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Growth curve model with covariance structures

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Contents

0. Introduction

Potthoff and Roy [33] first proposed the growth curve model which is a generalized multivariate analysis of variance model. The growth curve model is defined as

$$\underset{N\times p}{Y} = \underset{N\times k}{A}\ \underset{k\times q}{\Xi}\ \underset{q\times p}{B} + \underset{N\times p}{\varepsilon}, \tag{0.1}$$

where Y is an observed random matrix, A and B are known design matrices of ranks k and $q \leq p$, respectively, Ξ is an unknown parameter matrix, and the rows of ε are independent and identically distributed random vectors with distribution $N_p(\mathbf{0}, \Sigma)$. In most applications of the model, p is the number of time points observed on each of the N subjects, $(q-1)$ is the degree of the polynomial, and k is the number of groups. Especially, Rao [34] gave the analysis of such data for a single group of individuals by using a multivariate approach.

This model was studied by Potthoff and Roy [33] including Rao [34], [35], [37], Khatri [23], Grizzle and Allen [15], Timm [46], Lee [27], [28], Lee and Geisser [29], Reinsel [39], [40], Fujikoshi, Kanda and Tanimura [9], Kanda and Fujikoshi [22] and many others. A comprehensive review of growth curve analysis was given by Woolson and Leeper [50], and also has been recently given by von Rosen [42]. Many authors have considered the case where covariance structure is positive definite. However, Rao [36] has considered several covariance structure. Azzalini [3], [4], Glasbey [12], Hudson [17], Lee [27], Fujikoshi, Kanda and Tanimura [9] have investigated an autoregressive structure of the first order (or serial covariance structure), i.e.,

$$\begin{aligned} \Sigma &= \sigma^2 G_s(\rho) \\ &= \sigma^2(\rho^{|i-j|}), \qquad i, j = 1, 2, \dots, p, \end{aligned} \tag{0.2}$$

where $\sigma > 0$ and $-1 < \rho < 1$ are unknown.

Reinsel [40], Lee [27] and Kanda [19], [20], [21] have discussed a uniform covariance structure, i.e.,

$$\begin{aligned} \Sigma &= \sigma^2 G_u(\rho) \\ &= \sigma^2[(1-\rho)I_p + \rho \mathbf{1}_p \mathbf{1}_p'], \end{aligned} \tag{0.3}$$

where $\sigma > 0$ and $-1/(p-1) < \rho < 1$ are unknown, I_p is the identity matrix of order p and $\mathbf{1}_p = (1, \dots, 1)'$. Lee [27] has noted that the uniform covariance structure may not be particularly useful for time series data, but it could be useful for growth-curve data where the observations are a mixture of several populations.

On the other hand, when the coefficient parameter Ξ of polynomial is not uniform within group, it is necessary to consider the variation of individuals. One of such models is

$$\underset{N\times p}{Y} = \underset{N\times k}{A}\ \underset{k\times q}{\Xi}\ \underset{q\times p}{B} + \underset{N\times q}{\Theta}\ \underset{q\times p}{B} + \underset{N\times p}{\varepsilon}, \tag{0.4}$$

where Θ is an $N \times q$ matrix of unknown parameters, whose rows are independent and identically distributed random vectors with distribution $N_q(\mathbf{0}, \Delta)$, and the rows of ε are independent and identically distributed random vectors with distribution $N_p(\mathbf{0}, \sigma^2 I_p)$, Δ is an arbitrary positive semi-definite matrix and $\sigma^2 > 0$. This model is known as random effects model. In this case, let $E = \Theta B + \varepsilon$, model (0.4) is a usual growth curve model (0.1) with covariance matrix

$$\Sigma = B'\Delta B + \sigma^2 I_p, \tag{0.5}$$

which is called random effects covariance matrix. Random effects covariance structure has advantages of decreasing the number of unknown covariance elements and leads to a simple structure. This model was introduced by Rao [35] as a family of two-stage models for serial measurements, and was considered by Azzalini [4], Rao [36], [37], Reinsel [38], [41], Laird and Ware [26] and Ware [49].

Problem of prediction was considered by Lee and Geisser [29], Reinsel

[40], Rao [37], Lee [27], Kanda [21] and others. In particular, Lee [27] has studied prediction problems where Σ has a uniform covariance structure and a serial covariance structure, Reinsel [40] has noted prediction problems where Σ has an arbitrary covariance structure and a uniform covariance structure, and Kanda [21] has discussed prediction problems where Σ has arbitrary covariance structure, uniform covariance structure and serial covariance structure. Prediction problem for random effects covariance structure has been considered by Reinsel [39].

Testing hypotheses for mean parameter in the growth curve model were given by Khatri [23] and Gleser and Olkin [14] when the covariance structure is positive definite. Fujikoshi [7] has obtained asymptotic expansions of the non-null distributions of three statistics for testing hypotheses for mean parameter in GMANOVA model.

Confidence intervals or regions in growth curve model have been derived by Potthoff and Roy [33], Khatri [23], Rao [35], [36], Grizzle and Allen [15], Fujikoshi and Nishii [10], Srivastava and Carter [45] and many other authors. Fujikoshi and Nishii [10] gave some asymptotic comparisons of confidence regions.

Study of growth curve model in the case of missing data has been done by Kleinbaum [24]. Srivastava [44] obtained the MLE's (maximum likelihood estimators) by writing down the likelihood function and derived the LR (likelihood ratio) test using these estimators. Liski [30] estimated the parameters by using EM-algorithm, which was given by Dempster, Laird and Rubin [6], in a growth curve problem. However, these analyses are investigated under positive definite covariance structure.

In this paper, we consider the cases where Σ has arbitrary covariance structure, uniform covariance structure, and serial covariance structure. Random effects covariance structures are important but not considered here. In part I, we discuss the case of complete data in growth curve model with covariance structures. Part II deals with the case of missing data. In Section 1, the properties of the MLE's of (0.1) with the covariance structures, and their density functions are given. In Section 2, we derive asymptotic expansions of the distributions of MLE's and the LR statistics for testing the uniform covariance structure (0.3). In Section 3, testing hypotheses for mean parameters with covariance structures are considered. Confidence intervals and regions are obtained in Section 4. Sections 5–7 are concerned with the case of missing data. In Section 5, we derive the MLE's in the growth curve model with covariance structures. Asymptotic comparisons of the MLE's in the cases of complete data and missing data are discussed in Section 6. In the last section we consider testing hypotheses for mean parameters in the case of missing data.

Part I. Growth curve model with complete data

1. MLE's and their density functions

In what follows, three types of covariance structures are considered: they are positive definite covariance structure, uniform covariance structure and serial covariance structure. The MLE's of (0.1) under each covariance structure are derived. Our asymptotic distribution theory is based on perturbation method. We shall use stochastic expansions of the MLE's in terms of

$$U = (A'A)^{-1/2}A'(Y - A\Xi B)\Sigma^{-1/2}, \tag{1.1}$$

$$V = \sqrt{n}(\Sigma^{-1/2}S\Sigma^{-1/2} - I_p), \tag{1.2}$$

where $S = n^{-1}Y'(I_N - A(A'A)^{-1}A')Y$ and $n = N - k$. Then, U and V are independent, the rows of U are distributed as $N_p(\mathbf{0}, I_p)$. For the probability density function of V, we need the following lemma which was proved by Fujikoshi [8].

LEMMA 1.1. *Let V be a symmetric random matrix defined by* (1.2), *where nS is distributed as a Wishart distribution $W_p(\Sigma, n)$. Then, the probability density function of V can be expanded as*

$$f(V) = f_0(V)[1 + n^{-1/2}q_1(V) + n^{-1}q_2(V)] + O(n^{-3/2}), \tag{1.3}$$

where

$$f_0(V) = \pi^{-p(p+1)/4}2^{-p(p+3)/4}\,\mathrm{etr}\left(-\frac{1}{4}V^2\right), \tag{1.4}$$

$$q_1(V) = -\frac{1}{2}(p+1)\,\mathrm{tr}\,V + \frac{1}{6}\mathrm{tr}\,V^3, \tag{1.5}$$

$$q_2(V) = \frac{1}{2}\{q_1(V)\}^2 - \frac{1}{24}p(2p^2 + 3p - 1) + \frac{1}{4}(p+1)\,\mathrm{tr}\,V^2 - \frac{1}{8}\mathrm{tr}\,V^4. \tag{1.6}$$

We will derive the following stochastic expansion of the MLE $\hat{\Xi}$ represented in terms of U and V:

$$\hat{\Xi} = \Xi + n^{-1/2}\Xi_1 + n^{-1}\Xi_2 + n^{-3/2}\Xi_3 + O_p(n^{-2}). \tag{1.7}$$

Throughout this paper, for simplicity, we denote $\tilde{B} = B\Sigma^{-1/2}$ and $P_B = \tilde{B}'(\tilde{B}\tilde{B}')^{-1}\tilde{B}$. We also use the notations

$$\Psi(X) = (\tilde{B}\tilde{B}')^{-1/2}\tilde{B}X(I_p - P_B)X\tilde{B}'(\tilde{B}\tilde{B}')^{-1/2},$$

and

$$\Phi(X)=(2\pi)^{-kq/2}\exp\left\{-\frac{1}{2}\operatorname{tr}X'X\right\},$$

which is the kq-dimensional standard normal density function. Each covariance structure is studied in the following.

1.1. Positive definite covariance structure

When Σ is positive definite matrix, it is well known (see, e.g., Khatri [23]) that the MLE's $\hat{\Xi}$ and $\hat{\Sigma}$ of Ξ and Σ, respectively, are given in the following forms.

$$\hat{\Xi}=(A'A)^{-1}A'YS^{-1}B'(BS^{-1}B')^{-1}, \tag{1.8}$$

$$\hat{\Sigma}=N^{-1}(Y-A\hat{\Xi}B)'(Y-A\hat{\Xi}B). \tag{19}$$

It is easily seen that the Ξ_i's in (1.7) are given by

$$\Xi_1=\sqrt{n}(A'A)^{-1/2}U\tilde{B}'(\tilde{B}\tilde{B}')^{-1}, \tag{1.10}$$

$$\Xi_2=\sqrt{n}(A'A)^{-1/2}U(P_B-I_p)V\tilde{B}'(\tilde{B}\tilde{B}')^{-1}, \tag{1.11}$$

$$\Xi_3=\sqrt{n}(A'A)^{-1/2}U(P_B-I_p)V(P_B-I_p)V\tilde{B}'(\tilde{B}\tilde{B}')^{-1}. \tag{1.12}$$

Gleser and Olkin [13] obtained exact density function $f_1(X)$ of $(A'A)^{1/2}(\hat{\Xi}-\Xi)(\tilde{B}\tilde{B}')^{1/2}$ by using geometric distribution. However, this result is not simple expression. Fujikoshi and Shimizu [11] gave an asymptotic expansion of the density function $f_1(X)$ up to order n^{-1} and its error bound. The expansion is given in the following theorem.

THEOREM 1.1. *The density function $f_1(X)$ of $(A'A)^{1/2}(\hat{\Xi}-\Xi)(\tilde{B}\tilde{B}')^{1/2}$ can be expanded as*

$$f_1(X)=\Phi(X)\left[1+\frac{1}{2n}\cdot(p-q)(\operatorname{tr}X'X-kq)+O(n^{-2})\right].$$

1.2. Uniform covariance structure

The MLE's $\hat{\Xi}$, $\hat{\sigma}^2$ and $\hat{\rho}$ of Ξ, σ^2 and ρ under uniform covariance structure (0.3) are given as the solutions of the following equations (see, e.g., Lee [27] and Kanda [20]):

$$\hat{\Xi}=(A'A)^{-1}A'Y\hat{\Sigma}^{-1}B'(B\hat{\Sigma}^{-1}B')^{-1}, \tag{1.13}$$

$$\hat{\sigma}^2=ne/(pN), \tag{1.14}$$

$$d=\{1+(p-1)\hat{\rho}\}e, \tag{1.15}$$

where $\hat{\Sigma}=\hat{\sigma}^2G_u(\hat{\rho})$, $d=\mathbf{1}_p'R\mathbf{1}_p$, $e=\operatorname{tr}R$ and $R=n^{-1}(Y-A\hat{\Xi}B)'(Y-A\hat{\Xi}B)$.

It is easily seen that $\hat{\rho}$ with (1.13), (1.14) and (1.15) satisfies the condition $-1/(p-1) < \hat{\rho} < 1$.

We can write R in terms of U and V as

$$
\begin{aligned}
R &= S + n^{-1}\{I_p - \hat{\Sigma}^{-1}B'(B\hat{\Sigma}^{-1}B')^{-1}B\}' \\
&\quad \times \Sigma^{1/2}U'U\Sigma^{1/2}\{I_p - \hat{\Sigma}^{-1}B'(B\hat{\Sigma}^{-1}B')^{-1}B\} \qquad (1.16)\\
&= \Sigma^{1/2}\{I_p + n^{-1/2}V + n^{-1}W\}\Sigma^{1/2} + O_p(n^{-3/2}),
\end{aligned}
$$

where $W = (I_p - P_B)U'U(I_p - P_B)$. This implies stochastic expansions of d and e as follows:

$$d = d_0 + n^{-1/2}d_1 + n^{-1}d_2 + O_p(n^{-3/2}), \tag{1.17}$$

$$e = e_0 + n^{-1/2}e_1 + n^{-1}e_2 + O_p(n^{-3/2}), \tag{1.18}$$

where (d_0, d_1, d_2) and (e_0, e_1, e_2) are defined by $(\mathrm{tr}\,\tilde{D}_i, \mathrm{tr}\,\tilde{D}_iV, \mathrm{tr}\,\tilde{D}_iW)$, $i = 1, 2$, respectively, and $\tilde{D}_i = \Sigma^{1/2}D_i\Sigma^{1/2}$, $i = 1, 2$, $D_1 = \mathbf{1}_p\mathbf{1}_p'$, $D_2 = I_p$. Then we have

$$(1) \qquad \hat{\rho} = \rho + n^{-1/2}\rho_1 + n^{-1}\rho_2 + O_p(n^{-3/2}),$$

$$(2) \qquad \hat{\tau} = \tau + n^{-1/2}\tau_1 + n^{-1}\tau_2 + O_p(n^{-3/2}),$$

where

$$
\begin{aligned}
\rho_1 &= \{p(p-1)\sigma^2\}^{-1}(d_1 - se_1),\\
\rho_2 &= \{p(p-1)\sigma^2\}^{-1}\{d_2 - se_2 - (p-1)e_1\rho_1\},\\
\tau_1/\tau &= -(\tau/p)[p\sigma^2\{(p-1)(1-\rho) - s\}\rho_1 + (1-\rho)se_1],\\
\tau_2/\tau &= -(\tau_1/p)[p\sigma^2\{(p-1)(1-\rho) - s\}\rho_1 + (1-\rho)se_1]\\
&\quad -(\tau/p)[\{(p-1)(1-\rho) - s\}e_1\rho_1 + (1-\rho)se_2]\\
&\quad +\tau\sigma^2[(p-1)\rho_1^2 - \{(p-1)(1-\rho) - s\}\rho_2],
\end{aligned}
$$

$s = 1 + (p-1)\rho$, $\tau = \{\sigma^2(1-\rho)s\}^{-1}$ and $\hat{\tau} = [\hat{\sigma}^2(1-\hat{\rho})\{1 + (p-1)\hat{\rho}\}]^{-1}$.

Result (1) is obtained by substituting (1.1), (1.2) and (1.16) into equation (1.15) and finding the solution of $\hat{\rho}$ in an expanded form. This result and equation (1.14) yield result (2). Furthermore, these results and equation (1.7) yield

$$\Xi_1 = \sqrt{n}(A'A)^{-1/2}U\tilde{B}'(\tilde{B}\tilde{B}')^{-1}, \tag{1.19}$$

$$\Xi_2 = \sqrt{n}(A'A)^{-1/2}U(P_B - I_p)K\tilde{B}'(\tilde{B}\tilde{B}')^{-1}, \tag{1.20}$$

$$\Xi_3 = \sqrt{n}(A'A)^{-1/2}U(P_B - I_p)(KP_BK + L)\tilde{B}'(\tilde{B}\tilde{B}')^{-1}, \tag{1.21}$$

where

$$K = \mu I_p + \nu \Sigma,$$

$$L = \frac{\tau\rho_2 + \tau_1\rho_1}{\rho}\Sigma - \frac{\tau\rho_2 + \tau_1\rho_1 + \tau_2\rho}{\tau\rho} I_p,$$

$$\mu = -\left(\frac{\rho_1}{\rho} + \frac{\tau_1}{\tau}\right) \quad \text{and} \quad \nu = \frac{\tau\rho_1}{\rho}.$$

Let $\phi_2(T)$, where T is a $q \times k$ matrix, be the characteristic function of $\sqrt{n}(\hat{\Xi} - \Xi)$. Then $\phi_2(T)$ may be written as

$$\phi_2(T) = E\left[\exp(i \operatorname{tr} T\Xi_1)\left\{1 + \frac{i}{\sqrt{n}} \operatorname{tr} T\Xi_2 + \frac{1}{n}\left(i \operatorname{tr} T\Xi_3 + \frac{i^2}{2}(\operatorname{tr} T\Xi_2)^2\right)\right\}\right] + O(n^{-3/2}),$$

which will be evaluated as

$$\exp\left(-\frac{1}{2}\operatorname{tr} n(A'A)^{-1}T'(\tilde{B}\tilde{B}')^{-1}T\right)\left[1 + \frac{i^2}{2n}\cdot\frac{2}{p(p-1)\rho^2\sigma^4}\operatorname{tr} n(A'A)^{-1}T'\Psi(\Sigma)T\right] + O(n^{-2}).$$

Thus, the characteristic function of $(A'A)^{1/2}(\hat{\Xi} - \Xi)(\tilde{B}\tilde{B}')^{1/2}$ is

$$\exp\left(-\frac{1}{2}\operatorname{tr} T'T\right)\left[1 + \frac{i^2}{2n}\cdot\frac{2}{p(p-1)\rho^2\sigma^4}\operatorname{tr} T'\Psi(\Sigma)T\right] + O(n^{-2}).$$

Inverting this characteristic function, we can summarize in the following

THEOREM 1.2. *The density function $f_2(X)$ of $(A'A)^{1/2}(\hat{\Xi} - \Xi)(\tilde{B}\tilde{B}')^{1/2}$ can be expanded as*

$$f_2(X) = \Phi(X)\left[1 + \frac{1}{2n}\cdot\frac{2}{p(p-1)\rho^2\sigma^4}\{\operatorname{tr} X'\Psi(\Sigma)X - k\operatorname{tr}(\Psi(\Sigma))\}\right] + O(n^{-2}).$$

We note that the expansion remains valid even if $\rho = 0$ since $\operatorname{tr} X'\Psi(\Sigma)X - k\operatorname{tr}(\Psi(\Sigma))$ has the factor ρ^2. Other properties and results are given in Section 2.

1.3. Serial covariance structure

Anderson [2] has obtained MLE's $\hat{\rho}$ and $\hat{\sigma}^2$ in a time series setting, and Azzalini [3], [4] has derived the MLE's for the growth curve model for an AR(1) covariance structure. Lee [27] and Fujikoshi, Kanda and Tanimura [9] have also obtained the MLE's for the growth curve model with serial covariance structure (0.2). It is known (see, e.g., Fujikoshi, Kanda and Tanimura [9]) that the MLE's $\hat{\Xi}$, $\hat{\sigma}^2$ and $\hat{\rho}$ of Ξ, σ^2 and ρ based on Y are given as the solutions of the following equations:

$$\hat{\Xi} = (A'A)^{-1}A'Y\hat{\Sigma}^{-1}B'(B\hat{\Sigma}^{-1}B')^{-1}, \tag{1.22}$$

$$\hat{\sigma}^2 = \frac{n}{N}\{p(1-\hat{\rho}^2)\}^{-1}(a\hat{\rho}^2 - 2b\hat{\rho} + c), \tag{1.23}$$

$$(p-1)a\hat{\rho}^3 - (p-2)b\hat{\rho}^2 - (pa+c)\hat{\rho} + pb = 0, \tag{1.24}$$

where $\hat{\Sigma} = \hat{\sigma}^2 G_s(\hat{\rho})$, $a = \operatorname{tr} D_1 R$, $b = \operatorname{tr} D_2 R$, $c = \operatorname{tr} D_3 R$, $R = n^{-1}(Y - A\hat{\Xi}B)' \times (Y - A\hat{\Xi}B)$, $D_1 = \operatorname{diag}(0, 1, \dots, 1, 0)$, $D_3 = I_p$ and

$$D_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \tag{1.25}$$

Similar to the case of the uniform covariance structure, stochastic expansions of a, b and c are as follows:

$$a = a_0 + n^{-1/2}a_1 + n^{-1}a_2 + O_p(n^{-3/2}),$$
$$b = b_0 + n^{-1/2}b_1 + n^{-1}b_2 + O_p(n^{-3/2}),$$
$$c = c_0 + n^{-1/2}c_1 + n^{-1}c_2 + O_p(n^{-3/2}),$$

where (a_0, a_1, a_2), (b_0, b_1, b_2) and (c_0, c_1, c_2) are defined by $(\operatorname{tr}\tilde{D}_i, \operatorname{tr}\tilde{D}_i V, \operatorname{tr}\tilde{D}_i W)$, $i = 1, 2, 3$, respectively, and $\tilde{D}_i = \Sigma^{1/2}D_i\Sigma^{1/2}$, $i = 1, 2, 3$. Then we have (see, e.g., Kanda and Fujikoshi [22])

$$\hat{\rho} = \rho + n^{-1/2}\rho_1 + n^{-1}\rho_2 + O_p(n^{-3/2}), \tag{3}$$

$$\hat{\tau} = \tau + n^{-1/2}\tau_1 + n^{-1}\tau_2 + O_p(n^{-3/2}), \tag{4}$$

where

$$\begin{aligned}
\rho_1 &= -\{(p-1)r\sigma^2\}^{-1}\{(r-\rho^2)\rho a_1 - rb_1 + \rho c_1\}, \\
\rho_2 &= 2r^{-1}(p-2)\rho\rho_1^2 - \{(p-1)r\sigma^2\}^{-1}\rho_1[\{p-3(p-1)\rho^2\}a_1 - 2(p-2)\rho b_1 + c_1] \\
&\quad - \{(p-1)r\sigma^2\}^{-1}[(r-\rho^2)\rho a_2 - rb_2 + \rho c_2], \\
\frac{\tau_1}{\tau} &= \frac{1}{p}\left[\frac{2\rho\rho_1}{1-\rho^2} - \operatorname{tr} V\right], \\
\frac{\tau_2}{\tau} &= \frac{1}{p}\left[\frac{2\rho\rho_2}{1-\rho^2} - \frac{p-2}{1-\rho^2}\rho_1^2 - \frac{\rho_1}{\rho}(\operatorname{tr} V - \tau \operatorname{tr} QV) - \operatorname{tr} W\right]
\end{aligned}$$

$$+\frac{1}{p^2}\left(\operatorname{tr} V-\frac{2\rho\rho_1}{1-\rho^2}\right)^2,$$

$Q=\Sigma-\rho^2\tilde{D}_1$, $\tau=\{\sigma^2(1-\rho^2)\}^{-1}$, $\hat{\tau}=\{\hat{\sigma}^2(1-\hat{\rho}^2)\}^{-1}$ and $r=p-(p-2)\rho^2$. These results and equation (1.7) yield

$$\Xi_1=\sqrt{n}(A'A)^{-1/2}U\tilde{B}'(\tilde{B}\tilde{B}')^{-1}, \tag{1.26}$$

$$\Xi_2=\sqrt{n}(A'A)^{-1/2}U(P_B-I_p)K\tilde{B}'(\tilde{B}\tilde{B}')^{-1}, \tag{1.27}$$

$$\Xi_3=\sqrt{n}(A'A)^{-1/2}U(P_B-I_p)(KP_BK+L)\tilde{B}'(\tilde{B}\tilde{B}')^{-1}, \tag{1.28}$$

where

$$K=\mu I_p+\nu Q,$$

$$L=\frac{\tau\rho_2+\tau_1\rho_1+\tau_2\rho}{\tau\rho}I_p-\frac{\tau\rho_2+\tau_1\rho_1}{\rho}Q+\tau\rho_1^2\tilde{D}_1,$$

$$\mu=-\left(\frac{\rho_1}{\rho}+\frac{\tau_1}{\tau}\right) \quad \text{and} \quad \nu=\frac{\tau\rho_1}{\rho}.$$

Let $\phi_3(T)$ be the characteristic function of $\sqrt{n}(\hat{\Xi}-\Xi)$. Then we can easily obtain $\phi_3(T)$ by the same way as in the case of uniform covariance structure;

$$\exp\left(-\frac{1}{2}\operatorname{tr} n(A'A)^{-1}T'(\tilde{B}\tilde{B}')^{-1}T\right)\left[1+\frac{i^2}{2n}\cdot\frac{p}{(p-1)r\rho^2\sigma^4}\operatorname{tr} n(A'A)^{-1}T'\Psi(Q)T\right]$$
$$+O(n^{-2}).$$

Thus, the characteristic function of $(A'A)^{1/2}(\hat{\Xi}-\Xi)(\tilde{B}\tilde{B}')^{1/2}$ is

$$\exp\left(-\frac{1}{2}\operatorname{tr} T'T\right)\left[1+\frac{i^2}{2n}\cdot\frac{p}{(p-1)r\rho^2\sigma^4}\operatorname{tr} T'\Psi(Q)T\right]+O(n^{-2}).$$

This implies the following theorem similar to the case of the uniform covariance structure.

THEOREM 1.3. *The density function $f_3(X)$ of $(A'A)^{1/2}(\hat{\Xi}-\Xi)(\tilde{B}\tilde{B}')^{1/2}$ can be expanded as*

$$f_3(X)=\Phi(X)\left[1+\frac{1}{2n}\cdot\frac{p}{(p-1)r\rho^2\sigma^4}\{\operatorname{tr} X'\Psi(Q)X-k\operatorname{tr}(\Psi(Q))\}\right]+O(n^{-2}).$$

As is the case with the uniform covariance structure, we also note that this expansion remains valid even if $\rho=0$. For other properties and results, see [9] and [22].

2. Growth curve model with a uniform covariance structure

In this section, we consider the case when Σ has a uniform covariance structure (0.3). This structure has been studied by many authors, including Lee [27] and Kanda [21]. Lee [27] has considered the prediction problems where Σ has a uniform covariance structure and a serial covariance structure (0.2). He has noted that the uniform covariance structure may not be particularly useful for time series data, but it could be useful for growth-curve data when the observations are a mixture of several populations. Fujikoshi, Kanda and Tanimura [9] have studied the limiting distributions of the MLE's of ρ and σ^2 and the LR test for the model (0.1) with (0.2) in the situation where p and k are fixed and $N \to \infty$. Kanda and Fujikoshi [22] have extended their limiting results by finding the next term in the asymptotic expansion. In Subsection 2.1, some preliminary results on our asymptotic method are given. In Subsection 2.2, we obtain an asymptotic expansion of the distribution of the MLE's of ρ and σ^2 up to the order $N^{-1/2}$. We discuss the refinements of chi-square approximation to the null distribution of LR statistic for testing the uniform covariance structure (0.3) in Subsection 2.3. The data which were treated in [33] are examined in Subsection 2.4. Throughout this section it is assumed that Σ has the uniform covariance structure (0.3).

2.1. Preliminaries

Under (0.3), it is well known that

$$|\Sigma| = (\sigma^2)^p(1-\rho)^{p-1}s,$$
$$\Sigma^{-1} = \{\sigma^2(1-\rho)\}^{-1}(D_2 - \rho s^{-1}D_1),$$

where $D_1 = \mathbf{1}_p\mathbf{1}_p'$, $D_2 = I_p$ and $s = 1 + (p-1)\rho$. In this case, it is known (see, e.g., Lee [27], Kanda [21]) that the estimators $\hat{\Xi}$, $\hat{\sigma}^2$ and $\hat{\rho}$ of Ξ, σ^2 and ρ based on the likelihood of Y are given as the solutions of (1.13), (1.14) and (1.15). Note that the MLE's of Ξ, ρ and σ^2 are given by $\hat{\Xi}$, $\hat{\rho}$ and $(n/N)\hat{\sigma}^2$, respectively. Furthermore, in most applications of the growth curve model, B has the form $(\mathbf{1}_p, B_2')'$, where B_2 is any $(q-1)\times p$ matrix. It is known (see, e.g., Lee and Geisser [29]) that if $B = (\mathbf{1}_p, B_2')'$, then the equation (1.13) is replaced by

$$\hat{\Xi} = (A'A)^{-1}A'YB'(BB')^{-1}. \tag{2.1}$$

This result can be easily checked by inserting $B = (\mathbf{1}_p, B_2')'$ into (1.13). In fact, the equation (2.1) dose not involve $\hat{\sigma}^2$ and $\hat{\rho}$, and hence (1.13), (1.14) and (1.15) are solved exactly.

Our asymptotic distribution theory is based on perturbation method. In deriving stochastic expansions of the MLE's $\hat{\sigma}^2$ and $\hat{\rho}$ in terms of U and V in (1.1) and (1.2), we use (1.14), (1.15), (1.17) and (1.18).

The formulas in Lemma 2.1 are used in Subsections 2.2 and 2.3. These are easily obtained by straightforward calculations.

LEMMA 2.1. *Let Σ be the matrix defined by* (0.3). *Then*

(1) $\quad \operatorname{tr} \Sigma^2 = p\{1 + (p-1)\rho^2\}\sigma^4,$

(2) $\quad \operatorname{tr} \Sigma^3 = p\{1 + 3(p-1)\rho^2 + (p-1)(p-2)\rho^3\}\sigma^6.$

2.2. Asymptotic distributions of MLE's

We consider the asymptotic distributions of the estimators $\hat{\Xi}$, $\hat{\rho}$ and $\hat{\sigma}^2$ when p and k are fixed and $n \to \infty$. The following lemma represents further reductions for the expressions in Section 1.

LEMMA 2.2. *Let $\hat{\Xi}$, $\hat{\rho}$ and $\hat{\sigma}^2$ be the estimators defined by* (1.13), (1.14) *and* (1.15). *Then,*

(i) $\quad (A'A)^{1/2}(\hat{\Xi} - \Xi) = U\Sigma^{-1/2}B'(B\Sigma^{-1}B')^{-1} + O_p(n^{-1/2}),$

(ii) $\quad \hat{\rho} = \rho + n^{-1/2}\rho_1 + n^{-1}\rho_2 + O_p(n^{-3/2}),$

(iii) $\quad \hat{\sigma}^2 = \sigma^2 + n^{-1/2}\sigma_1 + n^{-1}\sigma_2 + O_p(n^{-3/2}),$

where

$$
\begin{aligned}
(2.2)\qquad & \rho_1 = \alpha_1 \operatorname{tr} DV, \qquad \rho_2 = \alpha_2 \operatorname{tr} \Sigma V \cdot \operatorname{tr} DV + \alpha_1 \operatorname{tr} DW, \\
& \alpha_1 = -s(1-\rho)/\{p(p-1)\}, \qquad \alpha_2 = s(1-\rho)/\{p^2(p-1)\sigma^2\}, \\
& \sigma_1 = p^{-1} \operatorname{tr} \Sigma V, \qquad \sigma_2 = p^{-1} \operatorname{tr} \Sigma W, \qquad D = I_p - \{s\sigma^2\}^{-1}\tilde{D}_1, \\
& \tilde{D}_1 = \Sigma^{1/2} D_1 \Sigma^{1/2} \quad \text{and} \quad W = (I_p - P_B)U'U(I_p - P_B).
\end{aligned}
$$

Using $\operatorname{vec}(\cdot)$ notation (see, e.g., Muirhead [31]), we have

THEOREM 2.1. *When p and k are fixed and $n \to \infty$, it holds that*

(i) $\quad \operatorname{vec}(\{(A'A)^{1/2}(\hat{\Xi} - \Xi)\}') \xrightarrow{d} N_{kq}(\mathbf{0}, I_k \otimes (B\Sigma^{-1}B')^{-1}),$

(ii) $\quad \sqrt{n}\begin{pmatrix} \hat{\rho} - \rho \\ \hat{\sigma}^2 - \sigma^2 \end{pmatrix} \xrightarrow{d} N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tau_\rho^2 & \tau_{\rho\sigma}^2 \\ \tau_{\rho\sigma}^2 & \tau_\sigma^2 \end{pmatrix}\right],$

(iii) $\quad \hat{\Xi}$ *and* $(\hat{\rho}, \hat{\sigma}^2)$ *are independent,*

where $\otimes$ is the Kronecker's product and

$$(2.3)\qquad \tau_\rho^2 = 2\{p(p-1)\}^{-1}s^2(1-\rho)^2,$$

$$\tau_{\rho\sigma}^2 = p^{-1}s\rho(1-\rho)\sigma^2,$$

$$\tau_\sigma^2 = 2p^{-1}\{1+(p-1)\rho^2\}\sigma^4. \tag{2.4}$$

PROOF. Lemma 2.2 (i) implies result (i). From Lemma 2.2 (ii) and (iii) it follows that the limiting distribution of $\sqrt{n}(\hat{\rho}-\rho, \hat{\sigma}^2-\sigma^2)$ is the same as that of (ρ_1, σ_1). Let $C(t_1, t_2)$ be the characteristic function of (ρ_1, σ_1), which is expressed as

$$\begin{aligned} E[\exp(it_1\rho_1 + it_2\sigma_1)] &= E[\exp(\operatorname{tr} MV)] \\ &= \exp(\operatorname{tr} M^2) + O(n^{-1/2}), \end{aligned}$$

where $M = it_1\alpha_1 D + it_2 p^{-1}\Sigma$. Result (ii) is proven by showing that

$$\operatorname{tr} M^2 = \frac{1}{2}\{\tau_\rho^2(it_1)^2 + 2\tau_{\rho\sigma}^2(it_1)(it_2) + \tau_\sigma^2(it_2)^2\}.$$

This identity follows from Lemma 2.1. Result (iii) follows from Lemma 2.2 and the independence of U and V.

We generalize Theorem 2.1 (ii) by finding the terms of $n^{-1/2}$ in the asymptotic expansions of the distributions. Let ϕ_ρ be the characteristic function of $\sqrt{n}(\hat{\rho}-\rho)$. Then we may write ϕ_ρ as

$$\phi_\rho(t) = E\left[\exp(it\rho_1)\left\{1 + \frac{1}{\sqrt{n}}it\rho_2\right\}\right] + O(n^{-1}),$$

which will be evaluated as

$$\exp\left(-\frac{1}{2}\tau_\rho^2 t^2\right)\left[1 + \frac{1}{\sqrt{n}}\{(it)g_1 + (it)^3 g_3\}\right] + O(n^{-1}). \tag{2.5}$$

An asymptotic expansion can be obtained by formally inverting (2.5). Using some formulas with respect to V (see, e.g., Siotani, Hayakawa and Fujikoshi [43]) and noting that $E[W] = k(I_p - P_B)$, it is seen that $\phi_\rho(t)$ can be expressed as (2.5) with

$$g_1 = 2\alpha_2 \operatorname{tr} D\Sigma + k\alpha_1 \operatorname{tr} D(I_p - P_B),$$

$$g_3 = \frac{4}{3}\alpha_1^3 \operatorname{tr} D^3 + 4\alpha_1^2\alpha_2 \operatorname{tr} D\Sigma \cdot \operatorname{tr} D^2.$$

The coefficients g_1 and g_3 can be simplified by using Lemma 2.1. The final result is given in Theorem 2.2.

THEOREM 2.2. *The distribution function of* $\sqrt{n}(\hat{\rho}-\rho)/\tau_\rho$ *can be expanded as*

$$P\left(\frac{\sqrt{n}(\hat{\rho}-\rho)}{\tau_\rho} \le x\right) = \Phi(x) - \frac{1}{\sqrt{n}}\left\{\frac{g_1}{\tau_\rho}\Phi^{(1)}(x) + \frac{g_3}{\tau_\rho^3}\Phi^{(3)}(x)\right\} + O(n^{-1}),$$

where $\Phi^{(j)}(x)$ *denotes the j-th derivative of the standard normal distribution function* $\Phi(x)$, τ_ρ *is given by* (2.3) *and*

$$g_1 = -2s\rho(1-\rho)/p,$$

$$g_3 = \frac{4}{3}s^3(1-\rho)^3\{p-2-3(p-1)\rho\}/\{p(p-1)\}^2.$$

Similarly we can derive an asymptotic expansion of the distribution of $\sqrt{n}(\hat{\sigma}^2 - \sigma^2)$ by expanding its characteristic function

$$\phi_\sigma(t) = E\left[\exp(it\sigma_1)\left\{1 + \frac{1}{\sqrt{n}}it\sigma_2\right\}\right] + O(n^{-1}). \tag{2.6}$$

The evaluation of (2.6) can be done by the same way as in the case of $\phi_\rho(t)$. We note only that

$$\sigma_1 = p^{-1}\operatorname{tr}\Sigma V, \qquad \sigma_2 = p^{-1}\operatorname{tr}\Sigma W.$$

THEOREM 2.3. *The distribution function of* $\sqrt{n}(\hat{\sigma}^2-\sigma^2)/\tau_\sigma$ *can be expanded as*

$$P\left(\frac{\sqrt{n}(\hat{\sigma}^2-\sigma^2)}{\tau_\sigma} \le x\right) = \Phi(x) - \frac{1}{\sqrt{n}}\left\{\frac{h_1}{\tau_\sigma}\Phi^{(1)}(x) + \frac{h_3}{\tau_\sigma^3}\Phi^{(3)}(x)\right\} + O(n^{-1}),$$

where τ_σ *is given by* (2.4), *and*

$$h_1 = k(p-q)\sigma^2/p, \qquad h_3 = \frac{4}{3}\{1 + 3(p-1)\rho^2 + (p-1)(p-2)\rho^3\}\sigma^6/p^2.$$

In the MANOVA case, $h_1 = 0$, since $P_B = I_p$.

2.3. The LR test for a uniform covariance structure

2.3.1. The LR test. We consider the problem of testing

$$H: \Sigma = \sigma^2 G_u(\rho) \quad \text{against} \quad K: \Sigma \text{ unrestricted}. \tag{2.7}$$

The maximum of the log likelihood $l(\Xi, \Sigma)$ when Σ is unrestricted, which was obtained by Khatri [23], is given by

$$\begin{aligned}\max l(\Xi, \Sigma) &= l(\tilde{\Xi}_\Omega, \tilde{\Sigma}_\Omega)\\ &= -\frac{1}{2}N[\log|\tilde{\Sigma}_\Omega| + p(1+\log 2\pi)],\end{aligned}$$

where $\tilde{\Xi}_\Omega = (A'A)^{-1}A'YS^{-1}B'(BS^{-1}B')^{-1}$, $\tilde{\Sigma}_\Omega = (n/N)\hat{\Sigma}_\Omega$ and $n\hat{\Sigma}_\Omega = (Y - A\tilde{\Xi}_\Omega B)'(Y - A\tilde{\Xi}_\Omega B)$. On the other hand, the maximum of the log likelihood $l(\Xi, \sigma^2 G_u(\rho))$ with respect to Ξ, ρ and σ^2 is given by

$$\max_H l(\Xi, \Sigma) = l(\tilde{\Xi}_\omega, \tilde{\Sigma}_\omega)$$

$$= -\frac{1}{2}N\left[p\log\left(\frac{n}{N}\hat{\sigma}^2\right) + (p-1)\log(1-\hat{\rho})\right.$$

$$\left. + \log\{1 + (p-1)\hat{\rho}\} + p(1 + \log 2\pi)\right],$$

where $\tilde{\Xi}_\omega = \hat{\Xi}$, $\tilde{\Sigma}_\omega = (n/N)\hat{\Sigma}_\omega$ and $\hat{\Sigma}_\omega = \hat{\sigma}^2 G_u(\hat{\rho})$. The LR test is equivalent to reject the hypothesis H if

$$T = -n\log\{|\hat{\Sigma}_\Omega|/|\hat{\sigma}^2 G_u(\hat{\rho})|\}$$
$$= -n\log[|\hat{\Sigma}_\Omega|/\{(\hat{\sigma}^2)^p(1-\hat{\rho})^{p-1}(1 + (p-1)\hat{\rho})\}] \tag{2.8}$$

is large.

LEMMA 2.3. (i) $\hat{\Sigma}_\Omega\hat{\Sigma}_\omega^{-1} - I_p = O_p(n^{-1/2})$,
(ii) $\mathrm{tr}(\hat{\Sigma}_\Omega\hat{\Sigma}_\omega^{-1} - I_p) = O_p(n^{-3/2})$.

PROOF. We can write

$$\hat{\Sigma}_\Omega = S + n^{-1}W(S)$$

and

$$\hat{\Sigma}_\omega^{-1} = \{\hat{\sigma}^2(1-\hat{\rho})\}^{-1}[D_2 - \hat{\rho}\{1 + (p-1)\hat{\rho}\}^{-1}D_1].$$

Result (i) follows from these expressions and Lemma 2.2. Using the equation of (1.9), we obtain

$$\mathrm{tr}\,\hat{\Sigma}_\Omega\hat{\Sigma}_\omega^{-1} = p\{d(1-\hat{\rho})\}^{-1}\,\mathrm{tr}\,\hat{\Sigma}_\Omega[\{1 + (p-1)\hat{\rho}\}I_p - \hat{\rho}D_1]. \tag{2.9}$$

From (1.16) we have $\hat{\Sigma}_\Omega = R + O_p(n^{-3/2})$. Substituting this result into the right-hand side of (2.9), we obtain result (ii).

THEOREM 2.4. *The asymptotic null distribution of the LR statistic T given by* (2.8) *when p and k are fixed and $n \to \infty$ is a central chi-square distribution with the degree of freedom $f = \frac{1}{2}p(p+1) - 2$.*

PROOF. Using Lemma 2.3, Lemma 2.2 (ii) and (iii), we can easily show the result by the same way as in the proof of Theorem 4.1 in [9].

We shall investigate refinement of the chi-square approximation.

2.3.2. The MANOVA case. The testing problem in the MANOVA case of $B = I_p$ is equivalent to test (2.7), based on S, where S is distributed as a Wishart distribution $W_p(\Sigma, n)$. The LR statistic T can be written as

$$T = -n \log [|S|/\{(\hat{\sigma}^2)^p (1-\hat{\rho})^{p-1}(1+(p-1)\hat{\rho})\}]. \tag{2.10}$$

Here $\hat{\sigma}^2$ is given by (1.14), but $\hat{\rho}$ is defined as the solution of

$$\{1+(p-1)\hat{\rho}\} \operatorname{tr} S = \operatorname{tr} D_1 S.$$

From Wakaki, Eguchi and Fujikoshi's [48] result, which is an asymptotic expansion of the null distribution of a class of tests for a general covariance structure based on a Wishart matrix, it follows that

$$P(T \le x) = P(\chi_f^2 \le x) + \frac{\ell}{n}\{P(\chi_{f+2}^2 \le x) - P(\chi_f^2 \le x)\} + O(n^{-3/2}),$$

where ℓ is a constant not depending on n. We can use a modified statistic $\tilde{T} = \{1 - 2\ell(fn)^{-1}\} T$ as a better chi-square approximation, since

$$P(\tilde{T} \le x) = P(\chi_f^2 \le x) + O(n^{-3/2}).$$

We shall determine a simple expression for ℓ by evaluating the expectation of T. Based on stochastic expansion of $\log|S|$, $\hat{\rho}$ and $\hat{\sigma}^2$, we can expand T as

$$T = T_0 + n^{-1/2} T_1 + n^{-1} T_2 + O_p(n^{-3/2}), \tag{2.11}$$

where

$$T_0 = \frac{1}{2}\left\{\operatorname{tr} V^2 - \frac{1}{p}(\operatorname{tr} V)^2 - \frac{1}{p(p-1)}(\operatorname{tr} DV)^2\right\},$$

$$T_1 = \frac{1}{3}\left\{-\operatorname{tr} V^3 + \frac{1}{p^2}(\operatorname{tr} V)^3 + \frac{3}{p^2(p-1)}\operatorname{tr} V \cdot (\operatorname{tr} DV)^2 - \frac{p-2}{p^2(p-1)^2}(\operatorname{tr} DV)^3\right\},$$

$$T_2 = \frac{1}{4}\operatorname{tr} V^4 - \frac{1}{4p^3}(\operatorname{tr} V)^4 - \frac{3}{2p^3(p-1)}(\operatorname{tr} V)^2(\operatorname{tr} DV)^2$$
$$+ \frac{p-2}{p^3(p-1)^2}\operatorname{tr} V \cdot (\operatorname{tr} DV)^3 - \frac{p^2-3p+3}{4p^3(p-1)^3}(\operatorname{tr} DV)^4.$$

Using Lemma 1.1, we can write

$$E[T] = E_V\left[T_0 + \frac{1}{\sqrt{n}}\{q_1(V)T_0 + T_1\} + \frac{1}{n}\{q_2(V)T_0 + q_1(V)T_1 + T_2\}\right] + O(n^{-3/2}), \tag{2.12}$$

where E_V denotes the expectation with respect to V with the probability density function $f_0(V)$ in (1.4). Since it is easily seen that

$$E_V[T_0] = f, \qquad E_V[q_1(V)T_0 + T_1] = 0,$$

we have

$$2\ell = E_V[q_2(V)T_0 + q_1(V)T_1 + T_2]. \tag{2.13}$$

To evaluate each of the expectations in (2.13), we use Lemma 2.2 in [22]. After much simplification, we obtain

$$\ell = \{24(p-1)\}^{-1}p(p+1)^2(2p-3). \tag{2.14}$$

In the special case of $p = 2$, we have $f = 1$ and $\ell = \frac{3}{4}$. This result agrees with the one in [22]. Thus the modified LR statistic is given by

$$\tilde{T} = \left(1 - \frac{3}{2n}\right)T.$$

2.3.3. The GMANOVA case. We shall obtain a constant $\tilde{\ell}$ such that

$$E[T] = f + 2(\ell + \tilde{\ell})n^{-1} + O(n^{-3/2}),$$

where ℓ is given by (2.14). We note that a modified LR statistic

$$\tilde{T} = \{1 - 2(\ell + \tilde{\ell})(fn)^{-1}\}T$$

gives a better chi-square approximation, in a sense that $E[\tilde{T}] = f + O(n^{-3/2})$.

The LR statistic T in the general model (0.1) with (0.3) can be expanded as

$$T = T_0 + n^{-1/2}(T_1 + \tilde{T}_1) + n^{-1}(T_2 + \tilde{T}_2) + O_p(n^{-3/2}), \tag{2.15}$$

where T_i's are given by (2.11),

$$\tilde{T}_1 = \mathrm{tr}\, VW - \frac{1}{p}\mathrm{tr}\, V \cdot \mathrm{tr}\, W - \frac{1}{p(p-1)}\mathrm{tr}\, DV \cdot \mathrm{tr}\, DW,$$

$$\begin{aligned}
\tilde{T}_2 = & -\mathrm{tr}\, V^2W + \frac{1}{2}\mathrm{tr}\, W^2 - \frac{1}{2p}(\mathrm{tr}\, W)^2 - \frac{1}{2p(p-1)}(\mathrm{tr}\, DW)^2 \\
& + \mathrm{tr}\, P_B(KWV + VWK) - \frac{1}{p}\mathrm{tr}\, V \cdot \mathrm{tr}\, P_B(WK + KW) \\
& - \frac{1}{p(p-1)}\mathrm{tr}\, DV \cdot \mathrm{tr}\, P_B(DWK + KWD) + \frac{1}{p^2}(\mathrm{tr}\, V)^2 \cdot \mathrm{tr}\, W \\
& + \frac{2}{p^2(p-1)}\mathrm{tr}\, V \cdot \mathrm{tr}\, DV \cdot \mathrm{tr}\, DW + \frac{1}{p^2(p-1)}(\mathrm{tr}\, DV)^2 \cdot \mathrm{tr}\, W
\end{aligned}$$

$$-\frac{p-2}{p^2(p-1)^2}(\operatorname{tr} DV)^2 \cdot \operatorname{tr} DW,$$

$K = -[(\rho_1/\rho) - (\tau/p)\{p\sigma^2((p-1)(1-\rho)-s)\rho_1 + s(1-\rho)e_1\}]I_p + (\tau\rho_1/\rho)\Sigma$ and $\tau^{-1} = s(1-\rho)\sigma^2$. By the same way as in (2.12), we have

$$2\tilde{\ell} = E_V E_W[q_1(V)\tilde{T}_1 + \tilde{T}_2].$$

Noting that W is distributed as a Wishart distribution $W_p(\Sigma, n)$ and using Lemma 2.2 in [22], we can easily obtain

$$\tilde{\ell} = \frac{k(p-q)}{4p(p-1)}[q\{(k-p)(p-1)+2\} - (p-1)(p^2+p+4)].$$

When $B = I_p$, we have $\tilde{\ell} = 0$ since $p = q$. Furthermore, it should be noted that $\tilde{\ell}$ does not depend on unknown parameter in this case, while $\tilde{\ell}$ under model (0.1) with serial covariance structure (see, e.g., Kanda and Fujikoshi [22]) does depend on unknown parameters.

2.4. Examples

We examine the well-known dental measurement data (see, e.g., Potthoff and Roy [33]) which are made on each of 11 girls and 16 boys at ages 8, 10, 12 and 14 years. Each measurement is the distance (in *mm*), from the center of the pituitary to the pteryomaxillary fissure. For the observation matrix Y: 27×4, we assume the model (0.1) with

$$(2.16)\qquad A = \begin{pmatrix} \mathbf{1}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{16} \end{pmatrix}, \quad \Xi = \begin{pmatrix} \xi_{10} & \xi_{11} \\ \xi_{20} & \xi_{21} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \end{pmatrix}$$

and the uniform covariance structure (0.3). Then, we obtain the MLE's of Ξ, ρ and σ^2 as follows:

$$\hat{\Xi} = \begin{pmatrix} \hat{\xi}_{10} & \hat{\xi}_{11} \\ \hat{\xi}_{20} & \hat{\xi}_{21} \end{pmatrix} = \begin{pmatrix} 22.6478 & 0.4795 \\ 24.9688 & 0.7844 \end{pmatrix}, \quad \hat{\rho} = 0.6178, \ \hat{\sigma}^2 = 4.9052.$$

These values are easily obtained. Let T and $\tilde{T}$ be the LR statistic and its modified LR statistic, respectively, in GMANOVA case. The values of T and $\tilde{T}$ for testing the uniform covariance structure are 8.48 and 8.46, respectively. These values are fairly below than the critical value of a chi-square distribution with the degree of freedom $f = 8$. Hence it seems reasonable to assume the uniform covariance structure in this example. On the other hand, if the covariance matrices are distinct, the design matrices for A, for girls and boys, are $\mathbf{1}_{11}$ and $\mathbf{1}_{16}$, respectively, and the matrices for Ξ are vectors (ξ_{i0}, ξ_{i1}), $i = 1, 2$, respectively. In these cases, the MLE's for the girls and the boys are

$$(\hat{\xi}_{10}, \hat{\xi}_{11}) = (22.6478, 0.4795), \qquad \hat{\rho} = 0.8680, \ \hat{\sigma}^2 = 4.4704,$$
$$(\hat{\xi}_{20}, \hat{\xi}_{21}) = (24.9688, 0.7844), \qquad \hat{\rho} = 0.4701, \ \hat{\sigma}^2 = 5.2041,$$

respectively. The values of the statistic T and $\tilde{T}$ for the girls, for testing the uniform covariance structure, are 6.72 and 6.16, respectively. The values of the statistic T and $\tilde{T}$ for the boys, for testing the uniform covariance structure, are 6.00 and 5.66, respectively. These values are also fairly below than the critical value of a chi-square distribution with the degree of fredom $f = 8$. The uniform covariance structure is not also rejected for these groups.

3. Testing hypotheses for mean parameters with covariance structures

In this section, we are interested in testing hypotheses for mean parameter. In the growth curve model (0.1) with positive definite covariance structure, Khatri [23] and Gleser and Olkin [14] considered LR tests for the following hypothesis:

$$\text{(3.1)} \qquad \mathrm{H}: C\Xi D = \mathrm{O} \quad \text{against} \quad \mathrm{K}: C\Xi D \neq \mathrm{O},$$

where C and D are $\ell \times k$ and $q \times m$ matrices of $\operatorname{rank}(C) = \ell \leq k$ and $\operatorname{rank}(D) = m \leq q$, respectively. This hypothesis contains many types of testing hypothesis for mean parameter. Under the hypothesis H, Khatri [23] obtained the likelihood ratio criterion by reducing the model (0.1) to a conditional model, and suggested the three test procedures in addition to the likelihood ratio criterion. The distribution of the likelihood ratio statistic

$$\Lambda = |V_e| / |V_H + V_e|$$

is $\Lambda_\ell(m, n)$, where $n = N - k - p + q$ and

$$V_e = D'(BS^{-1}B')^{-1}D, \quad V_H = (C\hat{\Xi}D)'R^{-1}(C\hat{\Xi}D),$$
$$R = C[(A'A)^{-1} + (A'A)^{-1}A'Y\{S^{-1} - S^{-1}B'(BS^{-1}B')^{-1}BS^{-1}\}Y'A(A'A)^{-1}]C'.$$

For details of these results, see also Siotani, Hayakawa and Fujikoshi [43]. Fujikoshi [7] gave asymptotic expansions of the non-null distributions of three statistics for testing hypotheses for mean parameter in GMANOVA model.

When Σ has a uniform covariance structure (0.3), Olkin and Shrikhande [32] gave the likelihood ratio test for the equality of means in a multivariate normal case using the property of uniform covariance structure. However, it is difficult to test the hypothesis H by using the above conditional model, when Σ has a uniform covariance structure or a serial covariance structure in the growth curve model (0.1). Therefore, we propose the following test procedure using asymptotic expansion.

3.1. Asymptotic expansions of the distributions of test statistics and their quantiles

Let $\eta = \mathrm{vec}\,(C\Xi D)'$, $\hat{\eta} = \mathrm{vec}\,(C\hat{\Xi}D)'$, where $\hat{\Xi}$ is the MLE of Ξ under the model (0.1) with covariance structure, and let $V(\eta)$ be the covariance matrix of η. In practice, the covariance matrix $V(\eta)$ involves an unknown covariance matrix Σ, so we consider an estimator $\hat{V}(\hat{\eta})$ of $V(\hat{\eta})$, which is obtained by replacing Σ by its estimator $\hat{\Sigma}$. Let $F(\eta)$ be $(\hat{\eta} - \eta)'\{\hat{V}(\hat{\eta})\}^{-1}(\hat{\eta} - \eta)$, then $F(\mathbf{0})$, say F, is a statistic for testing hypothesis H. This test statistic has been considered by Kleinbaum [24] in the case of missing data.

First, positive definite covariance structure is considered for simplicity and preparation of following section. In this case, Grizzle and Allen [15] (also, see Rao [36] and Gleser and Olkin [14]) have shown that the exact covariance matrix of the elements of $\hat{\Xi}'$ taken in a columnwise manner is

$$\frac{N-k-1}{N-k-(p-q)-1}(A'A)^{-1} \otimes (B\Sigma^{-1}B')^{-1}. \tag{3.2}$$

In practice, the covariance matrix Σ is unknown, so the estimator S is used in place of Σ. Using the expansion (1.7) of the MLE $\hat{\Xi}$, (1.10), (1.11), (1.12), and the following formula, see also Muirhead [31],

$$\mathrm{vec}\,(BXC) = (C' \otimes B)\,\mathrm{vec}\,(X),$$

$V(\hat{\eta}) = E[(\hat{\eta} - \eta)(\hat{\eta} - \eta)']$ can be represented as

$$\left(1 + \frac{p-q}{n}\right)C(A'A)^{-1}C' \otimes D'T^{-1}D + O(n^{-2}),$$

where $T = B\Sigma^{-1}B'$. Hence the estimator $\hat{V}(\hat{\eta})$ reduces to

$$\left(1 + \frac{p-q}{n}\right)C(A'A)^{-1}C' \otimes \left[D'T^{-1}D + \frac{1}{\sqrt{n}}D'T^{-1}\tilde{B}V\tilde{B}'T^{-1}D \right.$$
$$\left. + \frac{1}{n}D'T^{-1}\tilde{B}V(P_B - I)\tilde{B}'T^{-1}D\right] + O_p(n^{-3/2}),$$

where $\tilde{B} = B\Sigma^{-1/2}$ and $P_B = \tilde{B}'(\tilde{B}\tilde{B}')^{-1}\tilde{B}$. Noting $\hat{\eta} - \eta = n^{-1/2}\,\mathrm{vec}\,(D'\{\Xi_1' + n^{-1/2}\Xi_2' + n^{-1}\Xi_3' + O_p(n^{-3/2})\}C')$, and using the following formula, see also Muirhead [31],

$$(\mathrm{vec}\,(X))'(B'D' \otimes C)\,\mathrm{vec}\,(Y) = \mathrm{tr}\,BX'CYD, \tag{3.3}$$

after much simplification, we have

$$
\begin{aligned}
F = {} & \operatorname{tr} JU\tilde{P}_B U' + \frac{1}{\sqrt{n}}[2\operatorname{tr} JU\tilde{P}_B V(P_B - I)U' - \operatorname{tr} JU\tilde{P}_B V\tilde{P}_B U'] \\
& + \frac{1}{n}[\operatorname{tr} JU\tilde{P}_B V\{\tilde{P}_B - (P_B - I)\}V\tilde{P}_B U' - (p-q)\operatorname{tr} JU\tilde{P}_B U' \\
& - 2\operatorname{tr} JU\tilde{P}_B V\tilde{P}_B V(P_B - I)U' + 2\operatorname{tr} JU\tilde{P}_B V(P_B - I)V(P_B - I)U' \\
& + \operatorname{tr} JU(P_B - I)V\tilde{P}_B V(P_B - I)U'] + O_p(n^{-3/2}),
\end{aligned}
$$

where $J = (A'A)^{-1/2}C'(C(A'A)^{-1}C')^{-1}C(A'A)^{-1/2}$ and $\tilde{P}_B = \tilde{B}'T^{-1}D(D'T^{-1}D)^{-1} \times D'T^{-1}\tilde{B}$.

Let $\phi_n(t)$ be a characteristic function of F. After much simplification, we can easily obtain

$$
\begin{aligned}
& \phi_n(t) \\
& = (1-2it)^{-f/2}\left[1 + \frac{it}{n}\left\{f(p-q) + \frac{f(m+p-q+1)}{1-2it} - \frac{f(p-q)}{1-2it}\right\}\right. \\
& \quad \left. + \frac{(it)^2}{2n}\left\{\frac{4f(p-q)}{1-2it} + \frac{2f(m+2)}{(1-2it)^2}\right\}\right] + O(n^{-2}) \\
& = (1-2it)^{-f/2}\left[1 + \frac{f}{4n}\left\{\ell - m - 2(p-q) - 1 + \frac{2(p-q-\ell)}{1-2it} + \frac{\ell+m+1}{(1-2it)^2}\right\}\right] \\
& \quad + O(n^{-2}),
\end{aligned}
$$

where $f = \ell m$, by using the relations (3.3), (1.3), and integrating with respect to U and V in (1.1) and (1.2), respectively. Inverting this characteristic function, we have the following result.

THEOREM 3.1. *When the covariance structure is positive, the distribution function of the statistic F for testing H against K can be expanded asymptotically up to terms of order n^{-1} as*

$$
P(F \le x) = P(\chi_f^2 \le x) + \frac{f}{4n}\sum_{j=0}^{2} h_j P(\chi_{f+2j}^2 \le x) + O(n^{-2}), \tag{3.4}
$$

where $h_0 = \ell - m - 2(p-q) - 1$, $h_1 = 2(p-q-\ell)$ and $h_2 = \ell + m + 1$.

We consider a modified statistic

$$
\tilde{F} = \{1 + \kappa/n\}F
$$

such that $E[\tilde{F}] = f + O(n^{-2})$. The modified statistic is given in the following corollary.

COROLLARY 3.1. *When the covariance structure is positive, the distribution* *function of the modified statistic $\tilde{F}$ for testing H against K can be expanded asymptotically up to terms of order n^{-1} as*

$$P(\tilde{F} \le x) = P(\chi_f^2 \le x) + \frac{f}{4n} \sum_{j=0}^{2} \tilde{h}_j P(\chi_{f+2j}^2 \le x) + O(n^{-2}), \tag{3.5}$$

where $\tilde{h}_0 = h_0 - 2\kappa$, $\tilde{h}_1 = h_1 + 2\kappa$, $\tilde{h}_2 = h_2$ *and* $\kappa = -(m + p - q + 1)$.

Second, when the covariance structure is the uniform covariance structure (0.3), the variance $V(\hat{\eta})$ is represented as

$$C(A'A)^{-1}C' \otimes D'\left[T^{-1} + \frac{\gamma_1}{n}\Psi(\Sigma)\right]D + O(n^{-2}),$$

where $\gamma_1 = 2/\{p(p-1)\rho^2\sigma^4\}$ and $\Psi(\Sigma) = T^{-1}\tilde{B}\Sigma(I_p - P_B)\Sigma\tilde{B}'T^{-1}$. Hence, using the estimator $\hat{V}(\hat{\eta})$, which is given by substituting the MLE's $\hat{\sigma}^2$ and $\hat{\rho}$ for unknown parameters σ^2 and ρ in $V(\hat{\eta})$, respectively, the characteristic function $\phi_n(t)$ of F is obtained as

$$\begin{aligned}
&(1-2it)^{-f/2}\left[1 + \frac{it}{n}\left\{\frac{2f(p-q)}{p(p-1)} + \frac{1}{1-2it}\left(\frac{f}{p(p-1)}\{2(m-q)\right.\right.\right. \\
&\quad \left.\left. - (p-1)(k(p-q)-4)\} - \frac{2f(p-q)}{p(p-1)}\right)\right\} + \frac{(it)^2}{2n}\left\{\frac{4}{1-2it}\cdot\frac{2f(p-q)}{p(p-1)}\right. \\
&\quad \left.\left. + \frac{1}{(1-2it)^2}\cdot\frac{2f}{p(p-1)}((f+2)(p-1)+2(m-1))\right\}\right] + O(n^{-2}) \\
&= (1-2it)^{-f/2}\left[1 + \frac{f}{4n}\sum_{j=0}^{2} h_j'(1-2it)^{-j}\right] + O(n^{-2}),
\end{aligned}$$

where

$$h_0' = \frac{1}{p(p-1)}[m\{\ell(p-1)-2\} + 2(p-1)\{k(p-q)-3\} + 4q - 2], \tag{3.6}$$

$$h_1' = -\frac{2}{p(p-1)}[(p-1)\{k(p-q)+f-2\} + 2(q-1)], \tag{3.7}$$

$$h_2' = \frac{1}{p(p-1)}\{(f+2)(p-1) + 2(m-1)\}. \tag{3.8}$$

Inverting this characteristic function, we have

THEOREM 3.2. *When the covariance structure is uniform, the distribution function of the statistic F for testing H against K can be expanded asymptotically up to terms of order n^{-1} as*

$$P(F \le x) = P(\chi_f^2 \le x) + \frac{f}{4n} \sum_{j=0}^{2} h'_j P(\chi_{f+2j}^2 \le x) + O(n^{-2}), \tag{3.9}$$

where h'_0, h'_1 *and* h'_2 *are given in* (3.6), (3.7), *and* (3.8), *respectively.*

Furthermore, we obtain the following corollary similar to the case of positive definite covariance structure.

COROLLARY 3.2. *When the covariance structure is uniform, the distribution function of the modified statistic* $\tilde{F}$ *for testing* H *against* K *can be expanded asymptotically up to terms of order* n^{-1} *as*

$$P(\tilde{F} \le x) = P(\chi_f^2 \le x) + \frac{f}{4n} \sum_{j=0}^{2} \tilde{h}'_j P(\chi_{f+2j}^2 \le x) + O(n^{-2}), \tag{3.10}$$

where $\tilde{h}'_0 = h'_0 - 2\kappa$, $\tilde{h}'_1 = h'_1 + 2\kappa$, $\tilde{h}'_2 = h'_2$ *and* $\kappa = [(p-1)\{k(p-q)-4\} + 2(q-m)]/\{p(p-1)\}$.

Finally, we consider the case where the covariance structure is the serial covariance structure (0.2). In this case, the same result is obtained as the uniform covariance case except the following two points for the representations. One is to use (h''_0, h''_1, h''_2), γ_2 and $\Psi(Q)$ instead of (h'_0, h'_1, h'_2), γ_1 and $\Psi(\Sigma)$, respectively. Here, (h''_0, h''_1, h''_2) is more complicated than (h'_0, h'_1, h'_2);

$$h''_0 = \frac{1}{2}\lambda_1 - 2\lambda_2, \quad h''_1 = 2\lambda_2 - \lambda_1, \quad h''_2 = \frac{1}{2}\lambda_1, \tag{3.11}$$

where

$$\begin{aligned}
\lambda_1 &= \frac{m(f+2)(1+\rho^2)}{(p-1)\rho^2} - \frac{2(f+2)}{(p-1)\rho^2\sigma^2} \operatorname{tr} \tilde{P}_B Q + \gamma_2\{\ell(\operatorname{tr} \tilde{P}_B Q)^2 + 2 \operatorname{tr} (\tilde{P}_B Q)^2\}, \\
\lambda_2 &= \gamma_2\{\operatorname{tr} (\tilde{P}_B Q)^2 - \operatorname{tr} \tilde{P}_B Q P_B Q\} \\
&\quad + \frac{1}{(p-1)r}\left\{m(r+2) + \frac{2(p-2)}{\sigma^2} \operatorname{tr} \tilde{P}_B Q + \frac{p}{\sigma^2} \operatorname{tr} \tilde{P}_B \tilde{D}_1\right\} \\
&\quad + \frac{k(1-\rho^2)}{2(p-1)\rho^2}\left(q - \frac{p}{r\sigma^2} \operatorname{tr} P_B Q\right) \cdot (m - \tau \operatorname{tr} \tilde{P}_B Q) \\
&\quad + \frac{km}{p-1}\left(q - p + 1 - \frac{1}{r\sigma^2} \operatorname{tr} P_B Q\right),
\end{aligned}$$

$Q = \Sigma - \rho^2 \tilde{D}_1$ and $\gamma_2 = p/\{(p-1)r\rho^2\sigma^4\}$.

The other point is to change $f/4n$ in (3.9) by $\ell/4n$. Summarizing these results, we have

THEOREM 3.3. *When the covariance structure is serial, the distribution function of the statistic F for testing H against K can be expanded asymptotically up to terms of order* n^{-1} *as*

$$P(F \le x) = P(\chi_f^2 \le x) + \frac{\ell}{4n} \sum_{j=0}^{2} h_j'' P(\chi_{f+2j}^2 \le x) + O(n^{-2}), \tag{3.12}$$

where h_0'', h_1'' *and* h_2'' *are given in* (3.11).

Analogously, we obtain

COROLLARY 3.3. *When the covariance structure is serial, the distribution function of the modified statistic* $\tilde{F}$ *for testing H against K can be expanded asymptotically up to terms of order* n^{-1} *as*

$$P(\tilde{F} \le x) = P(\chi_f^2 \le x) + \frac{\ell}{4n} \sum_{j=0}^{2} \tilde{h}_j'' P(\chi_{f+2j}^2 \le x) + O(n^{-2}). \tag{3.13}$$

where $\tilde{h}_0'' = h_0'' - 2m\kappa$, $\tilde{h}_1'' = h_1'' + 2m\kappa$, $\tilde{h}_2'' = h_2''$ *and* $\kappa = -\lambda_2/m$.

Now we consider the quantiles of test statistic F. By the properties of p.d.f. of χ^2-distribution, the above result (3.4) can be written as

$$P(F \le x) = G_f\left(x - \frac{1}{2n} \cdot \frac{x}{f+2} \cdot \{h_2 x - h_0(f+2)\} + O(n^{-2})\right), \tag{3.14}$$

where $G_f(x) = P(\chi_f^2 \le x)$. In a similar way, (3.9) and (3.12) are rewritten as

$$P(F \le x) = G_f\left(x - \frac{1}{2n} \cdot \frac{x}{f+2} \cdot \{h_2' x - h_0'(f+2)\} + O(n^{-2})\right), \tag{3.15}$$

$$P(F \le x) = G_f\left(x - \frac{1}{2n} \cdot \frac{x}{m(f+2)} \cdot \{h_2'' x - h_0''(f+2)\} + O(n^{-2})\right). \tag{3.16}$$

The hypothesis H is rejected if $F > x_\alpha$, where $P(F > x_\alpha) = \alpha$. From these results, the quantiles x_α's are given as follows, according to positive definite covariance structure, uniform covariance structure and serial covariance structure, respectively,

$$u_f(\alpha)\left[1 + \frac{1}{2n} \cdot \frac{1}{f+2} \cdot \{h_2 u_f(\alpha) - h_0(f+2)\} + O(n^{-2})\right], \tag{3.17}$$

$$u_f(\alpha)\left[1 + \frac{1}{2n} \cdot \frac{1}{f+2} \cdot \{h_2' u_f(\alpha) - h_0'(f+2)\} + O(n^{-2})\right], \tag{3.18}$$

$$u_f(\alpha)\left[1 + \frac{1}{2n} \cdot \frac{1}{m(f+2)} \cdot \{h_2'' u_f(\alpha) - h_0''(f+2)\} + O(n^{-2})\right] \tag{3.19}$$

and $u_f(\alpha)$ is the upper $100\alpha\%$ point of a χ^2-distribution with f degrees of freedom. These expansions for the quantiles of F are also obtained by using Cornish-Fisher's inverse formula.

It should be noted that the equations (3.17) and (3.18) do not contain unknown parameter but (3.19) contains that. In practice, we need to use MLE's $\hat{\rho}$ and $\hat{\sigma}^2$ in place of unknown parameters ρ and σ^2.

3.2. Examples

From previous subsection, we can test many kinds of hypotheses by exchanging C and D for another matrices or vectors. As we have seen for dental measurement data in Subsection 2.4, growth curve model with uniform covariance structure is not rejected for the case of $q = 2$, $k = 2$. We can test hypothesis for the equality of mean parameter $\Xi' = (\xi_1, \xi_2) = ((\xi_{10}, \xi_{11})', (\xi_{20}, \xi_{21})')$. In this case, $k = q = m = 2$ and $\ell = 1$, i.e., C is $(1, -1)$ and D is I_2, the hypothesis is

$$\mathrm{H}:\ \xi_1 = \xi_2 \quad \text{against} \quad \mathrm{K}:\ \xi_1 \neq \xi_2.$$

Then, this hypothesis is rejected, since F is 16.50 and x_α, which value is the upper 5% point considered up to order n^{-1}, is 6.04. Moreover, when $q = k = 2$, $\ell = m = 1$, $C = (1, -1)$ and D is $(1, 0)'$, the hypothesis $\mathrm{H}:\ \xi_{10} = \xi_{20}$ against $\mathrm{K}:\ \xi_{10} \neq \xi_{20}$ is rejected as F is 374.61 and x_α is 3.832. Finally, when $q = k = 2$, $\ell = m = 1$, $C = (1, -1)$ and D is $(0, 1)'$, the hypothesis $\mathrm{H}:\ \xi_{11} = \xi_{21}$ against $\mathrm{K}:\ \xi_{11} \neq \xi_{21}$ is also rejected since F is 6.465 and x_α is 3.832.

4. Confidence regions of mean parameters

In the growth curve model, confidence intervals or confidence regions of mean parameter have been studied by Potthoff and Roy [33], Khatri [23], Grizzle and Allen [15], Gleser and Olkin [14], Fujikoshi and Nishii [10], Srivastava and Carter [45] and many other authors. In particular, Fujikoshi and Nishii [10] have given asymptotic expansions of the ratios of the expected volumes of the three confidence regions based on Rao [34], [35] in a one-sample "growth curves". Three types of covariance structures in the previous subsection are continuously considered: they are positive definite covariance structure, uniform covariance structure and serial covariance structure. In Subsection 4.1, confidence regions for mean parameters are obtained for the above three covariance structures. Examples are presented in Subsection 4.2.

4.1. Confidence regions

We study the asymptotic confidence regions of mean parameters up to order n^{-1}. Let C and D be $\ell \times k$ and $q \times m$ matrices of $\operatorname{rank}(C) = \ell \leq k$ and $\operatorname{rank}(D) = m \leq q$, respectively, and let $\boldsymbol{\eta} = \operatorname{vec}(C\Xi D)'$ and $\hat{\boldsymbol{\eta}} = \operatorname{vec}(C\hat{\Xi} D)'$, i.e., C, D, $\boldsymbol{\eta}$ and $\hat{\boldsymbol{\eta}}$ are the same ones as in Section 3.

Since we want to compare with the positive definite covariance structure, the uniform covariance structure, and the serial covariance structure in the same method, we use an asymptotic expansion. Because it is difficult to obtain the exact covariance for the later two covariance structures. Let $F = F(\boldsymbol{\eta})$ be $(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta})'\{\hat{V}(\hat{\boldsymbol{\eta}})\}^{-1}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta})$, which is as the same as the one in the Subsection 3.1. Then the distribution of $F(\boldsymbol{\eta})$ is the same as the one of $F(\mathbf{0})$ under $C\Xi D = \mathrm{O}$. Therefore, we obtain the distribution of F in the cases of positive definite, uniform covariance structure, and serial covariance structure as (3.14), (3.15), and (3.16), respectively.

From the above results, a confidence region for $\boldsymbol{\eta}$ with confidence coefficient $1 - \alpha + O(n^{-2})$ is obtained as

$$A_\alpha(\boldsymbol{\eta}) = \{\boldsymbol{\eta} \mid (\hat{\boldsymbol{\eta}} - \boldsymbol{\eta})'\{\hat{V}(\hat{\boldsymbol{\eta}})\}^{-1}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}) \leq x_\alpha\},$$

where x_α is (3.17), (3.18), and (3.19), according to positive definite covariance structure, uniform covariance structure, and serial covariance structure, respectively.

4.2. Examples

From Subsection 4.1, we can obtain many kinds of confidence regions or intervals by exchanging C and D for another matrices or vectors. In this subsection, confidence intervals and regions for $\boldsymbol{\xi} = \Xi'$ with confidence coefficient $1 - \alpha + O(n^{-2})$ mean the ones as in previous subsection.

The well-known data, which have been studied by many authors, are also considered here; one is a dental measurement data in Subsection 2.4, and another is ramus heights data which are made on 20 boys at ages 8, $8\frac{1}{2}$, 9, $9\frac{1}{2}$ (see, e.g., Potthoff and Roy [33] and Grizzle and Allen [15]). In the both data, the design matrix B is of (2.16). Since it is known that the best value of q is two (see, e.g., Fujikoshi, Kanda and Tanimura [9]), we study the case $q = 2$ in the following.

From Subsection 4.1, confidence intervals of for ξ_0 and ξ_1 with confidence coefficient $95\% + O(n^{-2})$ on the positive covariance structure (exact type), the positive covariance structure, the uniform covariance structure and the serial covariance structure are presented in Table 1. Table 1 shows that the uniform covariance structure is fairly good structure.

On the other hand, confidence regions for $\boldsymbol{\xi}$ with confidence coefficient $95\% + O(n^{-2})$ are presented in Table 2.

From Table 2, the uniform covariance structure and the serial covariance

Table 1. Confidence intervals for ξ_0 and ξ_1 with confidence coefficient $95\% + O(n^{-2})$

Dental Measurement Data: Boys group

	Positive*	Positive	Uniform	Serial
ξ_0	25.0018 ± 0.9843	25.0018 ± 1.0574	24.9688 ± 0.8802	25.0267 ± 0.7694
ξ_1	0.8340 ± 0.2135	0.8340 ± 0.2293	0.7844 ± 0.1846	0.7729 ± 0.2538

Dental Measurement Data: Girls group

	Positive*	Positive	Uniform	Serial
ξ_0	22.7275 ± 1.2795	22.7275 ± 1.4305	22.6477 ± 1.2117	22.6382 ± 1.2553
ξ_1	0.4823 ± 0.1400	0.4823 ± 0.1565	0.4795 ± 0.1037	0.4849 ± 0.1560

Ramus heights Data

	Positive*	Positive	Uniform	Serial
ξ_0	50.0500 ± 1.2020	50.0500 ± 1.2707	50.0750 ± 1.0814	50.0568 ± 1.1492
ξ_1	0.4628 ± 0.1129	0.4628 ± 0.1193	0.4665 ± 0.0809	0.4650 ± 0.0970

NOTE: Positive* is the result for the exact case of positive covariance structure.

Table 2. Confidence regions for ξ with confidence coefficient $95\% + O(n^{-2})$

Dental Measurement Data: Boys group

P^* $5.1322\xi_0^2 + 1.4132\xi_0\xi_1 + 109.0850\xi_1^2 - 257.8048\xi_0 - 217.2787\xi_1 + 3305.0077 \le 0$

P $4.4479\xi_0^2 + 1.2248\xi_0\xi_1 + 94.5403\xi_1^2 - 223.4309\xi_0 - 188.3082\xi_1 + 2863.2219 \le 0$

U $5.1023\xi_0^2 + 116.0407\xi_1^2 - 254.7950\xi_0 - 182.0400\xi_1 + 3246.0504 \le 0$

S $6.6515\xi_0^2 + 61.3633\xi_1^2 - 332.9301\xi_0 - 94.8615\xi_1 + 4196.1494 \le 0$

Dental Measurement Data: Girls group

P^* $3.3833\xi_0^2 - 23.7753\xi_0\xi_1 + 282.5783\xi_1^2 - 142.3209\xi_0 + 267.7655\xi_1 + 1543.1409 \le 0$

P $2.7066\xi_0^2 - 19.0202\xi_0\xi_1 + 226.0628\xi_1^2 - 113.8567\xi_0 + 214.2123\xi_1 + 1232.5962 \le 0$

U $2.7310\xi_0^2 + 372.8227\xi_1^2 - 123.7017\xi_0 - 357.5705\xi_1 + 1480.0504 \le 0$

S $2.7503\xi_0^2 + 150.9810\xi_1^2 - 124.5227\xi_0 - 146.4241\xi_1 + 1438.0433 \le 0$

Ramus heights Data

P^* $3.2771\xi_0^2 - 8.2449\xi_0\xi_1 + 371.8654\xi_1^2 - 324.2176\xi_0 + 68.4576\xi_1 + 8089.8222 \le 0$

P $2.9321\xi_0^2 - 7.3770\xi_0\xi_1 + 332.7217\xi_1^2 - 290.0894\xi_0 + 61.2516\xi_1 + 7237.4324 \le 0$

U $3.3604\xi_0^2 + 600.2041\xi_1^2 - 336.5442\xi_0 - 559.9904\xi_1 + 8550.6028 \le 0$

S $3.2831\xi_0^2 + 406.1933\xi_1^2 - 328.6840\xi_0 - 377.7630\xi_1 + 8307.7854 \le 0$

NOTE: P^*, P, U, and S represent the confidence regions of the positive covariance structure (exact), the positive covariance structure, the uniform covariance structure, and the serial covariance structure, respectively.

structure are fairly better than the positive covariance structure to obtain the confidence regions for ξ. Furthermore, the uniform covariance structure is more better than the serial covariance structure since the calculation of the uniform covariance structure's case is fairly easier than that of the serial one, and their confidence regions are contained in the regions of the positive covariance structure's case and are far narrower than that. This fact coincides with Lee's [27] indication that the uniform covariance structure seems to be one of the most natural covariance matrices for growth-curve data.

Part II. Growth curve model with missing data

5. MLE's and their asymptotic properties

5.1. Introduction

We often encounter missing data when we analyze longitudinal or repeated measurement data. The data are missing at chance or by design. To analyze the missing data is more complicated than to handle the complete data. Many authors have studied the problems which contain missing values. Especially, Anderson [1] has considered a problem of monotone type for the bivariate case, Bhargava [5] has obtained the monotone type of the missing observations in various multivariate models. Trawinski and Bargmann [47] have dealt with models that involve incomplete data obtained by design. Hartley and Hocking [16] have obtained results for the one sample case.

Dempster, Laird and Rubin [6] have given an applicable algorithm for computing maximum likelihood estimators from incomplete data. This powerful computational procedure is known as EM (Expectation and Maximization) algorithm. Jennrich and Schluchter [18] addressed the problem through maximum likelihood analysis using a general linear model for some covariance structures (see, e.g., Rao [36]), and described three algorithms for computing MLE's of the regression and covariance parameters including EM-algorithm.

In the growth curve models in which the data are missing either by chance or by design, Kleinbaum [24] has given a BAN (Best Asymptotically Normal) estimator assuming that a consistent estimator can be found for unknown covariance matrix Σ and the usual normality assumption. Kleinbaum [24] has also proposed a method for testing linear hypotheses, and has also noted the possibilities of another consistent estimators of the covariance matrix in small samples (see, e.g., Kleinbaum [25]). Srivastava [44] has obtained the consistency of the estimators in a multivariate regression model, and derived the likelihood ratio tests for a growth curve model. Liski [30] has applied EM-algorithm to the estimation of parameters in a growth curve problem.

However, these authors have considered only the case where the covariance structure is positive-definite.

On the other hand, Lee [27] has studied the cases where the covariance structure is both uniform covariance structure and serial covariance structure in the growth curve model for complete data, and has showed that these covariance structures are fitted for a growth curve analysis. In this section, we consider a growth curve model in the case of missing data, where, especially, the covariance structure is a uniform covariance structure or a serial covariance structure. The estimators are easily obtained for missing data which are monotone and general type, when the covariance structure is uniform covariance structure. However, in the case of serial covariance structure, it is difficult to handle the data, since the covariance structure corresponding to the incomplete data is not kept serial one for general type of missing data. Thus we only consider monotone type of incomplete data in this section. However, this monotone type of incomplete data is important since such missing data are found in many applications.

In the following model of monotone type, which has been studied by Bhargava [5],

$$\underset{N_i \times p_i}{Y_i} = \underset{N_i \times k}{A_i}\ \underset{k \times q}{\Xi}\ \underset{q \times p}{B}\ \underset{p \times p_i}{M_i} + \underset{N_i \times p_i}{\varepsilon_i}, \qquad i = 1, \dots, u, \tag{5.1}$$

where Y_i is an observation matrix, A_i and B are known design matrices of ranks k and $q \leq p$, respectively, Ξ is an unknown parameter, M_i, whose (j, j)-th elements are 1 and others are 0, is an incidence matrix of rank p_i, the rows of ε_i are independent and identically distributed random vectors with distribution $N_{p_i}(\mathbf{0}, M_i'\Sigma M_i)$, and

$$p_1 > q, \ p_i > p_{i+1}, \qquad i = 1, 2, \dots, u-1.$$

Moreover, ε_i and ε_j are independent if $i \neq j$. For simplicity, we use the notations $B_i = BM_i$ and $\Sigma_i = M_i'\Sigma M_i$.

In Subsection 5.2, we study the properties of MLE's in the model (5.1) with serial and uniform covariance structures.

5.2. MLE's under covariance structures

This subsection deals with growth curve model in the case of missing data. In what follows, we shall use following notations. Let $N = \sum_{i=1}^u N_i$, $m_1 = \sum_{i=1}^u N_i p_i$, $v_1 = \sum_{i=1}^u \delta_i$, $v_2 = \sum_{i=1}^u \delta_i p_i$, $n = N - k$, $n_i = N_i - k$ and

$$U_i = (A_i'A_i)^{-1/2} A_i'(Y_i - A_i \Xi B_i)\Sigma_i^{-1/2},$$

$$V_i = \sqrt{n_i}\,(\Sigma_i^{-1/2} S_i \Sigma_i^{-1/2} - I_{p_i}),$$

$$S_i = (1/n_i) Y_i'(I_{N_i} - A_i(A_i'A_i)^{-1}A_i') Y_i.$$

These U_i, V_i and S_i have similar properties as U, V and S, respectively, in Section 1.

5.2.1. Positive definite covariance structure

When the Σ is positive definite, the log likelihood $l(\Xi, \Sigma)$ of Ξ and Σ based on Y_i's is

$$-2l(\Xi, \Sigma) = \sum_{i=1}^{u} N_i p_i \log(2\pi) + \sum_{i=1}^{u} N_i \log|M_i' \Sigma M_i| + \sum_{i=1}^{u} \mathrm{tr}\,(M_i' \Sigma M_i)^{-1}(Y_i - A_i \Xi B_i)'(Y_i - A_i \Xi B_i).$$

The MLE's $\hat{\Xi}_\Omega$ and $\hat{\Sigma}_\Omega$ of Ξ and Σ, respectively, in the model (5.1) with positive definite covariance structure are the solutions of the following equations (1)–(2) (see e.g., Srivastava [44]):

$$(1) \qquad \sum_{i=1}^{u} A_i' Y_i \hat{\Sigma}_{\Omega i}^{-1} B_i' = \sum_{i=1}^{u} A_i' A_i \hat{\Xi}_\Omega B_i \hat{\Sigma}_{\Omega i}^{-1} B_i',$$

$$(2) \qquad \sum_{i=1}^{u} N_i M_i \hat{\Sigma}_{\Omega i}^{-1} M_i' = \sum_{i=1}^{u} M_i \hat{\Sigma}_{\Omega i}^{-1}(Y_i - A_i \hat{\Xi}_\Omega B_i)'(Y_i - A_i \hat{\Xi}_\Omega B_i)\hat{\Sigma}_{\Omega i}^{-1} M_i'.$$

5.2.2. Serial covariance structure

When the Σ has a serial covariance structure (0.2), the log likelihood $l(\Xi, \sigma^2 G_s(\rho))$ of Ξ, ρ and σ^2 based on Y_i's is

$$\begin{aligned} -2l(\Xi, \sigma^2 G_s(\rho)) &= \sum_{i=1}^{u} N_i p_i \log(2\pi) + \sum_{i=1}^{u} N_i p_i \log \sigma^2 + \sum_{i=1}^{u} N_i (p_i - 1)\log(1-\rho^2) \\ &+ \frac{1}{\sigma^2(1-\rho^2)} \sum_{i=1}^{u} \mathrm{tr}\,(\rho^2 D_{1i} - 2\rho D_{2i} + I_{p_i})(Y_i - A_i \Xi B_i)'(Y_i - A_i \Xi B_i), \end{aligned}$$

where D_{1i} and D_{2i} are $p_i \times p_i$ matrices such that $D_{1i} = \mathrm{diag}(0, 1, \ldots, 1, 0)$ and D_{2i} is the same matrix as D_2 in (1.25) except the dimension.

By the same way as in Fujikoshi, Kanda and Tanimura [9], we obtain the following theorem.

THEOREM 5.1. *The MLE's $\hat{\Xi}$, $\hat{\rho}$ and $\hat{\sigma}^2$ of Ξ, ρ and σ^2, respectively, in the model* (5.1) *with covariance structure* (0.2) *are the solutions of the following equations* (3)–(5):

$$(3) \qquad \hat{\Xi} = \Xi(\hat{\sigma}^2, \hat{\rho}) \colon \sum_{i=1}^{u} A_i' Y_i \hat{\Sigma}_i^{-1} B_i' = \sum_{i=1}^{u} A_i' A_i \hat{\Xi} B_i \hat{\Sigma}_i^{-1} B_i',$$

(4) $$\hat{\sigma}^2 = \frac{n}{m_1} \cdot \frac{a\hat{\rho}^2 - 2b\hat{\rho} + c}{1 - \hat{\rho}^2},$$

(5) $$(m_1 - N)a\hat{\rho}^3 - (m_1 - 2N)b\hat{\rho}^2 - (m_1 a + Nc)\hat{\rho} + m_1 b = 0,$$

where $\hat{\Sigma}_i = M_i'\hat{\Sigma}M_i$, $\hat{\Sigma} = \hat{\sigma}^2 G_s(\hat{\rho})$, $R_i = (1/n)(Y_i - A_i\hat{\Xi}B_i)'(Y_i - A_i\hat{\Xi}B_i)$, $a_i = \mathrm{tr}\,(D_{1i}R_i)$, $b_i = \mathrm{tr}\,(D_{2i}R_i)$, $c_i = \mathrm{tr}\,R_i$, $a = \sum_{i=1}^u a_i$, $b = \sum_{i=1}^u b_i$ and $c = \sum_{i=1}^u c_i$.

These formulas clearly coincide with (1), (2) and (3) of Theorem 2.1 in [9], respectively, when $u = 1$. Moreover, note that $\hat{\rho}$ which satisfies the equations (3)–(5) has property $-1 < \hat{\rho} < 1$. Using $\mathrm{vec}(\cdot)$ notation, (3) is replaced by

(3)′ $$\mathrm{vec}\,(\hat{\Xi}) = [\sum_{i=1}^u (B_i\hat{\Sigma}_i^{-1}B_i' \otimes A_i'A_i)]^{-1} \cdot \sum_{i=1}^u (B_i\hat{\Sigma}_i^{-1} \otimes A_i')\,\mathrm{vec}\,(Y_i).$$

This means that $\hat{\Xi}$ in (3) or (3)′ is the best unbiased estimator when Σ is known.

We consider the asymptotic distribution of the MLE's $\hat{\Xi}$, $\hat{\rho}$ and $\hat{\sigma}^2$ when p and k are fixed, and $N_i/N_1 \to \delta_i \geq 0$ (equality holds if N_i is fixed) as $N_i \to \infty$, $i = 1, \ldots, u$.

LEMMA 5.1.

$$\hat{\rho} = \rho + n^{-1/2}\rho_1 + n^{-1}\rho_2 + O_p(n^{-3/2}),$$
$$(N/n)\hat{\sigma}^2 = \sigma^2 + n^{-1/2}\sigma_1 + n^{-1}\sigma_2 + O_p(n^{-3/2}),$$

where

$$\rho_1 = -\frac{N\sum_{i=1}^u N_i\{(r_i - \rho^2)\rho a^{(1)} - r_i b^{(1)} + \rho c^{(1)}\}}{\sigma^2\sum_{i,j=1}^u N_iN_j(p_i - 1)r_j},$$

$$\begin{aligned}\rho_2 = \frac{1}{\sigma^2\sum_{i,j=1}^u N_iN_j(p_i - 1)r_j}&[2\sum_{i,j=1}^u N_iN_j(p_i - 1)(p_j - 2)\sigma^2\rho\rho_1^2\\ &- N\sum_{i=1}^u N_i\{3(r_i - \rho^2)a^{(1)} + 2(p_i - 2)\rho b^{(1)} + c^{(1)}\}\rho_1\\ &- N\sum_{i=1}^u N_i\{(r_i - \rho^2)\rho a^{(2)} - r_i b^{(2)} + \rho c^{(2)}\}\\ &+ 2N\sum_{i=1}^u N_ip_ia^{(1)}\rho_1 - kN(u\sum_{i=1}^u N_ip_i - N\sum_{j=1}^u p_j)\sigma^2\rho(1 - \rho^2)],\end{aligned}$$

$$\rho\sigma_1 = b^{(1)} - \rho a^{(1)} - N^{-1}\sum_{i=1}^u N_i(p_i - 1)\sigma^2\rho_1,$$

$$\rho\sigma_2 = b^{(2)} - \rho a^{(2)} - (a^{(1)} + \sigma_1)\rho_1 - N^{-1}\sum_{i=1}^u N_i(p_i - 1)\sigma^2\rho_2,$$

$$r_i = p_i - (p_i - 2)\rho^2, \qquad (i = 1,\dots,u),$$

$$a^{(1)} = \sum_{i=1}^{u} \sqrt{N_i/N}\ \mathrm{tr}\, D_{1i}\Sigma_i^{1/2} V_i \Sigma_i^{1/2}, \qquad a^{(2)} = \sum_{i=1}^{u} \mathrm{tr}\, D_{1i}\Sigma_i^{1/2} W_i \Sigma_i^{1/2},$$

$$b^{(1)} = \sum_{i=1}^{u} \sqrt{N_i/N}\ \mathrm{tr}\, D_{2i}\Sigma_i^{1/2} V_i \Sigma_i^{1/2}, \qquad b^{(2)} = \sum_{i=1}^{u} \mathrm{tr}\, D_{2i}\Sigma_i^{1/2} W_i \Sigma_i^{1/2},$$

$$c^{(1)} = \sum_{i=1}^{u} \sqrt{N_i/N}\ \mathrm{tr}\, \Sigma_i^{1/2} V_i \Sigma_i^{1/2}, \qquad c^{(2)} = \sum_{i=1}^{u} \mathrm{tr}\, \Sigma_i^{1/2} W_i \Sigma_i^{1/2},$$

$$W_i = (I - P_{B_i})U_i' U_i (I - P_{B_i}), \quad P_{B_i} = \tilde{B}_i'(\tilde{B}_i\tilde{B}_i')^{-1}\tilde{B}_i' \quad \textit{and} \quad \tilde{B}_i = B_i \Sigma_i^{-1/2}.$$

We can reduce the representations of ρ_1 and σ_1 in Lemma 5.1 to

$$\rho_1 = \frac{-N(1-\rho^2)}{2\rho \sum_{i,j=1}^{u} N_i N_j (p_i - 1) r_j} \cdot \sum_{i,j=1}^{u} N_i r_i \sqrt{\frac{N_j}{N}}\ \mathrm{tr}\, C_j V_j, \tag{5.2}$$

$$\sigma_1 = \frac{N}{\sum_{j=1}^{u} N_j r_j} \cdot \sum_{i=1}^{u} \sqrt{\frac{N_i}{N}}\ \mathrm{tr}\, Q_i V_i, \tag{5.3}$$

where

$$C_j = C_j(i) = I_{p_j} - \frac{p_i}{r_i \sigma^2} Q_j \quad \text{and} \quad Q_i = \Sigma_i - \rho^2 \Sigma_i^{1/2} D_{1i} \Sigma_i^{1/2}.$$

Next, we give asymptotic distributions of MLE's under assumption

$$\lim_{N_i \to \infty} N_i^{-1} A_i' A_i = \Gamma_i, \qquad i = 1,\dots,u. \tag{5.4}$$

THEOREM 5.2. *When p and k are fixed and $N_i/N_1 \to \delta_i \geq 0$ as $N_i \to \infty$,* $i = 1,\dots,u$, under the assumption (5.4), *it holds that*

(i) $$\mathrm{vec}\,(\sqrt{n}(\hat{\Xi} - \Xi)') \xrightarrow{d} N_{kq}\left(\mathbf{0}, \left[\frac{\sum_{i=1}^{u} \delta_i \Gamma_i \otimes B_i \Sigma_i^{-1} B_i'}{\sum_{j=1}^{u} \delta_j}\right]^{-1}\right),$$

(ii) $$\sqrt{n}\begin{pmatrix} \hat{\rho} - \rho \\ \dfrac{N}{n}\hat{\sigma}^2 - \sigma^2 \end{pmatrix} \xrightarrow{d} N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha_{Ms} & \gamma_{Ms} \\ \gamma_{Ms} & \beta_{Ms} \end{pmatrix}\right],$$

(iii) *$\hat{\Xi}$ and $(\hat{\rho}, \hat{\sigma}^2)$ are independent,*

where

$$\alpha_{Ms} = \frac{\nu_1 \nu_2 (1-\rho^2)^2}{\sum_{l=1}^{u} \delta_l (p_l - 1) \cdot \sum_{k=1}^{u} \delta_k r_k}, \quad \beta_{Ms} = \frac{2\nu_1(1+\rho^2)\sigma^4}{\sum_{k=1}^{u} \delta_k r_k}$$

and $$\gamma_{Ms} = \frac{2\nu_1 \rho (1-\rho^2)\sigma^2}{\sum_{k=1}^{u} \delta_k r_k}.$$

PROOF. From Lemma 5.1 it follows that the limiting distribution of $\sqrt{n}(\hat{\rho}-\rho, (N/n)\hat{\sigma}^2-\sigma^2)$ is the same as that of (ρ_1, σ_1), which is expressed as

$$\begin{aligned} C(t_1, t_2) &= E[\exp(it_1\rho_1 + it_2\sigma_1)] \\ &= E[\exp \sum_{j=1}^{u} \operatorname{tr} M_j V_j] \\ &= \exp(\sum_{j=1}^{u} \operatorname{tr} M_j^2), \end{aligned}$$

where

$$\begin{aligned} M_j = &- it_1 \cdot \frac{N(1-\rho^2)}{2\rho\sum_{l=1}^{u} N_l(p_l-1)\sum_{k=1}^{u} N_k r_k} \cdot \sum_{s=1}^{u} N_s r_s \sqrt{\frac{N_j}{N}} C_j \\ &+ it_2 \cdot \frac{N}{\sum_{k=1}^{u} N_k r_k} \sqrt{\frac{N_j}{N}} Q_j. \end{aligned}$$

Using equations

$$\operatorname{tr} Q_j = r_j \sigma^2 \quad \text{and} \quad \operatorname{tr} Q_j^2 = r_j(1+\rho^2)\sigma^4,$$

result (ii) is verified by showing that

$$\sum_{j=1}^{u} \operatorname{tr} M_j^2 = \frac{1}{2}\{\alpha_{Ms}(it_1)^2 + 2\gamma_{Ms}(it_1)(it_2) + \beta_{Ms}(it_2)^2)\}.$$

These results agree with the case of complete data when $u = 1$ (see, e.g., [9]).

5.2.3. Uniform covariance structure

When Σ has a uniform covariance structure (0.3), the log likelihood $l(\Xi, \sigma^2 G_u(\rho))$ of Ξ, ρ and σ^2 based on Y_i's is

$$\begin{aligned} -2l(\Xi, \sigma^2 G_u(\rho)) = &\sum_{i=1}^{u} N_i p_i \log(2\pi) + \sum_{i=1}^{u} N_i p_i \log \sigma^2 \\ &+ \sum_{i=1}^{u} N_i(p_i-1)\log(1-\rho) + \sum_{i=1}^{u} N_i \log s_i \\ &+ \frac{1}{\sigma^2(1-\rho)} \sum_{i=1}^{u} \operatorname{tr}\left[I_{p_i} - \frac{\rho}{s_i}\mathbf{1}_{p_i}\mathbf{1}_{p_i}'(Y_i - A_i\Xi B_i)'(Y_i - A_i\Xi B_i)\right], \end{aligned}$$

where $s_i = 1 + (p_i - 1)\rho$.

By the same way as in the case of serial, we obtain the following theorem.

THEOREM 5.3. *The MLE's $\hat{\Xi}$, $\hat{\rho}$ and $\hat{\sigma}^2$ of Ξ, ρ and σ^2, respectively, in the model* (5.1) *with covariance structure* (0.3) *are the solutions of the following equations* (6)–(8):

$$(6)\qquad \hat{\Xi} = \Xi(\hat{\sigma}^2, \hat{\rho}):\ \sum_{i=1}^{u} A_i' Y_i \hat{\Sigma}_i^{-1} B_i' = \sum_{i=1}^{u} A_i' A_i \hat{\Xi} B_i \hat{\Sigma}_i^{-1} B_i',$$

$$(7)\qquad \hat{\sigma}^2 = \frac{n}{m_1}\cdot \sum_{i=1}^{u} \frac{1}{1-\hat{\rho}}\left(e_i - \frac{\hat{\rho}}{\hat{s}_i} d_i\right),$$

$$(8)\qquad \left[\frac{N}{1-\hat{\rho}} + \sum_{i=1}^{u} \frac{N_i(p_i-1)}{\hat{s}_i}\right]\cdot \sum_{j=1}^{u} (e_j - \hat{\rho} d_j/\hat{s}_j) = m_1 \cdot \sum_{i=1}^{u} d_i/\hat{s}_i^2,$$

where $\hat{\Sigma}_i = M_i'\hat{\Sigma}M_i$, $\hat{\Sigma} = \hat{\sigma}^2 G_u(\hat{\rho})$, $R_i = (1/n)(Y_i - A_i\hat{\Xi}B_i)'(Y_i - A_i\hat{\Xi}B_i)$, $d_i =$ tr $\mathbf{1}_{p_i}\mathbf{1}_{p_i}' R_i = \sum_{j=0} d_i^{(j)} n^{-j/2}$, $e_i = \operatorname{tr} R_i = \sum_{j=0} e_i^{(j)} n^{-j/2}$ *and* $\hat{s}_i = 1 + (p_i - 1)\hat{\rho}$.

These formulas clearly coincide with (1.13), (1.14) and (1.15), respectively, in Section 1, when $u = 1$. Furthermore, note that $\hat{\rho}$ which satisfies the equations (6)–(8) has property $-1/(p-1) < \hat{\rho} < 1$. We can also write a representation of (6) as the one similar to (3)$'$ in this case.

We consider the asymptotic distributions of the MLE's $\hat{\Xi}$, $\hat{\rho}$ and $\hat{\sigma}^2$ by using U_i, V_i and S_i when p and k are fixed, and $N_i/N_1 \to \delta_i \geq 0$ (equality holds if N_i is fixed) as $N_i \to \infty$, $i = 1, \ldots, u$.

LEMMA 5.3.

$$\hat{\rho} = \rho + n^{-1/2}\rho_1 + n^{-1}\rho_2 + O_p(n^{-3/2}),$$
$$(N/n)\hat{\sigma}^2 = \sigma^2 + n^{-1/2}\sigma_1 + n^{-1}\sigma_2 + O_p(n^{-3/2}),$$

where

$$\rho_1 = [Nm_1(1-\rho)\sum_{j=1}^{u} d_j^{(1)}/s_j^2 - m_4 \sum_{j=1}^{u} (e_j^{(1)} - \rho d_j^{(1)}/s_j)]/(m_3\sigma^2),$$

$$\begin{aligned}\rho_2 = \Bigg[&\frac{\sigma^2}{N(1-\rho)}\left\{m_4 \sum_{i=1}^{u} \frac{N_i p_i(3s_i - 2p_i)}{s_i^2} - m_1 \sum_{i=1}^{u} \frac{N_i p_i^2(3s_i - 2p_i)}{s_i^3}\right\}\rho_1^2 \\ &+ \left\{m_4 \sum_{i=1}^{u} \frac{d_i^{(1)}}{s_i^2} - \frac{1}{1-\rho}\sum_{i=1}^{u} \frac{N_i p_i(2s_i - p_i)}{s_i^2} \sum_{j=1}^{u}\left(e_j^{(1)} - \frac{\rho d_j^{(1)}}{s_j}\right)\right. \\ &\left. - 2m_1(1-\rho)\sum_{i=1}^{u} \frac{(p_i-1)d_i^{(1)}}{s_i^3}\right\}\rho_1 \\ &+ \left\{m_1 \sum_{j=1}^{u} \frac{d_j^{(2)}}{s_j^2}(1-\rho) - m_4 \sum_{j=1}^{u}\left(e_j^{(2)} - \frac{\rho d_j^{(2)}}{s_j}\right)\right\}\Bigg] \Bigg/ \left(\frac{m_3\sigma^2}{N}\right),\end{aligned}$$

$$\sigma_1 = \frac{1}{m_1(1-\rho)}\left[m_5\sigma^2\rho\rho_1 + N\sum_{i=1}^{u}\left(e_i^{(1)} - \frac{\rho d_i^{(1)}}{s_i}\right)\right],$$

$$\sigma_2 = \frac{N}{m_1}\left[\frac{m_5\sigma^2\rho\rho_2}{N(1-\rho)} + \frac{\sigma^2\rho_1^2}{N(1-\rho)^2}\sum_{i=1}^{u}\frac{N_ip_i(p_i-1)}{s_i^2}\{1+(p_i-1)\rho^2\}\right.$$
$$+ \frac{\rho_1}{(1-\rho)^2}\sum_{i=1}^{u}\left(e_i^{(1)} + \frac{1+(p_i-1)\rho^2}{s_i^2}d_i^{(1)}\right) + \frac{1}{1-\rho}\sum_{i=1}^{u}\left(e_i^{(2)} - \frac{\rho d_i^{(2)}}{s_i}\right)$$
$$\left. + k\sigma^2\sum_{i=1}^{u}\left(\frac{N_i}{N} - 1\right)p_i\right],$$

$$m_3 = m_1\sum_{i=1}^{u}\frac{N_ip_i^2}{s_i^2} - m_4^2,\ m_4 = \sum_{i=1}^{u}\frac{N_ip_i}{s_i}\quad \textit{and}\quad m_5 = \sum_{i=1}^{u}\frac{N_ip_i(p_i-1)}{s_i}.$$

The representation of ρ_1 and σ_1 in Lemma 5.3 can be replaced by

$$\rho_1 = \frac{N(1-\rho)}{m_3}\sum_{j=1}^{u}\sqrt{\frac{N_j}{N}}\cdot \operatorname{tr}\left(\frac{m_1}{s_j^2\sigma^2}\tilde{D}_{1j} - m_4 I_{p_j}\right)V_j, \tag{5.5}$$

$$\sigma_1 = \frac{N\sigma^2}{m_1}\sum_{j=1}^{u}\sqrt{\frac{N_j}{N}}\operatorname{tr}\left[\frac{m_1m_5\rho}{m_3s_j^2\sigma^2}\tilde{D}_{1j} + \left(1 - \frac{m_4m_5\rho}{m_3}\right)I_{p_j}\right]V_j, \tag{5.6}$$

where

$$\tilde{D}_{1j} = \Sigma_j^{1/2}\mathbf{1}_{p_j}\mathbf{1}_{p_j}'\Sigma_j^{1/2}.$$

Next, we give asymptotic distributions of MLE's.

THEOREM 5.4. *When p and k are fixed and $N_i/N_1 \to \delta_i \ge 0$ as $N_i \to \infty$, $i = 1,\dots,u$, under the assumption* (5.4), *it holds that*

(i) $$\operatorname{vec}(\sqrt{n}(\hat{\Xi} - \Xi)') \xrightarrow{d} N_{kq}\left(\mathbf{0}, \left[\frac{\sum_{i=1}^{u}\delta_i\Gamma_i \otimes B_i\Sigma_i^{-1}B_i'}{\sum_{j=1}^{u}\delta_j}\right]^{-1}\right),$$

(ii) $$\sqrt{n}\begin{pmatrix}\hat{\rho} - \rho \\ \dfrac{N}{n}\hat{\sigma}^2 - \sigma^2\end{pmatrix} \xrightarrow{d} N_2\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\alpha_{Mu} & \gamma_{Mu}\\ \gamma_{Mu} & \beta_{Mu}\end{pmatrix}\right],$$

(iii) *$\hat{\Xi}$ and $(\hat{\rho}, \hat{\sigma}^2)$ are independent,*

where

$$\alpha_{Mu} = \frac{2(1-\rho)^2v_1v_2}{v_3},\quad \beta_{Mu} = \frac{2\sigma^4v_1}{v_2v_3}(v_3 + v_5^2\rho^2),\quad \gamma_{Mu} = \frac{\rho(1-\rho)\sigma^2v_1v_5}{v_3},$$

$$v_1 = \sum_{j=1}^{u}\delta_j,\quad v_2 = \sum_{j=1}^{u}\delta_jp_j,\quad v_3 = \sum_{j=1}^{u}v_2\frac{\delta_jp_j^2}{s_j^2} - v_4^2,\quad v_4 = \sum_{j=1}^{u}\frac{\delta_jp_j}{s_j},$$

and $v_5 = \dfrac{\delta_j p_j(p_j - 1)}{s_j}$.

PROOF. In this case, noting

$$M_j = it_1 \cdot \frac{N(1-\rho)}{m_3}\sqrt{\frac{N_j}{N}}\left(\frac{m_1}{s_j^2\sigma^2}\tilde{D}_{1j} - m_4 I_{p_j}\right)$$

$$+ it_2 \cdot \frac{N\sigma^2}{m_1}\sqrt{\frac{N_j}{N}}\left[\frac{m_1 m_5 \rho}{m_3 s_j^2 \sigma^2}\tilde{D}_{1j} + \left(1 - \frac{m_4 m_5 \rho}{m_3}\right) I_{p_j}\right]$$

and using

$$\operatorname{tr}\tilde{D}_{1j} = p_j s_j \sigma^2 \quad \text{and} \quad \operatorname{tr}\tilde{D}_{1j}^2 = p_j^2 s_j^2 \sigma^4,$$

result (ii) follows.

These results agree with the case of complete data when $u = 1$ (see, e.g., [20]).

6. Asymptotic comparison of the MLE's in the cases of complete data and missing data

6.1. Criterion of comparison

In this section, we study on the effects of missing data in the cases of serial and uniform covariance structures. Let T_1 and T_2 be the MLE's of a parameter or parameter vector θ in the cases of complete data and missing data, respectively. We define a criterion μ on the effects of missing data in the estimation of θ by

$$\mu(T_2 \mid T_1) = \text{A-Var}(T_1)/\text{A-Var}(T_2), \tag{6.1}$$

where A-Var (T_i) denotes asymptotic variance of T_i. In multivariate case, the variances on the right-hand-side of (6.1) are replaced by the generalized variances. We call $\mu(T_2 \mid T_1)$ an efficiency of T_2 relative to T_1. The quantity $1 - \mu(T_2 \mid T_1)$ denotes how much accuracies of the estimator are reduced when there are missing data. In the following, let the symbols $\hat{}$ and $\tilde{}$ denote the MLE's in the cases of complete data (0.1) and missing data (5.1), respectively.

(i) Serial case

When p and k are fixed, and $N_i/N_1 \to \delta_i \geq 0$ as $N_i \to \infty$, $i = 1, \ldots, u$, the criterion μ of $\tilde{\rho}$ relative to $\hat{\rho}$ and the criterion μ of $\tilde{\sigma}^2$ relative to $\hat{\sigma}^2$ are

$$\mu(\tilde{\rho} \mid \hat{\rho}) = p \sum_{i=1}^{u} \delta_i (p_i - 1) \cdot \sum_{j=1}^{u} \delta_j r_j / \{(p-1) r \sum_{i=1}^{u} \delta_i \sum_{j=1}^{u} \delta_j p_j\}, \tag{6.2}$$

and

(6.3) $$\mu(\tilde{\sigma}^2|\hat{\sigma}^2) = \sum_{i=1}^{u} \delta_i r_i/(r \sum_{j=1}^{u} \delta_j),$$

respectively. Furthermore, for the parameter matrix Ξ, we have

(6.4) $$\mu(\tilde{\Xi}|\hat{\Xi}) = |\hat{\Xi}|/|\tilde{\Xi}|.$$

(ii) Uniform case

Similarly to the serial case, we obtain the criterion μ of $\tilde{\rho}$ relative to $\hat{\rho}$ and $\tilde{\sigma}^2$ relative to $\hat{\sigma}^2$ as

(6.5) $$\mu(\tilde{\rho}|\hat{\rho}) = s^2 v_3/\{p(p-1)v_1 v_2\},$$

(6.6) $$\mu(\tilde{\sigma}^2|\hat{\sigma}^2) = v_2 v_3\{1 + (p-1)\rho^2\}/\{pv_1(v_3 + v_5^2\rho^2)\},$$

respectively. The criterion μ of $\tilde{\Xi}$ relative to $\hat{\Xi}$ is given by the same representation as (6.4).

6.2. Examples

We examine the effects of missing data (5.1) based on the criterion μ in the case $u = 2$, $p_1 = p$ and $p_2 = p - 1$ under the assumption of $N_2/N_1 \to \delta > 0$ as $N_i \to \infty$, $i = 1, 2$. In this case, (6.2), (6.3), (6.5) and (6.6) are respectively given as

(6.7) $$\mu(\tilde{\rho}|\hat{\rho}) = \frac{p\{p-1+\delta(p-2)\}[p-(p-2)\rho^2+\delta\{p-1-(p-3)\rho^2\}]}{(1+\delta)\{p+\delta(p-1)\}(p-1)r},$$

(6.8) $$\mu(\tilde{\sigma}^2|\hat{\sigma}^2) = \frac{p-(p-2)\rho^2+\delta\{p-1-(p-3)\rho^2\}}{(1+\delta)\{p-(p-2)\rho^2\}},$$

(6.9) $$\mu(\tilde{\rho}|\hat{\rho}) = s^2 v_3/\{p(p-1)v_1 v_2\},$$

(6.10) $$\mu(\tilde{\sigma}^2|\hat{\sigma}^2) = v_2 v_3\{1 + (p-1)\rho^2\}/\{pv_1(v_3 + v_5^2\rho^2)\},$$

where δ_2 in v_1, v_2, v_3 and v_5 are replaced by δ, respectively. Numerical values of these results are given in Tables 3 and 4 in the cases $p = 4, 7, 10$, $\rho = 0.1\,(0.2)\,0.9$, and $\delta = 1, \frac{1}{3}$.

The criterion (6.4) in the cases of serial and uniform for the parameter Ξ in the case $u = 2$, $p_1 = 4$ and $p_2 = 3$ under the assumption of $N_2/N_1 \to \delta > 0$ as $N_i \to \infty$, $i = 1, 2$, are given by

(6.11) $$\mu(\tilde{\Xi}|\hat{\Xi}) = [\{2(2-\rho)+\delta(3-\rho)\}\{2(10-5\rho+\rho^2)+\delta(11-4\rho+\rho^2)\} - \delta^2(1-\rho)(3-\rho)^2]/[4(1+\delta)^2(2-\rho)(10-5\rho+\rho^2)],$$

(6.12) $$\mu(\tilde{\Xi}|\hat{\Xi}) = [\{4(1+2\rho)+3\delta(1+3\rho)\}\{20(1+2\rho)+\delta(11+13\rho)\} - 9\delta^2(1-\rho)(1+3\rho)]/[80(1+\delta)^2(1+2\rho)^2],$$

Table 3. The values of the criterion μ of $\tilde{\rho}$ relative to $\hat{\rho}$

Serial structure

ρ	$\delta = 1$					$\delta = 1/3$				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$p = 4$	0.8339	0.8389	0.8503	0.8720	0.9144	0.9170	0.9195	0.9254	0.9365	0.9583
$p = 7$	0.9169	0.9186	0.9228	0.9319	0.9554	0.9584	0.9593	0.9614	0.9660	0.9778
$p = 10$	0.9445	0.9454	0.9476	0.9525	0.9673	0.9723	0.9727	0.9738	0.9762	0.9837

Uniform structure

ρ	$\delta = 1$					$\delta = 1/3$				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$p = 4$	0.7939	0.8551	0.8951	0.9232	0.9439	0.8970	0.9280	0.9484	0.9628	0.9734
$p = 7$	0.9065	0.9486	0.9673	0.9778	0.9846	0.9533	0.9744	0.9838	0.9891	0.9925
$p = 10$	0.9458	0.9739	0.9843	0.9896	0.9929	0.9729	0.9870	0.9922	0.9949	0.9965

Table 4. The values of the criterion μ of $\tilde{\sigma}^2$ relative to $\hat{\sigma}^2$

Serial structure

ρ	$\delta = 1$					$\delta = 1/3$				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$p = 4$	0.8756	0.8809	0.8929	0.9156	0.9601	0.9378	0.9404	0.9464	0.9578	0.9800
$p = 7$	0.9288	0.9305	0.9348	0.9440	0.9678	0.9644	0.9653	0.9674	0.9720	0.9839
$p = 10$	0.9501	0.9510	0.9531	0.9581	0.9730	0.9751	0.9755	0.9766	0.9790	0.9865

Uniform structure

ρ	$\delta = 1$					$\delta = 1/3$				
	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
$p = 4$	0.8787	0.9041	0.9382	0.9681	0.9910	0.9394	0.9521	0.9692	0.9841	0.9955
$p = 7$	0.9327	0.9555	0.9765	0.9895	0.9973	0.9664	0.9778	0.9883	0.9948	0.9987
$p = 10$	0.9542	0.9738	0.9876	0.9949	0.9987	0.9771	0.9869	0.9938	0.9974	0.9994

respectively. When ρ and δ are the same as the above, these results are given in Table 5.

Tables 3, 4 and 5 show that for both serial and uniform cases the efficiencies of a parameter or a parameter vector increase as p increases, δ decreases, or ρ increases.

Table 5. The values of the criterion μ of $\tilde{\hat{\Xi}}$ relative to $\hat{\Xi}$ ($p = 4$)

Serial structure

ρ	0.1	0.3	0.5	0.7	0.9
$\delta = 1$	0.6605	0.6849	0.7151	0.7531	0.8026
$\delta = 1/3$	0.8302	0.8424	0.8575	0.8766	0.9013

Uniform structure

ρ	0.1	0.3	0.5	0.7	0.9
$\delta = 1$	0.6625	0.6781	0.6875	0.6937	0.6982
$\delta = 1/3$	0.8313	0.8391	0.8438	0.8469	0.8491

7. Testing hypotheses for mean parameters

In this section, we consider again test for mean parameters along the similar line as in Section 3 except the point that the model contains missing data. Here we have the following hypothesis:

(7.1) $$\mathrm{H}: C\Xi D = \mathrm{O} \quad \text{against} \quad \mathrm{K}: C\Xi D \neq \mathrm{O},$$

where C and D are $\ell \times k$ and $q \times m$ matrices of $\mathrm{rank}(C) = \ell \leq k$ and $\mathrm{rank}(D) = m \leq q$, respectively. When the covariance structure is positive definite, Kleinbaum [24] proposed a test statistic. Srivastava [44] obtained the likelihood ratio test for the hypothesis $H: C\Xi = \mathrm{O}$ against $\mathrm{K}: C\Xi \neq \mathrm{O}$. He gave an asymptotic distribution of the likelihood ratio statistic when $M_1 = I_p$, $N_1 \to \infty$ while other N_i's are fixed under positive definite covariance structure.

Here we extend the test statistics for the case of complete data in Section 3 to the case of missing data when the covariance structure is serial or uniform covariance structure. In serial case, MLE $\hat{\Xi}$ of mean parameter matrix Ξ is given by (3) in Theorem 5.1. Let $\boldsymbol{\eta} = \mathrm{vec}(C\Xi D)'$, $\hat{\boldsymbol{\eta}} = \mathrm{vec}(C\hat{\Xi}D)'$, and let $V(\boldsymbol{\eta})$ be the covariance matrix of $\boldsymbol{\eta}$. Then the test statistic in the case of missing data is defined by the same way as in the case of complete data, i.e.,

(7.2) $$F = \hat{\boldsymbol{\eta}}\{\hat{V}(\hat{\boldsymbol{\eta}})\}^{-1}\hat{\boldsymbol{\eta}},$$

where $\hat{V}(\hat{\boldsymbol{\eta}})$, which is obtained by replacing Σ by its estimator $\hat{\Sigma}$, is the estimator of $V(\hat{\boldsymbol{\eta}})$. The MLE $\hat{\Xi}$ satisfies

(7.3) $$\sum_{i=1}^{u} (B_i\hat{\Sigma}_i^{-1}B_i' \otimes A_i'A_i) \cdot \mathrm{vec}(\hat{\Xi} - \Xi) = \sum_{i=1}^{u} [B_i\hat{\Sigma}_i^{-1}B_i' \otimes (A_i'A_i)^{1/2}]\, \mathrm{vec}(U_i).$$

From (7.3) we have

$$\begin{aligned}\operatorname{vec}(\Xi_1) &= \sqrt{n}\,[\sum B_i\Sigma_i^{-1}B_i' \otimes A_i'A_i]^{-1}\sum(B_i \otimes (A_i'A_i)^{1/2})\cdot \operatorname{vec}(U_i),\\ \operatorname{vec}(\Xi_2) &= [\sum B_i\Sigma_i^{-1}\tilde{B}_i' \otimes A_i'A_i]^{-1}\cdot[\sum(\tilde{B}_iK_i\tilde{B}_i' \otimes A_i'A_i)\cdot \operatorname{vec}(\Xi_1)\\ &\quad - \sqrt{n}\sum(\tilde{B}_iK_i \otimes (A_i'A_i)^{1/2})\cdot \operatorname{vec}(U_i)],\end{aligned}$$

where $K_i = -(\rho_1/\rho + \tau_1/\tau)I_{p_i} + (\tau\rho_1/\rho)Q_i$, $Q_i = \Sigma_i - \rho^2\Sigma_i^{1/2}D_{1i}\Sigma_i^{1/2}$, and $\sum$ stands for $\sum_{i=1}^{u}$. Further, $V(\hat{\eta})$ satisfies the following equation:

$$V(\hat{\eta}) = n[\{\sum B_i\Sigma_i^{-1}B_i' \otimes A_i'A_i\}^{-1} + O_p(n^{-2})].$$

Therefore, F can be expanded as

$$F = \sum_{i,j}(\operatorname{vec} U_i)'J_{ij}(\operatorname{vec} U_j) + \frac{1}{\sqrt{n}}Z_1 + O_p(n^{-1}), \tag{7.4}$$

where J_{ij} is $(\tilde{B}_i' \otimes (A_i'A_i)^{1/2})Z(D \otimes C')[(D' \otimes C)Z(D \otimes C')]^{-1}(D' \otimes C)Z(\tilde{B}_j \otimes (A_j'A_j)^{1/2})$, $Z = [\sum B_i\Sigma_i^{-1}B_i' \otimes A_i'A_i]^{-1}$, and Z_1 is a linear function of V_i's. Let $\phi_\eta(t)$ be the characteristic function of F. After simplification, we have

$$\phi_\eta(t) = |I - 2itJ|^{-1/2} + O(n^{-1}),$$

where $J = (J_{ij})$. It is easily seen that $J^2 = J$, i.e., J is idempotent, and $\operatorname{tr} J = \operatorname{rank}(J) = \ell m$. From these, it follows that

$$\phi_\eta(t) = (1 - 2it)^{-f/2} + O(n^{-1}),$$

where $f = \ell m$.

Similarly we can derive an expansion of the characteristic function of F when the covariance structure is uniform. These results give the following theorem.

THEOREM 7.1. *When the covariance structure is serial or uniform, under the assumption* (5.4) *the distribution function of the statistic F for testing H against K can be expanded asymptotically as*

$$P(F \le x) = P(\chi_f^2 \le x) + O(n^{-1}), \tag{7.5}$$

where $f = \ell m$.

Hypothesis H is rejected if $F > x_\alpha$, where $P(F > x_\alpha) = \alpha$ when covariance structure is serial or uniform. From Theorem 7.1, we can use the upper $100\alpha\%$ point of a χ^2-distribution with ℓm degrees of freedom as an approximation for x_α.

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References

[1] T. W. Anderson, Maximum likelihood estimates for a multivariate normal distribution when some observations are missing, J. Amer. Statist. Assoc., **52** (1957), 200–203.

[2] T. W. Anderson, The Statistical Analysis of Time Series, Wiley, New York, 1971.

[3] A. Azzalini, Estimation and hypothesis testing for collections of autoregressive time series, Biometrika, **71** (1984), 85–90.

[4] A. Azzalini, Growth curves analysis for patterned covariance matrices, New Perspectives in Theoretical and Applied Statistics (M. L. Puri, J. P. Vilaplana and W. Wertz, eds.), Wiley, New York, 1987, 63–73.

[5] R. P. Bhargava, Multivariate tests of hypotheses with incomplete data, Tech. Rep. No. 3, Statist. Stanford University, 1962.

[6] A. P. Dempster, N. M. Laird and D. B. Rubin, Maximum likelihood from incomplete data via the EM algorithm, J. Roy. Statist. Soc., Ser. B **39** (1977), 1–22.

[7] Y. Fujikoshi, Asymptotic expansions of the non-null distributions of three statistics in GMANOVA, Ann. Inst. Statist. Math., **26** (1974), 289–297.

[8] Y. Fujikoshi, An asymptotic expansion for the distributions of the latent roots of the Wishart matrix with multiple population roots, Ann. Inst. Statist. Math., **29** (1977), 379–387.

[9] Y. Fujikoshi, T. Kanda and N. Tanimura, The growth curve model with an autoregressive covariance structure, Ann. Inst. Statist. Math., **42** (1990), 533–542.

[10] Y. Fujikoshi and R. Nishii, Asymptotic comparison of three confidence regions in analysis of growth curves, J. Japan Statist. Soc., **13** (1983), 151–155.

[11] Y. Fujikoshi and R. Shimizu, Asymptotic expansions of some mixtures of the multivariate normal distribution and their error bounds, Ann. Statist., **17** (1989), 1124–1132.

[12] C. A. Glasbey, Examples of regression with serially correlated errors, The Statistician, **37** (1988), 277–291.

[13] L. J. Gleser and I. Olkin, Linear model in multivariate analysis, Essays in Probability and Statistics (R. C. Bose, ed.), Univ. North Carolina Press, Chapel Hill, N.C., 1970, 267–292.

[14] L. J. Gleser and I. Olkin, Estimation for a regression model with an unknown covariance matrix, Proc. Sixth Berkeley Symp. Math. Statist. Prob., **1** (1972), 541–568.

[15] J. E. Grizzle and D. M. Allen, Analysis of growth and dose response curves, Biometrics, **25** (1969), 357–381.

[16] H. O. Hartley and R. R. Hocking, The analysis of incomlete data, Biometrics, **27** (1971), 783–808.

[17] I. L. Hudson, Asymptotic tests for growth curve models with autoregressive errors, Austral. J. Statist., **25** (1983), 413–424.

[18] R. I. Jennrich and M. D. Schluchter, Unbalanced repeated-measures models with structured covariance matrices, Biometrics, **42** (1986), 805–820.

[19] T. Kanda, Confidence region of growth curve model with covariance structures, Tech. Rep. No. 276, Statist. Research Group, Hiroshima University, 1990.

[20] T. Kanda, Growth curve model with a uniform covariance structure, Tech. Rep. No. 278, Statist. Research Group, Hiroshima University, 1990.

[21] T. Kanda, MSE's of prediction in growth curve model with covariance structures, Ann. Inst. Statist. Math., **44** (1992), 519–528.

[22] T. Kanda and Y. Fujikoshi, Asymptotic distributions of the MLE's and the LR test in the growth curve model with a serial covariance structure, Hiroshima Math. J., **20** (1990), 285–295.

[23] C. G. Khatri, A note on a MANOVA model applied to problems in growth curve, Ann. Inst. Statist. Math., **18** (1966), 75–86.

[24] D. G. Kleinbaum, A generalization of the growth curve model which allows missing data, J. Multiv. Anal., **3** (1973), 117–124.

[25] D. G. Kleinbaum, Testing linear hypotheses in generalized multivariate linear models, Commun. Statist.-Theor. Meth., **A1** (1973), 433–457.

[26] N. M. Laird and J. H. Ware, Random-effects models for longitudinal data, Biometrics, **38** (1982), 963–974.

[27] J. C. Lee, Prediction and estimation of growth curves with special covariance structures, J. Amer. Statist. Assoc., **83** (1988), 432–440.

[28] J. C. Lee, Tests and model selection for the general growth curve model, Biometrics, **47** (1991), 147–159.

[29] J. C. Lee and S. Geisser, Growth curve prediction, Sankhyā Ser. A, **34** (1972), 393–412.

[30] E. P. Liski, Estimation from incomplete data in growth curves models, Commun. Statist.-Simula. Computa., **14** (1985), 13–27.

[31] R. J. Muirhead, Aspects of Multivariate Statistical Theory, Wiley, New York, 1982.

[32] I. Olkin and S. S. Shrikhande, An extension of Wilks' test for the equality of means, Ann. Math. Statist., **41** (1970), 683–687.

[33] R. F. Potthoff and S. N. Roy, A generalized multivariate analysis of variance model useful especially for growth curve problems, Biometrika, **51** (1964), 313–326.

[34] C. R. Rao, Some problems involving linear hypothesis in multivariate analysis, Biometrika, **46** (1959), 49–58.

[35] C. R. Rao, The theory of least squares when the parameters are stochastic and its application to the analysis of growth curves, Biometrika **52** (1965), 447–458.

[36] C. R. Rao, Least squares theory using an estimated dispersion matrix and its application to measurement of signals, Proc. Fifth Berkeley Symp. Math. Statist. Prob. **1** (1967), 355–372.

[37] C. R. Rao, Prediction of future observations in growth curve models, Statist. Sci., **2** (1987), 434–471.

[38] G. Reinsel, Multivariate repeated-measurement or growth curve models with multivariate random-effects covariance structure, J. Amer. Statist. Assoc., **77** (1982), 190–195.

[39] G. Reinsel, Estimation and prediction in a multivariate random effects generalized linear model, J. Amer. Statist. Assoc., **79** (1984), 406–414.

[40] G. C. Reinsel, Effects of the estimation of covariance matrix parameters in the generalized multivariate linear model, Commun. Statist.-Theor. Meth., **13** (1984), 639–650.

[41] G. C. Reinsel, Mean squared error properties of empirical Bayes estimators in a multivariate random effects general linear model, J. Amer. Statist. Assoc., **80** (1985), 642–650.

[42] D. von Rosen, The growth curve model: a review, Commun. Statist.-Theor. Meth., **20** (1991), 2791–2822.

[43] M. Siotani, T. Hayakawa and Y. Fujikoshi, Modern Multivariate Statistical Analysis: A Graduate Course and Handbook, American Sciences Press, Columbus, Ohio, 1985.

[44] M. S. Srivastava, Multivariate data with missing observations, Commun. Statist.-Theor. Meth., **14** (1985), 775–792.

[45] M. S. Srivastava and E. M. Carter, An Introduction to Applied Multivariate Statistics, North-Holland, New York · Amsterdam · Oxford, 1983.

[46] N. H. Timm, Multivariate analysis of variance of repeated measurements, In Handbook of Statistics, Analysis of Variance, Vol. 1, (P. R. Krishnaiah, ed.) North-Holland, Amsterdam, 1980, 41–87.

[47] I. M. Trawinski and R. E. Bargmann, Maximum likelihood estimation with incomplete multivariate data, Ann. Math. Statist., **35** (1964), 647–657.

[48] H. Wakaki, S. Eguchi and Y. Fujikoshi, A class of tests for a general covariance structure, J. Multiv. Anal., **32** (1990), 313–325.

[49] J. H. Ware, Linear models for the analysis of longitudinal studies, Amer. Statist., **39** (1985), 95–101.

[50] R. F. Woolson and J. D. Leeper, Growth curve analysis of complete and incomplete longitudinal data, Commun. Statist.-Theor. Meth., **9** (1980), 1491–1513.

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