HIROSHIMA MATH. J.
22 (1992), 461–467

Hypergeometric series over finite fields and Apéry numbers

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(Received May 20, 1991)

0. Introduction

Beukers proved some congruences for the numbers $b_n = \sum \binom{n}{k}^2 \binom{n+k}{n}$ which were introduced by Apéry. We shall give another proof of these congruences by using Greene's result on hypergeometric series over finite fields.

1. Some facts on hypergeometric series over finite fields

Let p be a prime, ≥ 3 and let $\boldsymbol{F}_p$ denote the finite field with p elements. Throughout this paper, capital letters A, B, C and χ, ψ, ρ will denote multiplicative characters of $\boldsymbol{F}_p$. Given any multiplicative character A of $\boldsymbol{F}_p$, we extend A to all of $\boldsymbol{F}_p$ by defining $A(0) = 0$. ε and ϕ denote the trivial multiplicative character and the character of order 2 respectively.

Finite analogue of binomial coefficients are defined as follows*):

$$\binom{A}{B}^* = \frac{B(-1)}{p-1} J(A, \bar{B})$$

where $J(A, \bar{B})$ denotes the *Jacobi sum.*

Then we have

$$A(1 + x) = \delta(x) + \sum_B \binom{A}{B}^* B(x),$$

where $\delta(x) = 0$ if $x \neq 0$ and $\delta(x) = 1$ if $x = 0$.

We may consider that these values are all in the p-adic number field $\boldsymbol{Q}_p$. We identify $\boldsymbol{F}_p$ with $\boldsymbol{Z}_p/p\boldsymbol{Z}_p$. Then we denote by ω the *Teichmuller character* of $\boldsymbol{F}_p$, i.e.

$$\omega(x) \bmod p = x \qquad \text{for all } x \in \boldsymbol{F}_p.$$

*) In [3], Greene defined the finite analogue $\binom{A}{B}$ of binomial coefficients by $\binom{A}{B} = \frac{p-1}{p}\binom{A}{B}^*$. In considering the reduction modulo p of special values of hypergeometric series over finite fields, our definition seems to be more convenient. It is clear that our $\binom{A}{B}^*$ also satisfy formulae (2.6), (2.7) and (2.8) in [3], and other formulae can be easily modified.

Then the set of all multiplicative characters of $\boldsymbol{F}_p$ is

$$\{\omega^k;\, 0 \le k \le p-2\}.$$

Lemma 1. *We have*

$$\binom{\omega^k}{\omega^i}^* \equiv \begin{cases} \binom{k}{i} & (\mathrm{mod}\ p) \quad \textit{for}\ 0 \le i \le k, \\ 0 & (\mathrm{mod}\ p) \quad \textit{for}\ k < i \le p-2. \end{cases}$$

Proof. The proof is obtained by comparing expansions of both sides of congruences: $\omega^k(1+x)\ (\mathrm{mod}\ p) = (1+x)^k$.

The hypergeometric series over the finite field $\boldsymbol{F}_p$ is defined by

$$_{n+1}F_n^*\left(\begin{matrix} A_0, & A_1, \dots, A_n \\ & B_1, \dots, B_n \end{matrix} \,\Big|\, x\right) = \sum_\chi \binom{A_0\chi}{\chi}^* \binom{A_1\chi}{B_1\chi}^* \cdots \binom{A_n\chi}{B_n\chi}^* \chi(x).$$

Our definition is different from Greene's $_{n+1}F_n$: the relation is $_{n+1}F_n = \left(\frac{p-1}{p}\right)^n {}_{n+1}F_n^*$.

In [3], Greene evaluated certain special values of these hypergeometric series:

(1) $$_2F_1^*\left(\begin{matrix} A, & B \\ & C \end{matrix} \,\Big|\, 1\right) = A(-1)\binom{B}{\bar{A}C}^*.$$

(2) $$_2F_1^*\left(\begin{matrix} A, & B \\ & \bar{A}B \end{matrix} \,\Big|\, -1\right) = \begin{cases} 0 & \textit{if } B \textit{ is not square}, \\ \binom{C}{A}^* + \binom{\phi C}{A}^* & \textit{if } B = C^2. \end{cases}$$

(3) If we assume that A, B and $AB\bar{C}$ are not trivial, then we have

$$_3F_2^*\left(\begin{matrix} A, & B, & C \\ & \bar{A}C, & \bar{B}C \end{matrix} \,\Big|\, 1\right)$$

$$= AB(-1) \begin{cases} 0 & \textit{if } C \textit{ is not square}, \\ \binom{D}{A}^* \binom{B\bar{D}}{AB\bar{D}}^* + \binom{\phi D}{A}^* \binom{\phi B\bar{D}}{\phi AB\bar{D}}^* & \textit{if } C = D^2. \end{cases}$$

In [4], Greene and Stanton evaluated $_3F_2\left(\begin{matrix} \phi, & \phi, & \phi \\ & \varepsilon, & \varepsilon \end{matrix} \,\Big|\, -1\right)$:

(4) $$_3F_2^*\left(\begin{matrix} \phi, & \phi, & \phi \\ & \varepsilon, & \varepsilon \end{matrix} \,\Big|\, -1\right) = \frac{1}{(p-1)^2} \begin{cases} -p\phi(2) & \textit{if } p \equiv 5, 7\ (\mathrm{mod}\ 8), \\ \phi(2)(4c^2 - p) & \textit{if } p \equiv 1, 3\ (\mathrm{mod}\ 8), \end{cases}$$

where $p = c^2 + 2d^2$, for $p \equiv 1, 3\ (\mathrm{mod}\ 8)$.

2. Congruences of Apéry numbers

For any pair (m, ℓ) of non negative integer m and ℓ, we define *the Apéry number of type* (m, ℓ) by

$$a_n^{(m,\ell)} = \sum_{k=0}^{n} \binom{n+k}{k}^m \binom{n}{k}^\ell.$$

Put $w = m + \ell$, which is called the *weight* of the Apéry number.

PROPOSITION 1. *Let $a_n^{(m,\ell)}$ be the Apéry number of weight w. For any prime $p \geq 3$, put $p = 2f + 1$. Then we have*

$$a_f^{(m,\ell)} \equiv {}_wF_{w-1}^*\left(\begin{matrix}\phi, \phi, \ldots, \phi \\ \varepsilon, \ldots, \varepsilon\end{matrix} \middle| (-1)^\ell\right) \pmod{p}.$$

PROOF.

$$a_f^{(m,\ell)} = \sum_{k=0}^{f} \binom{f+k}{k}^m \binom{f}{k}^\ell \equiv \sum_{k=0}^{f} \binom{\phi\omega^k}{\omega^k}^{*m} \binom{\phi}{\omega^k}^{*\ell} \pmod{p},$$

$$\equiv \sum_{\chi} \binom{\phi\chi}{\chi}^{*m} \binom{\phi}{\chi}^{*\ell} \pmod{p},$$

by Lemma 1. Since $\binom{\phi\chi}{\chi}^* = \chi(-1)\binom{\phi}{\chi}^*$, we have

$$a_f^{(m,\ell)} \equiv \sum_{\chi} \binom{\phi\chi}{\chi}^{*m} \binom{\phi\chi}{\chi}^{*\ell} \chi(-1)^\ell \pmod{p},$$

$$\equiv {}_wF_{w-1}^*\left(\begin{matrix}\phi, \phi, \ldots, \phi \\ \varepsilon, \ldots, \varepsilon\end{matrix} \middle| (-1)^\ell\right) \pmod{p}.$$

COROLLARY 1. *If $m + \ell = m' + \ell'$ and $\ell \equiv \ell'$* (mod 2), *then*

$$a_f^{(m,\ell)} \equiv a_f^{(m',\ell')} \pmod{p}.$$

Combining this result with (1), (2), (3) and (4) in the preceding section, we get

COROLLARY 2. *The notation being as above and if $p \equiv 1$* (mod 4) *we put $p = 4f' + 1$. Then we have*

(1) $$a_f^{(2,0)} \equiv \phi(-1) \pmod{p}.$$

(2) $$a_f^{(1,1)} \equiv \begin{cases} 0 & \pmod{p} \text{ if } p \equiv 3 \pmod{4}, \\ \binom{3f'}{2f'} & \pmod{p} \text{ if } p \equiv 1 \pmod{4}. \end{cases}$$

$$a_f^{(1,2)} \equiv \begin{cases} 0 & (\mathrm{mod}\ p) \ \text{if} \ p \equiv 3 \,(\mathrm{mod}\ 4), \\ \binom{3f'}{2f'}^2 & (\mathrm{mod}\ p) \ \text{if} \ p \equiv 1 \,(\mathrm{mod}\ 4). \end{cases} \tag{3}$$

$$a_f^{(2,1)} \equiv \begin{cases} 0 & (\mathrm{mod}\ p) \ \text{if} \ p \equiv 5, 7 \,(\mathrm{mod}\ 8), \\ \phi(2)4c^2 & (\mathrm{mod}\ p) \ \text{if} \ p \equiv 1, 3 \,(\mathrm{mod}\ 8). \end{cases} \tag{4}$$

REMARK 1. The following result is easily proved; If the weight is odd and ℓ is even, then $a_f^{(m,\ell)} \equiv 0 \ (\mathrm{mod}\ p)$ if $p \equiv 3 \ (\mathrm{mod}\ 4)$ and if the weight is even and ℓ is odd, then $a_f^{(m,\ell)} \equiv 0 \ (\mathrm{mod}\ p)$ if $p \equiv 3 \ (\mathrm{mod}\ 4)$.

The following congruences of binomial coefficients are well known: for any prime p, $p \equiv 1 \ (\mathrm{mod}\ 4)$, $p = a^2 + b^2$ with $a \equiv 1 \ (\mathrm{mod}\ 4)$, then

$$\binom{2f'}{f'} \equiv 2a \qquad (\mathrm{mod}\ p).$$

Since $\binom{2f'}{f'} \equiv \binom{\phi}{\bar{\rho}}^* \equiv \phi\rho(-1)\binom{\rho}{\phi}^* \ (\mathrm{mod}\ p)$ with $\bar{\rho} = \omega^{f'}$, so we get

$$\binom{3f'}{2f'} \equiv (-1)^{f'} 2a \qquad (\mathrm{mod}\ p).$$

Combining the above result with Corollary 2, we obtained the congruences of Apéry numbers proved in [1] and [6]: for example

THEOREM 1. *The notation being as above, we have*

$$a_f^{(1,2)} \equiv \begin{cases} 0 & (\mathrm{mod}\ p) \ \text{if} \ p \equiv 3 \,(\mathrm{mod}\ 4), \\ 4a^2 & (\mathrm{mod}\ p) \ \text{if} \ p \equiv 1 \,(\mathrm{mod}\ 4). \end{cases}$$

Beukers proved this congruence by knowing the zeta functions of a singular K3-surface. Our argument is different from his and may be convenient to further generalizations.

3. Variant 1

In [6], Stienstra and Beukers gave the congruences for the following numbers c_n:

$$c_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

We shall show that these numbers also are related to special values of hypergeometric series.

LEMMA 2. *For any multiplicative character χ of $\boldsymbol{F}_p$, we have*

$$\binom{\chi^2}{\chi}^* = \binom{\phi\chi}{\chi}^* \chi(4) + \delta(\chi),$$

where $\delta(\chi) = 0$ *if* $\chi \neq \varepsilon$ *and* $\delta(\chi) = 1$ *if* $\chi = \varepsilon$.

PROOF. This is proved by using the formula (2.16) in [3].

PROPOSITION 2. *The notation being as above, we have*

$$c_f \equiv {}_3F_2^*\left(\begin{matrix}\phi, \phi, \phi \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, 4\right) + 1 \qquad (\mathrm{mod}\ p).$$

PROOF. We have

$$\begin{aligned} c_f &= \sum_{k=0}^{f} \binom{f}{k}^2 \binom{2k}{k} \equiv \sum_{k=0}^{f} \binom{\phi}{\omega^k}^{*2} \binom{\omega^{2k}}{\omega^k}^* && (\mathrm{mod}\ p) \\ &\equiv \sum_{\chi} \binom{\phi}{\chi}^{*2} \binom{\chi^2}{\chi}^* \equiv \sum_{\chi} \binom{\phi}{\chi}^{*2} \binom{\phi\chi}{\chi}^* \chi(4) + \binom{\phi}{\varepsilon}^{*2} && (\mathrm{mod}\ p) \\ &\equiv {}_3F_2^*\left(\begin{matrix}\phi, \phi, \phi \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, 4\right) + 1 \qquad (\mathrm{mod}\ p). \end{aligned}$$

In contrast to the preceding section, we get the following congruences by using Theorem (13.1) in [6].

COROLLARY.

$${}_3F_2^*\left(\begin{matrix}\phi, \phi, \phi \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, 4\right) \equiv \begin{cases} -1 & (\mathrm{mod}\ p) \text{ if } p \equiv 2 \,(\mathrm{mod}\ 3), \\ -1 + 4e^2 & (\mathrm{mod}\ p) \text{ if } p \equiv 1 \,(\mathrm{mod}\ 3), \end{cases}$$

where $p = e^2 + 3g^2$ *for* $p \equiv 1$ (mod 3).

The following problem arises from this result:

PROBLEM. *Can we evaluate* ${}_3F_2\left(\begin{matrix}\phi, \phi, \phi \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, 4\right)$?

4. Variant 2

The result in §2 shows why the Apéry number b_n at $n = (p-1)/2$ satisfy interesting congruences modulo p. It is because such b_n is connected to ${}_3F_2\left(\begin{matrix}\phi, \phi, \phi \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, 1\right)$. Hence it is natural to consider whether ${}_3F_2\left(\begin{matrix}\psi, \psi, \psi \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, 1\right)$ and ${}_3F_2\left(\begin{matrix}\rho, \rho, \rho \\ \varepsilon, \varepsilon\end{matrix} \,\middle|\, 1\right)$ with ψ and ρ characters of order 3 and 4 respectively are connected to the Apéry numbers too.

Let p be a prime, $p \equiv 1$ (mod 3). Put $p = 3f + 1 = 6f' + 1$ and put $\psi = \omega^{2f}$

and $\bar{\psi} = \omega^f$. Then we have

$$
{}_2F_1^*\left(\begin{matrix}\psi, \psi \\ \varepsilon\end{matrix} \mid 1\right) = \sum_\chi \binom{\psi\chi}{\chi}^{*2} \equiv \sum_{k=0}^{f} \binom{2f+k}{k}^2 \pmod p,
$$

$$
\equiv a_{2f}^{(2,0)} \pmod p.
$$

Since $\binom{\psi\chi}{\chi} = \chi(-1)\binom{\bar{\psi}}{\chi}$, we have

$$
a_f^{(0,2)} \equiv a_{2f}^{(2,0)} \pmod p.
$$

On the other hand, by (1) in § 1, we see that

$$
{}_2F_1^*\left(\begin{matrix}\psi, \psi \\ \varepsilon\end{matrix} \mid 1\right) = \psi(-1)\binom{\psi}{\bar{\psi}} \equiv \binom{2f}{f} \pmod p,
$$

and

$$
\binom{2f}{f} \equiv -e \pmod p
$$

where $p = e^2 + 3g^2$, $e \equiv 1 \pmod 3$.

Hence we have

THEOREM 2. *The notation being as above, we have*

$$
a_f^{(0,2)} \equiv a_{2f}^{(2,0)} \equiv -e \pmod p.
$$

To obtain the congruences for the Apéry number $a_{2f}^{(1,1)}$ we have to evaluate ${}_2F_1\left(\begin{matrix}\bar{\psi}, \psi \\ \varepsilon\end{matrix} \mid -1\right)$, but this is not yet done.

By the same argument, we get

THEOREM 3. *The notation being as above, we get*

$$
a_{2f}^{(3,0)} \equiv \binom{5f'}{2f'}\binom{5f'}{2f'} \pmod p,
$$

$$
\equiv 2eh \pmod p,
$$

where $4p = h^2 + 3i^2$ *such that if* 2 *is a cubic non residue* mod p *then* $h \equiv -1 \pmod 3$, *h is odd and* $i \not\equiv 0 \pmod 3$ *and if* 2 *is a cubic residue* mod p *then* $h \equiv -1 \pmod 3$ *and h is even.*

Let $p = 4f + 1$ and put $\rho = \omega^{3f}$ and $\bar{\rho} = \omega^f$. We consider ${}_3F_2^*\left(\begin{matrix}\rho, \rho, \rho \\ \varepsilon, \varepsilon\end{matrix} \mid 1\right)$. By (3) in § 1, this is equal to zero if $p \equiv 5 \pmod 8$. So we may assume that $p \equiv 1$

(mod 8) and $p = 8f' + 1$. Then we have

$$
\begin{aligned}
{}_3F_2^*\left(\begin{matrix}\rho, \rho, \rho \\ \varepsilon, \varepsilon\end{matrix} \Big| 1\right) &\equiv \binom{\phi\omega^{3f'}}{\rho}^* \binom{\phi\omega^{3f'}}{\phi\omega^{f'}}^* \qquad (\mathrm{mod}\ p), \\
&\equiv \binom{7f'}{6f'}\binom{7f'}{5f'} \qquad (\mathrm{mod}\ p).
\end{aligned}
$$

THEOREM 4. *The notation being as above, we have*

$$
a_{3f}^{(3,0)} \equiv \begin{cases} 0 & (\mathrm{mod}\ p)\ \textit{if}\ p \equiv 5\ (\mathrm{mod}\ 8), \\ \binom{7f'}{6f'}\binom{7f'}{5f'} & (\mathrm{mod}\ p)\ \textit{if}\ p \equiv 1\ (\mathrm{mod}\ 8). \end{cases}
$$

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