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Calculation of the Stokes' multipliers for a polynomial system of rank 1 having distinct eigenvalues at infinity

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0. Introduction

Computation of the Stokes' multipliers and / or central connection matrices (hereafter referred to as connection data) for systems of the form

$$zx' = (zA_o + A_1)x, \tag{0.1}$$

with $n \times n$ constant matrices A_o, A_1, or equivalently computation of connection constants for the so-called *hyper-geometric system* (compare below), has attracted considerable attention lately. In case $n = 2$ and A_o having two distinct eigenvalues, the connection data can be explicitly computed using Gamma functions in the parameters of (0.1); see Jurkat, Lutz, and Peyerimhoff [4], and Kohno and Yokoyama [10], e.g. The same holds true for general n and A_o being a diagonal matrix with zeros and a single one along the diagonal; see Balser [3] and Okubo, Takano, and S. Yoshida [6]. Other cases of (0.1) have been treated by Yokoyama [8], [9], [10], who obtained under various assumptions upon n and / or the eigenvalues of A_o and A_1, together with other generic restrictions, explicit formulas in terms of classical special functions.

Aside from very special situations as the ones described above, no such explicit formulas for the connection data of (0.1) have been found and, to the author's opinion, may not exist. Instead, it appears reasonable to regard these data as being "new" special functions in the parameters of (0.1) and look for representations of them in terms of infinite series, or integrals, etc. In [2], such representations for the Stokes' multipliers of (0.1) (in case A_o has n distinct eigenvalues) are given. The terms of these series involve functions which are recursively defined and, although interesting in their own right, are relatively complicated. In the present paper, we obtain representations which are much simpler and (aside from explicit rational terms) involve solutions of a difference equation closely related to a system of the form (0.1), but of dimension $n - 1$. To do so, we use results of R. Schäfke [7], who showed (for A_o having distinct eigenvalues) that the connection data of (0.1) can be computed from certain connection constants of the so-called *hypergeometric system*

$$(A_o - t)\frac{d}{dt}y = (A_1 - s)y \tag{0.2}$$

(with t being a complex variable and s a parameter), and instead of (0.2) one may also regard the difference equation

$$(A_o - t)\zeta(s, t) = (s - A_1)\zeta(s + 1, t), \tag{0.3}$$

where now s acts as a variable and t becomes a parameter. The same results follow from a (more general) discussion by Balser, Jurkat, and Lutz [1], and for convenience we use the notation and terminology introduced there.

1. A difference equation

Throughout, let $A_o = diag[\lambda_1, \ldots, \lambda_n]$ denote a (complex) diagonal matrix of *distinct* diagonal entries $\lambda_1, \ldots, \lambda_n$, and let A_1 be an arbitrary $n \times n$ matrix. By $A' = diag[\lambda'_1, \ldots, \lambda'_n]$ we denote the diagonal matrix consisting of the diagonal elements of A_1. Moreover, let B_1, a, b be so that

$$A_1 - \lambda'_n = \left[\begin{array}{c|c} B_1 & a \\ \hline b^T & 0 \end{array}\right],$$

and let

$$B_o = diag[\lambda_n - \lambda_1, \ldots, \lambda_n - \lambda_{n-1}],$$
$$A'_B = diag[\lambda'_1 - \lambda'_n, \ldots, \lambda'_{n-1} - \lambda'_n].$$

With these data, we consider the first order system of difference equations in dimension $n - 1$

$$G(s + 1)(s + B_1 - s^{-1}ab^T) = G(s)B_o. \tag{1.1}$$

Since

$$\left[\begin{array}{c|c} s + B_1 - s^{-1}ab^T & a \\ \hline 0 & s \end{array}\right]\left[\begin{array}{c|c} I & 0 \\ \hline s^{-1}b^T & 1 \end{array}\right] = s + A_1 - \lambda'_n,$$

we find

$$s \det(s + B_1 - s^{-1}ab^T) = \det(s + A_1 - \lambda'_n) = \prod_{j=1}^{n} (s + \mu_j - \lambda'_n), \tag{1.2}$$

if $\mu_1, \dots, \mu_n$ are the eigenvalues of A_1, repeated according to their multiplicity. Let η be an arbitrarily fixed admissible number, and choose

$$\eta - \pi < \arg(\lambda_n - \lambda_j) < \eta + \pi, \qquad 1 \leq j \leq n-1. \tag{1.3}$$

Then we have

THEOREM 1. *The system* (1.1) *admits a solution* $G(s)$, *uniquely characterized by the following conditions*:
i) The matrix $G(s)$ *is analytic for* $s = x + iy$, *provided* x *is sufficiently large, and*

$$B_o^{-s}\Gamma(s + \Lambda_B')G(s) = I + O_y(x^{-1}) \quad as \quad x \longrightarrow \infty, \tag{1.4}$$

where B_0^{-s} *is defined according to* (1.3).
ii) For every s *as in i) and every* k, $1 \leq k \leq n-1$, *let* $g_k^T(s)$ *denote the* k^{th} *row of* $G(s)$. *Then* $(\lambda_n - \lambda_k)^{-s} g_k^T(s)$, *regarded as a function in any one of the parameters*

$$w_j^{(k)} = \frac{\lambda_n - \lambda_k}{\lambda_n - \lambda_j}, \quad j \neq k, \quad 1 \leq j \leq n-1,$$

is analytic in a complex plane with a cut from 1 *to* ∞ *along the positive real axis, and extends continuously to the lower border of the cut, excluding* 1 *and* ∞.

This unique solution $G(s)$ *then is meromorphic throughout the complex s-plane with possible poles only at points*

$$s = 0, \; -1, \; -2, \dots, \tag{1.5}$$

and

$$\det G(s) = b_0^s \Gamma(s) \{ \prod_{j=1}^{n} \Gamma(s + \mu_j - \lambda_n') \}^{-1}, \tag{1.6}$$

with $b_o = \det B_o$; *hence* $G(s)$ *is a fundamental solution of* (1.1).

PROOF. Suppose that $G(s)$ with i), ii) exists, then (1.1) implies $G(s)$ meromorphic with poles as stated, and using (1.2), (1.1), and (1.4) one can easily obtain (1.6). Now let $C(s)$ be a one-periodic matrix so that $C(s)G(s)$ again satisfies i), ii). Let $c_k^T(s)$ be the k^{th} row of $C(s)$, $1 \leq k \leq n-1$, then

$$\frac{\Gamma(s + \lambda_k' - \lambda_n')}{(\lambda_n - \lambda_k)^s} c_k^T(s) B_0^s \{\Gamma(s + \Lambda_B')\}^{-1} = e_k + O_y(x^{-1}) \quad \text{as} \quad x \longrightarrow \infty$$

follows. If $|w_j^{(k)}| < 1$, $j \neq k$, $1 \leq j \leq n-1$, then this implies $c_k^T(s) \equiv e_k$. Since $(\lambda_n - \lambda_k)^{-s} c_k^T(s) G(s)$ is analytic in $w_j^{(k)}$, this gives $c_k^T(s) \equiv e_k$ in any case, and hence $G(s)$ is uniquely characterised by i), ii).
To show existence, let

$$\hat{A}_1 = \lambda_n' - A_1^T, \quad \hat{A}_o = - A_o.$$

According to [1], part II, section 3, the difference equation

$$(s - \hat{A}_1)\xi(s + 1, t) = (\hat{A}_0 - t)\xi(s, t), \quad t \in \mathscr{P}_{k,\eta}, \quad 1 \leq k \leq n - 1 \text{ (fixed)},$$

has a solution vector $\xi_k(s, t)$ which is an entire function of s, is given by a power series expansion for $|t - \lambda_k|$ small and is analytic for $t \in \mathscr{P}_{k,\eta}$. For $s = x + iy$, and x sufficiently large, $\xi_k(s, t)$ has a limit $\xi_k(s, - \lambda_n)$ as $t \to - \lambda_n$ in $\mathscr{P}_{k,\eta}$. The proof of Proposition 4 in [1], part II, may be seen to give

$$\xi_k(s, - \lambda_n) = e_k \frac{(\lambda_n - \lambda_k)^{s + \lambda_k' - \lambda_n' - 1}}{\Gamma(s + \lambda_k' - \lambda_n')}$$
$$- \int_{-\lambda_k}^{-\lambda_n} \frac{(u + \lambda_n)^{s - \tilde{s} - 1}}{\Gamma(s - \tilde{s})} \left\{ \xi_k(\tilde{s}, u) - e_k \frac{(- \lambda_k - u)^{\tilde{s} + \lambda_k' - \lambda_n' - 1}}{\Gamma(\tilde{s} + \lambda_k' - \lambda_n')} \right\} du$$

provided $Re(s - \tilde{s}) > 0$, $Re(\tilde{s} + \lambda_k' - \lambda_n') > - 1$, $Re\,\tilde{s} > 0$, is we integrate in $\mathscr{P}_{k,\eta}$ along some fixed, but arbitrary, path from $- \lambda_j$ to $- \lambda_n$. For $\tilde{s} = p + \lambda_n' - \lambda_k'$, with sufficiently large integer p, a direct estimate of the integral, observing

$$\xi_k(p + \lambda_k' - \lambda_n', u) - e_k \frac{(- \lambda_k - u)^{p - 1}}{\Gamma(p)} = O((- \lambda_k - u)^p) \quad (u \longrightarrow - \lambda_k),$$

gives

$$\frac{\Gamma(s + \lambda_k' - \lambda_n')}{(\lambda_n - \lambda_k)^{s + \lambda_k' - \lambda_n' - 1}} \xi_k(s, - \lambda_n) = e_k + O_y(x^{-1}) \quad \text{as} \quad x \longrightarrow \infty.$$

If we define $g_k(s)$ by dropping the last component from $(\lambda_n - \lambda_k)^{1 - \lambda_k' + \lambda_n'}$ $\xi_k(s, - \lambda_n)$, then $G(s) = [g_1(s), \dots, g_n(s)]^T$ satisfies i). To show ii), we restrict for notational convenience to $k = 1$; for other k, the proof follows the same lines. Defining

$$\tilde{\xi}_1(s, t) = (\lambda_1 - \lambda_n)^{1 + \lambda_n' - \lambda_1' - s} \xi_1(s, t(\lambda_1 - \lambda_n) + \lambda_1),$$

one can see that

$$(s - \hat{A}_1)\tilde{\xi}_1(s + 1, t) = (\tilde{A}_o - t)\tilde{\xi}_1(s, t),$$
$$\tilde{A}_o = diag\left[0, \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_n}, \dots, \frac{\lambda_1 - \lambda_{n-1}}{\lambda_1 - \lambda_n}, 1\right],$$
$$\frac{d}{dt} \tilde{\xi}_1(s + 1, t) = - \tilde{\xi}_1(s, t).$$

From the expansion formulas in [2], one finds that $\tilde{\xi}_1(s, 1)$ (for sufficiently large x) is analytic in the parameters $u_j = \dfrac{\lambda_1 - \lambda_j}{\lambda_1 - \lambda_n}$, $2 \leq j \leq n - 1$, for u_j in a complex plane including ∞, with a cut from 0 to 1 along the real axis, and continuous, e.g., on the lower border of this cut (excluding the endpoints). The mapping $w = (1-u)^{-1}$ maps this cut plane onto the w-plane with a cut from 1 to ∞ (and the lower borders of the cuts onto each other), and u_j is mapped onto $w_j^{(1)}$, $2 \leq j \leq n-1$. Hence this completes the proof.

2. A series representation for $G(s)$

In this section, we represent $g_1^T(s)$ by a convergent series, provided the parameters $w_j^{(n)}$ are all inside the unit disc ($j = 2,\dots,n-1$). A similar representation holds for the other rows of $G(s)$.

Let $h_1(s)$ be a solution of

$$(s + B_1^T)h_1(s+1) = B_o h_1(s), \tag{2.1}$$

satisfying (with $s = x + iy$)

$$h_1(s)\frac{\Gamma(s + \lambda_1' - \lambda_n')}{(\lambda_n - \lambda_1)^s} = e_1 + O_y(x^{-1}), \quad \text{as} \quad x \longrightarrow \infty, \tag{2.2}$$

and being an entire function in s. Such an $h_1(s)$ exists, according to [1], Proposition 4 of part II. Let

$$B(s) = B_0^{-1}(s + B_1 - s^{-1}ab^T), \tag{2.3}$$

and use the notation

$$(B(s))_o \equiv I, \ (B(s))_j = B(s+j-1)\dots B(s), \ j \geq 1.$$

Then we have

THEOREM 2. *For*

$$|w_j^{(1)}| < 1, \quad 2 \leq j \leq n-1, \tag{2.4}$$

the first row $g_1^T(s)$ of $G(s)$ of Theorem 1 is given by

$$g_1^T(s) = h_1^T(s) - \sum_{j=0}^{\infty} (s+j)^{-1} h_1^T(s+j+1)ab^T(B(s))_j B_0^{-1}, \tag{2.5}$$

for every s in G, with

$$G = \mathbb{C} - \{0, -1, -2, \cdots\},$$

and the series converges compactly with respect to s in G.

PROOF. For $s \in G$ *we have*

$$B_o(B(s))_{j+1} = (s + j + B_1 - (s + j)^{-1} ab^T)(B(s))_j, \quad j \geq 0,$$

hence from (1.1) we conclude that $G(s + j)B_o(B(s))_j$ is independent of j, and therefore (observe $j = 0$)

$$G(s)B_o = G(s + j)B_o(B(s))_j, \quad j \geq 0.$$

Hence from (1.4), (2.2), (2.4)

$$\begin{aligned} h_1^T(s + j + 1)\, ab^T(B(s))_j B_0^{-1} &= h_1^T(s + j + 1)\, ab^T B_0^{-1} G^{-1}(s + j) G(s) \\ &= O_s(j^{-1}) \quad \text{as} \quad j \longrightarrow \infty, \end{aligned}$$

with a O_s-constant that may be seen to be locally uniform with respect to $s \in G$. This implies the convergence of (2.5). If we momentarily make (2.5) the definition of $g_1^T(s)$, then an immediate computation shows that $g_1^T(s)$ is a solution of (1.1), hence $c^T(s) = g_1^T(s)G^{-1}(s)$ is one-periodic in s. But from (2.5),

$$c_1^T(s) = h_1^T(s)G^{-1}(s) - \sum_{j=0}^{\infty} (s + j)^{-1} h_1^T(s + j + 1)\, ab^T G^{-1}(s + j) B_o.$$

If we replace s by $s + k$, k a natural number, then the series may be easily seen to vanish for $k \to \infty$.
Hence

$$c_1^T(s) = \lim_{k \to \infty} h_1^T(s + k) G^{-1}(s + k) \equiv e_1^T.$$

This completes the proof.

3. A series representation for the characteristic constants

We now turn our attention to the computation of the characteristic constants of (0.2) (or equivalently, the Stokes' multipliers of (0.1)). Here we concentrate upon the constants in the n^{th} columns of the Stokes' multipliers; an easy variation of the resulting formulas gives the other constants as well. So we are to find a constant vector of length $n - 1$, say

$$v = [v_1, \ldots, v_{n-1}]^T,$$

which, according to [1], part II, Corollary 1 in section 5, or the (earlier) results of R. Schäfke [7], satisfies

$$f_n(k) = \sum_{j=1}^{n-1} v_j \alpha_j(\lambda_n' - k + 1), \quad k \geq 1 \tag{3.1}$$

with

$$\alpha_j(s) = \xi_j^*(s, \alpha_n),$$

and $f_n(k)$ the coefficients of the n^{th} formal solution of (0.1). For this vector, we obtain the following representation:

THEOREM 3 *With $G(s)$ as in Theorem* 1, *and*

$$D = 2\pi i \; diag\,[(\lambda_n - \lambda_1)^{\lambda_1' - \lambda_n' - 1}, \ldots, (\lambda_n - \lambda_{n-1})^{\lambda_{n-1}' - \lambda_n' - 1}], \tag{3.2}$$

we have

$$v = DG(1)a. \tag{3.3}$$

In particular, if (2.4) *holds, then*

$$v_1 = h(1) - \sum_{j=0}^{\infty} \frac{h(j+2)}{j+1} p(j), \tag{3.4}$$

with

$$h(j) = 2\pi i(\lambda_n - \lambda_1)^{\lambda_1' - \lambda_n' - 1} h_1^T(j)a, \quad j \geq 1$$
$$p(j) = b^T(B(1))_j B_o^{-1} a, \qquad j \geq 0.$$

PROOF. It may be seen to follow from [2], Part II, Expansion Theorem + Theorem 2, that $(\lambda_n - \lambda_1)^{\lambda_n' - \lambda_1'} v_1$ is an analytic function of $w_j^{(1)}$, $2 \leq j \leq n-1$, in a complex plane with a cut from 1 to ∞ along the positive real axis, and extends continously to the lower border of this cut. Hence in view of Theorem 1 it is sufficient to show

$$v_1 = 2\pi i(\lambda_n - \lambda_1)^{\lambda_1' - \lambda_n' - 1} g_1^T(1)a \tag{3.5}$$

in the situation of (2.4). Under this assumption, let $\beta_j(s)$ be the $(n-1)$-vector obtained from $\alpha_j(\lambda_n' - s + 1)$ by dropping the last component. Since $\xi_j^*(s, t)$ is a solution vector of (0.3), we find

$$(s + B_1 - s^{-1} ab^T)\beta_j(s) = B_o \beta_j(s+1), \quad 1 \leq j \leq n-1,$$

and from [1], Part II we conclude

$$2\pi i \beta_j(s) = (\lambda_n - \lambda_j)^{\lambda_n' - \lambda_j' - s} \Gamma(s + \lambda_j' - \lambda_n')(e_j + 0_y(x^{-1})); \quad \text{as} \quad x \longrightarrow \infty.$$

The matrix

$$\tilde{G}(s) = \{B_o[\beta_1(s), \ldots, \beta_{n-1}(s)]\}^{-1}$$

may be seen to exist for x sufficiently large and is a solution of (1.1). The first row $\tilde{g}_1^T(s)$ of $\tilde{G}(s)$ then satisfies

$$\Gamma(s + \lambda_1' - \lambda_n')\tilde{g}_1^T(s) = 2\pi i(\lambda_n - \lambda_1)^{s + \lambda_1' - \lambda_n' - 1}(e_1^T + O_y(x^{-1})) \quad \text{as} \quad x \longrightarrow \infty$$

and due to assumption (2.4) (under which the first row of $G(s)$ is uniquely determined by (1.4)) we conclude

$$\tilde{g}_1^T(s) = 2\pi i(\lambda_n - \lambda_1)^{\lambda_1' - \lambda_n' - 1} g_1^T(s).$$

If $\tilde{f}(k)$ is obtained from $f_n(k)$ by dropping the last component, then (3.1) implies

$$v = \tilde{G}(k) B_o \tilde{f}(k) \quad (k \geq 1),$$

and since (compare the recursion formulas for formal solutions)

$$\tilde{f}(1) = B_o^{-1} a,$$

this proves (3.5).

REMARK. In case $n = 2$ it is easy to find $h(j)$ explicitly, and to see that (3.4) becomes the expansion of a hypergeometric function with the variable equal to one. Since such a series may be summed in terms of Gamma functions, one can easily rediscover the known formulas for the Stokes' multipliers in this case.

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