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A theorem of Hardy-Littlewood and removability for polyharmonic functions satisfying Hölder's condition

Dedicated to Professor M. Nakai on the occasion of his sixtieth birthday

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Abstract

One of our aims in this note is to give an extension of a result of Hardy-Littlewood [3, Theorems 40 and 41] for holomorphic functions on the unit disc. In fact we show that a polyharmonic function u on the unit ball $\boldsymbol{B}$ satisfies Hölder's condition of exponent α, $0 < \alpha \leqq 1$, if and only if

$$|\mathrm{grad}\, u(x)| \leqq M(1 - |x|^2)^{\alpha - 1} \qquad \text{for any } x \in \boldsymbol{B}$$

by appealing to a mean-value inequality for polyharmonic functions.

Next we discuss removable singularities for polyharmonic functions u satisfying

$$|D^j u(x + y) + D^j u(x - y) - 2D^j u(x)| \leqq M|y|^{\alpha - k}$$

for all $x \in G$, y with $x \pm y \in G$ and j with $|j| = k$, where G is an open set in R^n and k is the nonnegative integer such that $k < \alpha \leqq k + 1$. Our goal is to derive a generalization of the recent result of Ullrich [12, Theorem 1].

1. Introduction

Let G be an open set in R^n. An infinitely differentiable function u on G is called polyharmonic of order m in G if $\Delta^m u = 0$ holds in G; we say that u is polyharmonic in G if it is polyharmonic of order m in G for some positive integer m. In case $0 < \alpha \leqq 1$, if a continuous function u on G satisfies

$$(1) \qquad |u(x) - u(y)| \leqq M|x - y|^\alpha \qquad \text{whenever } x, y \in G$$

for some constant M, then we say that u satisfies Hölder's condition of exponent α in G

In this paper let M denote various constants, whose value may change from one occurrence to the next. We denote by $\boldsymbol{B}$ the unit ball of R^n.

Our first aim in this paper is to prove

THEOREM 1. *Let u be a polyharmonic function on $\boldsymbol{B}$ and* $0 < \alpha \leqq 1$. *Then* *u satisfies Hölder's condition of exponent* α *in* $\boldsymbol{B}$ *if and only if*

$$(2) \qquad |\nabla u(x)| \leqq M(1-|x|^2)^{\alpha-1} \qquad \textit{for any } x \in \boldsymbol{B},$$

where ∇ *denotes the gradient and* M *is a positive constant.*

This is an extension of Hardy-Littlewood's result [3, Theorems 40 and 41] for holomorphic functions on the unit disc (see also [2, Theorem 5.1]). It is easy to show that Theorem 1 is true for harmonic functions u on R^n. Further, as noted in section 15 of Krantz [7], harmonic functions can be replaced by solutions to any second order uniformly elliptic homogeneous partial differential equation.

We say that a continuous function h on the interval $[0, \infty)$ is a measure function if $h(0)=0$, $h(r)>0$ for $r>0$, h is nondecreasing on $[0, \infty)$ and

$$h(2r) \leqq Mh(r) \qquad \text{for any } r>0.$$

Let H_h denote the Hausdorff measure with a measure function h. If $h(r)=r^\alpha$, then we write H_α for H_h.

Let $1 \leqq p \leqq \infty$ and $1/p+1/p^*=1$. We use $B(x, r)$ to denote the open ball centered at x with radius r. For a measure function h and a locally integrable function u on G, we define

$$U_m(x) = \sup_B r^{-2m-n/p} h(r)^{-1/p^*} \inf_v \int_B |u(y)-v(y)|\,dy,$$

where the supremum is taken over all open balls $B=B(z, r)$ such that $x \in B \subseteqq G$, and the infimum is taken over all functions v polyharmonic of order m in B. Consider the set $S_m(u)$ of all points $x \in G$ such that

$$\limsup_{r \to 0} r^{-2m-n} \int_{B(x,r)} |u(y)-v(y)|\,dy > 0$$

for any function v polyharmonic of order m in a neighborhood of x. Note here that if u is polyharmonic of order m in a neighborhood of x, then $x \notin S_m(u)$.

Now we state a result of removability for polyharmonic functions, as a generalization of the results of Kaufman-Wu [6] and Mizuta [8] for harmonic functions.

THEOREM 2. *Let* G *be a bounded open set in* R^n *and let* h *be a measure function. For a given locally integrable function* u *on* G, *suppose* $U_m \in L^p(G)$.

(i) *If* $p<\infty$, $\lim_{r \to 0} r^{-n} h(r) = \infty$ *and* $H_h(S_m(u))<\infty$, *then* u *can be corrected on a set of measure zero to be polyharmonic of order* m *in* G;

(ii) *if* $p>1$ *and* $H_h(S_m(u))=0$, *then* u *can be corrected on a set of measure zero to be polyharmonic of order* m *in* G;

(iii) *if* $p = 1$ *and* $H_n(S_m(u)) = 0$, *then* u *can be corrected on a set of measure zero to be polyharmonic of order* m *in* G.

For a positive number α, let k be the integer such that

$$k < \alpha \leqq k + 1.$$

We denote by $\Lambda_\alpha(G)$ the Hölder space of all functions $f \in C^k(G)$ such that in case $\alpha < k + 1$,

$$|D^j f(x) - D^j f(y)| \leqq M|x - y|^{\alpha - k} \qquad \text{whenever } x, y \in G \text{ and } |j| = k;$$

in case $\alpha = k + 1$,

$$|D^j f(x + y) + D^j f(x - y) - 2D^j f(x)| \leqq M|y| \quad \text{whenever } x \in G,\ x \pm y \in G \text{ and } |j| = k,$$

where $D^j = (\partial/\partial x)^j = (\partial/\partial x_1)^{j_1} \cdots (\partial/\partial x_n)^{j_n}$ for $j = (j_1, \ldots, j_n)$ and $x = (x_1, \ldots, x_n)$.

As a special case, Theorem 2 implies the following result, which gives a generalization of the recent result of Ullrich [12, Theorem 1 (i)].

THEOREM 3. *Let* K *be a compact subset of* G, *and let* u *be polyharmonic of order* m *in* $G - K$. *If* $u \in \Lambda_\alpha(G)$, $2m - n < \alpha < 2m$ *and* $H_{n+\alpha-2m}(K) = 0$, *then* u *can be corrected on the set* K *to be polyharmonic of order* m *in* G.

2. Proof of Theorem 1

For a proof of Theorem 1, we prepare two lemmas.

LEMMA 1 (see Krantz [7, Theorem 15.7]). *If* u *is a differentiable function on* $\boldsymbol{B}$ *satisfying* (2) *with* $0 < \alpha \leqq 1$, *then* u *satisfies* (1).

LEMMA 2 (cf. [9, Lemma 2.2]). *If* u *is a polyharmonic function on* $B(x, r)$, *then*

$$|\nabla_k u(x)| \leqq M_k r^{-n-k} \int_{B(x,r)} |u(y)|\, dy$$

for any nonnegative integer k, *where* ∇_k *denotes the gradient iterated* k *times, that is,*

$$|\nabla_k u(x)| = \left(\sum_{|j|=k} \frac{k!}{j!} |D^j u(x)|^2 \right)^{1/2}.$$

Lemma 2 is well known in the harmonic case (see e.g. Stein [11, Appendix C.3]).

PROOF OF LEMMA 2. For a function v and $r > 0$, set

$$A_1(v, x, r) = \frac{1}{|B(x, r)|} \int_{B(x,r)} v(y)dy,$$

where $|B(x, r)|$ denotes the volume of the ball $B(x, r)$. Further, for each positive integer i, define

$$A_{i+1}(v, x, r) = \frac{1}{r^{n+2i}} \int_0^r A_i(v, x, t)t^{n+2i-1}dt,$$

inductively. Suppose $\Delta^m u = 0$ on $B(x, r_0)$. Then, in view of Théorème 1 in [10], u is of the form

$$u(y) = \sum_{j=1}^{m} |y - x|^{2j-2} v_j(y),$$

where v_j are harmonic in $B(x, r_0)$; note here that $u(x) = v_1(x)$. Consequently,

$$A_1(u, x, r) = n \sum_{j=1}^{m} b_j r^{2j-2} v_j(x)$$

with $b_j = 1/(n + 2j - 2)$. We integrate both sides repeatedly and obtain

$$A_i(u, x, r) = n \sum_{j=1}^{m} c_{j,i} r^{2j-2} v_j(x),$$

where $c_{j,i} = b_j \cdot b_{j+1} \cdots b_{j+i-1}$. Hence

$$v_1(x) = \frac{1}{nc} \begin{vmatrix} A_1(u, x, r) & c_{2,1} & \cdots & c_{m,1} \\ A_2(u, x, r) & c_{2,2} & \cdots & c_{m,2} \\ \cdots & \cdots & \cdots & \cdots \\ A_m(u, x, r) & c_{2,m} & \cdots & c_{m,m} \end{vmatrix} \quad \text{with}$$

$$c = \begin{vmatrix} c_{1,1} & c_{2,1} & \cdots & c_{m,1} \\ c_{1,2} & c_{2,2} & \cdots & c_{m,2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1,m} & c_{2,m} & \cdots & c_{m,m} \end{vmatrix} \neq 0,$$

so that u is of the form

$$u(x) = \sum_{i=1}^{m} c_i A_i(u, x, r)$$

for $0 < r < r_0$, where c_i are constants. It follows that

(3) $$|u(x)| \leqq MA_1(|u|, x, r) \quad \text{whenever } 0 < r < r_0.$$

For simplicity, let $D_i = \partial/\partial y_i$. Since $\Delta^m(D_i u) = 0$ on $B(x, r_0)$, we find

$$D_i u(x) = \sum_{j=1}^{m} c_j A_j(D_i u, x, t)$$

for $0 < t < r_0$. Noting that

$$A_1(D_i u, x, t) = \frac{1}{|B(x, t)|} \int_{S(x,t)} u(y) \frac{x_i - y_i}{|x - y|} dS(y),$$

we have

$$|D_i u(x)| \leqq M_1 \frac{1}{|B(x, t)|} \left(\int_{S(x,t)} |u(y)| dS(y) + t^{-1} \int_{B(x,t)} |u(y)| dy \right).$$

Hence, multiplying both sides by t^{n+1} and integrating them with respect to t, we have

$$|D_i u(x)| \leqq M_2 r^{-n-1} \int_{B(x,r)} |u(y)| dy.$$

This proves also

$$\int_{B(x,r)} |D_i u(y)| dy \leqq M_3 r^{-1} \int_{B(x,2r)} |u(y)| dy$$

for $0 < r < r_0/2$. Using this repeatedly, we establish

$$\int_{B(x,r)} |D^\lambda u(y)| dy \leqq (M_3 r^{-1})^{|\lambda|} \int_{B(x,2^{|\lambda|}r)} |u(y)| dy$$

for any multi-index λ and any $r \in (0, 2^{-|\lambda|} r_0)$. Hence (3) gives

$$|\nabla_i u(x)| \leqq M(i) r^{-i} A_1(|u|, x, r)$$

for $0 < r < r_0$, where $M(i)$ is a positive constant independent of x and r. Thus Lemma 2 is proved.

PROOF OF THEOREM 1. First assume that (1) holds. For fixed $x \in \boldsymbol{B}$, consider the function

$$u(\cdot) - u(x),$$

which is polyharmonic in $\boldsymbol{B}$. Applying Lemma 2, we have

$$|\nabla u(x)| \leqq M r^{-n-1} \int_{B(x,r)} |u(y) - u(x)| dy$$

whenever $B(x, r) \subseteqq \boldsymbol{B}$. If we take $r = (1 - |x|)/2$, then (1) gives (2).

The if part follows from Lemma 1, without assuming the polyharmonicity.

PROPOSITION 1. *Let u be a polyharmonic function on* $\boldsymbol{B}$, $0 < \alpha \leqq 1$ *and* $1 \leqq p < \infty$. *If*

$$\left(\frac{1}{|B|}\int_B |\nabla u(y)|^p dy\right)^{1/p} \leqq Mr^{\alpha-1} \tag{4}$$

for any open ball $B = B(x, r) \subseteqq \boldsymbol{B}$, *then u satisfies Hölder's condition of exponent* α *in* $\boldsymbol{B}$, *where* $|B|$ *denotes the n-dimensional Lebesgue measure of B.*

In fact, we have only to see that (4) implies (2) on account of Lemma 2.

In view of the proof of Theorem 1, we can establish

PROPOSITION 2. *Let u be a polyharmonic function on* $\boldsymbol{B}$ *and* $0 < \alpha < 2$. *Then*

$$|u(x+y) + u(x-y) - 2u(x)| \leqq M|y|^{\alpha} \quad \text{for all } x \in \boldsymbol{B} \text{ and } y \text{ with } x \pm y \in \boldsymbol{B} \tag{5}$$

if and only if

$$|\nabla_2 u(x)| \leqq M(1 - |x|^2)^{\alpha-2} \quad \text{for any } x \in \boldsymbol{B}. \tag{6}$$

In fact, if (5) holds, then we apply Lemma 2 with $k = 2$ and the function $v(y) = u(x+y) + u(x-y) - 2u(x)$ on $B(0, 1 - |x|)$ to establish (6). For a proof of the implication (6) ⇒ (5), see Krantz [7, Theorem 15.7].

3. Proofs of Theorems 2 and 3

For the proofs of Theorems 2 and 3, we need the following lemma.

LEMMA 3 (cf. [4, Lemma 3.1], [11, p. 174]). *Let* $\{B(x_i, r_i)\}$ *be a finite collection of open balls such that* $\{B(x_i, r_i/5)\}$ *is mutually disjoint. Then there exists a family* $\{\psi_i\} \subseteqq C_0^\infty$ *with the following properties:*

(a) $\psi_i = 0$ *outside* $B(x_i, 2r_i)$;
(b) $\psi_i \geqq 0$ *on* R^n;
(c) $\sum_i \psi_i \leqq 1$ *on* R^n;
(d) $\sum_i \psi_i = 1$ *on* $\bigcup_i B(x_i, r_i)$;
(e) $|D^j \psi_i| \leqq M_j r_i^{-|j|}$ *on* R^n *for any multi-index j.*

PROOF OF THEOREM 2. First we are concerned with the case (i). Now suppose $U_m \in L^p(G)$, $p < \infty$, $\lim_{r\to 0} r^{-n}h(r) = \infty$ and $H_h(S_m(u)) < A < \infty$. Let

$0 < \varepsilon < 1$. By the definition of Hausdorff measure, there exists a countable covering $\{B(x_i, r_i)\}$ of $S_m(u)$ such that

$$\sum_i h(r_i) < A \tag{7}$$

and, since $\lim_{r\to 0} r^{-n}h(r) = \infty$,

$$\sum_i r_i^n < \varepsilon. \tag{8}$$

For each $z \in G - S_m(u)$ take $r(z) > 0$ and a function v_z polyharmonic of order m in $B(z, 10r(z))$ such that

$$\int_{B(z,10r(z))} |u(y) - v_z(y)|\, dy \leqq \varepsilon r(z)^{n+2m}.$$

Let $\varphi \in C_0^\infty(G)$ and denote the support of φ by K. Since $K \subseteqq (\bigcup_i B(x_i, r_i)) \cup (\bigcup_{z\in G-S_m(u)} B(z, r(z)))$, we can find a finite family $\{B_\ell\} \subseteqq \{B(x_i, r_i)\} \cup \{B(z, r(z));\ z \in G - S_m(u)\}$ such that $K \subseteqq \cup B_\ell$. By a covering lemma we can choose a mutually disjoint subfamily $\{B_{\ell'}\}$ such that $K \subseteqq \cup 5B_{\ell'}$; here $5B = B(x, 5r)$ when $B = B(x, r)$. Now take $\{\psi_{\ell'}\}$ for $\{5B_{\ell'}\}$ in Lemma 3. If $B_{\ell'} = B(z_{i'}, r(z_{i'}))$ for $z_{i'} \in G - S_m(u)$, then

$$\left|\int u(y)[\Delta^m(\psi_{\ell'}\varphi)(y)]\, dy\right| = \left|\int [u(y) - v_{z_{i'}}(y)]\,[\Delta^m(\psi_{\ell'}\varphi)(y)]\, dy\right|$$
$$\leqq M\varepsilon r(z_{i'})^n$$

since $v_{z_{i'}}$ is polyharmonic of order m in $B(z_{i'}, 10r(z_{i'}))$; similarly, if $B_{\ell'} = B(x_{i''}, r_{i''})$, then

$$\left|\int u(y)\,[\Delta^m(\psi_{\ell'}\varphi)(y)]\, dy\right| \leqq M r_{i''}^{n/p} h(r_{i''})^{1/p^*} \inf_{y\in B(x_{i''},r_{i''})} U_m(y)$$
$$\leqq M h(r_{i''})^{1/p^*}\left(\int_{B(x_{i''},r_{i''})} U_m(y)^p dy\right)^{1/p}.$$

Hence it follows from Hölder's inequality that

$$\left|\int u(y)\,[\Delta^m\varphi(y)]\, dy\right| = \left|\sum_{\ell'}\int u(y)\,[\Delta^m(\psi_{\ell'}\varphi)(y)]\, dy\right|$$
$$\leqq M\varepsilon\sum_{i'} r(z_{i'})^n + MA^{1/p^*}\left(\int_{\cup_{i''}B(x_{i''},r_{i''})} U_m(y)^p dy\right)^{1/p}.$$

Since $\sum_{i'} r(z_{i'})^n \leqq M|G|$ and $|\bigcup_{i''} B(x_{i''}, r_{i''})| \leqq M\varepsilon$ by (8), this shows that

(9) $$\int u(y)\,[\Delta^m \varphi(y)]\,dy = 0.$$

In case of (ii) we replace (7) by

(7′) $$\sum_i h(r_i) < \varepsilon$$

and obtain

$$\left|\int u(y)\,[\Delta^m \varphi(y)]\,dy\right| \leqq M\varepsilon|G| + M\varepsilon^{1/p^*}\left(\int_G U_m(y)^p dy\right)^{1/p}.$$

Thus (9) follows.

In case of (iii) we establish

$$\left|\int u(y)\,[\Delta^m \varphi(y)]\,dy\right| \leqq M\varepsilon|G| + M\int_{\cup_{i''}B(x_{i''},r_{i''})} U_m(y)\,dy.$$

Instead of (7′), we need to note

$$\sum_{i''} |B(x_{i''}, r_{i''})| < M\varepsilon,$$

so that (9) also holds.

Since (9) implies that $\Delta^m u = 0$ on G in the distribution sense, one sees, from the regularity for the Laplace operator, that u is equal almost everywhere to a function polyharmonic of order m in G.

For a proof of Theorem 3, we note the following result, which is a part of alternative characterization of the space Λ_α.

Lemma 4 (cf. [5, Proposition 3 and Theorem 2 in Chapter III]). *Let G be a bounded open set in R^n. If $u \in \Lambda_\alpha(G)$, then for any open ball B of radius r with closure in G, there exists a polynomial P_B of degree at most $[\alpha]$ such that*

$$|u(y) - P_B(y)| \leqq Mr^\alpha \qquad \textit{for all } y \in B.$$

Let u be as in Theorem 3, and consider the measure function $h(r) = r^{n+\alpha-2m}$. In the present case, $p = \infty$ and $p^* = 1$. It follows from Lemma 4 that $U_m \in L^\infty(G)$. Further note that $S_m(u) \subseteqq K$. Therefore (ii) of Theorem 2 for $p = \infty$ yields the required conclusion of Theorem 3.

Finally we discuss the converse of Theorem 3.

Proposition 3. *Let $\alpha < 2m \leqq n + k$ and K be a compact set in R^n such that $H_{n+\alpha-2m}(K) > 0$. Then there exists $u \in \Lambda_\alpha(R^n)$ such that u is polyharmonic* *of order m in $R^n - K$ but u is not polyharmonic of order m in all of R^n.*

One should compare this result with an example given by Uy [13].

PROOF OF PROPOSITION 3. Let R_{2m} denote the Riesz kernel of order $2m$ (see [11]). In view of [1, Theorem 1 in Section II], we can find a nonnegative measure μ on K such that $\mu(K) = 1$ and

$$\mu(B(x, r)) \leqq Mr^{n+\alpha-2m} \qquad \text{for any } x \in R^n \text{ and any } r > 0. \tag{10}$$

Case 1: $2m < n$ or $2m > n$ and n is odd. Consider the potential

$$R_{2m}\mu(x) = \int R_{2m}(x-y)\,d\mu(y).$$

If $|j| = k$, then we have for $z \in R^n$ and $r > 0$,

$$\int_{B(x,r)} |D^j R_{2m}(x+z-y)|\,d\mu(y) \leqq M \int_{B(x,r)} |x+z-y|^{2m-k-n}\,d\mu(y)$$

$$\leqq M \int_{B(x+z,r)} |x+z-y|^{2m-k-n}\,d\mu(y) + M \int_{B(x,r)} |x-y|^{2m-k-n}\,d\mu(y).$$

With the aid of (10) we find

$$\int_{B(x,r)} |x-y|^{2m-k-n}\,d\mu(y) = \mu(B(x,r))r^{2m-k-n} + \int_0^r \mu(B(x,t))\,d(-t^{2m-k-n})$$

$$\leqq Mr^{\alpha-k},$$

so that

$$\int_{B(x,r)} |D^j R_{2m}(x+z-y)|\,d\mu(y) \leqq Mr^{\alpha-k};$$

here recall that $k < \alpha \leqq k+1$. Further, letting $r = 2|z|$, we see that

$$\int_{R^n - B(x,r)} |D^j R_{2m}(x+z-y) + D^j R_{2m}(x-z-y) - 2D^j R_{2m}(x-y)|\,d\mu(y)$$

$$\leqq Mr^2 \int_{R^n - B(x,r)} |x-y|^{2m-n-k-2}\,d\mu(y)$$

$$\leqq Mr^2 \int_r^\infty \mu(B(x,t))\,d(-t^{2m-n-k-2})$$

$$\leqq Mr^{\alpha-k}.$$

In case $\alpha < k+1$, we obtain

$$\begin{aligned}\int_{R^n - B(x,r)} & |D^j R_{2m}(x + z - y) - D^j R_{2m}(x - y)| \, d\mu(y) \\ & \leqq Mr \int_{R^n - B(x,r)} |x - y|^{2m-n-k-1} \, d\mu(y) \\ & \leqq Mr^{\alpha - k}.\end{aligned}$$

Thus it follows that $R_{2m}\mu \in \Lambda^{\alpha}(R^n)$. Now it is easy to show that $R_{2m}\mu$ satisfies all the required conditions.

Case 2: $2m \geqq n$ and n is even. In this case, $D^j R_{2m}(x)$, $|j| = k$, is of the form

$$D^j R_{2m}(x) = a_j \log |x| + b_j$$

with constants a_j and b_j. Now we need to modify the measure μ given as above. Since μ has no point mass by (10), the support of μ contains at least two points. Hence we can find two disjoint compact subsets K_1, K_2 of K such that $\mu(K_\ell) > 0$ for $\ell = 1, 2$. Define

$$\nu = \frac{\mu|_{K_1}}{\mu(K_1)} - \frac{\mu|_{K_2}}{\mu(K_2)}$$

and consider

$$R_{2m}\nu(x) = \int R_{2m}(x - y) \, d\nu(y),$$

where $\mu|_{K_\ell}$ denotes the restriction of μ to K_ℓ. If $|j| = k$, $x \in R^n$ and $r > 0$, then, writing

$$D^j R_{2m}\nu(x) = - a_j \int \log (r/|x - y|) \, d\nu(y),$$

we have by the above considerations

$$\left| \int_{B(x,r)} \log (r/|x + z - y|) \, d\nu(y) \right| \leqq M\mu(B(x, r)) \leqq Mr^{\alpha - k}$$

for any $z \in R^n$. The integration over $R^n - B(x, r)$ with $r = 2|z|$ can be estimated in the same way as in Case 1. Thus $R_{2m}\nu$ has all the required properties.

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