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# A linearization of $S(U(1) \times U(2)) \backslash SU(1, 2)$ $\sigma$-model

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## Introduction

The stationary axisymmetric Einstein-Maxwell equations have been closely investigated by many mathematicians and physicists since Weyl solved completely the static axially symmetric class of vacuum gravitational fields in 1917 [12]. In particular this subject experienced a drastic scene in the years 1978–1980 with the development of solution generating methods. Geroch has found that each given stationary axisymmetric solution of the Einstein field equations is accompanied by an infinite family of potentials [5]. This fact has led to the fact that there exists an action of some infinite dimensional group, so called Geroch group, on the space of (local) solutions. This symmetry is sometimes called the hidden symmetry. He also conjectured that this action is transitive up to gauge transformations. This conjecture, called the Geroch conjecture, was proved affirmatively by I. Hauser and F. J. Ernst [7]. They derived a non-linear differential equation for matrix-valued functions from the field equations and generalized their results to the case which has $N$ Abelian gauge potentials interacting with the gravitational field. In [3] H. Doi and K. Okamoto generalized the results of [7] to the case that the field equations take their values in an affine symmetric space, so that a "Kac-Moody" Lie group acts transitively on the space of solutions.

The purpose of this paper is to give a recipe for constructing solutions, following the method explored by P. Breitenlohner and D. Maison [1], who dealt with the Einstein equations there. But in this paper we treat the gravitational field interacting with electro-magnetic fields. The essential point of [1] was that the Ernst equations derived from the Einstein field equations [4] should be formulated as a $\sigma$-model which takes its value in an affine symmetric space or a Riemannian symmetric space. (The former is transformed into the latter by Kramer-Neugebauer transformation.) A linearization is, as in [1], carried out after introducing a 1-form taking its value in some Lie algebra with a spectral parameter.

In this paper we discuss in the category of formal power series. We treat only a $\sigma$-model with values in a Riemannian symmetric space which is derived from the Einstein-Maxwell field equations. In order to do that, we have to employ a different formulation of the Ernst equations from that in [3], where the fields take their value in an affine symmetric space.

The contents of this paper are as follows: In § 1, we introduce the stationary axisymmetric Einstein-Maxwell equations in matrix form and write down the Ernst equations. For derivation of the Ernst equations from the stationary axisymmetric Einstein-Maxwell equations, we refer to [3]. In § 2, we show that the Ernst equations are equivalent to a $\sigma$-model which takes its value in $S(U(1) \times U(2)) \backslash SU(1, 2)$ and construct a 1-form so that the integrability condition of this 1-form is equivalent to the equation of motion for that $\sigma$-model. In § 3 we provide a recipe for constructing solutions of the Ernst equation. In § 4 we follow the recipe described above and give some examples of solutions, the first one of which was provided by Nagatomo [10]. We have, however, not succeeded in constructing an action of the Geroch group $G^{(\infty)}$ on the space of the solutions, and restricted ourselves to letting it act only on the identity which correspends to the vacuum field.

We remark that all these results can be generalized immediately to the case of the interaction between $N$ Abelian gauge potentials and the gravitational field which corresponds to $S(U(1) \times U(N+1)) \backslash SU(1, N+1)$, although we deal only with the case $N = 1$.

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## 1. Ernst equations

In this section, following [1] we shall introduce the Ernst equations through the stationary axisymmetric Einstein-Maxwell field equations. Let $g$ be a metric on a Lorentzian manifold given by

$$g = \begin{pmatrix} h_{00} & h_{01} & & \\ h_{10} & h_{11} & & \\ & & -\lambda & 0 \\ & & 0 & -\lambda \end{pmatrix}$$

with $\lambda > 0$ and $h_{01} = h_{10}$. And let $A = \sum A_i dx^i$ be an abelian gauge potential. Here we adopt the coordinates $(x^0, x^1, x^2, x^3) = (t, \varphi, z, \rho)$ with $t$ being time and $(\varphi, z, \rho)$ the cylindrical coordinates of $\mathbf{R}^3$. Since we deal with the stationary axisymmetric fields, the functions $h = (h_{ij})$, $\lambda$ and $A_i$ depend only on $z$ and $\rho$. Moreover, we assume that $h_{00} \neq 0$, $\det h = -\rho^2$ and $A_2 = A_3 = 0$, which is physically reasonable.

Then the stationary axisymmetric Einstein-Maxwell field equations are given, in matrix form, as follows:

$$d(\rho h^{-1} * dA) = 0 \tag{1.1}$$

$$d\{\rho h^{-1} * dh - 2(\rho h^{-1} * dA) A^T - 2\varepsilon A(\rho h^{-1} * dA)^T \varepsilon\} = 0, \tag{1.2}$$

$$\frac{\partial_z \lambda}{\lambda} = \frac{\rho}{4} \operatorname{tr}(h^{-1}\partial_\rho h h^{-1}\partial_z h) - 2\rho\partial_\rho A^T h^{-1}\partial_z A, \tag{1.3A}$$

$$\begin{aligned}\frac{\partial_\rho \lambda}{\lambda} = &- \frac{1}{2\rho} + \frac{\rho}{8} \operatorname{tr}\{(h^{-1}\partial_\rho h)^2 - (h^{-1}\partial_z h)^2\} \\ &- \rho(\partial_\rho A^T h^{-1}\partial_\rho A - \partial_z A^T h^{-1}\partial_z A),\end{aligned} \tag{1.3B}$$

where $A = (A_0, A_1)^T$, $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $*$ = Hodge operator for the metric $dz^2 + d\rho^2$. Since $h_{00} \neq 0$ and $\det h = -\rho^2$, we can parametrize $h$ as

$$h = \begin{pmatrix} f & f\omega \\ f\omega & f\omega^2 - \rho^2/f \end{pmatrix}.$$

Now we can write down the Ernst equations which are described by Ernst potentials $u$, $v$ constructed from $h$ and $A$ by the standard method. The following proposition is well-known:

PROPOSITION 1.1. *The stationary axisymmetric Einstein-Maxwell equations* (1.1), (1.2) *are equivalent to the Ernst equations given by*:

$$f(d * du + \rho^{-1}d\rho \wedge * du) = (du + 2\bar{v}dv) \wedge * du, \tag{1.4}$$

$$f(d * dv + \rho^{-1}d\rho \wedge * dv) = (du + 2\bar{v}dv) \wedge * dv, \tag{1.5}$$

*where* $f = \operatorname{Re} u + |v|^2$.

In terms of the Ernst potentials $u$ and $v$, (1.3A) and (1.3B) turn into:

$$\begin{aligned}\frac{\partial_z \lambda}{\lambda} = &- \frac{\partial_z f}{2f} + \frac{\rho}{2f^2}(\partial_z f \partial_\rho f) \\ &- \frac{\rho}{2f^2}(\partial_\rho u - \partial_\rho f + 2\bar{v}\partial_\rho v)(\partial_z u - \partial_z f + 2\bar{v}\partial_z v) \\ &- \frac{\rho}{f}(\partial_z v\partial_\rho \bar{v} + \partial_z \bar{v}\partial_\rho v),\end{aligned} \tag{1.6A}$$

$$\frac{\partial_\rho \lambda}{\lambda} = - \frac{\partial_\rho f}{2f} + \frac{\rho}{4f^2}\{(\partial_\rho f)^2 - (\partial_z f)^2\} \tag{1.6B}$$

$$+\frac{\rho}{4f^2}\{(\partial_z u-\partial_z f+2\bar{v}\partial_z v)^2-(\partial_\rho u-\partial_\rho f+2\bar{v}\partial_\rho v)^2\}$$

$$+\frac{\rho}{f}(\partial_z v\partial_z\bar{v}-\partial_\rho\bar{v}\partial_\rho v).$$

Note that the integrability condition of the equations (1.3), or (1.6) follows easily from (1.1) and (1.2), or (1.4) and (1.5) respectively [3]. But we employ a different definition $f=\operatorname{Re}u-|v|^2$ so that (1.4) and (1.5) turn into:

$$f(d*du+\rho^{-1}d\rho\wedge *du)=(du-2\bar{v}dv)\wedge *du, \tag{1.7}$$

$$f(d*dv+\rho^{-1}d\rho\wedge *dv)=(du-2\bar{v}dv)\wedge *dv. \tag{1.8}$$

Similarly (1.6A) and (1.6B) turn, by our definition, into:

$$\begin{aligned}\frac{\partial_z\lambda}{\lambda}=&-\frac{\partial_z f}{2f}+\frac{\rho}{2f^2}(\partial_z f\partial_\rho f)\\&-\frac{\rho}{2f^2}(\partial_\rho u-\partial_\rho f-2\bar{v}\partial_\rho v)(\partial_z u-\partial_z f-2\bar{v}\partial_z v)\\&+\frac{\rho}{f}(\partial_z v\partial_\rho\bar{v}+\partial_z\bar{v}\partial_\rho v),\end{aligned}\tag{1.9A}$$

$$\begin{aligned}\frac{\partial_\rho\lambda}{\lambda}=&-\frac{\partial_\rho f}{2f}+\frac{\rho}{4f^2}\{(\partial_\rho f)^2-(\partial_z f)^2\}\\&+\frac{\rho}{4f^2}\{(\partial_z u-\partial_z f-2\bar{v}\partial_z v)^2-(\partial_\rho u-\partial_\rho f-2\bar{v}\partial_\rho v)^2\}\\&-\frac{\rho}{f}(\partial_z v\partial_z\bar{v}-\partial_\rho\bar{v}\partial_\rho v).\end{aligned}\tag{1.9B}$$

Since the integrability condition of the equations (1.9) follows from (1.7) and (1.8) as before, we shall be concerned with (1.7) and (1.8) henceforth.

## 2. Linearization

Let $G=\{g\in SL_3(\mathbf{C});\ g^*Jg=J\}\cong SU(1,2)$, where $J=\begin{pmatrix} & & i\\ & 1 & \\ -i & & \end{pmatrix}$,
and $K$ its maximal compact subgroup, i.e., $K=\{g\in G;\ g^*g=1\}$. We define the Cartan involution $\tau$ by $\tau(g)=(g^*)^{-1}$ for $g\in G$. In this section, from the Ernst equations (1.7) and (1.8), we derive an equation of motion for a $\sigma$-model (2.2) which takes its value in the Riemannian symmetric space $K\backslash G$. The reason

why we have changed the Ernst equations in § 1 is that we want to treat a $\sigma$-model with values in a Riemannian symmetric space, not in an affine symmetric space. Then we linearize the equation, or we introduce a 1-form with a spectral parameter. The integrability condition of this 1-form is equivalent to the equation mentioned above.

Let $G = KAN$ be an Iwasawa decomposition (see [9]) with

$$A = \left\{ \begin{pmatrix} a & & \\ & 1 & \\ & & 1/a \end{pmatrix} ; a > 0 \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & & \\ v & 1 & \\ x + i|v|^2/2 & i\bar{v} & 1 \end{pmatrix} ; x \in \mathbf{R},\ v \in \mathbf{C} \right\}.$$

Thanks to this decomposition, we can choose uniquely an element in $AN$ as a representative of an element in $K \backslash G$. Now we parametrize an element $P$ in $AN$ as follows [6]:

$$P = \begin{pmatrix} f^{1/2} & 0 & 0 \\ \sqrt{2}v & 1 & 0 \\ (x + i|v|^2)/f^{1/2} & \sqrt{2}i\bar{v}/f^{1/2} & 1/f^{1/2} \end{pmatrix}. \tag{2.1}$$

where $f$ and $v$ are the ones in (1.7), (1.8), and $x = \operatorname{Im} u$.

It is well-known that $(u, v)$ is a solution of (1.7), (1.8) if and only if $P$ is a solution of the following equation:

$$d(\rho * dMM^{-1}) = 0 \quad \text{with} \quad M = \tau(P)^{-1}P. \tag{2.2}$$

Let $\mathfrak{g}$ the Lie algebra of $G$, i.e.,

$$\mathfrak{g} = \{X \in sl_3(\mathbf{C});\ X^*J + JX = O\},$$

where $J$ is as above. We also denote by $\tau$ the involution of $\mathfrak{g}$ induced from the involution of $G$.

DEFINITION. *Let $\mathscr{A}$ and $\mathscr{I}$ be $\mathfrak{g}$-valued 1-forms defined by*

$$\mathscr{A} = \frac{1}{2}(dPP^{-1} + \tau(dPP^{-1})), \qquad \mathscr{I} = \frac{1}{2}(dPP^{-1} - \tau(dPP^{-1})).$$

*We define a $\mathfrak{g}$-valued 1-form $\Omega$ with a spectral parameter to be*

$$\Omega = \Omega(s) = \mathscr{A} + \frac{1 - 2sz - 2z\rho*}{\Lambda}\mathscr{I}, \tag{2.3}$$

*where we put* $\Lambda = \{(1 - 2sz)^2 + 4s^2\rho^2\}^{1/2}$.

Since we discuss in the category of formal power series, we expand $\Lambda$ in the form

$$\Lambda = 1 + \sum_{k \geq 1} \frac{\frac{1}{2}(\frac{1}{2} - 1) \cdots (\frac{1}{2} - k + 1)}{k!}(-4sz + 4s^2z^2 + 4s^2\rho^2)^k \tag{2.4}$$

$$= 1 - 2sz + 2s^2\rho^2 + O(s^3).$$

In view of (2.4) each coefficient of $s^k$ in the above expansion of $\Lambda$ belongs to $\mathfrak{m}^k$, where $\mathfrak{m}$ is the maximal ideal of $\mathbf{C}[[z, \rho]]$, a ring of formal power series in $z, \rho$. Note that $\Omega(0) = \mathscr{A} + \mathscr{I} = dPP^{-1}$.

PROPOSITION 2.1. *$\Omega$ satisfies the integrability condition, i.e.,*

$$d\Omega - \Omega \wedge \Omega = 0 \tag{2.5}$$

*if and only if $P$ is a solution of* (2.2).

PROOF: We first note that (2.2) is equivalent to

$$\begin{aligned} & d(\rho * dPP^{-1}) - d(\rho\tau(* dPP^{-1})) \\ & \quad + \rho(dPP^{-1} \wedge \tau(* dPP^{-1}) - \tau(dPP^{-1}) \wedge * dPP^{-1}) = 0. \end{aligned} \tag{2.6}$$

Next we introduce a scalar function $t$ defined by

$$t = \frac{2sz - 1 + \Lambda}{2s\rho} \tag{2.7}$$

with $\Lambda$ expanded as (2.4). Then $t$ satisfies the following equation:

$$(1 - t^2)dt + 2t * dt = t(1 + t^2)\frac{d\rho}{\rho}. \tag{2.8}$$

$\Omega$ can be rewritten, using $t$, in the form

$$\Omega = \mathscr{A} + \frac{1 - t^2}{1 + t^2}\mathscr{I} - \frac{2t}{1 + t^2} * \mathscr{I},$$

from which, it follows that

$$\Omega = \frac{1}{1 + t^2}((1 - t *)dPP^{-1} + t(* + t)\tau(dPP^{-1})). \tag{2.9}$$

Using (2.9), we obtain

(2.10)

$$
\begin{aligned}
d\Omega - \Omega \wedge \Omega = &-\frac{t}{(1+t^2)\rho}(d(\rho * dPP^{-1}) - d(\rho\tau(* dPP^{-1}))) \\
&-\frac{t}{1+t^2}(dPP^{-1} \wedge \tau(* dPP^{-1}) - \tau(dPP^{-1}) \wedge * dPP^{-1}) \\
&+ d\left(\frac{-t}{(1+t^2)\rho}\right) \wedge \rho * dPP^{-1} + d\left(\frac{t}{(1+t^2)\rho}\right) \wedge \rho\tau(* dPP^{-1}) \\
&+ d\left(\frac{1}{1+t^2}\right) \wedge dPP^{-1} + d\left(\frac{t^2}{1+t^2}\right) \wedge \rho\tau(dPP^{-1}).
\end{aligned}
$$

The last four terms of (2.10) can be rewritten in the form

$$
\frac{1}{(1+t^2)^2}\left((1-t^2) * dt - 2t dt - \frac{t(1+t^2)}{\rho} * d\rho\right) \wedge (dPP^{-1} - \tau(dPP^{-1})).
$$

Now the result follows immediately from (2.6) and (2.8). □

For any solution $P$ of the equation (2.2), by Proposition 2.1, there exists $\mathscr{P} = \mathscr{P}(s; z, \rho) \in SL(3, \mathbf{C}[[z, \rho, s]])$ which satisfies

$$
d\mathscr{P} = \Omega\mathscr{P}, \qquad \mathscr{P}|_{s=0} = P
$$

where $\mathbf{C}[[z, \rho, s]]$ is a ring of formal power series in $z$, $\rho$, $s$ and $SL(3, \mathbf{C}[[z, \rho, s]])$ is a group consisting of all matrices of determinant 1 whose entries are the elements of $\mathbf{C}[[z, \rho, s]]$.

## 3. Constructing solutions

In this section we give a recipe for constructing exact local-solutions of the equations (2.2) around the origin $(z, \rho) = (0, 0)$. This method was originally explored in [1].

Let $G^{(\infty)}$ be an infinite dimensional group

$$
\{g(s) \in SL(3, \mathbf{C}[[s^{-1}]]);\ g(s)^* J g(s) = J\},
$$

where $\mathbf{C}[[s^{-1}]]$ is a ring of formal power series in $s^{-1}$ and $g(s)^* = \overline{g(\bar{s})}^T$.

Next we introduce a formal loop group $\mathscr{G}_R$, following [11]. Let $R$ be a ring of formal power series $\mathbf{C}[[z, \rho]]$ and $I$ an ideal of $R$ generated by $\rho$, i.e., $I = (\rho)$. We put $R_n = I^n$ for $n > 0$ and $R_n = R$ for $n \leq 0$. We regard $t$ as a parameter. Then we define

$$
\mathscr{G}_R = \{u = \sum_{n \in \mathbf{Z}} u_n t^n;\ u_n \in gl(3, R_n),\ u_0 \text{ is invertible}\},
$$

and its subgroups

$$\mathcal{N}_R = \{u = \sum_{n \in \mathbf{Z}} u_n t^n \in \mathcal{G}_R;\ u_n = 0\ (n > 0),\ u_0 = 1\},$$

$$\mathcal{P}_R = \{u = \sum_{n \in \mathbf{Z}} u_n t^n \in \mathcal{G}_R;\ u_n = 0\ (n < 0)\}.$$

REMARK. *If we define*

$$\mathcal{G}_R^{(0)} = \{u = \sum_{n \in \mathbf{Z}} u_n t^n;\ u_n \in gl(3, R_{-n}),\ u_0 \text{ is invertible}\},$$

*then $\mathcal{G}_R^{(0)}$ also forms a group. For any $g(s) \in G^{(\infty)}$,*

$$g((\rho/t + 2z - \rho t)^{-1} \in \mathcal{G}_R \cap \mathcal{G}_R^{(0)}.$$

Our main theorem is:

THEOREM 3.1. *For any $g(s) \in G^{(\infty)}$, there exists uniquely an element $k(t) \in \mathcal{G}_R$ which satisfies the following conditions:*
(i) *$\tau(k(-1/t)) = k(t)$, where $\tau$ is the Cartan involution of $G$;*
(ii) *$k(t)g((\rho/t + 2z - \rho t)^{-1})^{-1}$ is an element of $\mathcal{P}_R$;*
(iii) *The leading term of $k(t)g((\rho/t + 2z - \rho t)^{-1})^{-1}$ is an element of $AN$ and is a solution of the Ernst equation* (2.2).

For the proof we reduce the problem to Birkhoff decomposition (3.17) of formal loop groups established in [11]:

LEMMA 3.2. *Any element $u$ of $\mathcal{G}_R$ can be uniquely decomposed as*

$$u = w^{-1} v, \quad w \in \mathcal{N}_R, \quad v \in \mathcal{P}_R$$

PROOF OF THEOREM 3.1: First we show the uniqueness of $k(t)$. For a given $g(s)$, assume that there exist $k_1(t)$ and $k_2(t)$ which satisfy the conditions (i), (ii), (iii). We put

$$\mathcal{P}_i(t) = k_i(t) g((\rho/t + 2z - \rho t)^{-1})^{-1} \qquad (i = 1, 2)$$

and decompose $\mathcal{P}_i(t)$ as

$$\mathcal{P}_i(t) = u_0^{(i)} u_+^{(i)}(t),$$

where $u_0^{(i)}$ is the leading term of $\mathcal{P}_i(t)$ so that the leading term of $u_+^{(i)}(t)$ is 1. Then we have

$$\tau(\mathcal{P}_i(-1/t)^{-1}) \mathcal{P}_i(t) = \tau(u_+^{(i)}(-1/t))^{-1} \tau(u_0^{(i)})^{-1} u_0^{(i)} u_+^{(i)}(t),$$

and

$$\tau(u_+^{(i)}(-1/t))^{-1}\in\mathcal{N}_R,\ \tau(u_0^{(i)})^{-1}u_0^{(i)}u_+^{(i)}(t)\in\mathcal{P}_R.$$

Since

$$\tau(\mathcal{P}_i(-1/t)^{-1})\mathcal{P}_i(t)=\tau(g((\rho/t+2z-\rho t)^{-1}))g((\rho/t+2z-\rho t)^{-1})^{-1},$$

it follows from the uniqueness of Birkhoff decomposition that

$$\mathcal{P}_1(t)=\mathcal{P}_2(t),$$

hence

$$k_1(t)=k_2(t).$$

Now we show the existence of $k(t)$. Let

$$\mathcal{M}(t)=\tau(g((\rho/t+2z-\rho t)^{-1}))g((\rho/t+2z-\rho t)^{-1})^{-1}.$$

Then $\mathcal{M}(t)$ belongs to $\mathcal{G}_R\cap\mathcal{G}_R^{(0)}$. By Lemma 3.2, it can be uniquely decomposed as

$$\mathcal{M}(t)=v_-(t)\cdot v_+(t),\quad v_-(t)\in\mathcal{N}_R,\quad v_+(t)\in\mathcal{P}_R.$$

We rewrite $v_+(t)$ in the form $u_0\cdot u_+(t)$, where $u_0$ is the leading term of $v_+(t)$ so that the leading term of $u_+$ is 1. Noting that $\tau(\mathcal{M}(-1/t)^{-1})=\mathcal{M}(t)$, we have

$$\tau(u_+(-1/t))^{-1}\tau(u_0)^{-1}\tau(v_-(-1/t))=v_-(t)u_0u_+(t).$$

Since $\tau(u_+(-1/t))$ also belongs to $\mathcal{N}_R$, multiplying it to the right of the both sides, we obtain

$$\tau(u_0)^{-1}\tau(v_-(-1/t))\in\mathcal{P}_R.$$

Therefore, by the uniqueness of Birkhoff decomposition, it follows that

$$v_-(t)=\tau(u_+(-1/t)^{-1}),\quad \tau(u_0^{-1})=u_0.$$

Hence we can find uniquely $p\in SL(3,R)$ which takes its value in $AN$ and satisfies $u_0=\tau(p^{-1})p$. We shall show that $p$ is a solution of the equation (2.2).

Now we assume that $t$ satisfies

$$dt=\frac{t}{(1+t^2)\rho}((1-t^2)d\rho+2tdz), \tag{3.1}$$

which certainly holds if $t$ satisfies (2.7).

By Proposition 2.1, writing $\mathcal{P}(t)=p\cdot u_+(t)$, it is sufficient to show that

$$d\mathcal{P}(t)\mathcal{P}(t)^{-1}=\mathcal{A}+\left(\frac{1-t^2}{1+t^2}-\frac{2t*}{1+t^2}\right)\mathcal{I}, \tag{3.2}$$

where $\mathcal{A}$ and $\mathcal{I}$ are constructed from $p$ as in §2. It is easy to show that (3.2) is equivalent to

$$dUU^{-1} = \frac{t}{1+t^2} M^{-1}(-*dM - tdM) \tag{3.3}$$

where we put $U = u_+(t)$ and $M = u_0$.

If we define

$$S(t) = \left(\frac{1}{t} + t\right) dUU^{-1},$$

then (3.3) is rewritten with $S(t)$ in the form

$$S(t) = M^{-1}(-tdM - *dM). \tag{3.4}$$

Using (3.1), we obtain $d\mathscr{M}(t) = 0$. Namely, we have

$$0 = -S\left(-\frac{1}{t}\right)^* M + \left(\frac{1}{t} + t\right) dM + MS(t).$$

Therefore

$$S(t) = M^{-1}\left(S\left(-\frac{1}{t}\right)^* M - \left(\frac{1}{t} + t\right) dM\right). \tag{3.5}$$

Since the left hand side of (3.5) contains no negative power terms in $t$, it can be written in the form

$$S(t) = M^{-1}(-tdM + G), \tag{3.6}$$

where $G$ is a matrix whose entries are $R$-valued 1-forms.

Taking (2.8) into account, we obtain

$$*S(t) \equiv iS(t) \bmod t - i, \tag{3.7}$$

where "$\equiv \bmod t - i$" means that each entry of the coefficient matrix of $dz$ (resp. $d\rho$) in the left hand side is equivalent to that of $dz$ (resp. $d\rho$) in the right hand side, modulo the ideal generated by $t - i$ in $R[t]$, a ring of polynomials in $t$ with coefficients in $R$.

Similarly

$$*S(t) \equiv -iS(t) \bmod t + i. \tag{3.8}$$

On the other hand, from (3.6) we have

$$S(t) \equiv M^{-1}G \mp M^{-1}dM \bmod t \pm i. \tag{3.9}$$

It follows from (3.7), (3.8), (3.9) that

$$G = - * dM$$

which is nothing but (3.4).

We put $k(t) = \mathscr{P}(t)g((\rho/t + 2z - \rho t)^{-1})$. Then we have to show that $k(t)$ satisfies the conditions (i), (ii), (iii). We have already shown that $k(t)$ satisfies (ii) and (iii). Note that by definition

$$\mathscr{M}(t) = \tau(g((\rho/t + 2z - \rho t)^{-1}))g((\rho/t + 2z - \rho t)^{-1})^{-1},$$

and from the construction of $\mathscr{P}(t)$

$$\mathscr{M}(t) = \tau(\mathscr{P}(-1/t)^{-1})\mathscr{P}(t),$$

from which it follows that $k(t)$ satisfies (i). $\square$

## 4. Examples

In this section we give some examples of solutions of the equation (2.2) following the recipe (Theorem 3.1) given in the last section.

We note that $SL(2, \mathbf{R})$ can be embedded in $G$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix}.$$

We use this embedding when we deal with the fields without electro-magnetic fields.

The first two examples below deal with the case where the gravitational field does not interact with electro-magnetic fields.

EXAMPLE 1. For $g(s) = \begin{pmatrix} 1 & s^{-1} \\ 0 & 1 \end{pmatrix}^{-1}$, the element $k(t) \in \mathscr{G}_R$ in Theorem 3.1 is given by

$$k(t) = \begin{pmatrix} a_0 + a_1 t & b_0 + b_1 t^{-1} \\ -b_0 + b_1 t & a_0 - a_1 t^{-1} \end{pmatrix}$$

with

$$a_0 = \frac{\sqrt{1-\rho^2}}{\lambda}, \quad a_1 = \frac{2z\rho}{\lambda\sqrt{1-\rho^2}},$$

$$b_0 = \frac{-2z}{\lambda\sqrt{1-\rho^2}}, \quad b_1 = \frac{-\rho\sqrt{1-\rho^2}}{\lambda},$$

where we put $\lambda = \sqrt{(1-\rho^2)^2 + 4z^2}$.
Hence $k(t)g((\rho/t + 2z - \rho t)^{-1})^{-1} (=: \mathscr{P}(t))$ is given by

$$\mathscr{P}(t) = \frac{a_0}{1-\rho^2} \times \begin{pmatrix} 1-\rho^2+z\rho t & (-2\rho(1-\rho^2)+4z^2\rho)t - 2z\rho^2 t^2 \\ 2z - \rho(1-\rho^2)t & (1-\rho^2)^2 + 4z^2 - 2z\rho(2-\rho^2)t + \rho^2(1-\rho^2)t^2 \end{pmatrix}.$$

Therefore $M = \tau(\rho(0)^{-1})P(0)$ is given by

$$M = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & 2z \\ 2z & (1-\rho^2)^2 + 4z^2 \end{pmatrix}. \tag{4.1}$$

This is the first example given in [10] modulo scaling transformation of $(z, \rho)$. The second example in [10] is obtained by multiplying

$$\begin{pmatrix} \sqrt{2} - 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

to the right of (4.1) and its transpose to the left.

EXAMPLE 2. For $g(s) = \begin{pmatrix} 1 & 0 \\ c_0 + c_1/s + c_2/s^2 & 1 \end{pmatrix}^{-1}$, (where $c_0, c_1, c_2$ are arbitrary real numbers) $k(t)$ is given by

$$k(t) = \begin{pmatrix} a(t) & b(t) \\ -b(-1/t) & a(-1/t) \end{pmatrix}$$

with

$$a(t) = ((1 + c_2^2\rho^4) + (c_1c_2 + 4c_2^2 z)\rho^3/t)/\mu,$$
$$b(t) = (-(c_1 + 4zc_2)\rho t + (c_2\rho^2 + \rho^6 c_2^3)t^2)/\mu$$

where

$$\mu = \{(1 + c_2^2\rho^4 + (c_1 + 4c_2 z)\rho)(1 + c_2^2\rho^4 - (c_1 + 4c_2 z)\rho)(1 + c_2^2\rho^4)\}^{1/2}.$$

Hence $k(t)g((\rho/t + 2z - \rho t)^{-1})^{-1} (=: \mathscr{P}(t))$ is given by

$$\mathscr{P}(t) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

where

$$p_{11} = -\frac{1}{\mu}\{-1 - (1+t)^2(-1+t)^2\rho^8 c_2^4 - (2 - 2t^2 + t^4)\rho^4 c_2^2$$

$$+ ((1+t)(-1+t)\rho^4 c_2^2 + (-3+2t^2))(c_1 + 4zc_2)\rho^3 tc_2$$
$$- (-c_1^2 + (c_1^2 + c_0c_2)t^2 + 2(c_1 + 2zc_2)(-4+5t^2)zc_2)\rho^2$$
$$+ (-\rho^5 tc_2^3 + (c_1 + 4zc_2))(c_0 + 2zc_1 + 4z^2c_2)\rho t\}$$

$$p_{12} = -\frac{1}{\mu}\{(-\rho tc_2 - \rho^5 tc_2^3 + (c_1 + 4zc_2))\rho t\}$$

$$p_{21} = \frac{1}{\mu}\{c_0 + 2zc_1 + 4z^2c_2 - (c_0 + 2zc_1 + 4z^2c_2)(c_1 + 4zc_2)\rho^3 tc_2$$
$$+ (1 + \rho^4 c_2^2)(-2+t^2)\rho^2 c_2 - (1 + (1+t)(-1+t)\rho^4 c_2^2)(c_1 + 4zc_2)\rho t$$
$$+ (-c_1^2 + c_0c_2 + t^2c_1^2 + 2(c_1 + 2zc_2)(-3+4t^2)zc_2)\rho^4 c_2\}$$

$$p_{22} = -\frac{1}{\mu}\{-1 - \rho^4 c_2^2 + (c_1 + 4zc_2)\rho^3 tc_2\}.$$

Therefore $M = \tau(\mathscr{P}(0)^{-1})\mathscr{P}(0)$ is given by

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

where

$$m_{11} = \frac{1}{\mu^2}\{1 + c_0^2 + 4zc_0c_1 + 16z^3c_1c_2 + 16z^4c_2^2 + 7\rho^8c_2^4 + \rho^{12}c_2^6$$
$$+ 4(c_1^2 + 2c_0c_2)z^2 + 2(c_1^2 - 2c_0c_2 + 4zc_1c_2 + 8z^2c_2^2)\rho^6c_2^2$$
$$- 2(c_1^2 + 2c_0c_2 + 12zc_1c_2 + 24z^2c_2^2)\rho^2$$
$$- (-7c_2^2 - c_1^4 + 2c_0c_1^2c_2 - c_0^2c_2^2 - 144z^3c_1c_2^3 - 144z^4c_2^4$$
$$+ 12(-c_1^2 + c_0c_2)zc_1c_2 + 12(-5c_1^2 + 2c_0c_2)z^2c_2^2)\rho^4\}$$

$$m_{12} = \frac{1}{\mu^2}\{c_0 + 2zc_1 + 4z^2c_2 - 2\rho^2c_2 - 2\rho^6c_2^3$$
$$- (c_1^2 - c_0c_2 + 6zc_1c_2 + 12z^2c_2^2)\rho^4c_2\}$$
$$= m_{21}$$

$$m_{22} = \frac{1}{\mu^2}(1 + \rho^4c_2^2)$$

and $\mu$ is as above.

EXAMPLE 3. Similarly for $g(s) = \begin{pmatrix} 1 & & \\ cs^{-1} & 1 & \\ i|c|^2s^{-2}/2 & i\bar{c}s^{-1} & 1 \end{pmatrix}^{-1}$ (where $c$ is an arbitrary complex number), $k(t)$ is given by

$$k(t)=\begin{pmatrix} a & -\bar{c}\rho a t & -i|c|^2\rho^2 a t^2/2 \\ -2c\rho t^{-1}/(2-|c|^2\rho^2) & (2+|c|^2\rho^2)/(2-|c|^2\rho^2) & 2ic\rho t/(2-|c|^2\rho^2) \\ i|c|^2\rho^2 a t^{-2}/2 & -i\bar{c}\rho a t^{-1} & a \end{pmatrix}.$$

hence $k(t)g((\rho/t+2z-\rho t)^{-1})^{-1}\,(=:\mathscr{P}(t))$ is given by

$$\mathscr{P}(t)=\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix},$$

where

$$\begin{aligned}
p_{11} &= a\{t^4|c|^4\rho^4-4t^3|c|^4\rho^3 z+2t^2|c|^2\rho^2(-|c|^2\rho^2+2|c|^2z^2+2) \\
&\quad +4t|c|^2\rho z(|c|^2\rho^2-2)+|c|^4\rho^4-4|c|^2\rho^2+4\}/4 \\
p_{12} &= -a(t^3\bar{c}|c|^2\rho^3-2t^2\bar{c}|c|^2\rho^2 z+t\bar{c}\rho(-|c|^2\rho^2+2))/2 \\
p_{13} &= -t^2 a|c|^2 i\rho^2/2 \\
p_{21} &= -(t^3 c|c|^2\rho^3-4t^2 c|c|^2\rho^2 z+tc\rho(-|c|^2\rho^2+4|c|^2z^2+2) \\
&\quad +2cz(|c|^2\rho^2-2))/2a \\
p_{22} &= (2t^2|c|^2\rho^2-4t|c|^2\rho z-|c|^2\rho^2+2)/2a \\
p_{23} &= itc\rho/a \\
p_{31} &= a(t^2|c|^2 i\rho^2-4t|c|^2 i\rho z+4|c|^2 iz^2)/2 \\
p_{32} &= a(2i\bar{c}z-i\rho\bar{c}t) \\
p_{33} &= a
\end{aligned}$$

and

$$a=\frac{2}{2-|c|^2\rho^2}.$$

Therefore $M=\tau(\mathscr{P}(0)^{-1})\mathscr{P}(0)$ is given by

$$M=\begin{pmatrix} a^{-2}+4|c|^2z^2+4a^2|c|^4z^4 & 2\bar{c}z+4a^2\bar{c}|c|^2z^3 & -2ia^2|c|^2z^2 \\ 2cz+4a^2c|c|^2z^3 & 1+4a^2|c|^2z^2 & -2ia^2cz \\ 2ia^2|c|^2z^2 & 2ia^2\bar{c}z & a^2 \end{pmatrix}.$$

Example 3 gives a solution with nontrivial electro-magnetic potentials.

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