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On links whose complements have the Lusternik-Schnirelmann category one

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1. Introduction

The Lusternik-Schnirelmann category $\operatorname{cat} X$ of a space X is the least integer n such that X can be covered by $n+1$ open subsets each of which is contractible to a point in X. In particular, $\operatorname{cat} X$ is a homotopy type invariant and $\operatorname{cat}(\bigvee_i S^{n_i}) = 1$ where $\bigvee$ stands for the one point union. We know that $\pi_1(X)$ is a free group if X is a manifold and $\operatorname{cat} X = 1$ [4], [7].

A locally flat knot (S^{n+2}, S^n) is topologically unknotted if and only if the category of its complement is one [14]. So, a smooth (or PL locally flat) knot (S^{n+2}, S^n) is unknotted if and only if $\operatorname{cat}(S^{n+2} - S^n) = 1$ when $n \neq 2$ ([12], [25] for $n \geq 4$, [21] for $n = 3$ and [18] for $n = 1$). We know also that there exists a smooth knot (S^{n+2}, S^n) whose complement is of category m with $2 \leq m \leq n+1$ for any n [15], [16].

A smooth (resp. PL locally flat or locally flat) m-component link L stands for m smoothly (resp. PL locally flatly or locally flatly) embedded disjoint n-spheres $L_1 \cup \cdots \cup L_m$ in S^{n+2}. A smooth (resp. PL locally flat or locally flat) m-component link is called trivial if it bounds m smoothly (resp. PL locally flatly or locally flatly) embedded disjoint $(n+1)$-disks; boundary if it bounds a Seifert manifold which consists of m disjoint compact smooth (resp. PL locally flat or locally flat) $(n+1)$-submanifolds with connected boundary. Let $N_i = N(L_i)(i = 1, \ldots, m)$ be the tubular neighborhoods of L_i which do not intersect each other. The compact manifold $E = S^{n+2} - \bigcup \operatorname{Int} N(L_i)$ with boundary $\partial E = \bigcup \partial N_i$ is called link exterior and has the homotopy type of the link complement $S^{n+2} - L$.

A smooth boundary link (S^{n+2}, L) is trivial if $\operatorname{cat}(S^{n+2} - L) = 1$ when $n \neq 2$ [11]. In particular, the complement $S^{n+2} - L$ of a smooth boundary link L has the homotopy type of $(\bigvee_m S^1) \vee (\bigvee_{m-1} S^{n+1})$ if $\operatorname{cat}(S^{n+2} - L) = 1$ when $n \neq 2$.

The purpose of this paper is to show the following Theorems 1 and 2. Note that any smooth or PL locally flat link is locally flat. So, Theorem 1 gives an alternative proof of the main theorem of [11] by unlinking criterion of boundary links due to Gutiérrez ([8] for $n \geq 4$ and use the splitting theorem [2] for $n = 3$). Moreover, we see that a PL locally flat or locally flat boundary link is also trivial if $\operatorname{cat}(S^{n+2} - L) = 1$ when $n \neq 2$.

THEOREM 1. *Let L be a locally flat m-component link in S^{n+2}. Assume that* $\operatorname{cat}(S^{n+2} - L) = 1$. *Then the link exterior E has the homotopy type of* $(\bigvee_m S^1) \vee (\bigvee_{m-1} S^{n+1})$.

The tubular neighborhoods $N(L_i)$ of L_i are the images of the embeddings $\Psi_i : D^2 \times S^n \to S^{n+2}$ with $\Psi_i(0 \times S^n) = L_i$ for $i = 1, 2, \ldots, m$. Let $M = E \cup_{\Psi_1(S_1^1 \times S_1^n)} (S_1^1 \times D_1^{n+1}) \cup \cdots \cup_{\Psi_m(S_m^1 \times S_m^n)} (S_m^1 \times D_m^{n+1})$ denote the result of the spherical modifications of S^{n+2} with respect to all Ψ_i. We may call M manifold obtained by a surgery along L.

THEOREM 2. *Let L be a smooth (resp. PL locally flat or locally flat) m-component link and M the manifold obtained by a surgery along L. Suppose that $n \neq 2$. If* $\operatorname{cat}(S^{n+2} - L) = 1$, *then M is diffeomorphic (resp. PL homeomorphic or homeomorphic) to the connected sum of m copies of $S^1 \times S^{n+1}$.*

A classical link L is trivial if $\pi_1(S^3 - L)$ is a free group by the loop theorem [18]. See also [13] for a detailed proof. So, Theorems 1 and 2 are already proved for $n = 1$, since $\operatorname{cat}(S^{n+2} - L) = 1$ implies that $\pi_1(S^{n+2} - L)$ is a free group as mentioned before.

Theorem 1 has no dimensional restriction. On the other hand, as for Theorem 2 we know that M is homeomorphic to $S^1 \times S^3$ when $n = 2$ and $m = 1$ by [14], but there are more difficulties when $n = 2$ and $m \geq 2$ [6].

We will prove Theorem 1 in §2 and Theorem 2 for $n \geq 3$ in §3. We will be concerned also with the conjecture that any link L is trivial if and only if $\operatorname{cat}(S^{n+2} - L) = 1$ in §3. This conjecture is due to Professor T. Matumoto and I express my heartiest thanks to him for suggesting the interesting problem.

2. Homotopy type of the link exterior with category one

If $\operatorname{cat}(S^{n+2} - L) = 1$, then the fundamental group $\pi_1(S^{n+2} - L) = \pi_1(E)$ is a free group as we mentioned in the introduction. Since the abelianized group $H_1(E; \mathbf{Z})$ is a free abelian group of rank m by the Alexander duality, $\pi_1(E)$ is a free group F_m of rank m. Theorem 1 follows almost directly from the following lemma.

LEMMA 2.1. *Let L be a locally flat m-component link in S^{n+2} such that* $\operatorname{cat}(S^{n+2} - L) = 1$. *Then, the link exterior E satisfies*

(1) $\pi_j(E) = 0$ *for* $2 \leq j \leq n$ *and*

(2) $\pi_{n+1}(E)$ *is isomorphic to a free* $\mathbf{Z}[F_m]$*-module* $\mathbf{Z}[F_m]^{m-1}$ *of rank* $m - 1$ *as* $\mathbf{Z}[F_m]$*-module.*

Due to [26, p. 458, Chapter X (1.3)] E is a co-H-space, because cat $E = 1$ and E has the homotopy type of a CW-complex. So, we can use the following lemma to prove Lemma 2.1 (1).

LEMMA 2.2 ([9, p. 11, Lemma 6.2]). *Let Y be a connected co-*H*-space and K a field. Then, the homology group $H_i(\tilde{Y}; K)$ is a free $K[\pi_1 Y]$-module for $i \geq 1$ and $H_i(Y; K) = H_i(\tilde{Y}; K) \otimes_{\pi_1 Y} K$ for $i > 1$, where $\tilde{Y}$ denotes the universal covering of Y.*

To prove Lemma 2.1 (1) and (2) we prepare the following algebraic lemma, which will be also used in §3 to prove Lemma 3.3.

LEMMA 2.3. *Let F_m be a free group of rank m generated by $t_1, \ldots, t_m$ and A a $K[F_m]$-module, where K is a field or ring $\mathbf{Z}$ of integers. Let $\eta_1 : A^m \to A$ denote the map defined by $\eta_1(x_1, \ldots, x_m) = \sum_{j=1}^{m} (t_j - 1)x_j$. Assume that A is a free $K[F_m]$-module and $\eta_1 : A^m \to A$ is surjective. Then $A = 0$.*

PROOF OF LEMMA 2.3. We define a $\mathbf{Z}$-homomorphism $\eta_k : A^{m^k} \to A$ by $\eta_1(x_1, \ldots, x_m) = \sum_j (t_j - 1)x_j$ and $\eta_k = \eta_{k-1} \circ (\eta_1^{m^{k-1}})$ inductively, where A^r denotes r times direct sum of A and η_1^r denotes r times direct sum of η_1. Since η_1 is surjective by the assumption, we see that η_k is surjective for any k by the induction on k. Hence, $A \subset \operatorname{Im} \eta_k$ for any k. We can take a free basis $\{b_i\}$ of A as $K[F_m]$-module by the assumption. We define $(\eta_k)_i$ by the restriction $\eta_k | K[F_m]\langle b_i \rangle$. For each i we see that $\operatorname{Im}(\eta_1)_i = I\langle b_i \rangle$ for the augmentation ideal I of $K[F_m]$ and that $\operatorname{Im}(\eta_k)_i = I^k\langle b_i \rangle$ for the k-th power of the augmentation ideal I of $K[F_m]$. We have that $\bigcap_k \operatorname{Im} \eta_k = \bigcap_k (\bigoplus_i I^k\langle b_i \rangle) = \bigoplus_i (\bigcap_k I^k)\langle b_i \rangle$. By the residual nilpotency [11, p. 333, Lemma 3.2], [5, p. 556, (4.4) Corollary] we have that $\bigcap_k I^k = 0$. So, we obtain that $\bigcap_k \operatorname{Im} \eta_k = \bigoplus_i (\bigcap_k I^k)\langle b_i \rangle = 0$. This implies that $A = 0$, because we have shown that $A \subset \operatorname{Im} \eta_k$ for any k. q.e.d.

PROOF OF LEMMA 2.1(1). Let p be a prime. We fix m free generators $t_1, \ldots, t_m$ of the free group $\pi_1(E)$ of rank m. Let $\tilde{E}$ be the universal covering of E and due to N. Sato [20, p. 502, Proposition 2.4] we have a homology long exact sequence:

$$\cdots \to H_{j+1}(E; \mathbf{Z}_p) \to H_j(\tilde{E}; \mathbf{Z}_p)^m \xrightarrow{\eta_1} H_j(\tilde{E}; \mathbf{Z}_p) \to H_j(E; \mathbf{Z}_p) \to \cdots.$$

Suppose that $2 \leq j \leq n$. Then $\eta_1 : A^m \to A$ is surjective by taking $A = H_j(\tilde{E}; \mathbf{Z}_p)$ because $H_j(E; \mathbf{Z}_p) = 0$. By Lemma 2.2 $A = H_j(\tilde{E}; \mathbf{Z}_p)$ is a free $\mathbf{Z}_p[F_m]$-module. By applying Lemma 2.3 we get that $A = H_j(\tilde{E}; \mathbf{Z}_p) = 0$ for $2 \leq j \leq n$. Therefore we obtain that $H_j(\tilde{E}; \mathbf{Z}) = 0$ for $2 \leq j \leq n$ by the universal coefficient theorem and hence $\pi_j(E) = 0$ for $2 \leq j \leq n$ by Hurewicz theorem. q.e.d.

PROOF OF LEMMA 2.1(2). We will prove that $H_{n+1}(\tilde{E}; \mathbf{Z})$ is isomorphic to $\mathbf{Z}[F_m]^{m-1}$ as $\mathbf{Z}[F_m]$-module. Even when $(E, \partial E)$ admits no triangulation, $(E, \partial E)$ has the simple homotopy type of a finite Poincaré complex by [10, III, §4]. Let $(E, \partial E)$ denote this finite Poincaré complex instead of the original link exterior hereafter. Let $p: \tilde{E} \to E$ be the universal covering and put $\partial\tilde{E} = p^{-1}(\partial E)$. Let $H_c^*(\tilde{E}, \partial\tilde{E}; \mathbf{Z})$ be the cohomology of $(\tilde{E}, \partial\tilde{E})$ with compact support. Since the CW complex pair $(\tilde{E}, \partial\tilde{E})$ has the proper homotopy type of the universal covering of the original link exterior, we can apply the Poincaré duality theorem for the non-compact manifold and get that the left $\mathbf{Z}[F_m]$-module $H_{n+1}(\tilde{E}; \mathbf{Z})$ is anti-$\mathbf{Z}[F_m]$ isomorphic to the right $\mathbf{Z}[F_m]$-module $H_c^1(\tilde{E}, \partial\tilde{E}; \mathbf{Z})$. The cellular chain complex $C_\#(\tilde{E}, \partial\tilde{E})$ of $(\tilde{E}, \partial\tilde{E})$ is a chain complex of finitely generated free $\mathbf{Z}[F_m]$-modules and we define the cochain complex $C^\#(E, \partial E; \mathbf{Z}[F_m])$ by $\mathrm{Hom}_{\mathbf{Z}[F_m]}(C_\#(\tilde{E}, \partial\tilde{E}), \mathbf{Z}[F_m])$. We write the cellular cochain complex of $(\tilde{E}, \partial\tilde{E})$ with compact support by $C_c^\#(\tilde{E}, \partial\tilde{E})$, which has the right action $(f\cdot g)(c) = f(g^{-1}c)$ $(f \in C_c^\#(\tilde{E}, \partial\tilde{E}),\ g \in F_m,\ c \in C_\#(\tilde{E}, \partial\tilde{E}))$. We define also a right $\mathbf{Z}[F_m]$-homomorphism $\psi: C_c^\#(\tilde{E}, \partial\tilde{E}) \to C^\#(E, \partial E; \mathbf{Z}[F_m])$ by $\psi(f)(c) = \sum_{g\in\pi_1(E)} f(gc)g^{-1}(c \in C_\#(\tilde{E}, \partial\tilde{E}), f \in C_c^\#(\tilde{E}, \partial\tilde{E}))$. Then, ψ is a cochain equivalence. In fact, we can take a free basis $\{c_i\}$ of $C_\#(\tilde{E}, \partial\tilde{E})$ as $\mathbf{Z}[F_m]$-module and define a $\mathbf{Z}[F_m]$-homomorphism $\phi: C^\#(E, \partial E; \mathbf{Z}[F_m]) \to C_c^\#(\tilde{E}, \partial\tilde{E})$ by $\phi(h)(w_{ij}^{-1}c_i) = n_{ij}$ when $h(c_i) = \sum_j n_{ij}w_{ij}$ for $h \in C^\#(E, \partial E; \mathbf{Z}[F_m])$, $n_{ij} \in \mathbf{Z}$ and $w_{ij} \in F_m$ with $w_{ij} \neq w_{ij'}$ for $j \neq j'$. We see that ψ and ϕ are cochain maps and $\psi\circ\phi = \phi\circ\psi = \text{identity}$. Now consider the universal coefficient spectral sequence $E_2^{p,q} = \mathrm{Ext}^p_{\mathbf{Z}[F_m]}(H_q(\tilde{E}, \partial\tilde{E}; \mathbf{Z}), \mathbf{Z}[F_m])$ with $d_r^{p,q}: E_r^{p,q} \to E_r^{p+r,q-r+1}$ which converges to $H^{p+q}(C^\#(E, \partial E; \mathbf{Z}[F_m])) = H_c^{p+q}(\tilde{E}, \partial\tilde{E}; \mathbf{Z})$. Then, we obtain that $H_c^1(\tilde{E}, \partial\tilde{E}; \mathbf{Z}) \cong \mathrm{Hom}_{\mathbf{Z}[F_m]}(H_1(\tilde{E}, \partial\tilde{E}; \mathbf{Z}), \mathbf{Z}[F_m])$ by a standard argument of spectral sequences, because $H_0(\tilde{E}, \partial\tilde{E}; \mathbf{Z}) = 0$.

Note that the kernel of $\mathbf{Z}[F_m]$-homomorphism between finitely generated projective $\mathbf{Z}[F_m]$-modules is a finitely generated projective $\mathbf{Z}[F_m]$-module. In fact, it is projective because $\mathbf{Z}[F_m]$ has the global dimension two due to [17, p. 326, Corollary 2.7], and finitely generated because $\mathbf{Z}[F_m]$ is coherent [3, p. 137, Theorem (2.1)], [24, p. 158, Proposition]. Moreover, due to [1] a finitely generated projective $\mathbf{Z}[F_m]$-module is a free $\mathbf{Z}[F_m]$-module. Then, $H_1(\tilde{E}, \partial\tilde{E}; \mathbf{Z})$ is a finitely presented $\mathbf{Z}[F_m]$-module, that is, we can take an exact sequence $P_1 \to P_0 \to H_1(\tilde{E}, \partial\tilde{E}; \mathbf{Z}) \to 0$, where P_0 and P_1 are finitely generated free $\mathbf{Z}[F_m]$-modules. By applying $\mathrm{Hom}_{\mathbf{Z}[F_m]}(-, \mathbf{Z}[F_m])$ to this exact sequence, we see that $\mathrm{Hom}_{\mathbf{Z}[F_m]}(H_1(\tilde{E}, \partial\tilde{E}; \mathbf{Z}), \mathbf{Z}[F_m])$ is the kernel of $\mathrm{Hom}_{\mathbf{Z}[F_m]}(P_0, \mathbf{Z}[F_m]) \to \mathrm{Hom}_{\mathbf{Z}[F_m]}(P_1, \mathbf{Z}[F_m])$, and hence a finitely generated free $\mathbf{Z}[F_m]$-module. So, $H_c^1(\tilde{E}, \partial\tilde{E}; \mathbf{Z}) \cong \mathrm{Hom}_{\mathbf{Z}[F_m]}(H_1(\tilde{E}, \partial\tilde{E}; \mathbf{Z}), \mathbf{Z}[F_m])$ is a free right $\mathbf{Z}[F_m]$-module. Then, its Poincaré dual $H_{n+1}(\tilde{E}; \mathbf{Z})$ is also a free left $\mathbf{Z}[F_m]$-module.

Since $H_n(\tilde{E}; \mathbf{Z}) = H_{n+2}(E; \mathbf{Z}) = H_{n+1}(E; \mathbf{Z}) = 0$ in the homology long exact sequence due to N. Sato, we have a short exact sequence

$$0 \to H_{n+1}(\tilde{E};\mathbf{Z})^m \xrightarrow{\eta_1} H_{n+1}(\tilde{E};\mathbf{Z}) \xrightarrow{p_*} H_{n+1}(E;\mathbf{Z}) \to 0,$$

where $\eta_1(x_1, \dots, x_m) = \sum_j (t_j - 1)x_j$ for $(x_1, \dots, x_m) \in H_{n+1}(\tilde{E};\mathbf{Z})^m$ as before. This exact sequence implies that $H_{n+1}(\tilde{E};\mathbf{Z}) \cong \mathbf{Z}[F_m]^{m-1}$ as $\mathbf{Z}[F_m]$-module. In fact, since p_* is surjective and $H_{n+1}(E;\mathbf{Z}) \cong \mathbf{Z}^{m-1}$, we have $\ell = \mathrm{rank}_{\mathbf{Z}[F_m]} H_{n+1}(\tilde{E};\mathbf{Z}) \geq m-1$ so that we can take the basis b_1, b_2, ... b_ℓ of $H_{n+1}(\tilde{E};\mathbf{Z})$ as $\mathbf{Z}[F_m]$-module with $p_*(b_i) = 0$ for $m \leq i \leq \ell$. For the free $\mathbf{Z}[F_m]$-submodule $A = \bigoplus_{i=m}^{\ell} \mathbf{Z}[F_m]\langle b_i \rangle$ of $H_{n+1}(\tilde{E};\mathbf{Z})$ generated by b_m, b_{m+1}, ..., b_ℓ the $\mathbf{Z}$-homomorphism $\eta_1 | A^m : A^m \to A$ is an isomorphism by the above exact sequence. By applying Lemma 2.3 we get that $A = 0$ and hence $\ell = m-1$. Now by Lemma 2.1 (1) and Hurewicz theorem we see that $\pi_{n+1}(E) \cong H_{n+1}(\tilde{E};\mathbf{Z})$ as $\mathbf{Z}[F_m]$-module and hence $\pi_{n+1}(E)$ is isomorphic to $\mathbf{Z}[F_m]^{m-1}$ as $\mathbf{Z}[F_m]$-module. q.e.d.

We take a representative $\sigma_i : S_i^1 \to E$ for each generator t_i of $\pi_1(E)$ $(i = 1, \dots, m)$. We define $f : (\bigvee_m S_i^1) \to E$ by $f | S_i^1 = \sigma_i$. Since $\pi_{n+1}(E) \cong (\mathbf{Z}[F_m])^{m-1}$, we take a representative $\gamma_j : S_j^{n+1} \to E$ for each element of the basis of $\pi_{n+1}(E)$ $(j = 1, \dots, m-1)$. We define $f' : (\bigvee_{m-1} S_j^{n+1}) \to E$ by $f' | S_j^{n+1} = \gamma_j$, and $g : (\bigvee_m S^1) \vee (\bigvee_{m-1} S^{n+1}) \to E$ by $g | (\bigvee_m S^1) = f$ and $g | (\bigvee_{m-1} S^{n+1}) = f'$. Then, g induces an isomorphism of the k-th homotopy group for any $1 \leq k \leq n+1$. So, g is a homotopy equivalence by the theorem of Whitehead, because E has the homotopy type of a CW complex of homological dimension $n+1$. The proof of Theorem 1 is completed.

3. Surgery along the link and further comments

In §2 we have proved that the link exterior E has the homotopy type of $(\bigvee_m S^1) \vee (\bigvee_{m-1} S^{n+1})$ if $\mathrm{cat}\, E = 1$. So, it suffices to show the following proposition in order to prove Theorem 2 for $n \geq 3$.

Before stating the proposition we remark that an element of $\pi_1(E)$ is called meridian if it is conjugate to the generator of the fundamental group of some component of $\partial E = \bigcup_{i=1}^m S_i^1 \times S_i^n$. Moreover, in the case that $\pi_1(E)$ is a free group the link is boundary if and only if there exist m number of meridians m_1, ..., m_m which generate $\pi_1(E)$ [8, p. 493, Proposition (3)], [23, p. 178, 6.3 Theorem].

PROPOSITION 3.1. *Let L be a smooth (resp. PL locally flat or locally flat) m-component link in S^{n+2} and M the manifold obtained by a surgery along L. Suppose that $\pi_1(E)$ is a free group and $\pi_j(E) = 0$ for $2 \leq j \leq n-1$. Then, M is diffeomorphic (resp. PL homeomorphic or homeomorphic) to the manifold obtained by a surgery along a trivial link, provided that $n \geq 3$.*

PROOF. By the assumption $\pi_1(E)$ is a free group of rank m as before.

Since $\pi_1(E) \cong \pi_1(M)$ for the natural inclusion $E \hookrightarrow M$, we have m free generators $t_1, \dots, t_m$ of $\pi_1(M)$. The tubular neighborhoods $N(L_i)$ of L_i are the images of the disjoint embeddings $\Psi_i: D_i^2 \times S_i^n \to S^{n+2}$ for $i = 1, 2, \dots, m$. We take the manifold $U_0(L)$ obtained by adding m number of $(n+1)$-handles to $S^{n+2} \times [0, 1]$ under $\Psi_i \times 1$ and get $\partial U_0(L) = S^{n+2} \cup M$. Since $\partial_*: \pi_2(U_0(L), M) \to \pi_1(M)$ is surjective, there exists $\alpha_i \in \pi_2(U_0(L), M)$ such that each $\partial_*(\alpha_i)$ is the free homotopy class of t_i in $\pi_1(M)$. Because $4 < \dim U_0(L) = n + 3$, we can take disjoint embeddings $\phi_i': (D_i^2, S_i^1) \to (U_0(L), M)$ such that each $\phi_i'(D_i^2)$ represents the free homotopy class of α_i. Since the normal bundle of $\phi_i'(D_i^2)$ is trivial, ϕ_i' extend to the proper disjoint embeddings $\phi_i: (D_i^2 \times D_i^{n+1}, S_i^1 \times D_i^{n+1}) \to (U_0(L), M)$. We define $U_1(L)$ by the manifold obtained by adding m number of 2-handles $D_i'^2 \times D_i^{n+1}$ to $U_0(L)$ under $\phi_i | S_i^1 \times D_i^{n+1}$. A connected manifold X is defined by $\partial U_1(L) = S^{n+2} \cup X$. Since $n \geq 3$ and X is the result of a surgery on M along $\phi_i(S_i^1 \times 0)$ $(i = 1, \dots, m)$ which represent the generators of $\pi_1(M)$, X is simply connected. By the homology long exact sequences of (M, M_-), (X, M_-) for $M_- = M - \bigcup_{i=1}^m \operatorname{Int} \phi_i(S_i^1 \times D_i^{n+1})$ and the Poincaré duality we see that X is a homology $(n+2)$-sphere, and hence a homotopy $(n+2)$-sphere. Now we will do a surgery on $U_1(L)$ and get an h-cobordism between S^{n+2} and X. Of course, $U_1(L)$ is simply connected and the basis of $H_2(U_1(L), S^{n+2}; \mathbf{Z})$ is represented by $\phi_i(D_i^2 \times D_i^{n+1}) \cup D_i'^2 \times D_i^{n+1} = S_i^2 \times D_i^{n+1}$ $(i = 1, \dots, m)$, because $H_2(U_1(L), S^{n+2}; \mathbf{Z}) \cong H_2(U_1(L); \mathbf{Z}) \cong H_2(U_1(L), X; \mathbf{Z}) \cong H_2(U_1(L), M \cup (\bigcup_{i=1}^m D_i'^2 \times D_i^{n+1}); \mathbf{Z}) \cong H_2(U_0(L), M; \mathbf{Z}) \cong \mathbf{Z}^m$. We perform the spherical modifications with respect to the embeddings $S_i^2 \times D_i^{n+1} \subset U_1(L)$ and write the result by $U(L)$. Then, we see that $H_j(U(L), S^{n+2}; \mathbf{Z}) = 0$ for $j \neq n$, $n+1$ by the homology long exact sequences of $(U_1(L), U_-(L))$, $(U(L), U_-(L))$ for $U_-(L) = U_1(L) - \bigcup_{i=1}^m \operatorname{Int} S_i^2 \times D_i^{n+1}$. By the Poincaré duality we see $H_i(U(L), S^{n+2}; \mathbf{Z}) = 0$ for $i = n$, $n+1$. (For $n = 3$ we need a little more careful observation). So, we get a simply connected h-cobordism $U(L)$ between S^{n+2} and X. Since $n + 2 \geq 5$, the h-cobordism theorem implies that X is diffeomorphic (resp. PL homeomorphic or homeomorphic) to S^{n+2} [21]. We have now m embedded disjoint n-spheres $\bigcup_{i=1}^m (0 \times S_i^n)$ in $X = M \cup_{\phi_1(S_1^1 \times D_1^{n+1})} (D_1'^2 \times S_1^n) \cup \dots \cup_{\phi_m(S_m^1 \times D_m^{n+1})} (D_m'^2 \times S_m^n)$, by which we define a smooth (resp. PL locally flat or locally flat) link L' in $X = S^{n+2}$. Put $E' = X - \bigcup_{i=1}^m \operatorname{Int}(D_i'^2 \times S_i^n)$. Note that the spherical modification with respect to the embeddings $D_i'^2 \times S_i^n$ in X gives back M and get natural inclusions $i: E \to M$ and $i': E' \to M$. Since $i'_*: \pi_1(E') \to \pi_1(M)$ is an isomorphism, $\pi_1(E')$ is generated by $t_1, t_2, \dots, t_m$. Since $\partial E' = \bigcup_{i=1}^m \phi_i(S_i^1 \times S_i^n)$, the elements $t_1, t_2, \dots, t_m$ give m number of meridians of L' and hence L' is a boundary link by the remark stated above the proposition. On the other hand, M is obtained by adding $(n+1)$-handles and $(n+2)$-handles to E and hence $i_*: \pi_j(E) \to \pi_j(M)$ is an isomorphism for $2 \leq j \leq n-1$. Similarly we see that

$i'_*:\pi_j(E')\to\pi_j(M)$ is also an isomorphism for $2\le j\le n-1$. So, $\pi_j(E')=0$ for $2\le j\le n-1$. Since we have shown that L' is boundary, we get now that L' is trivial by the unlinking criterion of boundary links [2], [8]. Hence, M is diffeomorphic (resp. PL homeomorphic or homeomorphic) to the manifold obtained by a suitable surgery along the trivial link L'. q.e.d.

This completes a proof of Theorem 2. The proof of Theorem 2 reminds us of Poenaru's example. The example of an m-component link given by Poenaru [19] satisfies the assumption of Proposition 3.1 for any m. He took m words $m_1=t_1$ and $m_k=t_kt_1t_k^{-1}t_1^{-1}t_k$ $(2\le k\le m)$ for m free generators t_1, t_2, $\dots$, t_m of F_m. An m-component link $L=L_1\cup\ldots\cup L_m$ in $X=S^{n+2}$, $n\ge 3$, is constructed by making surgery on $\#_mS^1\times S^{n+1}$ along circles representing $m_1=t_1$ and $m_k=t_kt_1t_k^{-1}t_1^{-1}t_k$ $(2\le k\le m)$ of the free group $\pi_1(\#_mS^1\times S^{n+1})$ generated by t_1, t_2, $\dots$, t_m. Note that the Poenaru's link L with the exterior E has m meridians m_1, m_2, $\dots$, m_m in $\pi_1(E)\cong\pi_1(\#_mS^1\times S^{n+1})$. Since the system $z_1m_1z_1^{-1}$, $\dots$, $z_mm_mz_m^{-1}$ can not be a system of generators for F_m for any $z_i\in F_m$ due to an algebraic lemma in [19, p. 48, Lemma], the Poenaru's link is not boundary by the remark stated before Proposition 3.1. We will show the following proposition.

PROPOSITION 3.2. *The Poenaru's link L given above does not satisfy the result of Theorem* 1, *that is, $H_n(\tilde{E};\mathbf{Z})$ is non-trivial for the universal covering $\tilde{E}$ of E.*

This proposition gives also an alternative proof of the fact that the Poenaru's link is not boundary by the unlinking criterion of boundary links [2], [8].

To prove Proposition 3.2 we prepare the following lemma.

LEMMA 3.3. *Let L be a locally flat m-component link in S^{n+2} with the link exterior E. The universal covering of E is denoted by $p:\tilde{E}\to E$ and put $\partial\tilde{E}=p^{-1}(\partial E)$. If $\pi_1(E)$ is a free group and $H_n(\tilde{E};\mathbf{Z})=0$, then* $\mathrm{Ext}^1(i_*):\mathrm{Ext}^1_{\mathbf{Z}[F_m]}(H_0(\tilde{E};\mathbf{Z}),\mathbf{Z}[F_m])\to\mathrm{Ext}^1_{\mathbf{Z}[F_m]}(H_0(\partial\tilde{E};\mathbf{Z}),\mathbf{Z}[F_m])$ *is surjective for the inclusion $i:\partial\tilde{E}\to\tilde{E}$.*

PROOF OF LEMMA 3.3. We use the same notation as in the proof of Lemma 2.1(2). We consider the same universal coefficient spectral sequence, that is, $E_2^{p,q}=\mathrm{Ext}^p_{\mathbf{Z}[F_m]}(H_q(\tilde{E},\partial\tilde{E};\mathbf{Z}),\mathbf{Z}[F_m])$ which converges to $H^{p+q}(C^\#(E,\partial E;Z[F_m]))$ $=H_c^{p+q}(\tilde{E},\partial\tilde{E};\mathbf{Z})\cong H_{n+2-p-q}(\tilde{E};\mathbf{Z})$. Since $H_0(\tilde{E},\partial\tilde{E};\mathbf{Z})=0$, we have that $E_2^{2,0}=0$. We will see also that $E_2^{2,0}=\mathrm{Hom}_{\mathbf{Z}[F_m]}(H_2(\tilde{E},\partial\tilde{E};\mathbf{Z}),\mathbf{Z}[F_m])=0$ in this paragraph. Note first that $H_2(\tilde{E};\mathbf{Z})\cong H_2(\tilde{E},\partial\tilde{E};\mathbf{Z})$ as $\mathbf{Z}[F_m]$-module. We assume that $\mathrm{Hom}_{\mathbf{Z}[F_m]}(H_2(\tilde{E};\mathbf{Z}),\mathbf{Z}[F_m])\ne 0$, that is, there exists a non-zero left $\mathbf{Z}[F_m]$-homomorphism $f:H_2(\tilde{E};\mathbf{Z})\to\mathbf{Z}[F_m]$. Then, there exists a prime p such that $f_p = f \otimes_{\mathbf{Z}} id_{\mathbf{Z}_p}: H_2(\tilde{E}; \mathbf{Z}_p) \to \mathbf{Z}_p[F_m]$ is a non-zero left $\mathbf{Z}_p[F_m]$-homomorphism. Since $\mathbf{Z}_p[F_m]$ has a global dimension one due to [17, p. 326, Corollary 2.7], any finitely generated submodule of a finitely generated free $\mathbf{Z}_p[F_m]$-module is a finitely generated free $\mathbf{Z}_p[F_m]$-module. So, $\operatorname{Im} f_p$ is a finitely generated free $\mathbf{Z}_p[F_m]$-module. Since $H_2(E; \mathbf{Z}_p) = H_3(E; \mathbf{Z}_p) = 0$ in the homology long exact sequence due to N. Sato, $\eta_1: H_2(\tilde{E}; \mathbf{Z}_p)^m \to H_2(\tilde{E}; \mathbf{Z}_p)$ is an isomorphism. Since $\operatorname{Im} f_p$ is a direct summand of $H_2(\tilde{E}; \mathbf{Z}_p)$ as $\mathbf{Z}_p[F_m]$-module, we see also that $\eta_1|(\operatorname{Im} f_p)^m: (\operatorname{Im} f_p)^m \to \operatorname{Im} f_p$ is an isomorphism. So, by applying Lemma 2.3 we get $\operatorname{Im} f_p = 0$. Hence, $f_p = f \otimes_{\mathbf{Z}} id_{\mathbf{Z}_p}$ is a null map. This is a contradiction and hence $E_2^{0,2} = 0$. We have obtained that $E_2^{p,q} = 0$ for $p + q = 2$ and $(p, q) \neq (1, 1)$.

Then, we obtain that $\operatorname{Ext}^1_{\mathbf{Z}[F_m]}(H_1(\tilde{E}, \partial\tilde{E}; \mathbf{Z}), \mathbf{Z}[F_m])$ is anti-$\mathbf{Z}[F_m]$ isomorphic to $H_n(\tilde{E}; \mathbf{Z})$ by a standard argument of spectral sequences and hence vanishes by the assumption. Now apply $\operatorname{Ext}^1_{\mathbf{Z}[F_m]}(-, \mathbf{Z}[F_m])$ to the following short exact sequence;

$$0 \to H_1(\tilde{E}, \partial\tilde{E}; \mathbf{Z}) \to H_0(\partial\tilde{E}; \mathbf{Z}) \xrightarrow{i_*} H_0(\tilde{E}; \mathbf{Z}) \to 0\,,$$

which comes from the homology long exact sequence of $(\tilde{E}, \partial\tilde{E})$ and we obtain the desired result. q.e.d.

We will determine $\operatorname{Ext}^1(i_*)$ explicitly and prove that $\operatorname{Ext}^1(i_*)$ is not surjective for the Poenaru's example given above. Let $i_k: H_0(p^{-1}(\partial N(L_k)); \mathbf{Z}) \to H_0(\tilde{E}; \mathbf{Z})$ be a $\mathbf{Z}[F_m]$-homomorphism induced by the inclusion $p^{-1}(\partial N(L_k)) \hookrightarrow \tilde{E}$ for any k with $1 \leq k \leq m$. Then, $\operatorname{Ext}^1(i_*)$ decomposes into $(\operatorname{Ext}^1(i_1), \operatorname{Ext}^1(i_2), \ldots, \operatorname{Ext}^1(i_m)): \operatorname{Ext}^1_{\mathbf{Z}[F_m]}(H_0(\tilde{E}; \mathbf{Z}), \mathbf{Z}[F_m]) \to \bigoplus_{k=1}^m \operatorname{Ext}^1_{\mathbf{Z}[F_m]}(H_0(p^{-1}(\partial N(L_k)); \mathbf{Z}), \mathbf{Z}[F_m])$.

First, we note that $\operatorname{Ext}^1_{\mathbf{Z}[F_m]}(H_0(\tilde{E}; \mathbf{Z}), \mathbf{Z}[F_m]) = \mathbf{Z}[F_m]^m/\langle(t_1 - 1, t_2 - 1, \ldots, t_m - 1)\rangle\mathbf{Z}[F_m]$ as right $\mathbf{Z}[F_m]$-module, where $\langle(t_1 - 1, t_2 - 1, \ldots, t_m - 1)\rangle\mathbf{Z}[F_m]$ denotes a right submodule generated by a single element $(t_1 - 1, t_2 - 1, \ldots, t_m - 1)$ of $\mathbf{Z}[F_m]^m$ with the diagonal action and $\mathbf{Z}[F_m]^m/\langle(t_1 - 1, t_2 - 1, \ldots, t_m - 1)\rangle\mathbf{Z}[F_m]$ denotes the quoteint right $\mathbf{Z}[F_m]$-module. This follows directly from the free resolution of $H_0(\tilde{E}; \mathbf{Z}) \cong \mathbf{Z}$ given by

$$0 \to \mathbf{Z}[F_m]^m \xrightarrow{\partial} \mathbf{Z}[F_m] \xrightarrow{\varepsilon} \mathbf{Z} \to 0\,, \tag{3.4}$$

where $\varepsilon(x) = \varepsilon(\sum_\alpha n_\alpha w_\alpha) = \sum_\alpha n_\alpha \in \mathbf{Z}$ for $x = \sum_\alpha n_\alpha w_\alpha \in \mathbf{Z}[F_m]$ with $n_\alpha \in \mathbf{Z}$ and $w_\alpha \in F_m$, $\partial(x_1, \ldots, x_m) = \sum_j x_j(t_j - 1)$ for $(x_1, \ldots, x_m) \in \mathbf{Z}[F_m]^m$.

Secondly, we note that $\operatorname{Ext}^1_{\mathbf{Z}[F_m]}(H_0(p^{-1}(\partial N(L_k)); \mathbf{Z}), \mathbf{Z}[F_m]) = \mathbf{Z}[F_m]/\langle m_k - 1\rangle\mathbf{Z}[F_m]$ as right $\mathbf{Z}[F_m]$-module, where $\langle m_k - 1\rangle\mathbf{Z}[F_m]$ denotes a right submodule generated by $m_k - 1$ of $\mathbf{Z}[F_m]$ and $\mathbf{Z}[F_m]/\langle m_k - 1\rangle\mathbf{Z}[F_m]$ denotes the quotient right $\mathbf{Z}[F_m]$-module. This follows directly from the free resolution of $H_0(p^{-1}(\partial N(L_k)); \mathbf{Z}) \cong \mathbf{Z}[F_m]/\mathbf{Z}[F_m]^m\langle m_k - 1\rangle$ given by

$$(3.5) \qquad 0 \to \mathbf{Z}[F_m] \overset{\varphi_k}{\to} \mathbf{Z}[F_m] \overset{\varphi_k'}{\to} \mathbf{Z}[F_m]/\mathbf{Z}[F_m]\langle m_k - 1\rangle \to 0\,,$$

where $\mathbf{Z}[F_m]\langle m_k - 1\rangle$ denotes a left submodule generated by $m_k - 1$ of $\mathbf{Z}[F_m]$, $\mathbf{Z}[F_m]/\mathbf{Z}[F_m]\langle m_k - 1\rangle$ denotes the quotient left $\mathbf{Z}[F_m]$-module, $\varphi_k'(x) = [x]$ for $x \in \mathbf{Z}[F_m]$ and $[x]$ denotes the class of x in $\mathbf{Z}[F_m]/\mathbf{Z}[F_m]\langle m_k - 1\rangle$ and $\varphi_k(x) = x(m_k - 1)$ for $x \in \mathbf{Z}[F_m]$.

To determine $\mathrm{Ext}^1(i_k)\colon \mathbf{Z}[F_m]^m/\langle (t_1 - 1, t_2 - 1, \ldots, t_m - 1)\rangle \mathbf{Z}[F_m] \to \mathbf{Z}[F_m]/\langle m_k - 1\rangle \mathbf{Z}[F_m]$ we define a left $\mathbf{Z}[F_m]$-homomorphism $\xi_k\colon \mathbf{Z}[F_m] \to \mathbf{Z}[F_m]^m$ by $\xi_k(x) = (x, 0, \ldots, 0) \in \mathbf{Z}[F_m]^m$ for $x \in \mathbf{Z}[F_m]$ if $k = 1$, and $\xi_k(x) = (x(-t_k t_1 t_k^{-1} t_1^{-1} + t_k), 0, \ldots, 0, x(t_k t_1 t_k^{-1} t_1^{-1} \overset{k}{-} t_k t_1 t_k^{-1} + 1), 0, \ldots, 0) \in \mathbf{Z}[F_m]^m$ for $x \in \mathbf{Z}[F_m]$ if $k = 2, \ldots, m$. Note that $i_k\colon \mathbf{Z}[F_m]/\mathbf{Z}[F_m]\langle m_k - 1\rangle \to \mathbf{Z}$ is well-defined by $i_k([x]) = \sum_\alpha n_\alpha \in \mathbf{Z}$ for $x = \sum_\alpha n_\alpha w_\alpha \in \mathbf{Z}[F_m]$ with $n_\alpha \in \mathbf{Z}$ and $w_\alpha \in F_m$ and $[x]$ denotes the class of x in $\mathbf{Z}[F_m]/\mathbf{Z}[F_m]\langle m_k - 1\rangle$. Since $m_1 - 1 = t_1 - 1$ and $m_k - 1 = (-t_k t_1 t_k^{-1} t_1^{-1} + t_k)(t_1 - 1) + (t_k t_1 t_k^{-1} t_1^{-1} - t_k t_1 t_k^{-1} + 1)(t_k - 1) (k = 2, \ldots, m)$, the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}[F_m] \arrow[r, "\varphi_k"] \arrow[d, "\xi_k"] & \mathbf{Z}[F_m] \arrow[r, "\varphi_k'"] \arrow[d, "id"] & \mathbf{Z}[F_m]/\mathbf{Z}[F_m]\langle m_k - 1\rangle \arrow[r] \arrow[d, "i_k"] & 0 \\
0 \arrow[r] & \mathbf{Z}[F_m]^m \arrow[r, "\partial"] & \mathbf{Z}[F_m] \arrow[r, "\varepsilon"] & \mathbf{Z} \arrow[r] & 0\,.
\end{tikzcd}$$

Apply $\mathrm{Hom}_{\mathbf{Z}[F_m]}(-, \mathbf{Z}[F_m])$ to the above diagram. Then, since the horizontal sequences are free resolutions (3.4) and (3.5), we see that $\mathrm{Ext}^1(i_k)\colon \mathbf{Z}[F_m]^m/\langle (t_1 - 1, t_2 - 1, \ldots, t_m - 1)\rangle \mathbf{Z}[F_m] \to \mathbf{Z}[F_m]/\langle m_k - 1\rangle \mathbf{Z}[F_m]$ is given by $\mathrm{Ext}^1(i_k)([(x_1, \ldots, x_m)]) = [x_1]$ if $k = 1$, and $\mathrm{Ext}^1(i_k)([(x_1, \ldots, x_m)]) = [(-t_k t_1 t_k^{-1} t_1^{-1} + t_k)x_1 + (t_k t_1 t_k^{-1} t_1^{-1} - t_k t_1 t_k^{-1} + 1)x_k]$ if $k = 2, \ldots, m$, where $[(x_1, \ldots, x_m)]$ denotes the class of $(x_1, \ldots, x_m) \in \mathbf{Z}[F_m]^m$ in $\mathbf{Z}[F_m]^m/\langle (t_1 - 1, t_2 - 1, \ldots, t_m - 1)\rangle \mathbf{Z}[F_m]$ and $[x]$ denotes the class of $x \in \mathbf{Z}[F_m]$ in $\mathbf{Z}[F_m]/\langle m_k - 1\rangle \mathbf{Z}[F_m]$.

We are in a position to prove that $\mathrm{Ext}^1(i_*)$ is not surjective. We assume contrary that $\mathrm{Ext}^1(i_*)$ is surjective, in particular, $\mathrm{Ext}^1(i_2)|\mathrm{KerExt}^1(i_1)\colon \mathrm{KerExt}^1(i_1) \to \mathbf{Z}[F_m]/\langle m_k - 1\rangle \mathbf{Z}[F_m]$ is surjective. Let $\mathbf{Z}[\langle t_1\rangle]$ be the group ring of an infinite cyclic group $\langle t_1\rangle \subset F_m$ generated by t_1. We regard $\mathbf{Z}[\langle t_1\rangle]$ as the right $\mathbf{Z}[F_m]$-module with the action defined by $t_1(x) = t_1 x$ and $t_j(x) = x$ $(2 \le j \le m)$ for $x \in \mathbf{Z}[F_m]$. We define a surjective right $\mathbf{Z}[F_m]$-homomorphism $e\colon \mathbf{Z}[F_m]/\langle m_k - 1\rangle \mathbf{Z}[F_m] \to \mathbf{Z}[\langle t_1\rangle]$ by $e([t_1]) = t_1$ and $e([t_j]) = 1$ $(2 \le j \le m)$. We see easily that $\mathrm{KerExt}^1(i_1)$ is generated as right $\mathbf{Z}[F_m]$-module by m elements $[(t_1 - 1, 0, \ldots, 0)]$ and $[(0, 0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0)]$ $(j = 2, \ldots, m)$ where $[(x_1, \ldots, x_m)]$ denotes the class of $(x_1, \ldots, x_m) \in \mathbf{Z}[F_m]^m$ in $\mathbf{Z}[F_m]^m/\langle (t_1 - 1, t_2 - 1, \ldots, t_m - 1)\rangle \mathbf{Z}[F_m]$. Since $e \circ (\mathrm{Ext}^1(i_2)|\mathrm{KerExt}^1(i_1))$ is surjective, $\mathbf{Z}[F_m]/\langle m_k - 1\rangle \mathbf{Z}[F_m]$ is generated by m elements $e \circ (\mathrm{Ext}^1(i_2)|\mathrm{KerExt}^1(i_1))([(t_1 - 1, 0, \ldots, 0)])$ and $e \circ (\mathrm{Ext}^1(i_2)|\mathrm{KerExt}^1(i_1))([(0, 0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0)])$

$(j = 2, \ldots, m)$. By a simple calculation $e \circ (\mathrm{Ext}^1(i_2)|\mathrm{KerExt}^1(i_1))$ $([(t_1 - 1, 0, \ldots, 0)]) = 0$, $e \circ (\mathrm{Ext}^1(i_2)|\mathrm{KerExt}^1(i_1))$ $([(0, 1, 0, \ldots, 0)]) = 2 - t_1$ and $e \circ (\mathrm{Ext}^1(i_2)|\mathrm{KerExt}^1(i_1))$ $([(0, 0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0)]) = 0$ $(j = 3, \ldots, m)$. Hence, $\mathbf{Z}[\langle t_1 \rangle]$ must be generated by $2 - t_1$ as right $\mathbf{Z}[F_m]$-module. Then, there exists $\beta \in \mathbf{Z}[\langle t_1 \rangle]$ such that $(2 - t_1)\,\beta = 1$ because the actions of $t_2, t_3, \ldots, t_m$ are trivial in $\mathbf{Z}[\langle t_1 \rangle]$. Hence, $2 - t_1$ should be a unit in $\mathbf{Z}[\langle t_1 \rangle]$. This contradicts the fact that $2 - t_1$ is not a unit in $\mathbf{Z}[\langle t_1 \rangle]$ and we obtain that $\mathrm{Ext}^1(i_*)$ is not surjective. The proof of Proposition 3.4 is completed.

Proposition 3.1 means that a link which satisfies the assumption can be reconstructed in the same way as the Poenaru's example. Proposition 3.2 supports partially the following conjecture which is proposed by T. Matumoto.

CONJECTURE 3.6 (MATUMOTO). *Let L be an m-component link in S^{n+2}. Then, L is trivial if and only if* $\mathrm{cat}(S^{n+2} - L) = 1$ *provided that $n \geq 3$.*

In fact, the following simpler conjecture is equivalent to Conjecture 3.2 by Theorem 1.

CONJECTURE 3.7. *Let L be an m-component link such that $S^{n+2} - L$ has the homotopy type of $(\bigvee_m S^1) \vee (\bigvee_{m-1} S^{n+1})$. If $n \geq 3$, then L is trivial.*

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