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Estimates of the euclidean span for an open Riemann surface of genus one

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1. Introduction

Every open Riemann surface of finite genus can be embedded conformally into a compact Riemann surface of the same genus. On the basis of [5], M. Shiba and K. Shibata gave in [7] a new proof of this classical theorem, and introduced the notion of hydrodynamic continuation. Their proof was of global character.

In [6] M. Shiba studied the set of compact continuations of an open Riemann surface of genus one in detail. He proved, among others, that the moduli set of compact continuations of a fixed marked open Riemann surface R of genus one is precisely a closed disk (or a point) in the upper half plane and that there is a bijection between the boundary of the closed disk and the set of hydrodynamic continuations of R. These results considerably improved Heins' result [2, Theorem 2]. The euclidean (resp. noneuclidean) diameter of the closed disk is called the euclidean (resp. hyperbolic) span for R (cf. Shiba-Shibata [8]). These spans represent the size of the ideal boundary of R. For example, the hyperbolic (or euclidean) span vanishes if and only if $R \in O_{AD}$ (see [6, Theorem 6]).

It seems that only few quantitative results about the moduli set are known. Shiba-Shibata [8] have calculated, using Jacobi's elliptic functions, the hyperbolic span explicitly for a strongly symmetric marked torus with a horizontal slit, and applied the formulae to estimate the hyperbolic span for an arbitrary marked torus with a horizontal slit. The results are rather complicated, however.

In this paper we consider an open Riemann surface (of genus one) of the form $R = \tilde{R}/G$, where G is a group generated by two translations of $\mathbb{C}$ and $\tilde{R}$ is a G-invariant domain of $\mathbb{C}$. By applying the length-area method we will give simple estimates of the euclidean span for R.

In the next section, after summarizing Shiba's results [6], we will state our main results. One of the hydrodynamic continuations of R has the smallest (normalized) area among the compact continuations of R. In § 3 we will characterize the area in terms of the moduli of ring domains on R and the modulus of a curve family. Our main results will be proved in §4. In the final section we treat the case where R is a torus with horizontal slits.

2. Main results

Let T be a torus (i.e., a compact Riemann surface of genus one). An ordered pair $\{a_T, b_T\}$ of generators for the fundamental group of T is called a marking of T if the intersection number of a_T and b_T is 1. The modulus of T with respect to the marking $\{a_T, b_T\}$, or the modulus of the marked torus $(T, \{a_T, b_T\})$, is the period

$$\tau = \int_{b_T} \varphi^T$$

of φ^T along b_T, where φ^T is the holomorphic differential on T with period 1 along a_T. It is well known that $\operatorname{Im} \tau > 0$.

Now, we summarize some results of Shiba [6]. Let R be an open Riemann surface of genus one, and fix a canonical homology basis $\{a, b\}$ of R modulo dividing cycles. The system $\{a, b\}$ may be regarded as a set of generators for the fundamental group of the Kerékjártó-Stoïlow compactification of R (cf. Richards [3]). The pair $(R, \{a, b\})$ is called a marked open Riemann surface. If there is a conformal embedding j of R into a torus T with marking $\{a_T, b_T\}$ such that $j(a)$ and $j(b)$ are freely homotopic to a_T and b_T in T respectively, then the triple $(T, \{a_T, b_T\}, j)$ is said to be a marked realization of $(R, \{a, b\})$. Two marked realizations $(T, \{a_T, b_T\}, j)$ and $(T', \{a'_{T'}, b'_{T'}\}, j')$ are defined to be equivalent if there exists a conformal mapping f of T onto T' such that $f \circ j = j'$. An equivalence class $[T, \{a_T, b_T\}, j]$ is called a compact continuation of $(R, \{a, b\})$.

By the modulus of a marked realization $(T, \{a_T, b_T\}, j)$ we mean the modulus of T with respect to $\{a_T, b_T\}$. Then equivalent marked realizations have the same modulus so that we can speak of the modulus of a compact continuation. Let $M(R, \{a, b\})$ denote the set of the moduli of all compact continuations of $(R, \{a, b\})$. Then, by Shiba [6, Theorem 5], $M(R, \{a, b\})$ is a closed disk (or a point), say $\{\tau \in \mathbb{C} \mid |\tau - \tau^*| \leq r\}$, in the upper half plane. The euclidean (resp. noneuclidean) diameter of $M(R, \{a, b\})$ is called the euclidean (resp. hyperbolic) span for the marked open Riemann surface $(R, \{a, b\})$ (cf. Shiba-Shibata [8]).

For each $t \in (-1, 1]$ there is a unique holomorphic differential φ_t on R with $\int_a \varphi_t = 1$ such that $\operatorname{Im}[e^{-i\pi t/2} \varphi_t]$ is a distinguished harmonic differential of Ahlfors (cf. [1, V.21D]). Furthermore, there is a marked realization

$(T_t, \{a_t, b_t\}, j_t)$ of $(R, \{a, b\})$ such that the transplant of φ_t via j_t^{-1} extends to a holomorphic differential φ^{T_t} on T_t with period 1 along a_t and that $T_t \backslash j_t(R)$ is of zero area. We can choose a_t and b_t to be geodesics with respect to the metric $|\varphi^{T_t}|$. Then each component of $T_t \backslash j_t(R)$ is a point or a geodesic arc of inclination $\pi t/2$ with a_t. The compact continuation $[T_t, \{a_t, b_t\}, j_t]$ is called the hydrodynamic continuation of $(R, \{a, b\})$ with respect to φ_t. Its modulus is $\tau_t = \tau^* + r \exp[i\pi(t - 1/2)] \in \partial M(R, \{a, b\})$, and it is the unique compact continuation of $(R, \{a, b\})$ whose modulus is τ_t ([6, Theorems 3 and 4]).

When R is a strongly symmetric torus with a horizontal slit, Shiba-Shibata [8] have explicitly calculated the hyperbolic span using Jacobi's elliptic functions. Also, they have obtained an estimate of the hyperbolic span for an arbitrary marked torus with a horizontal slit. The purpose of this paper is to estimate the euclidean span for the marked open Riemann surface $(R, \{a, b\})$ given in the following way. Let G be the group generated by the translations $\alpha(z) = z + 1$ and $\beta(z) = z + \tau'$ of the complex plane $\mathbb{C}$, where $\operatorname{Im} \tau' > 0$. Let $\tilde{R}(\neq \emptyset)$ be a G-invariant *proper* subdomain of $\mathbb{C}$. Then the orbit space $R = \tilde{R}/G$ is an open Riemann surface of genus one. We choose a canonical homology basis $\{a, b\}$ of R modulo dividing cycles so that a (resp. b) is covered by a Jordan arc in $\tilde{R}$ from a point z_0 to $\alpha(z_0)$ (resp. $\beta(z_0)$). Let $T' = \mathbb{C}/G$. The inclusion map of $\tilde{R}$ into $\mathbb{C}$ induces an embedding $j'\colon R \to T'$. Setting $a' = j'(a)$ and $b' = j'(b)$, we obtain a marked realization $(T', \{a', b'\}, j')$ of $(R, \{a, b\})$. In particular, $\tau' \in M(R, \{a, b\})$. As before, we denote by φ_t the holomorphic differential on R with $\int_a \varphi_t = 1$ such that $\operatorname{Im}[e^{-i\pi t/2}\varphi_t]$ is distinguished. Also, we assume that $M(R, \{a, b\}) = \{\tau \in \mathbb{C} \mid |\tau - \tau^*| \leq r\}$ and set $\tau_t = \int_b \varphi_t = \tau^* + r \exp[i\pi(t - 1/2)]$.

We denote one and two dimensional Lebesgue measures by μ_1 and μ_2, respectively. They naturally induce measures on T', which will be denoted by the same letters. Also, we denote by I the orthogonal projection onto the imaginary axis: $I(z) = i \operatorname{Im} z$. The closed segment joining z_1 to z_2 is denoted by $[z_1, z_2]$.

THEOREM 1. *Let $l_1 = \sup \mu_1(I(E))$, where the supremum is taken over all components E of $\mathbb{C} \backslash \tilde{R}$. Set $L = \max\{l_1 - \operatorname{Im}\tau', 0\}$. Then*

$$\operatorname{Im} \tau_0 \leq \frac{\mu_2(R)}{L^2 + 1},$$

and the euclidean span σ for $(R, \{a, b\})$ satisfies

$$\sigma \geq \operatorname{Im} \tau' - \frac{\mu_2(R)}{L^2 + 1}.$$

Theorem 1 makes sense when $\mathbb{C}\setminus\tilde{R}$ is large, while the following theorem does when $\mathbb{C}\setminus\tilde{R}$ is small.

THEOREM 2. *Let* $l_2=\mu_1(I(\mathbb{C}\setminus\tilde{R})\cap[0, I(\tau')])$. *Then*

$$\operatorname{Im}\tau_0 \geq \operatorname{Im}\tau' - l_2.$$

In particular, if $[T', \{a', b'\}, j']$ *is the hydrodynamic continuation of* $(R, \{a, b\})$ *with respect to* φ_t, $t\in(-1, 1]$, *then the euclidean span* σ *for* $(R, \{a, b\})$ *satisfies*

$$\sigma \leq \frac{l_2}{\sin^2(\pi t/2)}.$$

In the next section we give a proposition which is useful in estimating $\operatorname{Im}\tau_0$. The above theorems will be proved in §4.

3. Extremal properties of φ_0

A linear density (or a metric) $\rho=\rho(z)|dz|$ on a Riemann surface R is an assignment of a Borel measurable nonnegative function $\rho(z)$ to each local coordinate z on R such that $\rho(z)|dz|$ is invariant under coordinate changes. For a linear density $\rho=\rho(z)|dz|$ we set

$$\|\rho\|=\left(\iint_R \rho(z)^2\,dx\,dy\right)^{1/2}.$$

If $\varphi=\varphi(z)dz$ is a holomorphic differential on R, then $\rho=|\varphi|=|\varphi(z)||dz|$ is a linear density on R. We define $\|\varphi\|=\|\rho\|$. Note that $2\|\varphi\|^2=\iint_R \varphi\wedge *\bar{\varphi}$.

Let D be a ring domain (i.e., a doubly connected domain) of R. It can be mapped conformally onto an annulus $\{z\in\mathbb{C}\,|\,r_1<|z|<r_2\}$, $0\leq r_1<r_2\leq+\infty$. The modulus of D is defined to be $(1/2\pi)\log(r_2/r_1)$, and denoted by $\operatorname{mod} D$. Let c be a closed Jordan curve on R. If a closed Jordan curve on D which is homotopically nontrivial in D is freely homotopic to c on R, then D is said to be of homotopy type c. More generally, let $\{c_n\}_n$ be a sequence of finitely or countably many closed Jordan curves on R. A sequence $\{D_n\}_n$ of non-overlapping ring domains of R is said to be of homotopy type $\{c_n\}_n$ if each D_n is of homotopy type c_n.

Now, let R be an open Riemann surface of genus one, and a a closed Jordan curve on R which is homotopically nontrivial. We denote by $\mathscr{C}(R, a)$ the family of all rectifiable closed Jordan curves on R that are homologous to a on R modulo dividing cycles. The modulus $\operatorname{mod}\mathscr{C}(R, a)$ of the family $\mathscr{C}(R, a)$ is defined by

$$\operatorname{mod} \mathscr{C}(R, a) = \inf_{\rho} \|\rho\|^2,$$

where the infimum is taken over all linear densities ρ on R such that $\int_c \rho|dz| \geq 1$ for all $c \in \mathscr{C}(R, a)$.

Next, let $\mathfrak{C}(R, a)$ be the set of all sequences of finitely or countably many curves in $\mathscr{C}(R, a)$. We denote by $\mathfrak{D}(R, a)$ the class of all sequences $\{D_n\}_n$ of non-overlapping ring domains of R such that $\{D_n\}_n$ is of homotopy type $\{c_n\}_n$ for some $\{c_n\}_n \in \mathfrak{C}(R, a)$.

PROPOSITION 1. *Let* $(R, \{a, b\})$ *be a marked open Riemann surface of genus one, and* φ_0 *the holomorphic differential on* R *with* $\int_a \varphi_0 = 1$ *such that* $\operatorname{Im} \varphi_0$ *is a distinguished harmonic differential of Ahlfors. Then*

$$\|\varphi_0\|^2 = \operatorname{mod} \mathscr{C}(R, a) = \sup \sum_n \operatorname{mod} D_n, \tag{1}$$

where the supremum is taken over all $\{D_n\}_n \in \mathfrak{D}(R, a)$.

PROOF. First, we have $\int_c |\varphi_0| \geq 1$ for all $c \in \mathscr{C}(R, a)$ since $\int_c \varphi_0 = \int_a \varphi_0 = 1$. Thus, by the definition of the modulus of the curve family $\mathscr{C}(R, a)$,

$$\|\varphi_0\|^2 \geq \operatorname{mod} \mathscr{C}(R, a). \tag{2}$$

Next, let ρ be a linear density on R such that $\int_c \rho \geq 1$ for all $c \in \mathscr{C}(R, a)$. Since the modulus of a ring domain D is identical with the modulus of the family of closed Jordan curves in D which are homotopically nontrivial in D, we have

$$\|\rho\|^2 \geq \sum_n \iint_{D_n} |\rho|^2 \, dx \, dy \geq \sum_n \operatorname{mod} D_n$$

for all $\{D_n\}_n \in \mathfrak{D}(R, a)$. Therefore, denoting by s the supremum in (1), we obtain

$$\operatorname{mod} \mathscr{C}(R, a) \geq s. \tag{3}$$

Finally, in order to show that $s \geq \|\varphi_0\|^2$, we choose a regular exhaustion $\{R_\nu\}$ of R such that the cycles a and b are contained in R_1. For each ν there is a unique holomorphic differential $\varphi_0^{(\nu)}$ on R_ν with $\int_a \varphi_0^{(\nu)} = 1$ such that

Im $\varphi_0^{(\nu)}$ is a distinguished harmonic differential on R_ν. Let $[T_\nu, \{a_\nu, b_\nu\}, j_\nu]$ be the hydrodynamic continuation of $(R_\nu, \{a, b\})$ with respect to $\varphi_0^{(\nu)}$. Let G_ν be the group generated by the translations $\alpha(z) = z + 1$ and $\beta_\nu(z) = z + \tau_\nu$, where τ_ν is the modulus of the continuation $[T_\nu, \{a_\nu, b_\nu\}, j_\nu]$. We can identify T_ν with $\mathbb{C}/G_\nu$, and assume that a_ν and b_ν are covered by the segments $[0, 1]$ and $[0, \tau_\nu]$, respectively. Let $\pi_\nu: \mathbb{C} \to T_\nu$ be the natural projection, and set $\tilde{R}_\nu = \pi_\nu^{-1}(j_\nu(R_\nu))$. Then $\tilde{R}_\nu$ is a G_ν-invariant horizontal slit domain, and $F_\nu = I(\mathbb{C} \setminus \tilde{R}_\nu)$ is a discrete set of the imaginary axis $i\mathbb{R}$, where $I(z) = i \operatorname{Im} z$. If J is a component of $(i\mathbb{R}) \setminus F_\nu$, then $D = j_\nu^{-1}(\pi_\nu(I^{-1}(J)))$ is a ring domain of $R_\nu (\subset R)$, whose modulus is equal to

$$\mu_1(J) = \iint_{[0,1] \times J} dx\, dy = \iint_D |\varphi_0^{(\nu)}|^2\, dx\, dy.$$

Now, let $D_m^{(\nu)}$, $m = 1, \dots, m_\nu$, be the ring domains on R_ν obtained as above from the components of $(i\mathbb{R}) \setminus F_\nu$. Then $\{D_m^{(\nu)}\}_m \in \mathfrak{D}(R_\nu, a) \subset \mathfrak{D}(R, a)$. Since $\bigcup_m D_m^{(\nu)}$ differs from R_ν by a set of zero area, we obtain

$$s \geq \sum_m \operatorname{mod} D_m^{(\nu)} = \sum_m \iint_{D_m^{(\nu)}} |\varphi_0^{(\nu)}|^2\, dx\, dy = \| \varphi_0^{(\nu)} \|^2.$$

Since $\| \varphi_0^{(\nu)} \|^2$ tends to $\| \varphi_0 \|^2$ as $\nu \to \infty$, we have

$$s \geq \| \varphi_0 \|^2. \tag{4}$$

Combining (2), (3) and (4), we obtain the desired result (1).

REMARKS. (i) We could apply [9, Theorem 20.3] to prove inequality (3).

(ii) Since the differential $\psi = \operatorname{Re} \varphi_0 / \| \varphi_0 \|^2$ is the Γ_{hse}-reproducer for a, it follows from Rodin's theorem [4] that

$$\frac{1}{\operatorname{mod} \mathscr{C}(R, a)} = \iint_R \psi \wedge * \bar{\psi} = \frac{1}{\| \varphi_0 \|^2}.$$

We have thus obtained a half of Proposition 1.

4. Proof of Theorems 1 and 2

We can now easily prove Theorems 1 and 2.

PROOF OF THEOREM 1. Let $\tilde{c}$ be a curve on $\tilde{R}$ whose projection belongs to $\mathscr{C}(R, a)$. Then $\tilde{c}$ is a rectifiable Jordan arc joining a point $z_0 \in \tilde{R}$ to $\alpha(z_0) = z_0 + 1$ in $\tilde{R}$. We claim that

$$\int_{\tilde{c}} |dz| \geq \sqrt{L^2 + 1}.$$

This inequality clearly holds when $L=0$. Thus we have only to consider the case $L>0$. For each $\varepsilon\in(0, L)$, there is a component E of $\mathbb{C}\setminus\tilde{R}$ such that $\mu_1(I(E))>l_1-\varepsilon$. Since $l_1-\varepsilon>\operatorname{Im}\tau'$, we can find $g\in G$ such that (i) $g(E)\cap[z_0, \alpha(z_0)]\neq\emptyset$, (ii) $\operatorname{Im}\zeta^+\geq\operatorname{Im}z_0+(L-\varepsilon)/2$ for some $\zeta^+\in E$, and (iii) $\operatorname{Im}\zeta^-\leq\operatorname{Im}z_0-(L-\varepsilon)/2$ for some $\zeta^-\in E$. Since $\tilde{c}\cap g(E)=\emptyset$, we see that

$$\begin{aligned}\int_{\tilde{c}}|dz| &\geq \min\{|z_0-\zeta^+|+|\alpha(z_0)-\zeta^+|, |z_0-\zeta^-|+|\alpha(z_0)-\zeta^-|\}\\ &\geq\sqrt{(L-\varepsilon)^2+1}.\end{aligned}$$

Letting $\varepsilon\downarrow 0$, we obtain our claim.

The linear density $|dz|/\sqrt{L^2+1}$ on $\tilde{R}$ induces a linear density ρ on R. It then follows from the above claim that $\int_c\rho\geq 1$ for all $c\in\mathscr{C}(R, a)$. Consequently, by Proposition 1,

$$\operatorname{Im}\tau_0=\|\varphi_0\|^2=\operatorname{mod}\mathscr{C}(R, a)\leq\|\rho\|^2=\frac{\mu_2(R)}{L^2+1}.$$

Since $\tau'\in M(R, \{a, b\})$, we have

$$\sigma\geq|\tau'-\tau_0|\geq\operatorname{Im}\tau'-\operatorname{Im}\tau_0\geq\operatorname{Im}\tau'-\frac{\mu_0(R)}{L^2+1}.$$

This completes the proof.

Proof of Theorem 2. The set $F=I(\mathbb{C}\setminus\tilde{R})$ is a closed subset of the imaginary axis $i\mathbb{R}$. Let $\pi:\tilde{R}\to R$ be the natural projection. If J is a component of $(i\mathbb{R})\setminus F$, then $D=\pi(I^{-1}(J))$ is a ring domain of R with $\operatorname{mod}D=\mu_1(J)$. Let $\{D_m\}_m$ be the collection of the ring domains of R so obtained from the components of $(i\mathbb{R})\setminus F$. Then $\{D_m\}_m\in\mathfrak{D}(R, a)$. Therefore, by Proposition 1, we have

$$\operatorname{Im}\tau_0=\|\varphi_0\|^2\geq\sum_m\operatorname{mod}D_m=\operatorname{Im}\tau'-l_2.$$

Next, assume that $[T', \{a', b'\}, j']$ is the hydrodynamic continuation of $(R, \{a, b\})$ with respect to φ_t. Then $\tau'=\tau_t=\tau^*+r\exp[i\pi(t-1/2)]$. Hence

$$l_2\geq\operatorname{Im}\tau'-\operatorname{Im}\tau_0=r\left[1+\sin\pi\left(t-\frac{1}{2}\right)\right]=2r\sin^2\frac{\pi t}{2},$$

which implies that

$$\sigma = 2r \leq \frac{l_2}{\sin^2(\pi t/2)}.$$

We have proved the theorem.

5. Tori with horizontal slits

In this section we consider the case where $[T', \{a', b'\}, j']$ is the hydrodynamic continuation of $(R, \{a, b\})$ with respect to φ_0. Then $\tau' = \tau_0$. In this case Theorem 1 or 2 gives us no information about the euclidean span. However, by changing homology bases, we can prove the following

THEOREM 3. *Set $\theta' = \arg \tau'$, $0 < \theta' < \pi$, and $l_3 = \mu_1(I'(\mathbb{C} \setminus \tilde{R}) \cap [0, I'(1)])$, where I' is the orthogonal projection of $\mathbb{C}$ onto the line which passes through the origin and is orthogonal to the segment $[0, \tau']$. If $[T', \{a', b'\}, j']$ is the hydrodynamic continuation of $(R, \{a, b\})$ with respect to φ_0, then the euclidean span σ for $(R, \{a, b\})$ satisfies*

$$\sigma \leq \frac{l_3}{\sin\theta' - l_3} \cdot \frac{|\tau'|}{\sin\theta'}. \tag{5}$$

PROOF. Let

$$t = \begin{cases} \dfrac{2\theta'}{\pi} & \text{if } 0 < \theta' \leq \dfrac{\pi}{2}, \\[2ex] \dfrac{2\theta'}{\pi} - 2 & \text{if } \dfrac{\pi}{2} < \theta' < \pi, \end{cases}$$

and consider the hydrodynamic continuation $[T_t, \{a_t, b_t\}, j_t]$ of $(R, \{a, b\})$ with respect to φ_t. Note that the origin 0, τ' and τ_t are collinear. Thus, denoting by ψ_0 the holomorphic differential on R with $\int_b \psi_0 = 1$ such that $\operatorname{Im} \psi_0$ is distinguished, we have $\varphi_t = \tau_t \psi_0$ so that

$$\int_{-a} \psi_0 = -\frac{1}{\tau_t}. \tag{6}$$

Moreover, Theorem 2 implies that

$$\operatorname{Im} \int_{-a} \psi_0 \geq \frac{\sin\theta' - l_3}{|\tau'|}. \tag{7}$$

Since $\arg \tau_t = \arg \tau' = \theta'$,

$$\operatorname{Im}\left(-\frac{1}{\tau_t}\right) = \frac{\sin\theta'}{|\tau_t|}. \tag{8}$$

By (6), (7) and (8) we obtain

$$|\tau_t| \le \frac{\sin\theta'}{\sin\theta' - l_3}|\tau'|,$$

and hence

$$|\tau_t - \tau'| = |\tau_t| - |\tau'| \le \left(\frac{\sin\theta'}{\sin\theta' - l_3} - 1\right)|\tau'| = \frac{l_3}{\sin\theta' - l_3}|\tau'|.$$

Since $|\tau_t - \tau'| = \sigma\sin\theta'$, we conclude that

$$\sigma \le \frac{l_3}{\sin\theta' - l_3}\cdot\frac{|\tau'|}{\sin\theta'}.$$

The proof is complete.

In particular, assume that $T'\backslash j'(R)$ is connected; let l be the length of the horizontal slit. Then, since $l = l_3/\sin\theta'$, inequality (5) can be rewritten as

$$\sigma \le \frac{l}{1-l}\cdot\frac{\operatorname{Im}\tau'}{\sin^2\theta'}.$$

Now, choose $k \in \mathbb{Z}$ such that $|\operatorname{Re}\tau' + k| \le 1/2$, and set $\theta = \arg(\tau' + k)$. Observe that $[T'\{a', ka' + b'\}, j']$ is the hydrodynamic continuation of $(R, \{a, ka + b\})$ with respect to φ_0, whose modulus is $\tau' + k$. Since the euclidean span for $(R, \{a, ka + b\})$ is the same as that for $(R, \{a, b\})$, we finally obtain

$$\sigma \le \frac{l}{1-l}\cdot\frac{\operatorname{Im}\tau'}{\sin^2\theta}.$$

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