\mathbf{x} * \mathbf{y}|$ is called the intersection number of $\mathbf{x}$ and $\mathbf{y}$, where $|\mathbf{z}|$ means the number of non-zero elements of a vector $\mathbf{z}$. In particular, $|\mathbf{x} * \mathbf{x}|$ is called the weight of $\mathbf{x}$.

The following fundamental result is due to Chan et al. [2].

PROPOSITION 2.1. *Let M be a weighing matrix of order n and weight k, and let $\mathbf{m}$ and $\mathbf{n}$ be different rows (columns) of M. Then $|\mathbf{m} * \mathbf{n}|$ is even. Further let x_{2l} be the number of rows (columns) of M having the intersection number $2l$ with $\mathbf{m}$. Then the set of such non-negative integers $\{x_{2l}\}$ satisfies the equations*:

$$\sum_{l=k_0}^{k_1} x_{2l} = n - 1 \qquad \text{and} \qquad \sum_{l=k_0}^{k_1} 2l x_{2l} = k(k-1)\,,$$

where $k_0 = \max\left\{0, \left[\dfrac{2k-n}{2}\right]\right\}$, $k_1 = \left[\dfrac{k}{2}\right]$, *and $[s]$ is the largest integer not exceeding s.*

DEFINITION 2.1. Denote the set of all weighing matrices of order n and weight k by $\Delta(n, k)$. Let $\mathbf{m}$ be a row (column) of $M \in \Delta(n, k)$ and $\mathbf{c} = (x_{2k_0}, x_{2k_0+2}, \ldots, x_{2k_1})$ be the vector whose elements are intersection numbers associated with $\mathbf{m}$, where $k_0 = \max\left\{0, \left[\dfrac{2k-n}{2}\right]\right\}$ and $k_1 = \left[\dfrac{k}{2}\right]$. In this case,

$\mathbf{c}$ is called the *intersection pattern* of $\mathbf{m}$, and M is said to have an intersection pattern $\mathbf{c}$.

DEFINITION 2.2. For given positive integers n and k ($n \geq k$), the following equations are called the *intersection pattern condition* (IPC) with parameters n and k:

$$x_{2l} \geq 0 \qquad (k_0 \leq l \leq k_1)\,, \tag{1}$$

$$\sum_{l=k_0}^{k_1} x_{2l} = n - 1\,, \tag{2}$$

$$\sum_{l=k_0}^{k_1} 2l x_{2l} = k(k-1)\,, \tag{3}$$

where $k_0 = \max\left\{0, \left[\dfrac{2k-n}{2}\right]\right\}$ and $k_1 = \left[\dfrac{k}{2}\right]$. A solution $\{x_{2l}\}$ satisfying (1), (2) and (3) is expressed as $(x_{2k_0}, x_{2k_0+2}, \dots, x_{2k_1})$. The set of solutions of IPC is denoted by $\Gamma(n, k)$.

REMARK 2.1. Let $\mathbf{m}$ be a row (column) of $M \in \Delta(n, k)$ and $\mathbf{c}$ be the intersection pattern of $\mathbf{m}$. Then Proposition 2.1 shows $\mathbf{c} \in \Gamma(n, k)$. Conversely, for $\mathbf{c} \in \Gamma(n, k)$, a matrix having an intersection pattern $\mathbf{c}$, however, may exist or may not in $\Delta(n, k)$.

Hereafter, we will deal with the case of $n = 8a - 2$ and $k = 4a$, where $a \geq 2$ (note that if $a = 2$, it corresponds to $\Delta(14, 8)$ which will be discussed in detail in Section 3). In this case, $k_0 = 1$ and $k_1 = 2a$, and hence IPC with parameters $8a - 2$ and $4a$ is stated as the following:

$$\sum_{l=1}^{2a} x_{2l} = 8a - 3\,, \qquad \sum_{l=1}^{2a} l x_{2l} = 2a(4a-1)\,, \quad x_{2l} \geq 0\,. \tag{2.1}$$

Also, $\Delta(n, k)$ and $\Gamma(n, k)$ are abbreviated as Δ and Γ, respectively.

A general solution of (2.1) will be obtained inductively in the following manner: First the lower and the upper bounds for x_{4a} in (2.1) are given. Secondly for $1 \leq i \leq 2a - 2$ and $0 \leq j \leq i - 1$, let $x_{4a-2j} = z_{4a-2j}$ be fixed. Then the lower and the upper bounds for x_{4a-2i}, say $\underline{w} \leq x_{4a-2i} \leq \overline{w}$, are given so that for $\underline{w} \leq z_{4a-2i} \leq \overline{w}$, there exists a solution of (2.1) having $x_{4a-2j} = z_{4a-2j}$ ($0 \leq j \leq i$). In the following it will be discussed in detail.

LEMMA 2.1. *Let* $y_\alpha^{(0)} = -8a^2 + 18a - 6$, $y_\beta^{(0)} = 8a^2 - 10a + 3$ *and* $y_\gamma^{(0)} = 8a - 3$. *Let* Γ_0 *be the set of solutions of the following*:

$$\sum_{l=1}^{2a} x_{2l} = y_\gamma^{(0)}, \qquad \sum_{l=1}^{2a} (l-1)x_{2l} = y_\beta^{(0)}, \quad x_{2l} \geq 0. \tag{2.2}$$

Then $\Gamma_0 = \Gamma$ *and for* $(x_2, \ldots, x_{4a}) \in \Gamma_0$

$$0 \leq x_{4a} \leq \left[\frac{y_\beta^{(0)}}{2a-1}\right], \tag{2.3}$$

$$y_\alpha^{(0)} + y_\beta^{(0)} = y_\gamma^{(0)}, \qquad y_\beta^{(0)} \geq 0, \qquad y_\gamma^{(0)} \geq 0. \tag{2.4}$$

PROOF. (2.2) follows from (2.1). (2.3) is obtained by the second equality of (2.2). (2.4) is obvious. □

Let $w_\alpha^{(0)} = 0$ and $w_\beta^{(0)} = \left[\frac{y_\beta^{(0)}}{2a-1}\right]$. Further let $x_{4a} = z_{4a}$ be fixed, where $w_\alpha^{(0)} \leq z_{4a} \leq w_\beta^{(0)}$. Denote $\Gamma_1(z_{4a}) = \{\mathbf{c}_1 = (x_2, \ldots, x_{4a-2}) | (\mathbf{c}_1, z_{4a}) \in \Gamma\}$, where $(\mathbf{c}_1, z_{4a})$ means $(x_2, \ldots, x_{4a-2}, z_{4a})$.

Analogously to Lemma 2.1 one can prove the following:

LEMMA 2.2. *Let* $y_\gamma^{(1)} = y_\gamma^{(0)} - z_{4a}$, $y_\alpha^{(1)} = y_\alpha^{(0)} + (2a-2)z_{4a}$, *and* $y_\beta^{(1)} = y_\beta^{(0)} - (2a-1)z_{4a}$. *Let* Γ_1 *be the set of solutions of the following equations*:

$$\sum_{l=1}^{2a-1} x_{2l} = y_\gamma^{(1)}, \qquad \sum_{l=1}^{2a-1} (l-1)x_{2l} = y_\beta^{(1)}, \quad x_{2l} \geq 0.$$

Then $\Gamma_1 = \Gamma_1(z_{4a})$ *and for* $(x_2, \ldots, x_{4a-2}) \in \Gamma_1$

$$0 \leq x_{4a-2} \leq \left[\frac{y_\beta^{(1)}}{2a-2}\right],$$

$$y_\alpha^{(1)} + y_\beta^{(1)} = y_\gamma^{(1)}, \qquad y_\beta^{(1)} \geq 0, \qquad y_\gamma^{(1)} \geq 0.$$

Next, let $w_\alpha^{(1)} = 0$ and $w_\beta^{(1)} = \left[\frac{y_\beta^{(1)}}{2a-2}\right]$. For $1 \leq i \leq 2a-2$, let $x_{4a} = z_{4a}$, $x_{4a-2} = z_{4a-2}, \ldots, x_{4a-2(i-1)} = z_{4a-2(i-1)}$ be fixed in order. Further let $y_\alpha^{(l)}$, $y_\beta^{(l)}$, $y_\gamma^{(l)}$, $w_\alpha^{(l)}$, $w_\beta^{(l)}$, Γ_l and $\Gamma_l(z_{4a}, z_{4a-2}, \ldots, z_{4a-2l})$ be defined inductively, and suppose that $w_\alpha^{(l)} \leq z_{4a-2l} \leq w_\beta^{(l)}$, $0 \leq y_\beta^{(l)}$, $y_\gamma^{(l)}$, $w_\alpha^{(l)}$, $w_\beta^{(l)}$, where $0 \leq l \leq i-1$. In this case, we now further define

$$y_\alpha^{(i)} = y_\alpha^{(i-1)} + (2a-i-1)z_{4a-2(i-1)},$$

$$y_\beta^{(i)} = y_\beta^{(i-1)} - (2a-i)z_{4a-2(i-1)},$$

$$y_\gamma^{(i)} = y_\gamma^{(i-1)} - z_{4a-2(i-1)},$$

$$\Gamma_i(z_{4a}, z_{4a-2}, \dots, z_{4a-2(i-1)}) = \{\mathbf{c}_i = (x_2, \dots, x_{4a-2i}) | (\mathbf{c}_i, z_{4a-2(i-1)}, \dots, z_{4a}) \in \Gamma\},$$

$$w_\beta^{(i)} = \left[\frac{y_\beta^{(i)}}{2a-i-1}\right] \quad \text{and}$$

$$w_\alpha^{(i)} = \begin{cases} -\{y_\alpha^{(i)} + (2a-i-3)y_\gamma^{(i)}\} & \text{if } y_\alpha^{(i)} + (2a-i-3)y_\gamma^{(i)} < 0 \text{ and} \\ & \quad y_\alpha^{(i)} + (2a-i-2)y_\gamma^{(i)} \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Under the above notations we may proceed further.

LEMMA 2.3. *Let $0 \leq i \leq 2a-2$ and Γ_i be the set of solutions of the following*:

$$\sum_{l=1}^{2a-i} x_{2l} = y_\gamma^{(i)}, \qquad \sum_{l=1}^{2a-i} (l-1)x_{2l} = y_\beta^{(i)}, \quad x_{2l} \geq 0. \tag{2.5}$$

Then $\Gamma_i = \Gamma_i(z_{4a}, \dots, z_{4a-2(i-1)})$, $y_\beta^{(i)} \geq 0$ and $y_\gamma^{(i)} \geq 0$.

PROOF. The first equality is straightforward. By the assumption

$$z_{4a-2(i-1)} \leq w_\beta^{(i-1)} = \left[\frac{y_\beta^{(i-1)}}{2a-i}\right],$$

which yields $y_\beta^{(i)} = y_\beta^{(i-1)} - (2a-i)z_{4a-2(i-1)} \geq 0$. Let $\mathbf{c}_i = (x_2, \dots, x_{4a-2i}) \in \Gamma_i(z_{4a}, \dots, z_{4a-2(i-1)})$. By the definition of Γ_{i-1}, $(\mathbf{c}_i, z_{4a-2(i-1)}) \in \Gamma_{i-1}$. Hence

$$\sum_{l=1}^{4a-2i} x_{2l} + z_{4a-2(i-1)} = y_\gamma^{(i-1)}.$$

Thus

$$y_\gamma^{(i)} = y_\gamma^{(i-1)} - z_{4a-2(i-1)} = \sum_{l=1}^{4a-2i} x_{2l} \geq 0. \qquad \square$$

LEMMA 2.4. *For $0 \leq i \leq 2a-2$, $0 \leq w_\alpha^{(i)} \leq w_\beta^{(i)}$.*

PROOF. By the definition of $w_\beta^{(i)}$, it is clear that $w_\beta^{(i)} \geq 0$. When $w_\alpha^{(i)} = 0$, the statement holds, and then suppose that $w_\alpha^{(i)} > 0$. If $y_\beta^{(i)} \equiv 0$ (mod$(2a-i-1)$), then

$$\begin{aligned}
&(2a-i-1)(w_\beta^{(i)} - w_\alpha^{(i)}) \\
&= y_\beta^{(i)} - (2a-i-1)y_\gamma^{(i)} + \{y_\alpha^{(i)} + (2a-i-2)y_\gamma^{(i)}\}(2a-i-1) \\
&= -y_\alpha^{(i)} - (2a-i-2)y_\gamma^{(i)} + \{y_\alpha^{(i)} + (2a-i-2)y_\gamma^{(i)}\}(2a-i-1) \\
&= \{y_\alpha^{(i)} + (2a-i-2)y_\gamma^{(i)}\}(2a-i-2) \geq 0.
\end{aligned}$$

Thus, $w_\alpha^{(i)} \leq w_\beta^{(i)}$.

If $y_\beta^{(i)} \not\equiv 0 \pmod{2a-i-1}$,

$$\left[\frac{y_\beta^{(i)}}{2a-i-1}\right] \geq \frac{y_\beta^{(i)} - (2a-i-2)}{2a-i-1}.$$

Thus

$$\begin{aligned}(2a-i-1)(w_\beta^{(i)} - w_\alpha^{(i)}) &\geq y_\beta^{(i)} - (2a-i-2) - (2a-i-1)y_\gamma^{(i)} \\ &\quad + \{y_\alpha^{(i)} + (2a-i-2)y_\gamma^{(i)}\}(2a-i-1) \\ &= \{y_\alpha^{(i)} + (2a-i-2)y_\gamma^{(i)} - 1\}(2a-i-2).\end{aligned}$$

Now $y_\alpha^{(i)} + (2a-i-2)y_\gamma^{(i)} \geq 1$. Because if $y_\alpha^{(i)} + (2a-i-2)y_\gamma^{(i)} = 0$, $y_\beta^{(i)} = y_\gamma^{(i)} - y_\alpha^{(i)} = (2a-i-1)y_\gamma^{(i)}$. Thus $y_\beta^{(i)} \equiv 0 \pmod{2a-i-1}$. This is a contradiction. Hence $w_\alpha^{(i)} \leq w_\beta^{(i)}$. □

LEMMA 2.5. *If $w_\alpha^{(i)} > 0$ for $0 \leq i \leq 2a-2$, then $\bar{\mathbf{c}} = (0, \ldots, 0, z_{4a-2(i+1)}, z_{4a-2i}) \in \Gamma_i$, where $z_{4a-2i} = w_\alpha^{(i)}$ and $z_{4a-2(i+1)} = y_\gamma^{(i)} - w_\alpha^{(i)}$.*

PROOF. It follows that $z_{4a-2(i+1)} = y_\alpha^{(i)} + (2a-i-2)y_\gamma^{(i)} \geq 0$,

$$z_{4a-2i} + z_{4a-2(i+1)} = y_\gamma^{(i)}$$

and

$$\begin{aligned}&(2a-i-1)z_{4a-2i} + (2a-i-2)z_{4a-2(i+1)} \\ &\quad = -(2a-i-1)\{y_\alpha^{(i)} + (2a-i-3)y_\gamma^{(i)}\} + (2a-i-2)\{y_\alpha^{(i)} + (2a-i-2)y_\gamma^{(i)}\} \\ &\quad = -y_\alpha^{(i)} + y_\gamma^{(i)} = y_\beta^{(i)}.\end{aligned}$$

Hence, $\bar{\mathbf{c}} \in \Gamma_i$ by Lemma 2.3. □

THEOREM 2.1. *For $0 \leq i \leq 2a-2$, let Γ_i be the set of solutions of (2.5). If $(x_2, \ldots, x_{4a-2i}) \in \Gamma_i$,* then $w_\alpha^{(i)} \leq x_{4a-2i} \leq w_\beta^{(i)}$.

PROOF. The second inequality is clear by the definition of $w_\beta^{(i)}$. If $w_\alpha^{(i)} = 0$, the result follows. Suppose that $w_\alpha^{(i)} > 0$ and let $\mathbf{c} = (x_2, \ldots, x_{4a-2(i+1)}, x_{4a-2i}) \in \Gamma_i$. By Lemma 2.5, $\bar{\mathbf{c}} = (0, \ldots, 0, z_{4a-2(i+1)}, z_{4a-2i}) \in \Gamma_i$, where $z_{4a-2i} = w_\alpha^{(i)}$ and $z_{4a-2(i+1)} = y_\gamma^{(i)} - w_\alpha^{(i)}$. Now, suppose that $z_{4a-2i} > x_{4a-2i}$. Then

$$\sum_{l=1}^{2a-(i+1)} x_{2l} + x_{4a-2i} = z_{4a-2(i+1)} + z_{4a-2i} = y_\gamma^{(i)}$$

and

$$\begin{aligned}&\sum_{l=1}^{2a-(i+1)} (l-1)x_{2l} + (2a-i-1)x_{4a-2i} \\ &\quad = (2a-i-2)z_{4a-2(i+1)} + (2a-i-1)z_{4a-2i}.\end{aligned}$$

Hence

$$\sum_{l=1}^{2a-(i+1)} (l-1)x_{2l} - (2a-i-2)z_{4a-2(i+1)} = (2a-i-1)(z_{4a-2i} - x_{4a-2i})$$

and

$$\sum_{l=1}^{2a-(i+1)} (l-1)x_{2l} - (2a-i-2)\left\{\sum_{l=1}^{2a-(i+1)} x_{2l} + x_{4a-2i}\right\} + (2a-i-2)w_\alpha^{(i)}$$
$$= (2a-i-1)(w_\alpha^{(i)} - x_{4a-2i}).$$

Consequently

$$-\sum_{l=1}^{2a-(i+1)} (2a-1-l-i)x_{2l} + x_{4a-2i} - w_\alpha^{(i)} = 0.$$

This is a contradiction, because

$$\sum_{l=1}^{2a-(i+1)} (l+1-2a+i)x_{2l} \leq 0 \qquad \text{and} \qquad x_{4a-2i} - w_\alpha^{(i)} < 0.$$

Thus, $x_{4a-2i} \geq z_{4a-2i} = w_\alpha^{(i)}$. This completes the proof. □

DEFINITION 2.3. Let $\mathbf{c} = (x_2, \ldots, x_{4a})$ and $\bar{\mathbf{c}} = (\bar{x}_2, \ldots, \bar{x}_{4a}) \in \Gamma$. When $x_{4a} < \bar{x}_{4a}$ or there is a positive integer i_0 such that $x_{4a-2(l-1)} = \bar{x}_{4a-2(l-1)}$ $(1 \leq l \leq i_0 - 1)$ and $x_{4a-2i_0} < \bar{x}_{4a-2i_0}$, $\bar{\mathbf{c}}$ is said to be *larger* than $\mathbf{c}$. This is denoted by $\bar{\mathbf{c}} > \mathbf{c}$.

The following corollary follows from Definition 2.3 and Theorem 2.1, along with the definition of $w_\alpha^{(i)}$.

COROLLARY 2.1. *Let* $\underline{x}_{2a} = 7a - 3$, $\underline{x}_{2a+2} = a$, $\bar{x}_2 = 4a$ *and* $\bar{x}_{4a} = 4a - 3$. *Then* $(0, \ldots, 0, \underline{x}_{2a}, \underline{x}_{2a+2}, 0, \ldots, 0)$ *and* $(\bar{x}_2, 0, \ldots, 0, \bar{x}_{4a})$ *are the smallest and the largest solutions in* Γ, *respectively.*

DEFINITION 2.4. Let $M \in \Delta$ and $\mathbf{c}$ be the largest one among intersection patterns of rows and columns of M. Then M is said to be of *Type* $\mathbf{c}$. When M is a matrix of Type $\mathbf{c}$ and $\bar{\mathbf{c}} \in \Gamma$, where $\bar{\mathbf{c}} < \mathbf{c}$, M is said to be of *larger type* than Type $\bar{\mathbf{c}}$.

Let A be an $s \times t$ matrix whose elements are ± 1 or 0. Define $A^*_{s\times t} = A * A$, the Hadamard product. If there is no zero element in A, $A^*_{s\times t}$ is denoted by $J_{s\times t}$. Then $s \times t$ zero matrix is denoted by $O_{s\times t}$. If $s = t$, $A^*_{s\times t}$ and $O_{s\times t}$ are abbreviated as A^*_s and O_s, respectively. For matrices X and Y, the Kronecker product of X and Y is denoted by $X \otimes Y$.

DEFINITION 2.5. Let $\Delta(z_{4a})$ be the set of matrices of Type $\mathbf{c}$, where $\mathbf{c} = (x_2, \ldots, x_{4a-2}, z_{4a})$. Let $M \in \Delta(z_{4a})$. Then it can be assumed, without loss of generality, that

$$M = \left[\begin{array}{c|c} M_U & O_{s\times t} \\ \hline M_L & M_R \end{array}\right],$$

where $s = z_{4a} + 1$, $t = 4a - 2$, $M_U^* = J_{s\times 4a}$, and M_L and M_R are $(8a - 3 - z_{4a}) \times 4a$ and $(8a - 3 - z_{4a}) \times (4a - 2)$ matrices, respectively. Submatrices M_L, M_R, M_U and $[M_L \mid M_R]$ are called an *L*-, an *R*-, a *U*- and a *D-matrix* of M, respectively.

Hereafter, for any matrix in $\Delta(z_{4a})$ the above form will be always assumed. The following lemma will be used to construct $W(14, 8)$'s in Section 3.

LEMMA 2.6. *Let A be a $3 \times m$ matrix whose elements are ± 1 or 0, where $m \geq 3$. If $AA^t = mI_3$ and $A^* = J_{3\times m}$, then $m \equiv 0 \pmod 4$.*

PROOF. This can be easily shown by considering the structure of three rows of A. □

REMARK 2.2. When $M \in \Delta$, Lemma 2.6 means that it is impossible that three rows (columns) (say $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$) in M exist such that $|\mathbf{n}_1 * \mathbf{n}_2 * \mathbf{n}_3| = |\mathbf{n}_1 * \mathbf{n}_2| = |\mathbf{n}_1 * \mathbf{n}_3| = |\mathbf{n}_2 * \mathbf{n}_3| = m$, where $m \equiv 2 \pmod 4$.

The following Theorems 2.2–2.5 are powerful to reduce the possibilities of existence when $W(8a - 2, 4a)$'s are constructed by using solutions of IPC. Note that for $\Delta(z_{4a})$, $0 \leq z_{4a} \leq 4a - 3$.

THEOREM 2.2. *There is no weighing matrix of Type $\mathbf{c}$ or Type $\bar{\mathbf{c}}$, where $\mathbf{c} = (x_2, \ldots, 4a - 3) \in \Gamma(4a - 3)$ and $\bar{\mathbf{c}} = (\bar{x}_2, \ldots, 4a - 4) \in \Gamma(4a - 4)$.*

PROOF. Let $M \in \Delta(4a - 3)$. By Corollary 2.1, M is of Type $\mathbf{c}$, where $\mathbf{c} = (4a, 0, \ldots, 0, 4a - 3)$. Let M_R, M_L and M_U be an *R*-, an *L*- and a *U*-matrix of M, respectively. By Definition 2.5, it can be assumed that $M_R^* = J_{4a\times(4a-2)}$, $M_U^* = J_{(4a-2)\times 4a}$ and $M_L^* = I_{2a} \otimes J_2$. This means that there exists a submatrix $A_{3\times(4a-2)}$ of M_R such that $A_{3\times(4a-2)}A_{3\times(4a-2)}^t = (4a - 2)I_3$ and $A_{3\times(4a-2)}^* = J_{3\times(4a-2)}$. This contradicts to Lemma 2.6. Next, let $M \in \Delta(4a - 4)$ and M_R be an *R*-matrix of M. Then, M_R is a $(4a + 1) \times (4a - 2)$ matrix satisfying $M_R^t M_R = 4aI_{4a-2}$. Thus it can be assumed that $M_R = [A_{(4a-2)\times 4a} \mid O_{(4a-2)\times 1}]^t$, where $A_{(4a-2)\times 4a}^* = J_{(4a-2)\times 4a}$. Hence, $M^t \in \Delta(4a - 3)$. This contradicts to $M \in \Delta(4a - 4)$. □

THEOREM 2.3. *Let $M \in \Delta(4a - 5)$ and M be of Type $\mathbf{c}$, where $\mathbf{c} = (x_2, \ldots, x_{4a-2}, 4a - 5) \in \Gamma(4a - 5)$ with $a \geq 2$. Then x_{4a-2} is 0 or 2.*

PROOF. By Theorem 2.1, $0 \leq x_{4a-2} \leq 2 + \left[\dfrac{1}{a-1}\right]$. Thus $0 \leq x_{4a-2} \leq 3$. Suppose that M is of Type $\mathbf{c} = (x_2, \ldots, x_{4a-4}, 1, 4a - 5)$. Then, an *R*-matrix

M_R of M can be assumed that

$$M_R^* = \left[\begin{array}{c|c} J_{s\times 2} & J_{s\times t} \\ \hline J_{1\times 2} & O_{1\times t} \\ \hline O_2 & J_{2\times t} \\ \hline J_{1\times 2} & O_{1\times t} \end{array}\right] \quad \text{or} \quad \left[\begin{array}{c|c} J_{t\times 2} & J_t \\ \hline J_{1\times 2} & O_{1\times t} \\ \hline I_2 \otimes J_{2\times 1} & J_{4\times t} \\ \hline J_{1\times 2} & O_{1\times t} \end{array}\right],$$

where $s = 4a - 2$ and $t = 4a - 4$. In any case, this means that M^t is of Type $\bar{\mathbf{c}} = (\bar{x}_2, \ldots, \bar{x}_{4a-4}, 2, 4a - 5)$ which means that $\bar{\mathbf{c}} > \mathbf{c}$. This contradicts to the assumption of Type $\mathbf{c}$. Next, let M be of Type $\mathbf{c} = (x_2, \ldots, x_{4a-4}, 3, 4a - 5)$. This case occurs only when $a = 2$. Thus M is of Type $\mathbf{c} = (7, 0, 3, 3)$. Then an R-matrix M_R of M can be assumed that $M_R^* = [J_{6\times 7} \mid I_3 \otimes J_{2\times 1}]^t$. Clearly, $M_R^t M_R \neq 8I_6$. Thus this case does not hold. $\square$

THEOREM 2.4. *Let $M \in \Delta(1)$ and M be of Type $\mathbf{c} = (x_2, \ldots, x_{4a-2}, 1)$. Then $x_2 \leq 4$.*

PROOF. Let $x_2 \geq 5$ and M_D be a D-matrix of M. Since $a \geq 2$, M_D contains a submatrix N such that $N^* = [N_L^* \mid J_{5\times(4a-2)}]$ and N_L^* is a $5 \times 4a$ matrix whose each row has just two 1's. Thus it can be assumed that

$$N_L^* = \left[\begin{array}{c|c|c|c} J_2 & O_2 & O_2 & \\ O_2 & J_2 & O_2 & O_{5\times(4a-6)} \\ O_{1\times 2} & O_{1\times 2} & J_{1\times 2} & \end{array}\right].$$

This, with Lemma 2.6 and Remark 2.2, shows that $N^t N \neq 4aI_5$. $\square$

THEOREM 2.5. *Let $M \in \Delta(0)$ and M be of Type $\mathbf{c} = (x_2, \ldots, x_{4a-2}, 0)$. Then $x_2 \leq 2$.*

PROOF. Let $x_2 \geq 3$ and M_D be a D-matrix of M. Then M_D has a submatrix $N = [N_1 \mid N_2]$, where N_1 is a $3 \times 4a$ matrix whose each row contains just two non-zero elements and $N_2^* = J_{3\times(4a-2)}$. In this case, let $N_3 = N_1 N_1^t - 2I_3$. Then it follows that elements of N_3 are either ± 2 or 0, in order to keep the orthogonality with respect to rows of N. If there exists a non-zero element in N_3, then $M \in \Delta(1)$, which contradicts to the assumption of M of Type $\mathbf{c}$. If $N_3 = O_3$, $N_2 N_2^t = (4a - 2)I_3$, which is impossible by Lemma 2.6. $\square$

3. Construction and classification of $W(14, 8)$'s

In this section, we only consider a case $a = 2$ in the previous section. This case has special interest as described in Section 1. By Theorem 2.1, there are 25 solutions of IPC with parameters 14 and 8. They are listed in the following:

	x_2	x_4	x_6	x_8		x_2	x_4	x_6	x_8
$\mathbf{c}_1 =$	8	0	0	5	$\mathbf{c}_2 =$	7	1	1	4
$\mathbf{c}_3 =$	6	3	0	4	$\mathbf{c}_4 =$	7	0	3	3
$\mathbf{c}_5 =$	6	2	2	3	$\mathbf{c}_6 =$	5	4	1	3
$\mathbf{c}_7 =$	4	6	0	3	$\mathbf{c}_8 =$	6	1	4	2
$\mathbf{c}_9 =$	5	3	3	2	$\mathbf{c}_{10} =$	4	5	2	2
$\mathbf{c}_{11} =$	3	7	1	2	$\mathbf{c}_{12} =$	2	9	0	2
$\mathbf{c}_{13} =$	6	0	6	1	$\mathbf{c}_{14} =$	5	2	5	1
$\mathbf{c}_{15} =$	4	4	4	1	$\mathbf{c}_{16} =$	3	6	3	1
$\mathbf{c}_{17} =$	2	8	2	1	$\mathbf{c}_{18} =$	1	10	1	1
$\mathbf{c}_{19} =$	0	12	0	1	$\mathbf{c}_{20} =$	5	1	7	0
$\mathbf{c}_{21} =$	4	3	6	0	$\mathbf{c}_{22} =$	3	5	5	0
$\mathbf{c}_{23} =$	2	7	4	0	$\mathbf{c}_{24} =$	1	9	3	0
$\mathbf{c}_{25} =$	0	11	2	0					

It follows from Theorems 2.2–2.5 that there is no weighing matrix of Type $\mathbf{c}_i$ for $i = 1, 2, 3, 4, 6, 13, 14, 20, 21, 22$.

DEFINITION 3.1. Let N and N_i be $s \times 6$ matrices whose elements are ± 1 or 0 and weights of columns are 8 for $i = 1, 2$. N^* is said to be *admissible* when all elements of $N^{*t}N^*$ are even. If $N^tN = 8I_6$, N is said to be *feasible*. When M is a weighing matrix of Type $\mathbf{c}$ and N is an R-matrix of M, both admissible matrix N^* and feasible matrix N are said to be of Type $\mathbf{c}$. For two admissible matrices, N_1^* and N_2^*, if there are permutation matrices Q_1 and Q_2 such that $N_2^* = Q_1 N_1^* Q_2$, N_2^* is said to be *equivalent* to N_1^*. For two feasible matrices, N_1 and N_2, if there are signed permutation matrices $\bar{Q}_1$ and $\bar{Q}_2$ such that $N_2 = \bar{Q}_1 N_1 \bar{Q}_2$, N_2 is said to be *equivalent* to N_1.

One can find many admissible and feasible matrices. For example, an admissible matrix, say A^*, and a feasible matrix, say F, are given as follows.

$$A^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}^t,$$

$$F = \left[\begin{array}{cccccc:cc:cc} 1 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 0 & 0 & 1 & 1 \\ \hdashline 1 & 1 & - & - & 1 & 1 & - & - & 0 & 0 \\ 1 & 1 & 1 & 1 & - & - & - & - & 0 & 0 \\ 1 & 1 & - & - & - & - & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]^t,$$

where the symbol "–" means -1. Throughout this paper, the symbol "–" is used instead of -1. It can easily be shown that A^* and F are of Type $\mathbf{c}_{10}$ and of Type $\mathbf{c}_5$, respectively.

DEFINITION 3.2. Let $M \in \Delta$ and M_R be an R-matrix of M. Without loss of generality, it can be assumed that $M_R = [L(6)^t \mid L(4)^t \mid L(2)^t]^t$, where the weights of all rows of $L(i)$ equal i for $i = 2, 4, 6$. In this case, $L(i)$ is called an *Ri-matrix* of M_R. Letting $\mathbf{m}$ be a column of M_R, the portion belonging to $L(i)$ of $\mathbf{m}$ is called the *Ri-part* of $\mathbf{m}$.

Note that the existence of a $W(14, 8)$ implies the admissibility of an R-matrix. The following theorem will be proved by showing the non-existence of an admissible matrix for each type.

THEOREM 3.1. *There is no weighing matrix of Type* $\mathbf{c}_i$ *for* $i = 8, 11, 12, 16$.

PROOF. (i) Type $\mathbf{c}_8$. Let M_R be an R-matrix of such a weighing matrix. Then, without loss of generality, it can be assumed that

$$M_R^* = \left[\begin{array}{c:c} J_{6\times 4} & J_{6\times 2} \\ \hdashline J_{1\times 4} & O_{1\times 2} \\ \hdashline N_1^* & N_2^* \end{array}\right],$$

where the 4×6 matrix $[N_1 \mid N_2]$ is an $R2$-matrix of M_R. Thus M_R^* is not admissible, because there exists at least one pair of columns having an odd intersection number in the first four columns of M_R^*.

(ii) Type $\mathbf{c}_{11}$. Let M_R be an R-matrix of such a weighing matrix of Type $\mathbf{c}_{11}$. Then, without loss of generality, it can be assumed that

$$M_R^* = \left[\begin{array}{c|c} J_{3\times 2} & J_{3\times 4} \\ \hline J_{1\times 2} & O_{1\times 4} \\ \hline N_1^* & N_2^* \end{array}\right],$$

where the 7×6 matrix $N = [N_1 \mid N_2]$ is an $R4$-matrix of M_R. Moreover, as N^*, two cases, say $N(1)^*$ and $N(2)^*$, can be considered, where

$$N(1)^* = \left[\begin{array}{c|c} J_{4\times 2} & K_1^* \\ \hline O_{3\times 2} & J_{3\times 4} \end{array}\right] \quad \text{and} \quad N(2)^* = \left[\begin{array}{c|c} L_2^* & K_2^* \\ \hline O_{1\times 2} & J_{1\times 4} \end{array}\right],$$

with $L_2^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}^t$. For both cases of $N(1)$ and $N(2)$, one cannot determine K_1^* and K_2^* so that M_R^* is admissible.

(iii) Type $\mathbf{c}_{12}$. Let M_R be an R-matrix of such a weighing matrix of Type $\mathbf{c}_{12}$. Then, without loss of generality, it can be assumed that

$$M_R^* = \left[\begin{array}{c|c} J_{2\times 1} & J_{2\times 5} \\ \hline J_{6\times 1} & K^* \\ \hline O_{3\times 1} & L^* \end{array}\right],$$

where K is a 6×5 matrix and the weight of a column of K is 6 or 4. Let x_i be the number of columns of K^* having weight i, where $i = 6$ or 4. Thus we have two equations similar to IPC: $x_4 + x_6 = 5$ and $4x_4 + 6x_6 = 3 \times 6$. But there does not exist a non-negative solution. Hence, M_R^* is not admissible.

(iv) Type $\mathbf{c}_{16}$. Let M_R be an R-matrix of such a weighing matrix of Type $\mathbf{c}_{16}$ and M_{R2} be an $R2$-matrix of M_R. If M_{R2} has the submatrix $O_{3\times 1}$, it can be assumed that for the first column $\mathbf{m}$ of M_R, $\mathbf{m} = [\mathbf{m}_6^t \mid \mathbf{m}_4^t \mid \mathbf{m}_2^t]^t$, where $\mathbf{m}_i$ is the Ri-part of $\mathbf{m}$, $\mathbf{m}_6 = J_{3\times 1}$, $\mathbf{m}_4 = [J_{1\times 5} \mid 0]^t$ and $\mathbf{m}_2 = O_{3\times 1}$. Let $\overline{\mathbf{m}}$ ($\neq \mathbf{m}$) be any column of M_R. Then the intersection number of $\overline{\mathbf{m}}$ and $\mathbf{m}$ in the $R4$-part of M_R must be odd. Thus there are two equations: $x_1 + x_3 + x_5 = 5$ and $x_1 + 3x_3 + 5x_5 = 15$, where x_i is the number of columns having the intersection number i with $\mathbf{m}$ in the $R4$-matrix of M. Only three solutions $(x_1, x_3, x_5) = (2, 1, 2)$, $(1, 3, 1)$ and $(0, 5, 0)$ are obtained. However, in each case, one cannot determine an $R4$-matrix of M_R so that M_R^* is admissible. Next, if M_{R2} does not have the submatrix $O_{3\times 1}$, it can be assumed that $M_{R2}^* = I_3 \otimes J_{1\times 2}$. Then it follows that

$$M_{R4}^* = \left[\begin{array}{cc|c} 1 & 1 & \\ 1 & 1 & K_1^* \\ 1 & 0 & \\ 1 & 0 & \\ \hline 0 & 1 & \\ 0 & 1 & K_2^* \end{array}\right] \quad \text{or} \quad \left[\begin{array}{c|c} J_{4\times 2} & K_3^* \\ \hline O_2 & J_{2\times 4} \end{array}\right].$$

But it can also be shown that it is impossible to make M_R^* to be admissible in each case. This completes the proof. □

Note that the existence of a $W(14, 8)$ also implies the existence of a feasible matrix. The following theorem will be proved by showing the non-existence of a feasible matrix.

THEOREM 3.2. *There is no weighing matrix of Type* $\mathbf{c}_{10}$.

PROOF. Let M_R be an R-matrix of a weighing matrix of Type $\mathbf{c}_{10}$. Then, without loss of generality, it can be assumed that

$$M_R^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}^t.$$

Let $\mathbf{m}_i$ be the i-th column of M_R for $1 \le i \le 6$. Then, without loss of generality, it can be assumed that $\mathbf{m}_1 = (1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 0)^t$. It follows that there are three inequivalent cases to consider in order to decide on the second row of M_R, say $\mathbf{m}_2^{(j)}$, $1 \le j \le 3$, where $\mathbf{m}_2^{(1)} = (1, 1, 1, 1, 0, -, -, -, -, 0, 0)^t$, $\mathbf{m}_2^{(2)} = (1, 1, -, -, 0, 1, 1, -, -, 0, 0)^t$, $\mathbf{m}_2^{(3)} = (1, 1, 1, -, 0, 1, -, -, -, 0, 0)^t$. But it is impossible to construct a feasible matrix based on the matrix $[\mathbf{m}_1 \mid \mathbf{m}_2^{(j)}]$ for $j = 2$ and 3, because there is no 6×3 matrix S such that $S^* = J_{6\times 3}$ and $S^tS = 6I_3$ by Lemma 2.6. There are exactly two inequivalent matrices, say X_1 and X_2, based on the matrix $[\mathbf{m}_1 \mid \mathbf{m}_2^{(1)}]$ so that they are enlarged as large as possible keeping on the orthogonality with respect to columns, where

$$X_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & - & - & - & - & 0 & 0 \\ 1 & 1 & - & - & 1 & 0 & 1 & 0 & - & 0 & 1 \\ 1 & - & 1 & - & 1 & 1 & 0 & - & 0 & 0 & - \end{bmatrix}^t,$$

$$X_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & - & - & - & - & 0 & 0 \\ 1 & 1 & - & - & 1 & 0 & 1 & 0 & - & 0 & 1 \\ 1 & - & 1 & - & - & 1 & 0 & - & 0 & 0 & 1 \\ 1 & - & - & 1 & - & 0 & 0 & 1 & - & 1 & 0 \end{bmatrix}^t.$$

However, they cannot be extended into a feasible matrix. □

Hereafter, it will be investigated successively in the following lemmas and theorems whether there are weighing matrices of the remaining types or not.

LEMMA 3.1. *There is the unique weighing matrix of Type* $\mathbf{c}_5$ *up to equivalence.*

PROOF. Let M be a weighing matrix of Type $\mathbf{c}_5$ and $M_R = [L(6)^t \mid L(4)^t \mid L(2)^t]^t$ be an R-matrix of M, where $L(i)$ is the Ri-matrix of M_R. Considering $L(4)^*$ and $L(2)^*$, one can show that M_R^* is equivalent to one of the following matrices:

$$\left[\begin{array}{c|c} J_{6\times 2} & J_{6\times 4} \\ \hline O_2 & J_{2\times 4} \\ \hline J_2 & O_{2\times 4} \end{array}\right], \quad \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{J_{6\times 3}} & J_{6\times 3} \\ \hline 0 & 1 & 0 & \\ 0 & 0 & 1 & J_{2\times 3} \\ \hline 1 & 1 & 0 & \\ 1 & 0 & 1 & O_{2\times 3} \end{array}\right], \quad \left[\begin{array}{cccc|c} \multicolumn{4}{c|}{J_{6\times 2}} & J_2 \\ \hline 1 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 1 & J_2 \\ \hline 1 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 1 & O_2 \end{array}\right].$$

Clearly, the last two matrices are not admissible. Thus one can assume that M_R^* is the first one.

Next, it will be shown that M_R is unique up to equivalence. Let $\mathbf{m}_i$ be the i-th column of $M_R = [X_1 \mid X_2]$, where $1 \le i \le 6$ and $X_1^* = [J_{2\times 6} \mid O_2 \mid J_2]^t$, $X_2^* = [J_{4\times 6} \mid J_{4\times 2} \mid O_{4\times 2}]^t$. Suppose that $\mathbf{m}_3$ and $\mathbf{m}_4$ are orthogonal in the $R6$-parts of them. Then $\mathbf{m}_1$ is not orthogonal to $\mathbf{m}_3$ and $\mathbf{m}_4$ by Lemma 2.6. Thus, for $3 \le i \le 6$, the number of positive elements of $\mathbf{m}_i$ is even. Hence, without loss of generality, it can be assumed that

$$X_2 = \left[\begin{array}{cccccc|cc|cc} 1 & 1 & - & - & 1 & 1 & - & - & 0 & 0 \\ 1 & 1 & 1 & 1 & - & - & - & - & 0 & 0 \\ 1 & 1 & - & - & - & - & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]^t.$$

Furthermore, two feasible matrices of Type $\mathbf{c}_5$, say S and T, can be constructed, where

$$S = \left[\begin{array}{cccccc|cc|cc} 1 & - & 1 & - & - & 1 & 0 & 0 & - & - \\ 1 & - & 1 & - & 1 & - & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & - & - & 1 & 1 & - & - & 0 & 0 \\ 1 & 1 & 1 & 1 & - & - & - & - & 0 & 0 \\ 1 & 1 & - & - & - & - & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]^t,$$

$$T = \left[\begin{array}{cccccc|cc|cc} 1 & - & - & 1 & - & 1 & 0 & 0 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & - & - & 1 & 1 & - & - & 0 & 0 \\ 1 & 1 & 1 & 1 & - & - & - & - & 0 & 0 \\ 1 & 1 & - & - & - & - & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]^t.$$

Let $\pi = (5, 6, 1, 2, 3, 4, 7, 8, 9, 10)$ and $\rho = (\underline{1}, 2, \underline{5}, 3, \underline{4}, 6)$ be two signed permutations. Then $S^{(\pi,\rho)} = T$, i.e., S is equivalent to T. For the notations π, ρ and $S^{(\pi,\rho)}$, refer to Remark 3.1. Thus it follows that a feasible matrix based on M_R^* can be uniquely constructed up to equivalence, say P_5^1, where $P_5^1 = S$.

Finally, one can show that there exists the unique weighing matrix of Type $\mathbf{c}_5$ up to equivalence. Let M_U be a U-matrix of M. Then, without loss of generality, it can be assumed that

$$M_U = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & 1 & - & - & - & - & 1 & 1 \end{bmatrix}.$$

The trial and error approach produces the unique weighing matrix up to equivalence, say $(U1, 1)$, based on P_5^1 and M_U, where

$$(U1, 1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & - & - & 1 & 1 & - & - & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & - & - & - & - & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 1 & 1 & 1 & 1 \\ - & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & - & 1 & - & 1 \\ 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & 1 & - & 1 \\ 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & - & 1 & 1 & - & - & 1 \\ 0 & 0 & - & 1 & 0 & 0 & 0 & 0 & 1 & - & 1 & - & - & 1 \\ 0 & 0 & 0 & 0 & 1 & - & 1 & - & 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & 0 & 0 & - & 1 & - & 1 & 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & - & 1 & 1 & - & - & 1 & - & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 1 & - & 1 & 1 & - & - & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $(U1, 1)$ refer to Remark 3.2. This completes the proof. $\square$

REMARK 3.1. The notation $\pi(i, \underline{j}, \ldots, k)$ $(\rho(i, \underline{j}, \ldots, k))$ means a row (column) signed permutation on a matrix as follows: move the i-th row (column) to the first row (column), the j-th row (column) to the second row (column) by multiplying -1 in addition, $\ldots$, the k-th row (column) to the last row (column). The notation $X^{(\pi,\rho)}$ means the matrix resulting from the operations by row and column signed permutations π and ρ, respectively, on a matrix X.

REMARK 3.2. Many weighing matrices are constructed in Lemma 3.1 and the forthcoming Lemmas 3.2–3.6. They are listed with the abbreviated forms in Table 2 of this section in the following manner: (i) the name of a weighing matrix (for example, $(U1, 1)$) is given; (ii) for each row of a weighing matrix, the number is corresponded, i.e. for the row $(m_1, \ldots, m_{14})$ the number $\sum_{i=1}^{14} \overline{m}_i 3^{i-1}$, where $m_i \equiv \overline{m}_i (\mathrm{mod}\ 3)$, $0 \le \overline{m}_i \le 2$; (iii) the number corresponding to each row of a weighing matrix is given in order starting from the second row, because the first row of the matrix is $(1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0)$ which is common to all weighing matrices. For example, the weighing matrix W_1, named $(U1, 1)$, is expressed as follows:

$(U1, 1)$	6520	6232	3640	2388211	2414453	2978699
3004945	3103416	3116520	2603826	2602206	37062	38358

Here, for example, the number 3004945 corresponds to the 8-th row $(1, -, 0, 0, 0, 0, 0, 0, -, -, -, 1, -, 1)$ of $(U1, 1)$.

In the following, one will obtain many matrices, in the order of admissible, feasible and weighing matrices for each type. But the methods to find them are not described in detail, because they can be obtained with the same way as in the proof of Lemma 3.1.

LEMMA 3.2. *There are three inequivalent feasible matrices, say* P_7^i, $1 \le i \le 3$, *of Type* $\mathbf{c}_7$. *At most* n_7^i *inequivalent weighing matrices based on* P_7^1 *can be constructed with* $n_7^1 = n_7^2 = n_7^3 = 1$.

PROOF. Let M be a weighing matrix of Type $\mathbf{c}_7$ and M_R be an R-matrix of M. Then, M_R^* is unique up to equivalence, i.e., $M_R^* = [J_{6\times 4} \mid J_6 - I_3 \otimes J_2]^t$. Moreover, there are only three inequivalent feasible matrices, say P_7^i, $i = 1, 2, 3$, based on M_R^*, where

$$P_7^1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 1 & 1 \\ - & - & - & - & 1 & 1 \\ 0 & 0 & - & 1 & - & 1 \\ 0 & 0 & 1 & - & - & 1 \\ - & 1 & 0 & 0 & - & 1 \\ 1 & - & 0 & 0 & - & 1 \\ - & 1 & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \end{bmatrix}, \quad P_7^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & 1 \\ - & 1 & - & 1 & - & 1 \\ 1 & - & 1 & - & - & 1 \\ 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & - & - & 1 & 1 \\ 1 & 1 & 0 & 0 & - & 1 \\ - & - & 0 & 0 & - & 1 \\ 1 & - & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \end{bmatrix},$$

$$P_7^3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & 1 \\ - & 1 & - & - & 1 & 1 \\ 1 & - & - & - & 1 & 1 \\ 0 & 0 & - & 1 & - & 1 \\ 0 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 0 & - & 1 \\ - & - & 0 & 0 & - & 1 \\ - & 1 & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \end{bmatrix}.$$

Thus a weighing matrix, say $(Vi, 1)$, based on P_7^i can be uniquely constructed up to equivalence by the trial and error. Such $\{(Vi, 1)\}$ are listed in Table 2. □

THEOREM 3.3. *There is no weighing matrix of Type* $\mathbf{c}_9$.

PROOF. Let M_R be an R-matrix of a weighing matrix of Type $\mathbf{c}_9$ and M_{R2} be an $R2$-matrix of M_R. Then M_{R2}^* is equivalent to one of the following matrices, say $K(i)^*$, $1 \le i \le 8$:

$$\underset{(1)}{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}} \quad \underset{(2)}{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}} \quad \underset{(3)}{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}}$$

$$\underset{(4)}{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}} \quad \underset{(5)}{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}} \quad \underset{(6)}{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}},$$

$$\underset{(7)}{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{(8)}{\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

where (i) corresponds to $K(i)^*$. It can be shown that there is no admissible matrix of Type $\mathbf{c}_9$ based on $K(i)^*$ except for $i = 4, 8$. Note that an admissible matrix based on $K(1)^*$ can be constructed, but it is not of Type $\mathbf{c}_9$. Furthermore, one can construct uniquely an admissible matrix based on $K(i)^*$, say K_i^*, $i = 4, 8$, up to equivalence, where

$$K_4^* = \left[\begin{array}{cccccc} \multicolumn{6}{c}{J_{5\times 6}} \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right], \quad K_8^* = \left[\begin{array}{cccccc} \multicolumn{6}{c}{J_{5\times 6}} \\ \hline 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

Repeated applications of Lemma 2.6 show that K_8^* only is transformed to a feasible matrix, say K_8, up to equivalence, where

$$K_8 = \begin{bmatrix} 1 & - & 1 & - & - & 0 & 1 & 1 & 0 & 0 & - \\ 1 & - & 1 & - & 1 & 0 & - & - & 0 & 0 & 1 \\ 1 & 1 & - & - & - & 1 & 0 & - & 0 & - & 0 \\ 1 & 1 & - & - & 1 & - & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & - & - & - & 0 & - & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}^t.$$

But it can be shown by computer calculation that a weighing matrix based on K_8 does not exist. □

LEMMA 3.3. *There are four inequivalent feasible matrices of Type* $\mathbf{c}_{15}$, *say* P_{15}^i, $1 \le i \le 4$. *At most* n_{15}^i *inequivalent weighing matrices of Type* $\mathbf{c}_{15}$ *based on* P_{15}^i *can be constructed with* $n_{15}^1 = 7$, $n_{15}^2 = 2$, $n_{15}^3 = 2$, $n_{15}^4 = 0$.

PROOF. Let M be a weighing matrix of Type $\mathbf{c}_{15}$ and M_R be an R-matrix of M. Let M_{R2} be an $R2$-matrix of M_R. Then M_{R2}^* is equivalent to one of the following matrices, say $K(i)^*$, $1 \le i \le 21$:

$$\underset{(1)}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{(2)}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{(3)}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}}$$

$$\underset{(4)}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}} \quad \underset{(5)}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}} \quad \underset{(6)}{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}}$$

$$\begin{bmatrix}1&1&0&0&0&0\\1&1&0&0&0&0\\1&0&1&0&0&0\\0&1&1&0&0&0\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0\\1&1&0&0&0&0\\1&0&1&0&0&0\\0&1&0&1&0&0\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0\\1&1&0&0&0&0\\1&0&1&0&0&0\\0&0&1&1&0&0\end{bmatrix}$$
$$(7) \qquad\qquad\qquad (8) \qquad\qquad\qquad (9)$$

$$\begin{bmatrix}1&1&0&0&0&0\\1&1&0&0&0&0\\1&0&1&0&0&0\\0&0&0&1&1&0\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0\\1&0&1&0&0&0\\1&0&0&1&0&0\\0&1&1&0&0&0\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0\\1&0&1&0&0&0\\1&0&0&1&0&0\\0&1&0&0&1&0\end{bmatrix}$$
$$(10) \qquad\qquad\qquad (11) \qquad\qquad\qquad (12)$$

$$\begin{bmatrix}1&1&0&0&0&0\\1&0&1&0&0&0\\1&0&0&1&0&0\\0&0&0&0&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0\\1&1&0&0&0&0\\0&0&1&1&0&0\\0&0&1&1&0&0\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0\\1&1&0&0&0&0\\0&0&1&1&0&0\\0&0&1&0&1&0\end{bmatrix}$$
$$(13) \qquad\qquad\qquad (14) \qquad\qquad\qquad (15)$$

$$\begin{bmatrix}1&1&0&0&0&0\\1&1&0&0&0&0\\0&0&1&1&0&0\\0&0&0&0&1&1\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&1&0&0\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0\\1&0&1&0&0&0\\0&1&1&0&0&0\\0&0&0&1&1&0\end{bmatrix}$$
$$(16) \qquad\qquad\qquad (17) \qquad\qquad\qquad (18)$$

$$\begin{bmatrix}1&1&0&0&0&0\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0\\1&0&1&0&0&0\\0&0&0&1&1&0\\0&0&0&1&0&1\end{bmatrix} \quad \begin{bmatrix}1&1&0&0&0&0\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&0&0&1&1\end{bmatrix},$$
$$(19) \qquad\qquad\qquad (20) \qquad\qquad\qquad (21)$$

where (i) corresponds to $K(i)^*$.

Suppose that $K(i)^*$ can be extended to an admissible matrix K^* of Type $\mathbf{c}_{15}$ so that the $R2$-matrix of K^* is $K(i)^*$. If there exists a column of weight 0 in $K(i)^*$, the weights of columns in the $R4$-matrix of K^* are 4. Consequently, weights of the other columns of K^* must be even in the $R2$-matrix. Thus the cases of $K(i)^*$ are removed for $i = 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 18, 19$. Furthermore, $K(i)^*$ for $i = 1, 3$ are also removed, because weighing matrices constructed based on these cases are of larger types than Type $\mathbf{c}_{15}$. In a similar way, it follows that for $i = 16, 20, 21$, $K(i)^*$ cannot be extended to the admissible matrices. From $K(i)^*$ for $i = 13, 14, 17$, one can uniquely construct an admissible matrix K_i^* up to equivalence, where

$$K_{13}^* = \left[\begin{array}{c} J_{4\times 6} \\ \hline 0\ 1\ 1\ 0\ 1\ 1 \\ 0\ 1\ 0\ 1\ 1\ 1 \\ 0\ 0\ 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 1\ 0\ 0 \\ \hline K(13)^* \end{array}\right], \quad K_{14}^* = \left[\begin{array}{c} J_{4\times 6} \\ \hline 1\ 1\ 0\ 0\ 1\ 1 \\ 1\ 1\ 0\ 0\ 1\ 1 \\ 0\ 0\ 1\ 1\ 1\ 1 \\ 0\ 0\ 1\ 1\ 1\ 1 \\ \hline K(14)^* \end{array}\right], \quad K_{17}^* = \left[\begin{array}{c} J_{4\times 6} \\ \hline 1\ 1\ 0\ 0\ 1\ 1 \\ 0\ 0\ 1\ 1\ 1\ 1 \\ 1\ 0\ 1\ 0\ 1\ 1 \\ 0\ 1\ 0\ 1\ 1\ 1 \\ \hline K(17)^* \end{array}\right].$$

But some repeated applications of Lemma 2.6 show that it is impossible to construct a feasible matrix based on K_{13}^* or K_{17}^*. Moreover, there are only four inequivalent feasible matrices, say P_{15}^i, $1 \leq i \leq 4$, based on K_{14}^*, where

$$\underset{(1)}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & 1 \\ 1 & - & - & 1 & - & 1 \\ 1 & - & 1 & - & - & 1 \\ 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & - & - & 1 & 1 \\ - & 1 & 0 & 0 & - & 1 \\ - & 1 & 0 & 0 & - & 1 \\ 0 & 0 & - & 1 & 0 & 0 \\ 0 & 0 & - & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}}, \quad \underset{(2)}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & 1 \\ - & 1 & - & - & 1 & 1 \\ 1 & - & - & - & 1 & 1 \\ 0 & 0 & - & 1 & - & 1 \\ 0 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 0 & - & 1 \\ - & - & 0 & 0 & - & 1 \\ 0 & 0 & - & 1 & 0 & 0 \\ 0 & 0 & - & 1 & 0 & 0 \\ - & 1 & 0 & 0 & 0 & 0 \\ - & 1 & 0 & 0 & 0 & 0 \end{bmatrix}}, \quad \underset{(3)}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & - & - & 1 & 1 \\ - & - & 1 & 1 & - & 1 \\ 1 & 1 & - & - & - & 1 \\ 0 & 0 & - & 1 & 1 & 1 \\ 0 & 0 & 1 & - & 1 & 1 \\ - & 1 & 0 & 0 & - & 1 \\ 1 & - & 0 & 0 & - & 1 \\ 0 & 0 & - & 1 & 0 & 0 \\ 0 & 0 & - & 1 & 0 & 0 \\ - & 1 & 0 & 0 & 0 & 0 \\ - & 1 & 0 & 0 & 0 & 0 \end{bmatrix}},$$

$$\underset{(4)}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 1 & 1 \\ - & - & - & - & 1 & 1 \\ 0 & 0 & - & 1 & - & 1 \\ 0 & 0 & 1 & - & - & 1 \\ - & 1 & 0 & 0 & - & 1 \\ 1 & - & 0 & 0 & - & 1 \\ 0 & 0 & - & 1 & 0 & 0 \\ 0 & 0 & - & 1 & 0 & 0 \\ - & 1 & 0 & 0 & 0 & 0 \\ - & 1 & 0 & 0 & 0 & 0 \end{bmatrix}},$$

and (i) corresponds to P_{15}^i. By computer search at most n_{15}^i inequivalent weighing matrices, say (Wi, l), $1 \leq l \leq n_{15}^i$, based on P_{15}^i can be constructed with $n_{15}^1 = 7$, $n_{15}^2 = 2$, $n_{15}^3 = 2$ and $n_{15}^4 = 0$. For the method of constructing weighing matrices with the aid of a computer, refer to Remark 3.3. Such $\{(Wi, l)\}$ are listed in Table 2. $\square$

Remark 3.3. The present algorithm for construction of weighing matrices of Type $\mathbf{c}$ is described as follows: (i) construct a set of column vectors of size 14 and weight 8 which are orthogonal to each column of a feasible matrix of Type $\mathbf{c}$, and choose eight vectors with the first elements being all ones which are orthogonal to each other in the set; (ii) remove weighing matrices obtained in (i) which are matrices of larger types than Type $\mathbf{c}$; (iii) remove equivalent matrices by using automorphism groups of feasible matrices and automorphism groups of the U-matrices of weighing matrices obtained. The computation was performed on a PC-9801 computer.

Our algorithm will be used for constructing weighing matrices of each type hereafter.

Lemma 3.4. *There are five inequivalent feasible matrices of Type* $\mathbf{c}_{17}$, *say* P_{17}^i, $1 \leq i \leq 5$. *At most* n_{17}^i *inequivalent weighing matrices* $\mathbf{c}$ *of Type* $\mathbf{c}_{17}$ *based on* P_{17}^i *can be constructed with* $n_{17}^1 = 1$, $n_{17}^2 = 2$, $n_{17}^3 = 1$, $n_{17}^4 = 0$ *and* $n_{17}^5 = 0$.

Proof. Let K be an R-matrix of a weighing matrix of Type $\mathbf{c}_{17}$. Then K^* is equivalent to one of three inequivalent admissible matrices, say K_i^*, $1 \leq i \leq 3$, of Type $\mathbf{c}_{17}$, as follows:

$$K_1^* = \left[\begin{array}{cccccc} \multicolumn{6}{c}{J_{2\times 6}} \\ \hdashline 1&1&0&0&1&1 \\ 1&1&0&0&1&1 \\ 0&0&1&1&1&1 \\ 0&0&1&1&1&1 \\ 1&1&1&1&0&0 \\ 1&1&1&1&0&0 \\ 1&1&1&1&0&0 \\ 1&1&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&1&1 \end{array}\right], \quad K_2^* = \left[\begin{array}{cccccc} \multicolumn{6}{c}{J_{2\times 6}} \\ \hdashline 1&0&1&1&0&1 \\ 1&0&1&1&0&1 \\ 0&1&1&1&0&1 \\ 0&1&1&1&0&1 \\ 1&1&1&0&1&0 \\ 1&1&1&0&1&0 \\ 1&1&0&1&1&0 \\ 1&1&0&1&1&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&1&1 \end{array}\right], \quad K_3^* = \left[\begin{array}{cccccc} \multicolumn{6}{c}{J_{2\times 6}} \\ \hdashline 0&1&0&1&1&1 \\ 0&1&0&1&1&1 \\ 1&0&1&1&0&1 \\ 1&0&1&1&0&1 \\ 1&1&1&0&1&0 \\ 1&1&1&0&1&0 \\ 1&1&1&1&0&0 \\ 1&1&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&1&1 \end{array}\right].$$

By Lemma 2.6, it is impossible to be extended to a feasible matrix of Type $\mathbf{c}_{17}$ based on K_2^*. However, there are only four inequivalent feasible matrices of Type $\mathbf{c}_{17}$ based on K_1^*, say P_{17}^i, $1 \leq i \leq 4$, and only one inequivalent feasible matrix based on K_3^*, say P_{17}^5, where

$$P^1_{17} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & - & - & 1 & 1 \\ 0 & 0 & - & 1 & - & 1 \\ 0 & 0 & 1 & - & - & 1 \\ - & 1 & 0 & 0 & - & 1 \\ 1 & - & 0 & 0 & - & 1 \\ - & - & 1 & 1 & 0 & 0 \\ - & - & 1 & 1 & 0 & 0 \\ - & 1 & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad P^2_{17} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & - & 1 \\ 0 & 0 & - & - & 1 & 1 \\ - & - & 0 & 0 & 1 & 1 \\ 0 & 0 & - & - & - & 1 \\ 1 & 1 & 0 & 0 & - & 1 \\ - & 1 & - & 1 & 0 & 0 \\ - & 1 & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & - & 1 \end{bmatrix},$$

$$P^3_{17} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & 1 \\ 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & - & - & 1 & 1 \\ - & 1 & 0 & 0 & - & 1 \\ 1 & - & 0 & 0 & - & 1 \\ 1 & 1 & - & 1 & 0 & 0 \\ - & 1 & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \\ - & - & - & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 1 \\ 0 & 0 & 0 & 0 & - & 1 \end{bmatrix}, \quad P^4_{17} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & 1 \\ 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & - & - & 1 & 1 \\ 1 & 1 & 0 & 0 & - & 1 \\ - & - & 0 & 0 & - & 1 \\ - & 1 & - & 1 & 0 & 0 \\ - & 1 & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 1 \\ 0 & 0 & 0 & 0 & - & 1 \end{bmatrix},$$

$$P^5_{17} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & - & 1 & 1 & 1 & 1 \\ 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & - & 1 & - & 1 \\ - & 1 & 0 & - & 0 & 1 \\ 1 & - & 0 & - & 0 & 1 \\ - & 1 & - & 0 & 1 & 0 \\ 1 & - & - & 0 & 1 & 0 \\ 1 & 1 & - & 1 & 0 & 0 \\ - & - & - & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 1 \\ 0 & 0 & 0 & 0 & - & 1 \end{bmatrix}.$$

By using a computer, at most n^i_{17} inequivalent weighing matrices, say (Xi, l),

$1 \le l \le n_{17}^i$, based on P_{17}^i can be constructed with $n_{17}^1 = 1$, $n_{17}^2 = 2$, $n_{17}^3 = 1$, $n_{17}^4 = 0$, $n_{17}^5 = 0$. Such $\{(Xi, l)\}$ are listed in Table 2. □

LEMMA 3.5. *There are two inequivalent feasible matrices of Type* $\mathbf{c}_{18}$, *say* P_{18}^i, $1 \le i \le 2$. *At most* n_{18}^i *inequivalent weighing matrices based on* P_{18}^i *can be constructed with* $n_{18}^1 = 1$ *and* $n_{18}^2 = 1$.

PROOF. Let K be an R-matrix of a weighing matrix of Type $\mathbf{c}_{18}$. Then K^* is equivalent to the following admissible matrix:

$$\left[\begin{array}{c|cccc} \multirow{2}{*}{J_2} & 1 & 1 & 1 & 1 \\ & 0 & 0 & 0 & 0 \\ \hline J_{6\times 2} & \multicolumn{4}{c}{L^*} \\ \hline O_{4\times 2} & \multicolumn{4}{c}{J_4} \end{array}\right],$$

where $L^{*t}L^* = 2I_4 + J_4$, i.e., L^{*t} is the incidence matrix of a BIBD with parameters $(4, 6, 3, 2, 1)$ (see Raghavarao (1971) for the definition of a BIBD). Without loss of generality, it can be expressed as

$$L^* = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}^t.$$

Then one can construct two inequivalent feasible matrices, say P_{18}^1 and P_{18}^2, based on the above admissible matrix, where

$$P_{18}^1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & - & - & 1 & 1 \\ - & - & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & - & 1 \\ 1 & 0 & 0 & - & - & 1 \\ 0 & - & 1 & 0 & - & 1 \\ - & 0 & - & 0 & - & 1 \\ - & 1 & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \\ - & - & 1 & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \end{bmatrix}, \quad P_{18}^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & - & 1 \\ 0 & 0 & - & 1 & 1 & 1 \\ 0 & - & 0 & - & 1 & 1 \\ - & 0 & 0 & - & 1 & 1 \\ 0 & 1 & 1 & 0 & - & 1 \\ - & 0 & - & 0 & - & 1 \\ 1 & - & 0 & 0 & - & 1 \\ - & - & 1 & 1 & 0 & 0 \\ - & - & 1 & 1 & 0 & 0 \\ - & 1 & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \end{bmatrix}.$$

By computer search, at most n_{18}^i inequivalent weighing matrices, say (Yi, l), $1 \le l \le n_{18}^i$, based on P_{18}^i can be constructed with $n_{18}^1 = 1$ and $n_{18}^2 = 1$. Such $\{(Yi, l)\}$ are listed in Table 2. □

LEMMA 3.6. *There are* 19 *inequivalent feasible matrices of Type* $\mathbf{c}_{19}$, say P_{19}^i, $1 \le i \le 19$. *At most* n_{19}^i *inequivalent weighing matrices based on* P_{19}^i *can be constructed with* $n_{19}^1 = 0$, $n_{19}^2 = 3$, $n_{19}^3 = 8$, $n_{19}^4 = 10$, $n_{19}^5 = 2$, $n_{19}^6 = 6$, $n_{19}^7 = 9$, $n_{19}^8 = 6$, $n_{19}^9 = 4$, $n_{19}^{10} = 5$, $n_{19}^{11} = 8$, $n_{19}^{12} = 6$, $n_{19}^{13} = 6$, $n_{19}^{14} = 1$, $n_{19}^{15} = 1$, $n_{19}^{16} = 1$, $n_{19}^{17} = 4$, $n_{19}^{18} = 1$, $n_{19}^{19} = 1$.

PROOF. Let K be an R-matrix of a weighing matrix of Type $\mathbf{c}_{19}$. Then K^* is equivalent to one of three inequivalent admissible matrices of Type $\mathbf{c}_{19}$ as follows:

$$K_1^* = \begin{bmatrix} 0&0&1&1&1&1\\ 0&0&1&1&1&1\\ 0&0&1&1&1&1\\ 0&0&1&1&1&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&1&1&0&0\\ 1&1&1&1&0&0\\ 1&1&1&1&0&0\\ 1&1&1&1&0&0 \end{bmatrix}, \quad K_2^* = \begin{bmatrix} 0&0&1&1&1&1\\ 0&0&1&1&1&1\\ 0&1&0&1&1&1\\ 0&1&0&1&1&1\\ 1&0&1&0&1&1\\ 1&0&1&0&1&1\\ 1&1&0&0&1&1\\ 1&1&0&0&1&1\\ 1&1&1&1&0&0\\ 1&1&1&1&0&0\\ 1&1&1&1&0&0\\ 1&1&1&1&0&0 \end{bmatrix},$$

$$K_3^* = \begin{bmatrix} 0&0&1&1&1&1\\ 0&0&1&1&1&1\\ 0&1&0&1&1&1\\ 0&1&0&1&1&1\\ 1&0&1&0&1&1\\ 1&0&1&0&1&1\\ 1&1&0&1&0&1\\ 1&1&0&1&0&1\\ 1&1&1&0&1&0\\ 1&1&1&0&1&0\\ 1&1&1&1&0&0\\ 1&1&1&1&0&0 \end{bmatrix}.$$

One can construct l_i inequivalent feasible matrices based on K_i^*, $1 \le i \le 3$, respectively, where $(l_1, l_2, l_3) = (5, 8, 6)$. They are numbered as P_{19}^l, $1 \le l \le 19$, where

$$
(1)\ \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & - & - & 1 & 1\\
0 & 0 & - & - & 1 & 1\\
1 & 1 & 0 & 0 & - & 1\\
1 & 1 & 0 & 0 & - & 1\\
- & - & 0 & 0 & - & 1\\
- & - & 0 & 0 & - & 1\\
- & 1 & - & 1 & 0 & 0\\
- & 1 & - & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0
\end{bmatrix},\quad
(2)\ \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & - & - & 1 & 1\\
0 & 0 & - & - & 1 & 1\\
1 & 1 & 0 & 0 & - & 1\\
- & 1 & 0 & 0 & - & 1\\
1 & - & 0 & 0 & - & 1\\
- & - & 0 & 0 & - & 1\\
1 & 1 & - & 1 & 0 & 0\\
- & 1 & - & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0\\
- & - & - & 1 & 0 & 0
\end{bmatrix},\quad
(3)\ \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & - & 1 & 1 & 1\\
0 & 0 & 1 & - & 1 & 1\\
0 & 0 & - & - & 1 & 1\\
1 & 1 & 0 & 0 & - & 1\\
- & 1 & 0 & 0 & - & 1\\
1 & - & 0 & 0 & - & 1\\
- & - & 0 & 0 & - & 1\\
1 & 1 & 1 & 1 & 0 & 0\\
- & - & 1 & 1 & 0 & 0\\
- & 1 & - & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0
\end{bmatrix},
$$

$$
(4)\ \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & - & 1 & 1 & 1\\
0 & 1 & 0 & - & 1 & 1\\
0 & - & 0 & - & 1 & 1\\
1 & 0 & 1 & 0 & - & 1\\
1 & 0 & - & 0 & - & 1\\
- & 1 & 0 & 0 & - & 1\\
- & - & 0 & 0 & - & 1\\
1 & 1 & 1 & 1 & 0 & 0\\
- & - & 1 & 1 & 0 & 0\\
- & 1 & - & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0
\end{bmatrix},\quad
(5)\ \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1\\
0 & 1 & 0 & 1 & 1 & 1\\
0 & 0 & - & - & 1 & 1\\
0 & - & 0 & - & 1 & 1\\
1 & 0 & 1 & 0 & - & 1\\
- & 0 & - & 0 & - & 1\\
- & 1 & 0 & 0 & - & 1\\
1 & - & 0 & 0 & - & 1\\
1 & 1 & - & 1 & 0 & 0\\
- & - & - & 1 & 0 & 0\\
- & - & 1 & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0
\end{bmatrix},\quad
(6)\ \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1\\
0 & 1 & 0 & 1 & 1 & 1\\
0 & 0 & - & - & 1 & 1\\
0 & - & 0 & - & 1 & 1\\
1 & 0 & 1 & 0 & - & 1\\
- & 0 & - & 0 & - & 1\\
1 & 1 & 0 & 0 & - & 1\\
- & - & 0 & 0 & - & 1\\
- & 1 & - & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0\\
- & - & 1 & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0
\end{bmatrix},
$$

$$
(7)\ \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & - & - & 1 & 1\\
1 & 1 & 0 & 0 & 1 & 1\\
- & - & 0 & 0 & 1 & 1\\
0 & 0 & - & 1 & - & 1\\
0 & 0 & 1 & - & - & 1\\
- & 1 & 0 & 0 & - & 1\\
1 & - & 0 & 0 & - & 1\\
1 & 1 & 1 & 1 & 0 & 0\\
- & - & 1 & 1 & 0 & 0\\
- & 1 & - & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0
\end{bmatrix},\quad
(8)\ \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & - & - & 1 & 1\\
1 & 1 & 0 & 0 & 1 & 1\\
- & - & 0 & 0 & 1 & 1\\
0 & 1 & 0 & 1 & - & 1\\
0 & - & 0 & - & - & 1\\
1 & 0 & 1 & 0 & - & 1\\
- & 0 & - & 0 & - & 1\\
- & 1 & - & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0\\
- & - & 1 & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0
\end{bmatrix},\quad
(9)\ \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & - & - & 1 & 1\\
1 & 1 & 0 & 0 & 1 & 1\\
- & - & 0 & 0 & 1 & 1\\
0 & 1 & 0 & 1 & - & 1\\
0 & - & 0 & - & - & 1\\
- & 0 & 1 & 0 & - & 1\\
1 & 0 & - & 0 & - & 1\\
- & 1 & - & 1 & 0 & 0\\
1 & - & - & 1 & 0 & 0\\
1 & - & 1 & 1 & 0 & 0\\
- & - & - & 1 & 0 & 0
\end{bmatrix},
$$

$$
(10)\begin{bmatrix}
0&0&1&1&1&1\\
0&0&-&-&1&1\\
1&1&0&0&1&1\\
-&-&0&0&1&1\\
0&0&-&1&-&1\\
0&0&1&-&-&1\\
-&1&0&0&-&1\\
1&-&0&0&-&1\\
-&1&1&1&0&0\\
1&-&1&1&0&0\\
1&1&-&1&0&0\\
-&-&-&1&0&0
\end{bmatrix},\quad
(11)\begin{bmatrix}
0&0&1&1&1&1\\
0&1&0&-&1&1\\
1&0&-&0&1&1\\
-&-&0&0&1&1\\
0&0&1&1&-&1\\
0&1&0&-&-&1\\
1&0&-&0&-&1\\
-&-&0&0&-&1\\
-&1&-&1&0&0\\
1&-&-&1&0&0\\
1&1&1&1&0&0\\
-&1&-&1&0&0
\end{bmatrix},\quad
(12)\begin{bmatrix}
0&0&1&1&1&1\\
0&1&0&-&1&1\\
1&0&-&0&1&1\\
-&-&0&0&1&1\\
0&0&-&1&-&1\\
0&-&0&-&-&1\\
1&0&1&0&-&1\\
-&1&0&0&-&1\\
1&1&1&1&0&0\\
-&-&1&1&0&0\\
-&1&-&1&0&0\\
1&-&-&1&0&0
\end{bmatrix},
$$

$$
(13)\begin{bmatrix}
0&0&1&1&1&1\\
0&1&0&-&1&1\\
1&0&-&0&1&1\\
-&-&0&0&1&1\\
0&0&-&1&-&1\\
0&-&0&-&-&1\\
1&0&1&0&-&1\\
-&1&0&0&-&1\\
-&1&1&1&0&0\\
1&-&1&1&0&0\\
1&1&-&1&0&0\\
-&-&-&1&0&0
\end{bmatrix},\quad
(14)\begin{bmatrix}
0&0&1&1&1&1\\
0&1&0&1&1&1\\
0&0&-&-&1&1\\
0&-&0&1&-&1\\
1&0&1&0&-&1\\
-&0&-&0&-&1\\
1&1&0&-&0&1\\
-&-&0&-&0&1\\
-&-&1&0&1&0\\
1&-&-&0&1&0\\
-&1&-&1&0&0\\
1&-&-&1&0&0
\end{bmatrix},\quad
(15)\begin{bmatrix}
0&0&1&1&1&1\\
0&1&0&1&1&1\\
0&0&-&-&1&1\\
0&-&0&1&-&1\\
1&0&1&0&-&1\\
-&0&-&0&-&1\\
-&1&0&-&0&1\\
1&-&0&-&0&1\\
-&-&1&0&1&0\\
1&-&-&0&1&0\\
1&1&-&1&0&0\\
-&-&-&1&0&0
\end{bmatrix},
$$

$$
(16)\begin{bmatrix}
0&0&1&1&1&1\\
0&1&0&-&1&1\\
1&0&-&0&1&1\\
0&1&0&1&-&1\\
0&0&1&-&-&1\\
-&0&-&0&-&1\\
-&-&0&1&0&1\\
1&-&0&-&0&1\\
-&-&1&0&1&0\\
-&1&-&0&1&0\\
1&1&1&1&0&0\\
1&-&-&1&0&0
\end{bmatrix},\quad
(17)\begin{bmatrix}
0&0&1&1&1&1\\
0&1&0&-&1&1\\
1&0&-&0&1&1\\
0&1&0&1&-&1\\
0&0&1&-&-&1\\
-&0&-&0&-&1\\
-&-&0&1&0&1\\
1&-&0&-&0&1\\
-&1&1&0&1&0\\
-&-&-&0&1&0\\
1&-&1&1&0&0\\
1&1&-&1&0&0
\end{bmatrix},\quad
(18)\begin{bmatrix}
0&0&1&1&1&1\\
0&0&-&-&1&1\\
0&1&0&1&1&1\\
0&-&0&1&-&1\\
1&1&0&0&-&1\\
-&-&0&0&-&1\\
1&0&1&-&0&1\\
-&0&-&-&0&1\\
-&-&1&0&1&0\\
1&-&-&0&1&0\\
-&1&-&1&0&0\\
1&-&-&1&0&0
\end{bmatrix},
$$

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & - & - & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & - & 0 & 1 & - & 1 \\ 1 & 1 & 0 & 0 & - & 1 \\ - & - & 0 & 0 & - & 1 \\ - & 0 & 1 & - & 0 & 1 \\ 1 & 0 & - & - & 0 & 1 \\ 1 & - & 1 & 0 & 1 & 0 \\ - & - & - & 0 & 1 & 0 \\ - & 1 & - & 1 & 0 & 0 \\ 1 & - & - & 1 & 0 & 0 \end{bmatrix}, \tag{19}$$

and (i) corresponds to P_{19}^i. By computer search, many weighing matrices based on P_{19}^i can be constructed. For example, by algorithm (i) described in Remark 3.3, one can construct 480 weighing matrices based on P_{19}^3. Let $G = \langle g_l \rangle$, $1 \leq l \leq 5$, be an automorphism group of P_{19}^3, having $\{g_l\}$ as generators, where

$$g_1 = (\pi(\underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}, \underline{7}, \underline{8}, \underline{9}, \underline{10}, \underline{11}, \underline{12}), \rho(\underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}))\,,$$

$$g_2 = (\pi(1, 3, 2, 4, 5, 6, 7, 8, 9, 10, \underline{12}, \underline{11}), \rho(1, 2, 4, 3, 5, 6))\,,$$

$$g_3 = (\pi(1, 2, 3, 4, \underline{8}, \underline{7}, \underline{6}, \underline{5}, 9, 10, 11, 12), \rho(1, 2, 3, 4, 6, 5))\,,$$

$$g_4 = (\pi(4, 3, 2, 1, 5, 6, 7, 8, \underline{10}, \underline{9}, \underline{12}, \underline{11}), \rho(1, 2, \underline{3}, \underline{4}, 5, 6))\,,$$

$$g_5 = (\pi(5, 6, 7, 8, 1, 2, 3, 4, 9, \underline{10}, 11, \underline{12}), \rho(3, 4, 1, 2, \underline{5}, 6))\,.$$

Using G in order to remove equivalent matrices, one can reduce from 480 matrices to 15 ones. Furthermore, by removing matrices being not of Type $\mathbf{c}_{19}$, at most $n_{19}^3 = 8$ inequivalent weighing matrices based on P_{19}^3 can be constructed. The same method can be performed for other feasible matrices, in order to construct weighing matrices. As a result, n_{19}^i weighing matrices based on P_{19}^i, say (Zi, l), can be constructed for $1 \leq i \leq 19$ and $1 \leq l \leq n_{19}^i$. Such $\{(Zi, l)\}$ are listed in Table 2. Note that the construction is performed in the order starting from P_{19}^1. □

THEOREM 3.4. *There is no weighing matrix of Type* $\mathbf{c}_{23}$.

PROOF. Let M be a weighing matrix of Type $\mathbf{c}_{23}$ and M_R be an R-matrix of M. For M_{R2} being an $R2$-matrix of M_R, M_{R2}^* is equivalent to one of 21 matrices presented in the proof of Lemma 3.3. If K^* is an admissible matrix based on $K(i)^*$, $1 \leq i \leq 21$, where $K(i)^*$ is one of the matrices as in the proof of Lemma 3.3, then it can be shown that the type of weighing matrix having K^* as an R-matrix is larger than Type $\mathbf{c}_{23}$. This contradicts to the assumption of the matrix M of Type $\mathbf{c}_{23}$. $\square$

THEOREM 3.5. *There are two inequivalent admissible matrices and four inequivalent feasible matrices of Type* $\mathbf{c}_{24}$, *say* P_{24}^i, $1 \leq i \leq 4$. *All weighing matrices constructed based on those matrices are of larger types than Type* $\mathbf{c}_{24}$.

PROOF. Let K be an R-matrix of a weighing matrix of Type $\mathbf{c}_{24}$. Then K^* is equivalent to one of two inequivalent admissible matrices of Type $\mathbf{c}_{24}$, say K_1^* and K_2^*. Moreover, it can be shown that there are one and three inequivalent feasible matrices based on K_1^* and K_2^*, say P_{24}^1 and P_{24}^i, $2 \leq i \leq 4$, respectively, where

$$K_1^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad K_2^* = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\underset{(1)}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ - & 0 & - & 0 & 1 & 1 \\ - & 0 & 1 & 0 & - & 1 \\ 0 & - & - & 1 & 0 & 1 \\ 0 & - & 1 & - & 0 & 1 \\ 1 & 1 & - & 0 & 0 & 1 \\ - & - & 0 & 1 & 1 & 0 \\ 1 & - & 0 & - & 1 & 0 \\ - & 1 & 1 & 0 & 1 & 0 \\ 1 & - & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 1 \\ 0 & 0 & 0 & - & 0 & 1 \\ 0 & 0 & 0 & - & 1 & 0 \end{bmatrix}}, \quad \underset{(2)}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & - & 1 & 1 & 1 \\ 0 & - & 0 & - & 1 & 1 \\ 0 & 1 & 0 & - & - & 1 \\ 1 & - & 0 & 0 & - & 1 \\ - & - & 1 & 0 & 0 & 1 \\ - & 1 & - & 0 & 0 & 1 \\ - & 0 & 1 & - & 1 & 0 \\ 1 & 0 & - & - & 1 & 0 \\ - & - & - & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ - & 1 & 0 & 0 & 0 & 0 \end{bmatrix}},$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & - & 1 & - & 1 \\ 0 & - & 0 & - & 1 & 1 \\ 0 & 1 & 0 & - & - & 1 \\ 1 & - & 0 & 0 & - & 1 \\ - & - & 1 & 0 & 0 & 1 \\ - & 1 & - & 0 & 0 & 1 \\ - & 0 & - & 1 & 1 & 0 \\ 1 & 0 & - & - & 1 & 0 \\ 1 & - & - & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & - & - & 1 \\ 0 & - & 0 & - & 1 & 1 \\ 0 & - & 0 & 1 & - & 1 \\ - & 1 & 0 & 0 & - & 1 \\ - & 1 & - & 0 & 0 & 1 \\ 1 & - & - & 0 & 0 & 1 \\ - & 0 & - & 1 & 1 & 0 \\ - & 0 & 1 & - & 1 & 0 \\ - & - & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(3) (4)

and (i) corresponds to P_{24}^i. The computer search shows that all weighing matrices constructed based on P_{24}^i are of larger types than Type $\mathbf{c}_{24}$. □

THEOREM 3.6. *There exists the unique admissible matrix of Type* $\mathbf{c}_{25}$ *and there are two inequivalent feasible matrices, say* P_{25}^1 *and* P_{25}^2, *based on the admissible matrix. All weighing matrices constructed based on* P_{25}^i, $1 \le i \le 2$, *are of larger types than Type* $\mathbf{c}_{25}$.

PROOF. Let K be an R-matrix of a weighing matrix of Type $\mathbf{c}_{25}$. Then K^* is equivalent to the admissible matrix $\underline{K}^*$. Furthermore, it can be shown that there are two inequivalent feasible matrices, say P_{25}^i, $i = 1, 2$, based on $\underline{K}^*$. Here

$$\underline{K}^* = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \quad P_{25}^1 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ - & 1 & 0 & 0 & - & 1 \\ 0 & - & - & 1 & 0 & 1 \\ 0 & 1 & - & - & 0 & 1 \\ - & - & 0 & - & 0 & 1 \\ 1 & - & 1 & 0 & 0 & 1 \\ - & 0 & 1 & - & 1 & 0 \\ 1 & 0 & - & - & 1 & 0 \\ - & 1 & 0 & 1 & 1 & 0 \\ - & - & - & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & - & 1 \\ 0 & 0 & - & 1 & 0 & 0 \end{bmatrix},$$

$$P_{25}^2 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & - & 1 \\ - & - & 0 & 0 & - & 1 \\ 0 & 1 & - & 1 & 0 & 1 \\ 0 & - & - & - & 0 & 1 \\ - & 1 & 0 & - & 0 & 1 \\ 1 & - & 1 & 0 & 0 & 1 \\ - & 0 & - & 1 & 1 & 0 \\ - & 0 & 1 & - & 1 & 0 \\ 1 & 1 & 0 & - & 1 & 0 \\ 1 & - & - & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Using a computer, it can be shown that all weighing matrices constructed based on P_{25}^i are of larger types than Type $\mathbf{c}_{25}$. □

Now, a set of $W(14, 8)$'s constructed in Lemmas 3.1–3.6 contains all inequivalent weighing matrices of order 14 and weight 8. Thus weighing matrices in the set will be classified into some inequivalent classes.

DEFINITION 3.3. Let $M \in \Delta$ and $\mathbf{C} = \mathbf{C}(\cdots i^{n_i} \cdots j^{n_j} \cdots)$ be the distribution of types of rows of M, for $1 < i < j \leq 25$, $n_i \geq 1$, $n_j \geq 1$, where n_l is the number of rows of M having Type $\mathbf{c}_l$. In this case, $\mathbf{C}$ is called the $\mathbf{C}$-*distribution* associated with M.

The following result is straightforward.

THEOREM 3.7. *Let $M_i \in \Delta$ and $\mathbf{C}_i$ be the $\mathbf{C}$-distribution of M_i, $i = 1, 2$. If $\mathbf{C}_1 \neq \mathbf{C}_2$, then M_1 is not equivalent to M_2. In particular, if M_2 is the transpose matrix of M_1 and $\mathbf{C}_1 \neq \mathbf{C}_2$, M_1 is not self-dual.*

There are many inequivalent weighing matrices having the same $\mathbf{C}$-distribution. Thus, another criterion is needed to determine whether two matrices are equivalent or not.

DEFINITION 3.4. Let $M \in \Delta$ and $\mathbf{m} = (m_1, m_2, \ldots, m_{14})$, $\mathbf{m}_i = (m_1^i, m_2^i, \ldots, m_{14}^i)$ be three different rows of M, where $i = 1, 2$. Define a 3×8 matrix $T = (t_{ij})$ associated with $\mathbf{m}$, where $t_{0l} = m_{j_l} \neq 0$ and $t_{il} = m_{j_l}^i$, $1 \leq l \leq 8$, $i = 1, 2$. T is called a *t-matrix* associated with $\mathbf{m}$ if $|\mathbf{t}_1 * \mathbf{t}_2| \geq |\mathbf{t}_1 * \mathbf{t}_3|$, $\mathbf{t}_1 = J_{1\times 8}$, and the first non-zero elements of $\mathbf{t}_2$ and $\mathbf{t}_3$ are ones, where $\mathbf{t}_i$ is the i-th row of T. Let T_1 and T_2 be two t-matrices associated with $\mathbf{m}$. If there are two signed matrices $\bar{P}$ and $\bar{Q}$ such that $T_2 = \bar{P} T_1 \bar{Q}$, then T_2 is said to be *equivalent* to T_1.

The following lemma is straightforward.

LEMMA 3.7. *Let $M \in \Delta$ and* **m** *be a row of M. Then a t-matrix associated with* **m** *is equivalent to one of following matrices.*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (1) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & - & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & - & 0 & 0 \end{bmatrix} \quad (3) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & 0 & - & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & - & - \end{bmatrix} \quad (6)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & - & - & 0 \end{bmatrix} \quad (7) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & - & 0 & 0 \end{bmatrix} \quad (8)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & 0 & - & 0 & 1 & - & 0 & 0 \end{bmatrix} \quad (9) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & - & - & 0 & 0 \end{bmatrix} \quad (10)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & - & 1 & 0 & - & 0 & 0 & 0 \end{bmatrix} \quad (11) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & 1 & - & 0 & - & 0 & 0 & 0 \end{bmatrix} \quad (12)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 & 0 & 0 \end{bmatrix} \quad (13) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \end{bmatrix} \quad (14)$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & - & - & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & - \end{bmatrix} \quad (15) \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & - & - & 0 & 0 \\ 1 & - & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (16)$$

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&0&0&-&0&0&0&0 \end{bmatrix}$$
(17)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&-&0&0&0&0&1&- \end{bmatrix}$$
(18)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&1&0&0&0&0&-&- \end{bmatrix}$$
(19)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&0&0&1&0&0&-&- \end{bmatrix}$$
(20)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&0&0&-&0&0&1&- \end{bmatrix}$$
(21)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&1&-&0&0&0&-&0 \end{bmatrix}$$
(22)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&-&0&1&0&0&-&0 \end{bmatrix}$$
(23)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&-&0&1&-&0&0&0 \end{bmatrix}$$
(24)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&1&-&-&0&0&0&0 \end{bmatrix}$$
(25)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&-&0&1&-&0&1&- \end{bmatrix}$$
(26)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&1&-&1&0&0&-&- \end{bmatrix}$$
(27)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&1&-&1&-&0&-&0 \end{bmatrix}$$
(28)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&-&-&-&0&0\\ 1&1&-&1&-&-&0&0 \end{bmatrix}$$
(29)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&-&-&-&-\\ 1&-&0&0&0&0&0&0 \end{bmatrix}$$
(30)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&-&-&-&-\\ 1&-&0&0&1&-&0&0 \end{bmatrix}$$
(31)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&-&-&-&-\\ 1&1&-&-&0&0&0&0 \end{bmatrix}$$
(32)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&-&-&-&-\\ 1&1&-&-&1&-&0&0 \end{bmatrix}$$
(33)

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&-&-&-&-\\ 1&1&-&-&1&1&-&- \end{bmatrix}$$
(34)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & - & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & - & - & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & - & 0 & 0 & 0 \end{bmatrix}$$
(35) (36)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & - & - & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & - & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & - & - & 0 & 0 \\ 1 & 1 & - & - & 0 & 0 & - & 0 \end{bmatrix}$$
(37) (38)

REMARK 3.4. For each of rows (columns) of weighing matrices obtained in Lemmas 3.1–3.6, t-matrices are searched. As a result, there is no weighing matrix having a t-matrix equivalent to the i-th matrix (i) for $35 \le i \le 38$.

For the i-th matrix (i) in Lemma 3.7, let $T_1(i) = (i)$ (for $1 \le i \le 2$), $T_2(i) = (i+2)$ (for $1 \le i \le 12$), $T_3(i) = (i+14)$ (for $1 \le i \le 15$), $T_4(i) = (i+29)$ (for $1 \le i \le 5$), for the sake of convenience.

Let $M \in \Delta$ and $\mathbf{m}$ be a row of M. Then one can make 78 t-matrices associated with $\mathbf{m}$, each of which is equivalent to one of the first 34 matrices given in Lemma 3.7. Hence the distribution of such t-matrices associated with $\mathbf{m}$ is obtained.

DEFINITION 3.5. The distribution of t-matrices associated with $\mathbf{m}$ is denoted by $\{\ldots, T_i(\ldots, j^{n_{ij}}, \ldots), \ldots\}$, where $T_i(\ldots, j^{n_{ij}}, \ldots)$ means that there are n_{ij} t-matrices associated with $\mathbf{m}$ equivalent to $T_i(j)$. In this case, the distribution is called the **T**-*distribution* associated with $\mathbf{m}$.

Note that $\sum_{i,j} n_{ij} = 78$. For all weighing matrices obtained in Lemmas 3.1–3.6, and then for all rows (columns) of each matrices, **T**-distributions are derived and hence 91 different **T**-distributions can be obtained. They are listed as $\mathbf{T}_i$, $1 \le i \le 91$, in Table 1 of this section.

DEFINITION 3.6. Let $M \in \Delta$ and $\mathbf{T} = \mathbf{T}(\ldots, i^l, \ldots)$ be the distribution of **T**-distributions associated with rows of M, where i^l means that there are l rows having the **T**-distribution $\mathbf{T}_i$ for $l \ge 1$. In this case, $\mathbf{T}$ is called the **T**-*distribution* associated with M.

The next is straightforward.

THEOREM 3.8. *Let $M_i \in \Delta$ and $\mathbf{T}(i)$ be the* **T**-*distribution associated with M_i for $i = 1, 2$. If $\mathbf{T}(1) \ne \mathbf{T}(2)$, then M_1 is not equivalent to M_2. In particular, if M_2 is the transpose of M_1 and $\mathbf{T}(1) \ne \mathbf{T}(2)$, M_1 is not self-dual.*

There are 103 $W(14, 8)$'s obtained in Lemmas 3.1–3.6. As a result, they can be classified into 65 inequivalent classes by using the **C**- or the **T**-

distribution associated with each matrix in the following manner. Let M be a weighing matrix obtained in Lemmas 3.1–3.6. Then M is divided into two cases.

Case I: The case being used as the representative matrix of the i-th inequivalent class. In this case, the **C**-distribution and/or the **T**-distribution associated with M are attached. Furthermore M is named as W_i in Table 2. See Remark 3.2 for the expression of W_i in Table 2. For W_i, other informations are also attached in Table 2 as follows: If W_i is self-dual, first the notation SD and two signed permutations, say π and ρ, and secondly the **C**-distribution and/or the **T**-distribution associated with W_i are attached. This means that $W_i = W_i^{t(\pi,\rho)}$. If W_i is not self-dual, W_i^t is used as the representative matrix of the $(i+1)$-th inequivalent class. Then the notation $W_{i+1} = W_i^t$ is used, and the **C**-distributions and/or the **T**-distributions associated with W_i and W_i^t are also attached.

Case II: The case being not used as the representative matrix of inequivalent class. In this case, only two signed permutations, say π and ρ, are attached with the notations W_l or P_α (P_α^t) together in Table 2. If W_l ($1 \le l \le 65$) is attached, it means that $W_l = M^{(\pi,\rho)}$. If P_α (P_α^t) is attached, M is of Type $\mathbf{c}_{19}$. Let $M = (m_{ij})$ be a weighing matrix based on P_{19}^β given in Lemma 3.6, and π^* and ρ^* be permutations ignoring signs of π and ρ, respectively. Further let $L = (l_{ab})$ be a submatrix of M, where $l_{ab} = m_{\pi^*(a)\rho^*(b)}$, and $\pi^*(a)$ and $\rho^*(b)$ be the a-th element of π^* and the b-th element of ρ^*, respectively. In this case, $L^{(\bar{\pi},\bar{\rho})} = P_\alpha$ (P_α^t) and $\alpha < \beta$, where $\bar{\pi}$ ($\bar{\rho}$) is the signed permutation defined from π (ρ) as follows: for $\pi = \pi(i_1, \underline{i_2}, \ldots, i_t)$ ($\rho = \rho(i_1, \underline{i_2}, \ldots, i_t)$), $\bar{\pi} = \pi(1, \underline{2}, \ldots, t)$ ($\bar{\rho} = \rho(1, \underline{2}, \ldots, t)$). This means that M is equivalent to one of weighing matrices constructed based on P_{19}^α ($P_{19}^{\alpha t}$) (see the proof of Lemma 3.6). Note that the notations A, B, C, D, E are used as elements of signed permutations in Table 2, where A, B, C, D, E correspond to 10, 11, 12, 13, 14, respectively.

Summarizing the previous discussion, we have obtained the following:

THEOREM 3.9. *There are* 65 *inequivalent weighing matrices of order* 14 *and weight* 8.

When $M \in \Delta(14, 8)$ and $N \in \Delta(n, k)$, it follows that $M \otimes N \in \Delta(14n, 8k)$. Thus the classification of weighing matrices of order 14 and weight 8 is useful for further classification of $\Delta(14n, 8k)$ and $\Delta(m, 8)$ for $m \ge 15$.

REMARK 3.5. All computer programs used in order to construct and classify weighing matrices are available on request. Matrices W_i, $1 \le i \le 65$, expressed with the exact forms, which are representative matrices of inequivalent classes, are also available on request.

TABLE 1. T-distribution of t-matrices.

$\mathbf{T}_1 = \{T_2(4^4 5^{12} 6^{16} 7^6 8^8 9^8 11^1), T_3(6^4 7^4 9^8 10^6 12^1)\}$

$\mathbf{T}_2 = \{T_2(4^{12} 6^{24} 7^{12} 8^{12} 11^6), T_4(2^{12})\}$

$\mathbf{T}_3 = \{T_2(4^8 5^8 6^{20} 7^{10} 8^{10} 9^8 11^2), T_4(2^{12})\}$

$\mathbf{T}_4 = \{T_2(4^4 5^{16} 6^{20} 7^{10} 8^{10} 11^6), T_4(2^{12})\}$

$\mathbf{T}_5 = \{T_2(4^4 5^8 6^{22} 7^{13} 8^3 9^4 11^1), T_3(4^4 9^{16} 11^2 13^1)\}$

$\mathbf{T}_6 = \{T_2(4^8 5^8 6^{24} 7^8 8^8 9^8 11^1 12^1), T_4(2^{12})\}$

$\mathbf{T}_7 = \{T_2(4^{12} 6^{24} 7^{20} 8^4 11^6), T_4(2^8 3^4)\}$

$\mathbf{T}_8 = \{T_2(4^6 5^8 6^{18} 7^{11} 8^5 9^4 11^3), T_3(4^4 9^{16} 11^2 13^1)\}$

$\mathbf{T}_9 = \{T_2(4^8 6^{32} 7^{16} 8^8 11^2), T_4(2^{12})\}$

$\mathbf{T}_{10} = \{T_2(4^4 5^{12} 6^{28} 7^{10} 8^6 9^4 11^1 12^1), T_4(2^{12})\}$

$\mathbf{T}_{11} = \{T_2(4^8 5^8 6^{32} 7^4 8^4 9^4 10^4 11^1 12^1), T_4(2^{12})\}$

$\mathbf{T}_{12} = \{T_2(4^2 5^{16} 6^{16} 7^{11} 8^3 9^4 11^3), T_3(4^6 5^2 9^8 10^4 11^2 13^1)\}$

$\mathbf{T}_{13} = \{T_2(4^8 6^{32} 7^{20} 8^4 11^2), T_4(2^{10} 3^2)\}$

$\mathbf{T}_{14} = \{T_2(1^6 2^3 4^6 6^{12} 7^3 8^{12} 11^3), T_3(3^3 6^9 7^6 10^{12} 12^3)\}$

$\mathbf{T}_{15} = \{T_2(4^4 5^{16} 6^{24} 7^{16} 11^5 12^1), T_4(2^8 3^4)\}$

$\mathbf{T}_{16} = \{T_2(1^6 2^1 3^2 4^2 5^8 6^{12} 7^8 8^3 11^3), T_3(2^2 3^1 4^6 5^4 6^1 9^8 10^6 11^2 12^2 13^1)\}$

$\mathbf{T}_{17} = \{T_2(4^6 5^8 6^{18} 7^8 8^8 9^4 11^3), T_3(6^2 7^2 9^{16} 10^2 12^1)\}$

$\mathbf{T}_{18} = \{T_2(4^8 6^{32} 7^{24} 11^2), T_4(2^8 3^4)\}$

$\mathbf{T}_{19} = \{T_2(4^4 5^{16} 6^{20} 7^{18} 8^2 11^6), T_4(2^8 3^4)\}$

$\mathbf{T}_{20} = \{T_2(4^8 5^8 6^{20} 7^{18} 8^2 9^8 11^2), T_4(2^8 3^4)\}$

$\mathbf{T}_{21} = \{T_2(4^4 5^{12} 6^{24} 7^{12} 8^8 9^4 11^2), T_4(2^{12})\}$

$\mathbf{T}_{22} = \{T_2(4^{12} 6^{24} 7^{24} 11^6), T_4(2^6 3^6)\}$

$\mathbf{T}_{23} = \{T_2(1^6 2^3 4^2 5^8 6^{10} 7^5 8^8 11^3), T_3(3^3 4^2 5^2 6^5 7^2 9^8 10^8 12^3)\}$

$\mathbf{T}_{24} = \{T_2(1^6 2^1 3^2 4^2 5^8 6^{12} 7^6 8^5 11^3), T_3(2^2 3^1 4^4 5^2 6^3 7^2 9^8 10^6 11^2 12^2 13^1)\}$

$\mathbf{T}_{25} = \{T_2(4^2 5^{16} 6^{16} 7^6 8^8 9^4 11^3), T_3(6^4 7^4 9^8 10^6 12^1)\}$

$\mathbf{T}_{26} = \{T_2(1^6 2^3 4^4 5^4 6^{10} 7^8 8^5 9^4 11^1), T_3(3^3 4^8 6^1 7^2 9^8 10^6 11^2 13^3)\}$

$\mathbf{T}_{27} = \{T_2(1^6 2^3 4^4 5^4 6^{10} 7^5 8^8 9^4 11^1), T_3(3^3 4^2 6^7 7^2 9^8 10^6 11^2 12^3)\}$

$\mathbf{T}_{28} = \{T_2(4^4 5^{12} 6^{20} 7^4 8^6 9^8 12^1), T_3(6^4 7^4 9^8 10^6 12^1)\}$

$\mathbf{T}_{29} = \{T_2(4^4 5^{12} 6^{16} 7^{11} 8^3 9^8 11^1), T_3(4^6 6^2 9^8 10^2 11^4 13^1)\}$

$\mathbf{T}_{30} = \{T_2(4^4 5^{12} 6^{16} 7^8 8^6 9^8 11^1), T_3(4^2 7^2 9^{16} 10^2 13^1)\}$

$\mathbf{T}_{31} = \{T_2(4^4 5^8 6^{26} 7^{11} 8^1 9^4 12^1), T_3(4^4 9^{16} 11^2 13^1)\}$

$\mathbf{T}_{32} = \{T_2(4^8 6^{40} 7^{12} 8^4 12^2), T_4(2^{12})\}$

$\mathbf{T}_{33} = \{T_2(5^{24} 6^{18} 7^9 8^9 11^6)^4 T_4(2^{12})\}$

$\mathbf{T}_{34} = \{T_2(1^6 3^3 5^{12} 6^9 7^6 8^3 9^3 11^3), T_3(2^3 4^6 5^3 9^{18} 14^3)\}$

$\mathbf{T}_{35} = \{T_2(4^{12} 6^{48} 12^6), T_4(2^{12})\}$

$\mathbf{T}_{36} = \{T_2(1^6 2^3 4^6 6^{12} 7^{12} 8^3 11^3), T_3(3^3 4^{12} 6^3 10^6 11^6 13^3)\}$

$\mathbf{T}_{37} = \{T_2(4^{12} 6^{32} 7^{16} 11^4 12^2), T_4(2^8 3^4)\}$

$\mathbf{T}_{38} = \{T_2(5^{20} 6^{17} 7^{10} 8^1 9^3 10^1 11^3), T_3(4^4 5^3 8^2 9^{10} 10^4 12^1)\}$

$\mathbf{T}_{39} = \{T_2(5^{20} 6^{15} 7^8 8^5 9^4 11^3), T_3(4^2 5^2 6^1 7^1 9^{12} 10^4 12^1)\}$

TABLE 1 (*continued*)

$\mathbf{T}_{40} = \{T_2(1^6 2^1 3^2 5^{12} 6^{11} 7^8 8^2 11^3), T_3(2^2 3^1 4^2 6^4 7^3 8^4 9^8 10^4 11^2 12^2 13^1)\}$

$\mathbf{T}_{41} = \{T_2(4^2 5^{16} 6^{16} 7^{10} 8^4 9^4 11^3), T_3(4^4 9^{16} 11^2 13^1)\}$

$\mathbf{T}_{42} = \{T_1(2^1), T_2(1^8 2^6 4^2 5^4 6^8 8^6 12^1), T_3(1^2 3^6 5^4 6^6 9^8 10^{10} 12^6)\}$

$\mathbf{T}_{43} = \{T_2(4^4 5^{12} 6^{18} 7^{10} 8^2 9^7 10^1 11^1), T_3(6^2 7^2 8^4 9^{12} 10^2 12^1)\}$

$\mathbf{T}_{44} = \{T_2(4^6 5^8 6^{32} 7^8 8^4 9^4 10^4), T_4(2^{12})\}$

$\mathbf{T}_{45} = \{T_2(1^6 3^3 5^{12} 6^9 7^8 8^1 9^3 11^3), T_3(2^3 4^4 5^1 6^2 7^2 8^4 9^{14} 14^3)\}$

$\mathbf{T}_{46} = \{T_2(4^8 5^8 6^{24} 7^{16} 9^8 11^1 12^1), T_4(2^8 3^4)\}$

$\mathbf{T}_{47} = \{T_2(4^4 5^{12} 6^{16} 7^7 8^7 9^8 11^1), T_3(6^2 7^2 9^{16} 10^2 12^1)\}$

$\mathbf{T}_{48} = \{T_2(4^6 5^8 6^{24} 7^{12} 8^8 9^8), T_4(2^{12})\}$

$\mathbf{T}_{49} = \{T_2(5^{20} 6^{17} 7^8 8^3 9^3 10^1 11^3), T_3(4^1 5^1 6^2 7^2 8^2 9^{10} 10^4 12^1)\}$

$\mathbf{T}_{50} = \{T_2(4^2 5^{16} 6^{18} 7^{10} 8^2 9^3 10^1 11^3), T_3(6^2 7^2 8^4 9^{12} 10^2 12^1)\}$

$\mathbf{T}_{51} = \{T_2(4^4 5^{12} 6^{24} 7^{20} 9^4 11^2), T_4(2^8 3^4)\}$

$\mathbf{T}_{52} = \{T_2(1^6 2^1 3^2 4^4 5^4 6^{16} 7^4 8^3 9^4 12^1), T_3(2^2 3^1 4^4 6^5 7^2 9^8 10^4 11^4 12^2 13^1)\}$

$\mathbf{T}_{53} = \{T_2(1^6 2^3 4^4 5^4 6^{14} 7^1 8^8 9^4 12^1), T_3(3^3 6^7 7^4 9^8 10^8 12^3)\}$

$\mathbf{T}_{54} = \{T_2(4^6 5^{12} 6^{22} 7^7 8^7 9^{10} 10^2), T_4(2^{12})\}$

$\mathbf{T}_{55} = \{T_1(2^1), T_2(1^8 2^4 3^2 4^2 5^4 6^{10} 7^1 8^3 12^1), T_3(1^2 2^2 3^4 4^4 5^2 6^2 7^2 9^8 10^8 11^2 12^3 13^3)\}$

$\mathbf{T}_{56} = \{T_1(2^1), T_2(1^8 2^6 4^2 5^4 6^8 8^6 12^1), T_3(1^2 3^6 6^{10} 9^8 10^6 11^4 12^6)\}$

$\mathbf{T}_{57} = \{T_2(4^4 5^{12} 6^{16} 7^9 8^5 9^8 11^1), T_3(4^2 6^2 9^{16} 11^2 12^1)\}$

$\mathbf{T}_{58} = \{T_2(4^6 5^8 6^{24} 7^{16} 8^4 9^8), T_4(2^{10} 3^2)\}$

$\mathbf{T}_{59} = \{T_2(4^2 5^{16} 6^{16} 7^{10} 8^1 9^8 10^1 11^1), T_3(4^2 6^2 7^2 8^4 9^{10} 10^2 14^1)\}$

$\mathbf{T}_{60} = \{T_2(1^6 2^2 3^1 4^2 5^8 6^{10} 7^7 8^2 9^5 10^1 11^1), T_3(2^1 3^2 4^3 6^4 7^2 8^4 9^{12} 10^2 12^1 14^2)\}$

$\mathbf{T}_{61} = \{T_2(4^8 5^8 6^{28} 7^{10} 8^2 9^4 10^4 11^2), T_4(2^{10} 3^2)\}$

$\mathbf{T}_{62} = \{T_2(4^4 5^8 6^{22} 7^{15} 8^1 9^4 11^1), T_3(4^1 6^2 7^1 8^4 9^{12} 10^1 11^1 12^1)\}$

$\mathbf{T}_{63} = \{T_2(4^4 5^8 6^{22} 7^{11} 8^5 9^4 11^1), T_3(4^2 7^2 9^{16} 10^2 13^1)\}$

$\mathbf{T}_{64} = \{T_2(4^4 5^{12} 6^{18} 7^7 8^5 9^7 10^1 11^1), T_3(4^1 6^2 7^1 9^{16} 10^1 11^1 12^1)\}$

$\mathbf{T}_{65} = \{T_2(4^6 5^8 6^{22} 7^9 8^3 9^4 11^2 12^1), T_3(4^4 9^{16} 11^2 13^1)\}$

$\mathbf{T}_{66} = \{T_1(1^2 2^4), T_2(1^2 2^4 6^4 12^2), T_3(1^4 2^4 3^8 4^4 6^8 10^4 12^4 15^2), T_4(1^4 2^2 3^2 4^4)\}$

$\mathbf{T}_{67} = \{T_1(1^1), T_2(1^{12} 3^4 4^8 6^4 7^8 11^7 12^1), T_3(2^4 4^4 6^4 7^4 10^4 15^1), T_4(1^2 2^2 3^6 4^2)\}$

$\mathbf{T}_{68} = \{T_1(1^1), T_2(1^8 2^4 3^4 4^8 6^{16} 11^1 12^3), T_3(1^4 6^4 7^4 10^8 15^1), T_4(1^2 2^4 3^4 4^2)\}$

$\mathbf{T}_{69} = \{T_1(1^2 2^4), T_2(1^8 2^8 4^4 12^2), T_3(1^8 3^8 6^8 10^8 12^4 15^2), T_4(1^4 3^4 4^4)\}$

$\mathbf{T}_{70} = \{T_1(1^1), T_2(1^{12} 3^4 4^8 6^{12} 11^5 12^3), T_3(2^4 4^8 6^4 11^4 15^1), T_4(1^2 2^6 3^2 4^2)\}$

$\mathbf{T}_{71} = \{T_1(1^2 2^4), T_2(1^{12} 3^4 7^4 11^2), T_3(1^4 2^8 3^4 4^{12} 11^4 13^4 15^2), T_4(1^4 2^4 4^4)\}$

$\mathbf{T}_{72} = \{T_1(1^1), T_2(1^8 2^4 3^4 4^8 6^8 7^8 11^3 12^1), T_3(1^4 4^4 6^4 10^4 11^4 15^1), T_4(1^2 2^4 3^4 4^2)\}$

$\mathbf{T}_{73} = \{T_1(1^2 2^4), T_2(1^8 3^8 4^4 11^2), T_3(1^8 2^8 7^8 10^8 13^4 15^2), T_4(1^4 3^4 4^4)\}$

$\mathbf{T}_{74} = \{T_1(1^2 2^4), T_2(1^{12} 3^4 7^4 11^2), T_3(1^4 2^8 3^4 4^8 7^4 10^4 13^4 15^2), T_4(1^4 2^2 3^2 4^4)\}$

$\mathbf{T}_{75} = \{T_1(1^1), T_2(1^{12} 3^4 4^4 6^{12} 7^8 11^3 12^1), T_3(2^4 4^8 6^4 11^4 15^1), T_4(1^2 2^6 3^2 4^2)\}$

$\mathbf{T}_{76} = \{T_2(4^{20} 6^{32} 11^8 12^6), T_4(2^8 3^4)\}$

$\mathbf{T}_{77} = \{T_1(1^1), T_2(1^{12} 2^4 4^4 6^8 7^{12} 11^3 12^1), T_3(3^4 4^8 6^4 10^4 15^1), T_4(1^2 2^4 3^4 4^2)\}$

$\mathbf{T}_{78} = \{T_1(1^2 2^4), T_2(1^8 3^8 4^4 11^2), T_3(1^8 2^8 4^8 11^8 13^4 15^2), T_4(1^4 2^4 4^4)\}$

TABLE 1 (*continued*)

$\mathbf{T}_{79} = \{T_2(4^{20}6^{16}7^{16}11^{12}12^2),\ T_4(2^4 3^8)\}$

$\mathbf{T}_{80} = \{T_2(4^2 5^{14}6^{18}7^{13}8^1 9^6 11^1),\ T_3(4^1 6^2 7^1 8^4 9^{12}10^1 11^1 12^1)\}$

$\mathbf{T}_{81} = \{T_2(4^2 5^{14}6^{20}7^8 8^4 9^5 10^1 11^1),\ T_3(4^1 6^2 7^1 9^{16}10^1 11^1 12^1)\}$

$\mathbf{T}_{82} = \{T_2(4^2 5^{14}6^{18}7^9 8^5 9^6 11^1),\ T_3(4^2 7^2 9^{16}10^2 13^1)\}$

$\mathbf{T}_{83} = \{T_2(4^3 5^{14}6^{17}7^5 8^6 9^9 10^1),\ T_3(6^3 7^3 9^{12}10^4 12^1)\}$

$\mathbf{T}_{84} = \{T_2(1^6 2^3 4^3 5^6 6^{13}7^2 8^6 9^4 10^2),\ T_3(3^3 4^1 6^6 7^2 9^{12}10^5 11^1 12^3)\}$

$\mathbf{T}_{85} = \{T_2(4^3 5^{14}6^{15}7^8 8^5 9^{10}),\ T_3(4^3 6^1 7^2 9^{12}10^3 11^1 13^1)\}$

$\mathbf{T}_{86} = \{T_2(4^2 5^{14}6^{20}7^{11}8^1 9^5 10^1 11^1),\ T_3(6^2 7^2 8^4 9^{12}10^2 12^1)\}$

$\mathbf{T}_{87} = \{T_2(4^6 5^{12}6^{30}7^3 8^3 9^6 10^6),\ T_4(2^{12})\}$

$\mathbf{T}_{88} = \{T_2(4^4 5^8 6^{22}7^{13}8^3 9^4 11^1),\ T_3(4^4 9^{16}11^2 13^1)\}$

$\mathbf{T}_{89} = \{T_2(1^6 2^3 4^3 5^6 6^9 7^6 8^6 9^6),\ T_3(3^3 4^3 6^6 9^{12}10^3 11^3 12^3)\}$

$\mathbf{T}_{90} = \{T_2(4^3 5^{14}6^{15}7^8 8^5 9^{10}),\ T_3(4^2 6^3 7^1 9^{12}10^2 11^2 12^1)\}$

$\mathbf{T}_{91} = \{T_2(1^6 2^3 4^3 5^6 6^9 7^6 8^6 9^6),\ T_3(3^3 4^6 7^3 9^{12}10^6 13^3)\}$

TABLE 2. Weighing matrices

$(U1, 1)$	6520	6232	3640	2388211	2414453	2978699	W_1
3004945	3103416	3116520	2603826	2602206	37062	38358	
SD 5, $\underline{6}$, 7, $\underline{8}$, E, C, B, D, 9, A, 1, $\underline{2}$, $\underline{3}$, 4: B, $\underline{C}$, $\underline{E}$, D, $\underline{2}$, 1, $\underline{4}$, 3, 9, A, 7, 6, 8, 5							
$\mathbf{C}(5^8 15^4 19^2)$							
$(V1, 1)$	6520	6232	3640	2388211	2414453	2624405	W_2
2650651	2958120	3074760	2690478	2703582	331740	344880	
$\mathbf{C}(7^8 19^6)$							
$\mathbf{C}(15^{12}19^2)$							$W_3 = W_2^t$
$(V2, 1)$	6520	6232	3640	2388211	2414453	2985318	W_4
3116538	2603826	2602206	2683499	2709745	345384	346680	
SD 5, $\underline{6}$, 8, $\underline{7}$, 9, A, D, E, 1, $\underline{2}$, B, C, 3, $\underline{4}$: D, $\underline{E}$, 9, $\underline{A}$, 1, $\underline{2}$, 3, $\underline{4}$, 8, $\underline{7}$, B, $\underline{C}$, $\underline{6}$, 5							
$\mathbf{C}(7^4 15^6 17^4)$							
$(V3, 1)$	6520	6232	3640	2388211	2414453	2631024	W_5
2644128	2958120	3074760	2683859	2710105	331740	344880	
$\mathbf{C}(7^4 15^4 17^4 19^2)$							
$\mathbf{C}(15^{10}17^4)$							$W_6 = W_5^t$
$(W1, 1)$	6520	2388211	2414453	3000726	3120930	2598200	W_7
2598232	2691270	2691774	301446	300978	31757	30461	
$\mathbf{C}(15^4 17^8 19^2)$	$\mathbf{T}(72^4 74^4 75^4 76^2)$						
$\mathbf{C}(15^4 17^8 19^2)$	$\mathbf{T}(66^4 68^4 77^4 79^2)$						$W_8 = W_7^t$
$(W1, 2)$	6520	2388211	2414453	3000726	3120930	2598200	W_9
2598232	2695518	2694222	300960	301464	27995	27527	
$\mathbf{C}(15^6 17^8)$	$\mathbf{T}(66^4 67^4 68^4 69^2)$						
$\mathbf{C}(15^6 17^8)$	$\mathbf{T}(70^4 71^4 72^4 73^2)$						$W_{10} = W_9^t$

TABLE 2 (*continued*)

($W1, 3$) 6520 2388211 2414453 3000726 3120930 2599416 W_9
2599920 2690054 2690086 301446 300978 31757 30461
3, 4, 1, $\underline{2}$, B, $\underline{C}$, D, $\underline{E}$, 8, $\underline{7}$, $\underline{5}$, 6, 9, $\underline{A}$: 1, $\underline{2}$, 9, A, B, E, D, C, 4, 3, 8, 5, 7, 6

($W1, 4$) 6520 2388211 2414453 3000726 3120930 2603664 W_{11}
2602368 2690054 2690086 300960 301464 27995 27527
$\mathbf{C}(15^{14})$ $\mathbf{T}(69^{14})$
$\mathbf{C}(15^{14})$ $\mathbf{T}(78^{14})$ $W_{12} = W_{11}^t$

($W1, 5$) 6520 2388211 2414453 3000726 3120930 2599416 W_{13}
2599920 2691756 2691288 299744 299776 31757 30461
SD E, D, 1, 2, 5, 8, 3, 4, 6, 7, $\underline{B}$, C, $\underline{9}$, $\underline{A}$: 3, 4, 7, 8, 5, 9, A, 6, $\underline{D}$, $\underline{E}$, $\underline{B}$, C, 2, 1
$\mathbf{C}(15^2 17^{12})$

($W1, 6$) 6520 2388211 2414453 3000726 3120930 2599416 W_7
2599920 2695518 2694222 299744 299776 27995 27527
5, 6, 1, 2, D, $\underline{E}$, B, $\underline{C}$, 7, $\underline{8}$, $\underline{3}$, 4, 9, $\underline{A}$: 8, $\underline{5}$, C, $\underline{B}$, $\underline{A}$, $\underline{D}$, E, 9, 6, 7, 1, 2, 3, 4

($W1, 7$) 6520 2388211 2414453 3000726 3120930 2603664 W_9
2602368 2691270 2691774 299744 299776 27995 27527
5, $\underline{6}$, 1, 2, D, $\underline{E}$, B, $\underline{C}$, 9, $\underline{A}$, $\underline{3}$, 4, 7, $\underline{8}$: $\underline{5}$, 8, $\underline{B}$, C, 9, $\underline{A}$, $\underline{D}$, E, 6, 7, 1, 2, 3, 4

($W2, 1$) 6520 2388211 2414453 2631006 2644146 2956014 W_8
3074598 2687909 2714155 296990 297022 38376 37080
6, $\underline{7}$, D, E, 9, A, C, B, 1, $\underline{2}$, $\underline{4}$, 3, 8, $\underline{5}$: $\underline{7}$, 8, B, C, D, $\underline{2}$, 1, $\underline{E}$, $\underline{A}$, $\underline{9}$, 5, 6, 4, 3

($W2, 2$) 6520 2388211 2414453 2633310 2648538 2953710 W_{10}
3071826 2685155 2711401 300816 299520 37304 37336
3, 4, 9, A, D, E, 1, $\underline{2}$, C, B, $\underline{6}$, 7, $\underline{8}$, 5: $\underline{E}$, D, 1, 2, 8, $\underline{7}$, B, C, 6, $\underline{5}$, 9, $\underline{A}$, $\underline{4}$, 3

($W3, 1$) 6520 2388211 2650651 2948238 3158190 2422229 W_{12}
2540813 2694510 2707650 296990 297022 38376 37080
1, 2, 8, 5, 9, A, E, D, 3, 4, C, B, 7, 6: 1, 2, $\underline{B}$, C, 3, $\underline{7}$, $\underline{8}$, 4, E, $\underline{D}$, 6, 5, $\underline{9}$, $\underline{A}$

($W3, 2$) 6520 2388211 2650651 2948238 3158190 2421059 W_{10}
2539175 2695680 2709288 300816 299520 34550 34582
3, 4, 1, $\underline{2}$, B, $\underline{C}$, $\underline{9}$, A, D, E, 6, 5, 8, $\underline{7}$: 1, $\underline{2}$, E, D, B, A, 9, C, 4, 3, 5, 8, 7, 6

($X1, 1$) 6520 2388211 2650651 2954844 3075048 2691288 W_{14}
2704896 288728 288760 334206 346842 2131277 2129981
SD 6, $\underline{7}$, B, $\underline{C}$, $\underline{D}$, E, $\underline{3}$, 4, 9, A, 2, $\underline{1}$, 5, 8: $\underline{C}$, B, 8, $\underline{7}$, D, 1, $\underline{2}$, $\underline{E}$, 9, A, 3, $\underline{4}$, $\underline{5}$, 6
$\mathbf{C}(17^8 19^6)$

($X2, 1$) 6520 2388211 2945934 2600550 2179530 3134059 W_{15}
2684669 328094 328126 346680 345384 2131925 2661318
$\mathbf{C}(17^4 19^2 22^4 24^4)$
$\mathbf{C}(17^8 19^6)$ $W_{16} = W_{15}^t$

($X2, 2$) 6520 2388211 2948238 2600550 2183922 3129667 W_{17}
2684669 328094 328126 346680 345384 2127533 2663406
$\mathbf{C}(17^6 22^6 24^2)$
$\mathbf{C}(17^{12} 19^2)$ $W_{18} = W_{17}^t$

TABLE 2 (*continued*)

(X3, 1) 6520 2388211 2414453 2604312 2958120 1984158 W_{16}
1997262 683280 696420 325139 351385 2657686 2657654
$\underline{9}$, A, $\underline{8}$, 7, 2, E, $\underline{D}$, 1, 4, 3, $\underline{B}$, $\underline{C}$, $\underline{5}$, $\underline{6}$: 3, $\underline{4}$, B, $\underline{C}$, 8, $\underline{E}$, $\underline{7}$, D, A, 9, $\underline{1}$, 2, $\underline{5}$, 6

(Y1, 1) 6520 2388211 2129333 2599902 2182770 2858657 W_{19}
3021768 2756232 2789323 332262 341216 292734 341248
$\mathbf{C}(18^4 19^2 23^2 25^6)$
$\mathbf{C}(19^6 24^8)$ $W_{20} = W_{19}^t$

(Y2, 1) 6520 2388211 2660814 2421419 2521008 2494458 W_{21}
2740397 2793049 2706840 288728 288760 332262 346842
SD 1, 2, 7, B, 8, $\underline{A}$, 9, $\underline{5}$, 6, $\underline{C}$, $\underline{3}$, 4, $\underline{E}$, D: 1, 2, $\underline{B}$, C, $\underline{8}$, 9, 3, $\underline{5}$, 7, $\underline{6}$, 4, $\underline{A}$, E, $\underline{D}$
$\mathbf{C}(18^4 19^2 22^2 24^6)$

(Z2, 1) 6520 2362534 2362370 2603322 2601846 2684359 W_{22}
2694494 2704848 2712423 324173 329806 345648 348657
SD 9, $\underline{A}$, E, D, $\underline{B}$, $\underline{C}$, $\underline{3}$, 7, 8, 4, $\underline{1}$, $\underline{5}$, 6, 2: 7, $\underline{A}$, B, $\underline{E}$, 9, $\underline{8}$, D, $\underline{C}$, $\underline{2}$, $\underline{1}$, 3, 4, 5, 6
$\mathbf{C}(19^6 25^8)$ $\mathbf{T}(12^8 15^4 32^2)$

(Z2, 2) 6520 2362534 2362370 2603322 2601846 2684359 W_{23}
2692694 2704728 2714343 325973 329806 343536 348969
SD 9, A, E, D, C, B, $\underline{5}$, $\underline{1}$, 2, 6, $\underline{7}$, $\underline{3}$, 4, 8: 7, $\underline{A}$, B, $\underline{E}$, 9, $\underline{8}$, D, $\underline{C}$, 1, 2, 5, 6, 3, 4
$\mathbf{C}(19^6 25^8)$ $\mathbf{T}(8^8 46^4 32^2)$

(Z2, 3) 6520 2362036 2362004 2604312 2601720 2683861 W_{24}
2693660 2703744 2714859 326597 328624 344880 348183
$\mathbf{C}(19^6 24^8)$ $\mathbf{T}(36^8 37^6)$
$\mathbf{C}(19^{14})$ $\mathbf{T}(15^{12} 35^2)$ $W_{25} = W_{24}^t$

(Z3, 1) 5578 2362012 2423981 2539767 2604312 2683493 W_{26}
2690436 2706832 2715363 266952 290396 330004 345738
SD E, D, 2, 5, 4, 8, $\underline{3}$, $\underline{1}$, $\underline{6}$, $\underline{7}$, 9, C, B, A: 5, 7, 3, 8, 9, 4, 6, A, $\underline{C}$, $\underline{D}$, E, B, 1, $\underline{2}$
$\mathbf{C}(19^6 24^8)$ $\mathbf{T}(2^8 6^4 7^2)$

(Z3, 2) 4282 2362012 2423981 2539767 2604312 2683493 W_{27}
2693676 2703592 2715363 266952 290396 332596 343146
$\mathbf{C}(19^2 24^4 25^8)$ $\mathbf{T}(12^4 16^4 1^2 13^2 28^2)$
$\mathbf{C}(19^6 25^8)$ $\mathbf{T}(38^8 10^4 19^2)$ $W_{28} = W_{27}^t$

(Z3, 3) 5578 2362012 2423369 2540379 2604312 2684267 W_{29}
2690436 2706832 2714589 268284 289064 328672 346296
$\mathbf{C}(19^6 25^8)$ $\mathbf{T}(8^8 3^4 9^2)$
$\mathbf{C}(19^6 25^8)$ $\mathbf{T}(17^8 6^4 18^2)$ $W_{30} = W_{29}^t$

(Z3, 4) 4282 2362012 2423369 2540379 2604312 2684267 W_{31}
2693676 2703592 2714589 268284 289064 333208 341760
$\mathbf{C}(19^2 24^4 25^8)$ $\mathbf{T}(1^2 8^2 13^2 26^2 27^2 29^2 65^2)$
$\mathbf{C}(19^6 25^8)$ $\mathbf{T}(50^8 10^2 11^2 51^2)$ $W_{32} = W_{31}^t$

TABLE 2 (*continued*)

$(Z3, 5)$	4282	2362012	2423369	2540379	2604312	2684267	W_{33}
2691984	2707678	2712195	264198	293150	333208	341760	
C$(19^2 23^6 25^6)$	**T**$(29^4 55^4 1^2 22^2 56^2)$						
C$(19^6 24^8)$	**T**$(45^8 11^4 20^2)$						$W_{34} = W_{33}^t$
$(Z3, 6)$	4282	2362012	2423981	2539767	2604312	2687123	W_{35}
2695128	2703592	2710281	266952	290396	329674	346068	
C$(19^2 24^4 25^8)$	**T**$(1^4 12^4 24^4 9^2)$						
C$(19^6 25^8)$	**T**$(39^8 10^4 19^2)$						$W_{36} = W_{35}^t$
$(Z3, 7)$	4282	2362012	2423369	2540379	2604312	2685737	W_{37}
2695128	2703592	2711667	268284	289064	328126	346842	
C$(19^2 24^{12})$	**T**$(26^4 27^4 52^4 7^2)$						
C$(19^6 24^8)$	**T**$(40^8 11^4 20^2)$						$W_{38} = W_{37}^t$
$(Z3, 8)$	4126	2362012	2423981	2539767	2604312	2687123	W_{39}
2695518	2703202	2710281	266952	290396	330004	345738	
C$(19^6 24^8)$	**T**$(24^8 3^4 2^2)$						
C$(19^6 24^8)$	**T**$(23^8 6^4 7^2)$						$W_{40} = W_{39}^t$
$(Z4, 1)$	4120	2362012	2423981	2501237	2523856	2724201	P_3^t
2786568	2690436	2713905	264036	293312	332920	342822	
B, E, C, D, 1, 2: 3, 1, 6, 8, $\underline{7}$, $\underline{5}$, $\underline{4}$, $\underline{2}$, C, $\underline{9}$, $\underline{B}$, A							
$(Z4, 2)$	5578	2362012	2423369	2504927	2520940	2726271	W_{41}
2783652	2690436	2714589	268284	289064	328672	346296	
C$(19^6 25^8)$	**T**$(41^8 21^4 3^2)$						
C$(19^2 24^4 25^8)$	**T**$(1^4 17^4 53^4 18^2)$						$W_{42} = W_{41}^t$
$(Z4, 3)$	4120	2362012	2423981	2501237	2523856	2725653	W_{43}
2783664	2695518	2710275	266958	290390	329998	345744	
C$(19^6 24^8)$	**T**$(34^8 3^6)$						
C$(19^2 23^6 25^6)$	**T**$(1^6 42^6 22^2)$						$W_{44} = W_{43}^t$
$(Z4, 4)$	5584	2362012	2423981	2501613	2524368	2724201	W_{45}
2786568	2690060	2713393	264036	293312	332920	342822	
C$(19^6 25^8)$	**T**$(49^8 21^4 4^2)$						
C$(19^2 24^4 25^8)$	**T**$(23^4 25^4 1^2 13^2 28^2)$						$W_{46} = W_{45}^t$
$(Z4, 5)$	5584	2362012	2423981	2501613	2524368	2725653	P_3^t
2783664	2695142	2709763	266958	290390	329998	345744	
B, E, C, D, 1, 2: 2, 1, 6, 7, $\underline{8}$, $\underline{5}$, $\underline{4}$, $\underline{3}$, C, $\underline{9}$, $\underline{B}$, A							
$(Z4, 6)$	5578	2362012	2423369	2500545	2525346	2726757	W_{41}
2783166	2694656	2710345	267312	289550	329644	345810	
D, C, 4, 6, 3, 5, 7, 9, 8, A, 1, $\underline{E}$, B, 2: C, 1, 5, $\underline{B}$, $\underline{7}$, $\underline{9}$, $\underline{4}$, A, 6, 2, 8, 3, D, E							
$(Z4, 7)$	4282	2362012	2423981	2501613	2524368	2725653	P_3^t
2783664	2695168	2709737	266906	290442	330024	345718	
C, D, B, E, 2, 1: 3, 1, 5, 8, 2, 4, 6, 7, C, 9, $\underline{B}$, $\underline{A}$							

TABLE 2 (*continued*)

$(Z4, 8)$ 5746 2362012 2423981 2501263 2523830 2724201 W_{41}
2786568 2690436 2713905 263984 293364 332946 342796
B, E, 3, 6, 5, 4, $\underline{7}$, $\underline{A}$, $\underline{9}$, $\underline{8}$, 1, $\underline{C}$, $\underline{D}$, 2: 2, 3, $\underline{7}$, $\underline{8}$, 9, A, B, C, 6, 5, 1, 4, E, D

$(Z4, 9)$ 4288 2362012 2423421 2500493 2525346 2726705 W_{41}
2783166 2694682 2710371 267312 289550 329618 345810
C, D, A, 8, 7, 9, 5, 3, 4, 6, 2, B, $\underline{E}$, 1: $\underline{9}$, $\underline{4}$, 5, $\underline{A}$, B, C, $\underline{1}$, 7, $\underline{3}$, 8, 2, $\underline{6}$, $\underline{D}$, E

$(Z4, 10)$ 4288 2362012 2423421 2504979 2520914 2726219 W_{41}
2783652 2690436 2714563 268284 289064 328646 346296
B, E, 3, 6, 5, 4, $\underline{7}$, $\underline{A}$, $\underline{9}$, $\underline{8}$, 1, $\underline{C}$, D, $\underline{2}$: 2, 3, $\underline{7}$, $\underline{8}$, 9, A, B, C, 6, 5, 1, 4, E, D

$(Z5, 1)$ 6226 2362012 2324846 2603664 2521452 2728001 W_{47}
2788953 2691732 2705536 326353 348615 293232 342902
$\mathbf{C}(19^2 25^{12})$ $\mathbf{T}(43^8 5^4 44^2)$
$\mathbf{C}(19^2 25^{12})$ $\mathbf{T}(30^4 57^4 5^2 31^2 58^2)$ $W_{48} = W_{47}^t$

$(Z5, 2)$ 6226 2362012 2324846 2603664 2521452 2727227 W_{49}
2789727 2691732 2705536 326353 349389 293844 341516
SD E, D, 5, 6, 3, 1, 7, 4, 8, 2, $\underline{A}$, $\underline{B}$, 9, C: 6, A, 5, 8, 3, 4, 7, 9, D, $\underline{B}$, $\underline{C}$, E, 2, 1
$\mathbf{C}(19^2 24^4 25^8)$ $\mathbf{T}(59^4 60^4 5^2 31^2 61^2)$

$(Z6, 1)$ 5740 2362012 2324846 2603288 2520940 2727603 P_3^t
2790213 2684037 2713257 332590 342216 293144 343152
1, 2, B, D, C, E: 9, A, B, C, 3, 4, 6, 7, 1, 2, 5, 8

$(Z6, 2)$ 6226 2362012 2324846 2603664 2521452 2728001 P_3^t
2788953 2685073 2712195 332526 342902 293232 342442
$\underline{2}$, 1, $\underline{D}$, $\underline{B}$, C, E: 9, $\underline{A}$, $\underline{B}$, C, 4, 3, $\underline{7}$, $\underline{6}$, 8, $\underline{2}$, 5, $\underline{1}$

$(Z6, 3)$ 4126 2362012 2324846 2603664 2521452 2723567 P_3^t
2791287 2689291 2710077 332526 342902 293232 342442
1, 2, B, D, C, E: C, $\underline{A}$, $\underline{B}$, 9, 1, $\underline{4}$, 7, $\underline{6}$, 2, $\underline{8}$, $\underline{5}$, 3

$(Z6, 4)$ 4288 2362012 2324846 2599068 2525346 2723333 P_3^t
2789727 2688121 2713743 333912 341516 293844 341830
1, 2, B, D, C, E: C, $\underline{A}$, $\underline{B}$, 9, 1, $\underline{6}$, 3, $\underline{5}$, 2, $\underline{8}$, $\underline{4}$, 7

$(Z6, 5)$ 5578 2362012 2324846 2599068 2525346 2724681 P_3^t
2788703 2689119 2712421 333886 341568 293792 341856
$\underline{1}$, 2, B, D, C, E: C, $\underline{A}$, $\underline{B}$, 9, $\underline{5}$, 3, $\underline{6}$, 4, $\underline{8}$, 2, 7, $\underline{1}$

$(Z6, 6)$ 4126 2362012 2324846 2603314 2520914 2727603 P_3^t
2790213 2684037 2713257 332616 342164 293196 343126
1, 2, B, C, D, E: C, $\underline{A}$, $\underline{B}$, 9, 1, $\underline{7}$, 3, $\underline{6}$, 2, $\underline{8}$, $\underline{4}$, 5

$(Z7, 1)$ 5578 2362012 2601720 2152388 2184096 2952494 W_{50}
3074598 2695134 2703754 263100 291656 331750 341598
SD 9, A, 2, 4, 5, 8, $\underline{3}$, $\underline{1}$, $\underline{7}$, $\underline{6}$, $\underline{E}$, $\underline{C}$, B, D: B, $\underline{9}$, $\underline{6}$, E, A, 5, $\underline{C}$, $\underline{D}$, $\underline{8}$, $\underline{4}$, $\underline{3}$, $\underline{7}$, 1, $\underline{2}$
$\mathbf{C}(19^{14})$ $\mathbf{T}(2^{14})$

TABLE 2 (*continued*)

$(Z7, 2)$	4282	2362012	2601720	2152388	2184096	2952494	W_{51}
3074598	2690598	2708290	263100	291656	328510	344838	
$\mathbf{C}(19^6 24^8)$	$\mathbf{T}(14^8 2^6)$						
$\mathbf{C}(19^{14})$	$\mathbf{T}(4^{12} 2^2)$						$W_{52} = W_{51}^t$
$(Z7, 3)$	5578	2362012	2601720	2153720	2182764	2953268	P_3^t
3073050	2694576	2705086	263712	291044	331750	341598	
6, 9, 5, A, 2, 1: 3, 1, 6, 7, 2, 4, 5, 8, E, 9, $\underline{D}$, $\underline{A}$							
$(Z7, 4)$	4282	2362012	2601720	2153720	2182764	2953268	P_4^t
3073050	2691984	2707678	263712	291044	328510	344838	
$\underline{6}$, $\underline{9}$, 5, A, 2, 1: 3, 1, 8, 5, 6, 7, 4, 2, 9, E, $\underline{A}$, $\underline{D}$							
$(Z7, 5)$	5578	2362012	2601720	2152388	2184096	2952884	W_{51}
3074208	2695680	2703208	263100	291656	331750	341754	
B, E, 1, C, $\underline{6}$, 4, $\underline{2}$, D, $\underline{9}$, 8, $\underline{5}$, 3, $\underline{A}$, 7: 3, $\underline{5}$, $\underline{6}$, 2, 9, A, B, C, $\underline{D}$, $\underline{E}$, 1, 4, 7, 8							
$(Z7, 6)$	4282	2362012	2601720	2152388	2184096	2956124	W_{53}
3070968	2695680	2703208	263100	291656	328510	346290	
$\mathbf{C}(19^2 24^{12})$	$\mathbf{T}(14^{12} 2^2)$						
$\mathbf{C}(19^{14})$	$\mathbf{T}(33^8 4^6)$						$W_{54} = W_{53}^t$
$(Z7, 7)$	5578	2362012	2601720	2153720	2182764	2953442	P_4^t
3072876	2694906	2704756	263712	291044	331750	341754	
$\underline{8}$, 4, 3, $\underline{7}$, 2, 1: 1, 3, 7, 6, 4, 2, 8, 5, D, B, $\underline{C}$, $\underline{E}$							
$(Z7, 8)$	4282	2362012	2601720	2153720	2182764	2954738	P_4^t
3071580	2694906	2704756	263712	291044	328510	346290	
8, 3, 4, 7, 2, 1: 8, 5, 1, 3, 7, 6, 4, 2, E, C, $\underline{B}$, $\underline{D}$							
$(Z7, 9)$	6520	2362012	2601720	2153720	2182764	2952500	P_3^t
3076056	2690592	2706832	263712	291044	328510	346290	
4, 8, 3, 7, 2, 1: 4, 2, 1, 3, 7, 6, 8, 5, E, C, $\underline{D}$, $\underline{B}$							
$(Z8, 1)$	5578	2362012	2601080	2153766	2183124	2859221	P_3^t
3051019	2725659	2790315	330778	342726	291170	342084	
1, $\underline{2}$, B, D, C, E: C, $\underline{A}$, $\underline{B}$, 9, 2, $\underline{3}$, 7, $\underline{5}$, 8, $\underline{4}$, $\underline{1}$, 6							
$(Z8, 2)$	4282	2362012	2598200	2154414	2182764	2855333	P_2^t
3052243	2728899	2792331	332596	342204	291044	343146	
B, D, 5, 6, 3, 4: D, E, $\underline{2}$, 4, 3, $\underline{1}$, $\underline{B}$, $\underline{C}$, $\underline{9}$, $\underline{A}$, $\underline{8}$, 5							
$(Z8, 3)$	4288	2362012	2601132	2153714	2183124	2855333	P_3^t
3055581	2723041	2792259	332722	342726	291170	342084	
$\underline{1}$, 2, B, D, C, E: C, $\underline{A}$, $\underline{B}$, 9, $\underline{3}$, 2, $\underline{5}$, 7, $\underline{4}$, 6, 8, $\underline{1}$							
$(Z8, 4)$	5578	2362012	2601720	2152388	2184096	2854235	P_3^t
3054759	2723359	2793627	331750	341754	291656	341598	
$\underline{1}$, 2, B, D, C, E: C, $\underline{A}$, $\underline{B}$, 9, $\underline{5}$, 6, $\underline{3}$, 2, $\underline{4}$, 8, 7, $\underline{1}$							
$(Z8, 5)$	4282	2362012	2601720	2153720	2182764	2855333	P_2^t
3055281	2723035	2792331	332596	342204	291044	343146	
E, D, 2, 4, $\underline{6}$, $\underline{7}$: E, $\underline{C}$, 2, $\underline{1}$, 5, $\underline{7}$, A, $\underline{4}$, 3, $\underline{9}$, 8, $\underline{6}$							

TABLE 2 (*continued*)

(Z8, 6) 5584 2362012 2601720 2153772 2182712 2856245 P^t_2
3052755 2728413 2788567 332622 342152 291096 343120
E, D, 1, 2, 6, 7: 1, 2, C, $\underline{E}$, 4, $\underline{8}$, 9, $\underline{5}$, 6, $\underline{A}$, 7, $\underline{3}$

(Z9, 1) 5578 2362012 2601720 2152388 2184096 2854235 P^t_5
3054759 2730726 2786260 331750 341754 284529 348725
$\underline{D}$, $\underline{C}$, $\underline{B}$, E, 2, 1: 7, 6, 1, 3, 4, 2, 8, 5, $\underline{9}$, C, $\underline{A}$, $\underline{B}$

(Z9, 2) 4282 2362012 2601720 2153720 2182764 2855333 P^t_5
3055281 2732022 2783344 332596 342204 282783 351407
C, D, $\underline{E}$, B, 2, 1: 1, 3, 8, 5, 7, 2, 4, 6, C, $\underline{9}$, A, $\underline{B}$

(Z9, 3) 4126 2362012 2601720 2153720 2182764 2854787 W_{55}
3055671 2731632 2783890 332596 342204 282783 351407
SD, E, $\underline{D}$, 7, 4, 3, 5, 1, 8, 6, 2, A, B, 9, $\underline{C}$: 7, A, 5, 4, 6, 9, 3, 8, D, B, C, $\underline{E}$, $\underline{2}$, 1
$\mathbf{C}(19^2 25^{12})$ $\mathbf{T}(47^8 5^4 48^2)$

(Z9, 4) 5578 2362012 2601720 2152388 2184096 2854235 W_{55}
3054759 2730180 2786806 331750 341598 284859 348551
3, $\underline{A}$, 5, $\underline{7}$, $\underline{9}$, 6, 4, $\underline{8}$, $\underline{C}$, D, $\underline{E}$, B, 2, 1: D, E, 1, 8, 2, 6, 3, 5, 7, 4, C, $\underline{9}$, A, $\underline{B}$

(Z10, 1) 5578 2362012 2601720 2152388 2184096 2952494 P^t_7
3074598 2695134 2703754 271737 283019 323113 350235
$\underline{4}$, 8, 3, $\underline{7}$, 2, 1: 1, 3, 6, 7, 2, 4, 8, 5, B, D, $\underline{C}$, $\underline{E}$

(Z10, 2) 4282 2362012 2601720 2152388 2184096 2952494 W_{56}
3074598 2690598 2708290 269955 284801 323113 350235
SD 9, A, E, C, 4, $\underline{2}$, $\underline{D}$, $\underline{B}$, 3, $\underline{1}$, 8, $\underline{5}$, 6, $\underline{7}$: 5, 9, $\underline{6}$, A, $\underline{E}$, $\underline{B}$, C, D, $\underline{2}$, 1, 4, 8, 3, 7
$\mathbf{C}(19^6 25^8)$ $\mathbf{T}(25^8 4^4 9^2)$

(Z10, 3) 5578 2362012 2601720 2153720 2182764 2953268 P^t_3
3073050 2694576 2705086 271737 283019 321565 351783
6, 9, 5, A, 2, 1: 3, 1, 6, 7, 2, 4, 5, 8, E, 9, $\underline{D}$, $\underline{A}$

(Z10, 4) 4282 2362012 2601720 2153720 2182764 2953268 P^t_4
3073050 2691984 2707678 269955 284801 321565 351783
A, 6, 5, 9, 2, 1: 1, 2, 3, 4, 5, 6, 7, 8, 9, A, D, E

(Z10, 5) 5578 2362012 2601720 2153720 2182764 2953442 P^t_4
3072876 2694906 2704756 271407 283349 322111 351393
8, 3, 4, 7, 2, 1: 7, 6, 1, 3, 8, 5, 4, 2, E, $\underline{B}$, C, $\underline{D}$

(Z11, 1) 5740 2362012 2501945 2251227 2184096 2893781 P_3
3035962 2783916 2713257 333182 341592 263100 331750
6, 4, 5, 3, 7, 8, 9, A, 1, D, C, 2: 5, 6, 2, 3, D, E

(Z11, 2) 5740 2362012 2501945 2255691 2179848 2894429 P^t_3
3037258 2784636 2710377 332534 342888 264396 330454
1, 2, C, D, B, E: C, A, $\underline{B}$, $\underline{9}$, 1, $\underline{4}$, 8, $\underline{5}$, 3, $\underline{7}$, $\underline{6}$, 2

(Z11, 3) 5584 2362012 2502821 2252259 2183124 2894267 P^t_3
3035572 2782398 2713743 332696 342078 264072 330778
$\underline{1}$, 2, C, D, B, E: C, A, $\underline{B}$, $\underline{9}$, $\underline{4}$, 1, $\underline{6}$, 8, $\underline{5}$, 3, 2, $\underline{7}$

TABLE 2 (*continued*)

$(Z11, 4)$	4120	2362012	2502821	2252457	2182926	2895725	P_3^t
3032662	2787216	2710377	329780	343542	266988	329320	
$\underline{1}$, 2, C, D, B, E: C, A, $\underline{B}$, $\underline{9}$, $\underline{4}$, 1, $\underline{6}$, 7, $\underline{5}$, 2, 3, $\underline{8}$							
$(Z11, 5)$	4120	2362012	2502821	2252457	2182926	2896131	P_2
3032982	2787130	2709737	328920	343622	267068	329700	
1, $\underline{2}$, B, $\underline{E}$, 4, 9, $\underline{8}$, $\underline{5}$, 3, A, $\underline{7}$, $\underline{6}$: D, E, 2, 4, 5, 7							
$(Z11, 6)$	5584	2362012	2502821	2251945	2182712	2896131	P_3^t
3032982	2787480	2710113	328920	343622	267068	329700	
1, 2, C, D, B, E: C, A, $\underline{B}$, $\underline{9}$, 2, $\underline{5}$, 4, $\underline{7}$, 6, $\underline{1}$, $\underline{8}$, 3							
$(Z11, 7)$	5746	2362012	2502873	2252457	2182874	2894627	P_3^t
3035886	2782074	2713393	330338	342204	264198	332596	
$\underline{1}$, 2, C, D, B, E: C, A, $\underline{B}$, $\underline{9}$, $\underline{8}$, 1, $\underline{5}$, 3, $\underline{4}$, 6, 2, $\underline{7}$							
$(Z11, 8)$	5746	2362012	2502873	2252457	2182874	2896079	P_2
3032982	2787156	2709763	328868	343674	267120	329674	
1, 2, B, $\underline{E}$, 4, 9, $\underline{8}$, $\underline{5}$, 6, 7, $\underline{A}$, $\underline{3}$: D, E, 1, 2, 5, 7							
$(Z12, 1)$	5578	2362012	2501945	2251227	2184096	2952884	P_{10}^t
3053701	2724651	2695680	263100	291656	331750	341754	
1, 2, C, D, B, E: C, 9, 3, $\underline{6}$, $\underline{B}$, $\underline{A}$, $\underline{2}$, 5, $\underline{4}$, 8, 1, $\underline{7}$							
$(Z12, 2)$	5740	2362012	2501945	2255691	2179848	2953046	P_3^t
3053701	2728899	2691054	265170	289586	329680	343662	
B, E, C, D, 2, 1: 3, 1, 7, 5, 8, 6, 2, 4, C, $\underline{9}$, $\underline{B}$, A							
$(Z12, 3)$	5578	2362012	2502821	2252259	2183124	2952494	P_7^t
3054673	2724165	2694648	264072	291170	330778	342084	
B, E, C, D, $\underline{2}$, 1: 8, 4, 2, 5, 1, 6, 7, 3, C, $\underline{9}$, $\underline{B}$, A							
$(Z12, 4)$	5578	2362012	2501945	2251227	2184096	2956500	P_3
3052269	2727515	2690632	263100	291656	333182	341754	
9, 7, A, 8, 6, 4, 5, 3, 1, $\underline{B}$, 2, $\underline{E}$: 2, 3, 5, 6, $\underline{D}$, E							
$(Z12, 5)$	5578	2362012	2502821	2252259	2183124	2955138	P_{10}^t
3052755	2728001	2690086	264072	291170	332696	342084	
B, $\underline{E}$, C, D, 2, 1: 6, 1, 3, 7, 4, 8, 5, 2, $\underline{9}$, C, A, $\underline{B}$							
$(Z12, 6)$	5584	2362012	2502821	2251945	2182712	2955144	P_3^t
3052755	2728413	2690394	264146	291096	332622	342152	
C, D, B, E, 2, 1: 6, 7, 2, 1, 3, 4, 8, 5, C, 9, $\underline{B}$, $\underline{A}$							
$(Z13, 1)$	5578	2362012	2501945	2251227	2184096	2952884	P_3^t
3053701	2724651	2695680	271407	283349	323443	350061	
B, E, C, D, 2, 1: 3, 1, 7, 6, 8, 5, 2, 4, C, 9, $\underline{B}$, A							
$(Z13, 2)$	5740	2362012	2501945	2255691	2179848	2953046	P_4
3053701	2728899	2691054	269661	285095	325189	348153	
3, 4, 6, 5, A, 9, 7, 8, $\underline{B}$, $\underline{1}$, 2, E: 6, 5, $\underline{3}$, $\underline{2}$, D, E							

TABLE 2 (*continued*)

(Z13, 3) 5578 2362012 2502821 2252259 2183124 2952494 P_7^t
3054673 2724165 2694648 272709 282533 322141 350721
1, $\underline{2}$, B, E, C, D: C, 9, 2, $\underline{5}$, $\underline{B}$, A, $\underline{6}$, 1, 8, $\underline{4}$, $\underline{3}$, 7

(Z13, 4) 5578 2362012 2501945 2251227 2184096 2956500 P_4
3052269 2727515 2690632 269897 284859 326385 348551
6, 5, 3, 4, 7, 8, A, 9, $\underline{E}$, 2, 1, B: $\underline{5}$, 1, $\underline{7}$, 2, D, E

(Z13, 5) 5578 2362012 2502821 2252259 2183124 2955138 P_3^t
3052755 2728001 2690086 270713 284529 326055 348725
B, E, C, D, $\underline{2}$, 1: 2, 4, 8, 5, 6, 7, 3, 1, C, 9, $\underline{B}$, A

(Z13, 6) 5584 2362012 2502821 2251945 2182712 2955144 P_4
3052755 2728413 2690394 270633 284609 326135 348639
4, 6, 3, 5, 7, 9, 8, A, $\underline{2}$, C, D, $\underline{1}$: $\underline{6}$, 4, 8, $\underline{1}$, D, E

(Z14, 1) 5740 2362012 2324846 2603664 2875098 2727515 W_{57}
2789925 1976809 2003445 648822 696062 332526 342928
C$(19^2 25^{12})$ **T**$(80^4 81^4 82^4 54^2)$
C$(24^2 25^{12})$ **T**$(62^4 83^4 63^2 84^2 85^2)$ $W_{58} = W_{57}^t$

(Z15, 1) 5740 2362012 2324846 2603664 2875098 2727515 W_{59}
2789925 1982010 1998244 648822 696062 327325 348129
SD 5, 6, A, C, $\underline{8}$, $\underline{1}$, 7, $\underline{9}$, 4, $\underline{E}$, $\underline{B}$, 2, D, 3: 1, 2, $\underline{5}$, $\underline{6}$, 7, 9, C, E, $\underline{8}$, 3, $\underline{B}$, 4, D, $\underline{A}$
C$(19^2 25^{12})$ **T**$(62^4 64^4 30^2 54^2 63^2)$

(Z16, 1) 5584 2362012 2502821 2252259 2859221 3070618 P_7^t
2790315 1826688 1999434 647606 687214 264072 342078
9, A, B, $\underline{C}$, 7, 3: B, 1, E, $\underline{2}$, $\underline{3}$, D, 4, C, $\underline{A}$, 6, 9, 8

(Z17, 1) 5584 2362012 2502821 2252259 2859221 3070618 W_{60}
2790315 1826688 1999434 629139 707145 282539 322147
C(19^{14}) **T**(33^{14})
C(24^{14}) **T**(14^{14}) $W_{61} = W_{60}^t$

(Z17, 2) 4126 2362012 2502821 2256507 2857763 3070618 W_{60}
2793789 1824618 1995132 627807 704355 283151 322759
9, A, E, 2, 7, 1, $\underline{D}$, 3, $\underline{6}$, $\underline{4}$, C, B, 8, 5: $\underline{6}$, $\underline{A}$, $\underline{9}$, C, 3, E, 2, $\underline{5}$, $\underline{D}$, 8, $\underline{B}$, 7, $\underline{4}$, 1

(Z17, 3) 4282 2362012 2502873 2252207 2859273 3070592 W_{60}
2790289 1826636 1999408 629139 707145 282591 322173
9, A, $\underline{E}$, 1, 7, $\underline{2}$, $\underline{D}$, 3, $\underline{6}$, $\underline{4}$, C, B, 8, $\underline{5}$: $\underline{6}$, $\underline{A}$, $\underline{9}$, C, $\underline{1}$, E, 2, 8, $\underline{D}$, 7, $\underline{B}$, $\underline{5}$, 3, 4

(Z17, 4) 4126 2362012 2502585 2252207 2858823 3070598 W_{60}
2790451 1826930 1999702 627807 706593 284079 322725
7, $\underline{A}$, $\underline{9}$, C, $\underline{1}$, E, $\underline{4}$, 8, $\underline{D}$, $\underline{5}$, $\underline{B}$, $\underline{6}$, 2, $\underline{3}$: 9, A, E, 2, 7, $\underline{1}$, $\underline{D}$, 3, $\underline{6}$, $\underline{4}$, C, B, 8, $\underline{5}$

(Z18, 1) 4282 2362012 2601720 2322902 2877528 2684045 W_{62}
2714811 2018659 2081637 643550 701334 332914 342054
C$(19^2 25^{12})$ **T**$(86^{12} 87^2)$
C$(24^2 25^{12})$ **T**$(85^6 88^6 89^2)$ $W_{63} = W_{62}^t$

TABLE 2 (*continued*)

$(Z19, 1)$	4282	2362012	2601720	2322902	2877528	2684045	W_{64}
2714811	2020944	2079352	641289	703595	332914	342054	
$\mathbf{C}(19^2 25^{12})$	$\mathbf{T}(86^{12} 87^2)$						
$\mathbf{C}(24^2 25^{12})$	$\mathbf{T}(5^6 90^6 91^2)$						$W_{65} = W_{64}^t$

Acknowledgments

The author is grateful to Professor Yasunori Fujikoshi, Hiroshima University, for his encouragement and guidance. The author also would like to express his deep gratitude to Professor Sanpei Kageyama, Hiroshima University, for his critical reading and valuable suggestions. Further thanks are due to Professor Hiroshi Kimura, Ehime University, for his helpful discussions while working on the subject. In addition, the author extends his warmest thanks to Professor Jennifer Seberry, University College, Australia, for her recommending studies on the subject.

References

[1] M. C. Chakrabarti, Mathematics of Design and Analysis of Experiments, Asia Publishing House, Bombay, 1962.

[2] H. C. Chan, C. A. Rodger and J. Seberry, On inequivalent weighing matrices, ARS Combin., A **21** (1986), 299–333.

[3] W. T. Federer, Experimental Design: Theory and Application, Macmillan, New York, 1955.

[4] A. V. Geramita and J. Seberry, Orthogonal Designs: Quadratic Forms and Hadamard Matrices, Marcel Dekker, New York, 1979.

[5] M. Hall, Hadamard matrices of order 16, J.P.L. Research Summary 32–10, **1** (1961), 21–26.

[6] M. Hall, Hadamard matrices of order 20, J.P.L. Technical Report 32–761, Pasadena, 1965.

[7] M. Jr. Hall, Combinatorial Theory, 2nd. ed., John Wiley, New York, 1986.

[8] Q. M. Hussain, On the totality of the solutions for the symmetrical incomplete block designs $\lambda = 2$, $k = 5$ or 6, Sankhyā, **7** (1945), 204–208.

[9] N. Ito, J. S. Leon, J. Q. Longyear, Classification of 3-(24, 12, 5) designs and 24-dimensional Hadamard matrices, J. Combin. Theory, A **31** (1981), 66–93.

[10] H. Kimura, Hadamard matrices of order 28 with automorphism groups of order two, J. Combin. Theory, A **43** (1986), 98–102.

[11] H. Kimura, New Hadamard matrix of order 24, Graphs Combin., **5** (1989), 235–242.

[12] H. Kimura, Classification of Hadamard matrices of order 28 with Hall sets and new matrices. Preprint.

[13] H. Kimura and H. Ohmori, A characterization of some partial geometric spaces, Osaka J. Math., **22** (1985), 401–410.

[14] H. Kimura and H. Ohmori, Construction of Hadamard matrices of order 28, Graphs Combin., **2** (1986), 247–257.

[15] H. Kimura and H. Ohmori, Hadamard matrices of order 28, Mem. Fac. Educ. Ehime Univ., Nat. Sci., **7** (1987), 7–57.

[16] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error-Correcting Codes, North-Holland, Amsterdam, 1977.

[17] H. Ohmori, On the classifications of weighing matrices of order 12, J. Combin. Math. Combin. Comput., **5** (1989), 161–216.

[18] H. Ohmori, Classification of weighing matrices of order 13 and weight 9, (to appear in Ann. Discrete Math.).

[19] V. Pless, Symmetry codes over GF(3) and new 5-designs, J. Combin. Theory, **12** (1972), 119–142.

[20] V. Pless, Symmetry codes and their invariant subcodes, J. Combin. Theory, A **18** (1975) 116–125.

[21] D. Raghavarao, Some aspects of weighing designs, Ann. Math. Stat. **31** (1960), 878–884.

[22] D. Raghavarao, Constructions and Combinatorial Problems in Design of Experiments, John Wiley, New York, 1971.

[23] J. Seberry and K. Wehrhahan, A class of codes generated by circulant weighing matrices, Lecture Notes in Mathematics, **686** (1978), 282–289, Springer, Berlin.

[24] O. Taussky, (1, 2, 4, 8)-sums of squares and Hadamard matrices, Proc. Symp. in Pure Mathematics, Amer. Math. Soc., Providence, R.I., 1971, 229–233.

[25] V. D. Tonchev, Hadamard matrices of order 28 with automorphisms of order 13, J. Combin. Theory, A **35** (1983), 43–57.

[26] V. D. Tonchev, Hadamard matrices of order 28 with automorphisms of order 7, J. Combin. Theory, A **40** (1985), 62–81.

[27] W. D. Wallis, Combinatorial Designs, Marcel Dekker, New York, 1988.

Department of Mathematics
Faculty of Education
Ehime University
Matsuyama, 790 Japan