HIROSHIMA MATH. J.
24 (1994), 359–363

A note on G-extensible regularity condition

Shyuichi IZUMIYA

(Received November 14, 1992)

0. Introduction

Let X and Y be smooth G-manifolds, where G is a finite group. Then the r-jet bundle $J^r(X, Y)$ is naturally a differentiable G-fibre bundle. Let $J^r_G(X, Y)$ be the subspace of $J^r(X, Y)$ consisting of all the r-jets of "equivariant local maps". Then $J^r_G(X, Y)$ is a G-invariant subspace of $J^r(X, Y)$.

Let $\Omega(X, Y)$ be an open G-subbundle of $J^r(X, Y) \to X$ which is invariant under the natural action by local equivariant diffeomorphisms of X on $J^r(X, Y)$. Then $\Omega(X, Y)$ is called *a natural stable regularity condition.* We say that a map $f: X \to Y$ is *Ω-regular* if $j^r f(X) \subset \Omega(X, Y)$. Now we assume that $\Omega(X, Y)$ be a natural stable regularity condition. We say that $\Omega(X, Y)$ is G-extensible if the following conditions hold: There exists a natural stable regularity condition $\Omega'(X \times \mathbb{R}, Y) \subset J^r(X \times \mathbb{R}, Y)$ (where G acts on $\mathbb{R}$ trivially) such that

$$\begin{cases} \pi(i^*(\Omega'(X \times \mathbb{R}, Y))) = \Omega(X, Y) \\ \pi(i^*(\Omega'(X \times \mathbb{R}, Y) \cap J^r(X \times \mathbb{R}, Y))) = \Omega(X, Y) \cap J^r_G(X, Y), \end{cases}$$

where $\pi: i^*(J^r(X \times \mathbb{R}, Y)) \to J^r(X, Y)$ is the natural projection defined by $\pi(j^r_{(x,0)} f) = j^r_x f \circ i$ for the canonical inclusion $i: X \to X \times \mathbb{R}$. The examples of the G-extensible regularity condition are given in ([2], [3]).

In this paper we will prove the following approximation theorem.

THEOREM 0.1. *Let $\Omega(X, Y)$ be a G-extensible regularity condition, and suppose that there is a continuous equivariant section $\sigma: X \to \Omega(X, Y)$ covering the map $f: X \to Y$. Then f may be fine C^0-approximated by smooth Ω-regular equivariant maps whose r-jets are G-homotopic to σ as sections of $\Omega(X, Y)$.*

This result is an equivariant generalization of the approximation theorem in Appendix of [4]. In [2] we have shown a theorem of homotopy classification on Ω-regular smooth equivariant maps. If we consider an open manifold X, Theorem 1.3 in [2] does not assert that the homotopy class of a proper equivariant map is represented by the jet of an Ω-regular proper smooth equivariant map. However, Theorem 0.1 guarantees this property, so that the theorem refines the previous result in [2].

In §1 we recall some elementary facts about transformation groups. Theorem 0.1 will be proved in §2 by using the usual extension technique for G-equivariant maps on equivariant simplicial complexes and the C^1-triangulation theorem for smooth G-manifolds which has been proved in [1].

1. Preliminaries

Let G be a finite group and X a G-manifold. For any $x \in X$, we denote Gx the orbit of x and G_x the isotropy subgroup of x. Let V, W be Riemannian G-vector bundles over Gx. The fibres V_x, W_x of V, W over the point x are G_x-modules and we have canonical isomorphisms $V \cong G \times_{G_x} V_x$, $W \cong G \times_{G_x} W_x$. The bundle

$$D(V) \oplus D(W) = \{(v, w) \in V \oplus W \mid \|v\| \leq 1, \|w\| \leq 1\}$$

with fibre $D(V_x) \times D(W_x)$ is called *a handle bundle* with index $= \dim V$. Let Z, Y be invariant submanifolds of X and $\phi\colon S(V) \oplus D(W) \to \partial Z$ be an equivariant embedding, where $S(V) \oplus D(W) = \{(v, w) \in V \oplus W \mid \|x\| = 1, \|w\| \leq 1\}$. If $Y = Z \bigcup_\phi (D(V) \oplus D(W))$, then we say that Y is obtained by attaching the handle bundle $D(V) \oplus D(W)$ to Z via $S(V) \oplus D(W)$. We may consider that $D(V) \oplus D(W) \cong G \times_{G_x} (D(V_x) \times D(W_x))$. Then there is an open G_x-equivariant C^∞-map

$$F\colon J^r(D_2(V) \oplus D(W), Y) | D_2(V_x) \times D(W_x) \longrightarrow J^r(D_2(V_x) \times D(W_x), Y)$$

defined by $F(j^r_y f) = j^r_y(f | D_2(V_x) \times D(W_x))$. Here, $D_2(V) = \{v \in V \mid \|v\| \leq 2\}$. Since G is a finite group, F is an isomorphism on fibre and maps

$$J^r_G(D_2(V) \oplus D(W), Y) | D_s(V_x) \times D(W_x)$$

isomorphically onto $J^r_{G_x}(D_2(V_x) \times D(W_x), Y)$. We define

$$\Omega_x(D_2(V_x) \times D(W_x), Y) = F(\Omega(D_2(V) \oplus D(W), Y) | D_2(V_x) \times D(W_x)).$$

If $\Omega(X, Y)$ is G-extensible, then $\Omega_x(D_2(V_x) \times D(W_x), Y)$ is G_x-extensible. Since G is a finite group, we can interpret Propositions 4.3 and 4.4 in [2] as follows:

Proposition 1.1. *Let G be a finite group. Suppose that $\Omega(X, Y)$ is G-extensible and G_x acts on V trivially. Then the restriction map*

$$\rho_\Omega\colon C^\infty_{G_x\Omega_x}(D_2(V_x) \times D(W_x), Y) \longrightarrow C^\infty_{G_x\Omega_x}(D_{[1\,2]}(V_x) \times D(W_x), Y)$$

is a Serre fibration, where $D_{[1\,2]}(V_x) = \{v \in V_x \mid 1 \leq \|v\| \leq 2\}$.

Proposition 1.2. *With the same assumption, the restriction map*

$$\rho\colon \Gamma^0_{G_x}(\Omega_x(D_2(V_x) \times D(W_x), Y)) \longrightarrow \Gamma^0_{G_x}(D_{[1\,2]}(V_x \times D(W_x), Y))$$

is a Serre fibration.

Here, the notation is the same as in [2].

We now review some results of equivariant simplicial complexes in [1]. By a simplicial complex we mean a geometric simplicial complex, that is, the topological realization of an abstract simplicial complex considered as a topological space together with the structure given by the simplexes. *A simlicial G-complex* consists of a simplicial complex K together with a G-action $\phi\colon G \times K \to K$ such that $g\colon K \to K$ is a simplicial homeomorphism for every $g \in G$. For a simplicial G-complex K we say that it is *an equivariant simplicial complex* if the following conditions are satisfied.

1. For any subgroup H of G we have that if $s = \langle v_0, \dots, v_n \rangle$ is a simplex of K and $s' = \langle h_0 v_0, \dots, h_n v_n \rangle$ is also a simplex of K for some $h_i \in H$ $(i = 0, \dots, n)$ then there exists $h \in H$ such that $hv_i = h_i v_i$ $(i = 0, \dots, n)$.
2. For any simplex s of K the vertices $v_0, \dots, b_n$ of s can be ordered in such a way that $G_{v_n} \subset \cdots \subset G_{v_0}$.

In condition 1 the vertices $v_0, \dots, v_n$ need not be distinct. We call G_{v_n} *the principal isotropy subgroup of s.* The above conditions are purely technical since the second barycentric subdivision of any simplicial G-complex is an equivariant simplicial complex. In [1] Illman has shown the C^1-triangulability theorem for smooth G-manifold when G is a finite group.

2. Proof of Theorem 0.1

Let ρ be any invariant smooth metric on Y, and let $\alpha\colon X \to (0\ \infty)$ be an invariant smooth function. We shall show that there exists an Ω-regular equivariant map $g\colon X \to Y$ such that $j^r g$ is Ω-regulary G-homotopic to σ (i.e. σ and $j^r g$ is G-homotopic as sections of $\Omega(X, Y)$) and that $\rho(f(x), g(x)) < \alpha(x)$ for each $x \in X$. For an open invariant submanifold W of Y, we define $\Omega(X, W) = J^r(X, W) \cap \Omega(X, Y)$. If $\Omega(X, Y)$ is G-extensible then $\Omega(X, W)$ is also G-extensible. We will use this fact to prove Theorem 0.1. Furthermore, we need the following simple lemma.

LEMMA 2.1. *Suppose that we have a commutative diagram*

$$\begin{tikzcd} E \arrow[r, "\bar{g}"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B \arrow[r, "g"] & B', \end{tikzcd}$$

where p, p' are Serre fibrations and g is a weak homotopy equivalence. Then $\bar{g}$ is a weak homotopy equivalence if and only if its restriction to each fibre of E is a weak homotopy equivalence.

Proof of Theorem 0.1. For each $x \in X$, let W_x be an open $G_{f(x)}$-invariant convex coordinate neighbourhood of $f(x)$ contained in $\{y \in Y \mid \rho(f(x), y) < 1/4\alpha(x)\}$ such that $gW_x = W_{gx}$ for each $g \in G$. Take an open covering $\{U_x\}$ of X such that each U_x is a G_x-invariant subset which is included in $\{y \in Y \mid \alpha(y) > 1/2\alpha(x)\} \cap f^{-1} W_x$. Choose a smooth equivariant triangulation of X so fine that each n-simplex A lies in one of the open sets U_x, say U_A, where $n = \dim X$. Suppose inductively that we have construct an invariant neighborhood X_{j-1} of $(j-1)$-skeleton on X, and an Ω-regular equivariant map $g_{j-1}\colon X_{j-1} \to Y$ such that $j^r g_{j-1}$ is Ω-regularly G-homotopic to $\sigma \mid X_{j-1}$ and such that $j^r g_{j-1}(X_{j-1} \cap U_A) \subset W_A$ for each n-simplex A (these constructions may clearly be made for $j = 1$).

Now let X'_{j-1} be an invariant neighbourhood of the $(j-1)$-skeleton in X_{j-1}, and for each j-simplex E let $X(E)$ be a G_e-invariant neighbourhood of E, where $e \in \operatorname{Int} E$, in $U_E = \cap \{U_A \mid E \prec A\}$ such that $X(E) - X'_{j-1}$ are disjoint and there is a G_e-diffeomorphism

$$(X(E), X(E) \cap X'_{j-1}) \cong (D^j_2(V_e) \times D^{n-j}(W_e), D^j_{[1\,2]} \times D^{n-j}(W_e)).$$

Here we may choose $X(E)$ with the following properties:

a) V_e and W_e are G_e-vector spaces and the action of G_e on V_e is trivial.

b) For any $h \in G$, $h(D^j_2(V_e) \times D^{n-j}(W_e)) = D^j_2(V_{he}) \times D^{n-j}(W_{he})$.

c) If $h(D^j_2(V_e) \times D^{n-j}(W_e)) \cap (D^j_2(V_e) \times D^{n-j}(W_e)) \neq \emptyset$, then $h \in G_e$.

We have a G-diffeomorphism from $G \times_{G_e} (D^j_2(V_e) \times D^{n-j}(W_e))$ to $G(D^j_2(V_e) \times D^{n-j}(W_e)) \cong GX(E)$.

Consider the following commutative diagram:

$$\begin{array}{ccc} C^\infty_{G_e\Omega_e}(X(E), W_A) & \xrightarrow{j^r} & \Gamma^0_{G_e}(\Omega_e(X(E), W_A)) \\ \downarrow{\scriptstyle \rho_{\Omega_e}} & & \downarrow{\scriptstyle \rho} \\ C^\infty_{G_e\Omega_e}(X(E) \cap X'_{j-1}, W_A) & \xrightarrow{j^r} & \Gamma^0_{G_e}(\Omega_e(X(E) \cap X'_{j-1}, W_A)) \end{array}$$

whose vertical maps are Serre fibrations (by Propositions 1.1 and 1.2) and whose horizontal maps are weak homotopy equivalence by Theorem 1.3 in [2].

Since $j^r g_{j-1} \mid X(E) \cap X'_{j-1}$ is Ω_e-regularly G_e-homotopic to $\sigma \mid X(E) \cap X'_{j-1}$, and so there is a section τ in the fibre over $j^r g_{j-1} \mid X(E) \cap X'j-1$ which is Ω_e-regularly G_e-homotopic to $\sigma \mid X(E)$ and there is an Ω_e-regular G_e-equivariant map $g_E\colon X(E) \to W_E$ in the fibre over $g_{j-1} \mid X(E) \cap X'_{j-1}$ (using Lemma 2.1) such that $j^r g_E$ is Ω_e-regularly G_e-homotopic to τ and hence σ.

Now we define $X(GE)=GX(E)$ and $g_{GE}\colon X(GE)\to GW_E$ by $g_{GE(hz)}=hg_E(z)$ for any $z\in X(E)$ and each $h\in G$. Then g_{GE} is an equivariant map and $j^r g_{GE}$ is Ω-regularly G-homotopic to σ. Define $X_j=\bigcap_{GE}X(GE)$, and $g_j\colon X_j\to Y$ by $g_j(z)=g_{GE}(z)$ for $z\in X(GE)$. This completes the induction step.

By this method, we can construct an Ω-regular equivariant map $g\colon X\to Y$ such that $j^r g$ is Ω-regularly G-homotopic to σ and such that $g(U_A)\subset W_A$ for each n-simplex A.

By the definition, $U_A=U_{x_A}$ for some $x_A\in X$. If $z\in U_A$, then $f(z)$, $g(z)\in W_{x_A}$, so that $\rho(f(z), g(z))\leq\rho(f(z), f(x_A))+\rho(g(z), f(x_A))<1/4(\alpha(x_A)+\alpha(x_A))=1/2\alpha(x_A)$. However, $z\in U_A$, then $\alpha(z)>1/2\alpha(x_A)$. It follows that $\rho(f(z), g(z))<\alpha(z)$. This completes the proof.

References

[1] S. Illman, *Smooth equivariant triangiulations of G-manifolds for G a finite group*, Math. Ann., **233** (1978), 199–220.

[2] S. Izumiya, *Homotopy classification of equivariant regular sections*, manuscripta math., **28** (1979), 337–360.

[3] S. Izumiya, *On G-extensible regularity condition and Thom-Boardman singularities*, J. Math. Soc. Japan, **33** (1981), 497–504.

[4] A. A. du Plessis, *Homotopy classification of regular sections*, Comps. Math., **32** (1976), 301–333.

Department of Mathematics
Hokkaido University
Sapporo 060, *Japan*