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On the Billingsley dimension on R^N

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1. Introduction

P. Billingsley has proved the following theorem (c.f. [1]): *Let ν be a Borel probability measure on* $[0, 1)$. *Assume that* $M \subseteq [0, 1)$ *is a Borel set satisfying the conditions*

$$\nu(M) > 0 \quad \textit{and} \quad M \subseteq \left\{ \omega \in [0, 1) : \liminf_{n \to \infty} -\frac{\log \nu(u_n(\omega))}{n \log r} \geq \delta \right\}$$

*then the Hausdorff dimension H-*dim (M) *of M is bounded from below as*

$$H\textit{-dim}\,(M) \geq \delta,$$

where $u_n(\omega)$ is the element containing ω, of the special covering of M in the form of $[j/r^n, j + 1/r^n)$ $j = 0, 1, \cdots, r^n - 1$.

On the other hand, L. S. Young has proved the following theorem (c.f. [8]): *Let ν be a Borel probability measure on $\mathbb{R}^N$ and suppose that there exists $\delta \geq 0$ such that*

$$\lim_{r \to +0} \frac{\log \nu(B(\omega, r))}{\log r} = \delta \qquad \textit{for} \ \ \nu\textit{-a.e.} \ \omega \in K$$

then

$$H\textit{-dim}\,(K) \geq \delta,$$

where $B(\omega, r)$ denotes the closed ball of radius r with center at ω. In this paper, we consider applying Billingsley's theorem to Euclidian space (see THEOREM 3.3). And then we intend to construct a useful method for calculating the Hausdorff dimension (see THEOREM 3.4 and THEOREM 3.5).

In Section 2, we will introduce a NET $\mathscr{R}$ for a given bounded subset K of $\mathbb{R}^N$ and a ν-NET $\mathscr{R}$ for K, which is associated with a finite Borel measure ν on $\mathbb{R}^N$. Then we will define the Billingsley measure $\mathscr{R}$-M_ν^α and the Billingsley dimension $\mathrm{H}_{\mathscr{R}}$-dim_ν which are induced from ν and $\mathscr{R}$. In Section 3 main results will be presented. We will study some relations between those and the Hausdorff measure, the Hausdorff dimension. And we construct a useful method for calculating the Hausdorff dimension. Furthermore in Section 4, we will show some strict results under a strict condition of $\mathscr{R}$. In Section 5, we introduce two examples.

2. Definition

In this paper, H^{α} and λ_N denote the α-dimensional Hausdorff measure and the N-dimensional Lebesgue measure, respectively ($\alpha \in \mathbb{R}$, $N \in \mathbb{N}$). And $|E|$ denotes the diameter of E.

For an arbitrary family $\mathscr{R}$ of bounded Borel subsets of $\mathbb{R}^N$ with a positive diameter and a constant $\lambda (> 1)$, we can classify the elements of $\mathscr{R}$ as

$$\mathscr{R}_{\lambda}^{(n)} = \{R \in \mathscr{R} : \lambda^{-n} < |R| \leq \lambda^{-(n-1)}\}, \qquad n \in \mathbb{Z}.$$

DEFINITION 2.1. For a given bounded set $K \subset \mathbb{R}^N$, a family $\mathscr{R}$ of bounded Borel subsets of $\mathbb{R}^N$ is called *a NET for K* if $\mathscr{R}$ satisfies the following conditions:

(1) If $R_1, R_2 \in \mathscr{R}$ then $R_1 \subseteq R_2$, $R_2 \subseteq R_1$ or $\lambda_N(R_1 \cap R_2) = 0$ holds.
(2) There exists a positive constant C such that

$$\lambda_N(R) \geq C \cdot |R|^N \qquad \text{for any } R \in \mathscr{R}.$$

(3) For any $\omega \in K$ and $n \geq N_{\mathscr{R}}$, there exists $R \in \mathscr{R}_{\lambda}^{(n)}$ with $\omega \in R$, furthermore for any R with $\omega \in R$, there exist $R' \in \mathscr{R}_{\lambda}^{(n+1)}$ and $R'' \in \mathscr{R}_{\lambda}^{(n-1)}$ (if $n > N_{\mathscr{R}}$) such that $\omega \in R' \subset R \subset R''$, for suitably fixed $\lambda > 1$ and $N_{\mathscr{R}} \in \mathbb{Z}$.

Let $\mathscr{R}$ be a NET for K, and let Ω_N be the volume of the unit ball in $\mathbb{R}^N$. For any $R_1 \in \mathscr{R}_{\lambda}^{(n)}$ and $R_2 \in \mathscr{R}_{\lambda}^{(m)}$ with $n < m$, the inclusion $R_1 \supseteq R_2$ holds if and only if $\lambda_N(R_1 \cap R_2) > 0$. For any $R \in \mathscr{R}_{\lambda}^{(n)}$, $C\lambda^{-(n+1)N} \leq \lambda_N(R) \leq \Omega_N \lambda^{-nN}$ holds. These simple remarks are useful. We have a suitable *sub-NET* $\tilde{\mathscr{R}}$ for K of $\mathscr{R}$, that is, $\tilde{\mathscr{R}} \subseteq \mathscr{R}$ and $\tilde{\mathscr{R}}$ is a NET for K, as follows.

PROPOSITION 2.2. *For a given NET $\mathscr{R}$ for $K \subset \mathbb{R}^N$, there exists a sub-NET $\tilde{\mathscr{R}}$ for K of $\mathscr{R}$, which has the following properties*:
(1) $\tilde{\mathscr{R}}_{\lambda}^{(n)} = \{R \in \tilde{\mathscr{R}} : \lambda^{-n} \leq |R| < \lambda^{-n+1}\}$ *consists of finite members.*
(2) *If* $R_1, R_2 \in \tilde{\mathscr{R}}_{\lambda}^{(n)}$, *then either* $R_1 = R_2$ *or* $\lambda_N(R_1 \cap R_2) = 0$ *holds.*
(3) *For any* $\omega \in K$ *and any* $R \in \mathscr{R}_{\lambda}^{(n)}$ *with* $\omega \in R$, *there exist* $R' \in \tilde{\mathscr{R}}_{\lambda}^{(n-1)}$, $R'' \in \tilde{\mathscr{R}}_{\lambda}^{(n+1)} (n > N_{\mathscr{R}})$ *such that* $\omega \in R' \subset R \subset R''$.
(4) *For any* $\omega \in K$, *there exists a sequence* $\{E_n(\omega) \in \tilde{\mathscr{R}}_{\lambda}^{(n)}\}_{n=N_{\mathscr{R}}}^{\infty}$ *such that* $\omega \in E_{n+1}(\omega) \subset E_n(\omega)$ *for* $n \geq N_{\mathscr{R}}$.

PROOF. Put $V_n = \bigcup_{R \cap K \neq \emptyset, R \in \mathscr{R}_{\lambda}^{(n)}} R$. Then by (3) of DEFINITION 2.1, $V_{n+1} \subseteq V_n$ holds for any $n \geq N_{\mathscr{R}}$. Let $R_1, R_2, \cdots, R_{m(n)}$ be a family of elements of $\mathscr{R}_{\lambda}^{(n)}$ such that

$$K \cap R_i \neq \emptyset, \ \lambda_N(R_i \cap R_j) = 0 \qquad \text{for } i \neq j, \ 1 \leq i, \ j \leq m(n),$$

and there does not exist $R \in \mathscr{R}_\lambda^{(n)}$ with $R \cap K \neq \emptyset$ and $\lambda_N(R_i \cap R) = 0$ for any i, $1 \leq i \leq m(n)$. We see that the volume of $\bigcup_i R_i$ is bounded by a constant depending on n, λ, N, $|K|$, and that the number m is also bounded by a constant depending n, λ, N, $|K|$, C in (2) of DEFINITION 2.1. Now suppose that there exists $\omega' \in V_{n+1} \setminus \bigcup_{i=1}^{m(n)} R_i$. Then there exists $R' \in \mathscr{R}_\lambda^{(n+1)}$ such that $\omega' \in R'$ with $K \cap R' \neq \emptyset$. Then R' is not included in $\bigcup_{i=1}^{m(n)} R_i$. Let ω be in $R' \cap K$. By (3) of DEFINITION 2.1, there exists an $R \in \mathscr{R}_\lambda^{(n)}$ with $R' \subset R$. By $\lambda_N(R \cap R_j) > 0$ for some j, $R_j \subset R$ holds, otherwise $R' \subset R \subset R_j$ implies a contradiction. The set R may include some of R_i's. Thus we have a new family $\{R_i' : 1 \leq i \leq m'(n)\}$ which consists of R and R_i's which are not included in R with $m'(n) \leq m(n)$. Then $\lambda_N(\bigcup_i R_i') \geq \lambda_N(\bigcup_i R_i) + \lambda_N(R')$. This shows that the procedure of the replacement of the family $\{R_j\}$ can be performed at most finitely many times. In the last stage we have $\{\tilde{R}_i^{(n)}\}_{i=1}^{m(n)} \subset \mathscr{R}$ for $n \geq N_{\mathscr{R}}$ such that

$$K \subseteq V_{n+1} \subseteq \bigcup_{j=1}^{m(n)} \tilde{R}_j^{(n)} \subseteq V_n, \ K \cap \tilde{R}_j^{(n)} \neq \emptyset, \ \lambda_N(\tilde{R}_i^{(n)} \cap \tilde{R}_j^{(n)}) = 0 \qquad (i = j).$$

Set $\tilde{\mathscr{R}} = \{\tilde{R}_i^{(n)} : 1 \leq i \leq m(n), n \geq N_{\mathscr{R}}\}$. The second assertion (2) is obvious by the construction and (1) of DEFINITION 2.1. Now we show (3). Let us suppose that $R \in \mathscr{R}_\lambda^{(n)} (n > N_{\mathscr{R}})$ and $\omega \in K \cap R$. Since $R \subset V_n \subset \bigcup_i \tilde{R}_i^{(n-1)}$, $\lambda_N(R \cap \tilde{R}_j^{(n-1)}) > 0$ with some j. Then $R \subset \tilde{R}_j^{(n-1)} = R''$. On the other hand, we can find $R''' \in \mathscr{R}_\lambda^{(n+2)}$ such that $\omega \in R''' \subset R$. Then just as above, there exists $R' \in \tilde{\mathscr{R}}_\lambda^{(n+1)}$ such that $R''' \subset R'$. Since $\lambda_N(R \cap R') \geq \lambda_N(R''') > 0$, $R' \subset R$ holds. Thus we get (3). This implies that $\mathscr{R}$ is a NET.

For $\omega \in K$ and $n = N_{\mathscr{R}} + 1$, let $E_n(\omega)$ be the first member of $\tilde{R}_i^{(n)}$, $1 \leq i \leq m(n)$, containing ω and $\tilde{E}_{N_{\mathscr{R}}}(\omega)$ be the first member of $\tilde{R}_i^{(N_{\mathscr{R}})}$, $1 \leq i \leq m(N_{\mathscr{R}})$, including $E_n(\omega)$. By (3) we can find a desired sequence in this way. □

Now we introduce *a* ν*-NET for* K, which is more loosely defined associated with a positive finite Borel measure ν on $\mathbb{R}^N$.

DEFINITION 2.3. Suppose that $K \subset \mathbb{R}^N$ is a bounded set, $\mathscr{R}$ is a family of bounded Borel subsets of $\mathbb{R}^N$ and ν is a positive finite Borel measure on $\mathbb{R}^N$ without atoms. Define a set K_ν by

$$K_\nu = K \cap \bigcap_{l=N_{\mathscr{R}}}^{\infty} \bigcup_{R \in \mathscr{R}_\lambda^{(l)}} R \setminus \bigcup_{R \in \mathscr{R}, \nu(R) = 0} R.$$

The family $\mathscr{R}$ is called a ν*-NET for* K, if $\mathscr{R}$ is a NET for K_ν.

In whole paper, C, $N_{\mathscr{R}}$, $\mathscr{R}_\lambda^{(n)}$ and $\tilde{\mathscr{R}}_\lambda^{(n)}$ mean the same meanings as in this section. If $\mathscr{R}$ is a ν-NET for $K \subset \mathbb{R}^N$, then for $\alpha \geq 0$, $\rho > 0$, $E \subseteq K_\nu$

$$\mathscr{R}\text{-M}_{\nu,\rho}^\alpha(E) = \inf\{\sum_{i=1}^\infty \nu^\alpha(R_i): R_i \in \mathscr{R}, E \subseteq \bigcup_{i=1}^\infty R_i, \nu(R_i) \leq \rho\}$$

and

$$\mathscr{R}\text{-M}_\nu^\alpha(E) = \lim_{\rho \downarrow 0} \mathscr{R}\text{-M}_{\nu,\rho}^\alpha(E)$$

are defined. Then $\mathscr{R}\text{-M}_\nu^\alpha$ has similar properties to Hausdorff measure. There exists $D \in [0, 1]$ such that

$$\mathscr{R}\text{-M}_\nu^\alpha(E) = \begin{cases} \infty & \text{if } \alpha < D, \\ 0 & \text{if } \alpha > D. \end{cases}$$

Therefore *the Billingsley dimension* $H_{\mathscr{R}}$*-dim*$_\nu$ *for E referring to* ν *and* $\mathscr{R}$ is defined by

$$\begin{aligned} H_{\mathscr{R}}\text{-dim}_\nu(E) &= \sup\{\alpha: \mathscr{R}\text{-M}_\nu^\alpha(E) = \infty\} \\ &= \inf\{\alpha: \mathscr{R}\text{-M}_\nu^\alpha(E) = 0\}. \end{aligned}$$

Furthermore, we can easily check the following facts.

PROPOSITION 2.4. *Suppose that* $\mathscr{R}$ *is a* ν*-NET for* $K \subset \mathbb{R}^N$, $E = \bigcup_i E_i \subseteq K$ *then we have the following properties*:
(1) $\mathscr{R}\text{-}M_\nu^\alpha$ *is a metric outer measure.*
(2) $0 \leq H_{\mathscr{R}}\text{-}dim_\nu(E) \leq 1$ *holds. Especially if* $\nu(E) > 0$ *then* $H_{\mathscr{R}}\text{-}dim_\nu(E) = 1$ *holds.*
(3) $H_{\mathscr{R}}\text{-}dim_\nu(E) = \sup_i H_{\mathscr{R}}\text{-}dim_\nu(E_i)$ *holds.*

3. Main results

In this section, we study the relation between the Billingsley dimension and the Hausdorff dimension and the relation between the Billingsley measure and the Hausdorff measure. First we will show that if $\mathscr{R}$ is a NET for K then we only have to refer to $\mathscr{R}$ substituting for an arbitrary covering of M in calculating the Hausdorff dimension of $M \subseteq K$.

LEMMA 3.1. *For a given* $M \subseteq K \subset \mathbb{R}^N$, *define*

$$\mathscr{R}\text{-}H_\rho^\alpha(M) = \inf\{\sum_{i=1}^\infty |U_i|^\alpha: U_i \in \mathscr{R}, M \subseteq \bigcup_{i=1}^\infty U_i, |U_i| \leq \rho\}$$

and

$$\mathscr{R}\text{-}H^\alpha(M) = \lim_{\rho \downarrow 0} \mathscr{R}\text{-}H^\alpha_\rho(M).$$

Then

$$\lambda^{-\alpha} L_{N,\lambda,C}^{-1} \mathscr{R}\text{-}H^\alpha(M) \le H^\alpha(M) \le \mathscr{R}\text{-}H^\alpha(M),$$

where $L_{N,\lambda,C} = (2\lambda)^N \Omega_N C^{-1}$ *and* $\Omega_N = \pi^{\frac{1}{2}N} / \Gamma(N/2 + 1)$ *is the volume of a unit ball in* $\mathbb{R}^N$.

PROOF. Let $\{U_i\}_{i=1}^\infty$ be an arbitrary ρ-covering of K for some sufficiently small $\rho > 0$, then we can find $n_i \in \mathbb{N}$ for every U_i such that

$$\lambda^{-n_i} < |U_i| \le \lambda^{-(n_i-1)}.$$

By Proposition 2.2, for every U_i and $\omega_i \in U_i$

$$K \cap U_i \subseteq \bigcup_{R \in \mathscr{R}_\lambda^{(n_i)},\, R \cap U_i \neq \emptyset} R \subset B(\omega_i, 2\lambda^{-(n_i-1)})$$

holds. By $\lambda_N(B(\omega_i, r)) = \Omega_N r^N$ and (2) of Definition 2.1, we have the estimate

$$^\#\{R \in \tilde{\mathscr{R}}_\lambda^{(n_i)} : R \cap U_i \neq \emptyset\} \le (2\lambda)^N \Omega_N C^{-1}$$

for any i. Therefore there exists $\{R_{i,j}^{(n_i)}\}_{j=1,\dots,m_i}^{i=1,2,\dots}$ such that

$$U_i \cap K \subseteq \bigcup_{j=1}^{m_i} R_{i,j}^{(n_i)},\ R_{i,j}^{(n_i)} \in \tilde{R}_\lambda^{(n_i)},\ m_i \le (2\lambda)^N \Omega_N C^{-1}.$$

Since $|U_i| > \lambda^{-n_i} \ge \lambda^{-1} |R_{i,j}^{(n_i)}|$,

$$\sum_{i=1}^\infty |U_i|^\alpha \ge \frac{1}{\lambda^\alpha} \sum_{i=1}^\infty \frac{1}{m_i} \sum_{j=1}^{m_i} |R_{i,j}^{(n_i)}|^\alpha.$$

This implies that

$$C(2^N \lambda^{N+\alpha} \Omega_N)^{-1} \mathscr{R}\text{-}\mathrm{H}^\alpha(M) \le \mathrm{H}^\alpha(M).$$

The estimate

$$\mathrm{H}^\alpha(M) \le \mathscr{R}\text{-}\mathrm{H}^\alpha(M)$$

is clear from the definition. □

From now on, we restrict the Lebesgue measure on a sufficiently large ball $V \subset \mathbb{R}^N$ such that

$$R \subseteq V \qquad \text{for any } R \in \mathscr{R}_\lambda^{(N_\mathscr{R})} \text{ with } R \cap K \neq \emptyset.$$

Then λ_N is a finite measure.

LEMMA 3.2. *Suppose that* $\mathscr{R}$ *is a NET for* $K \subset \mathbb{R}^N$. *Then for any Borel set* $M \subseteq K$,

$$H\text{-}dim\,(M) = N \cdot H_{\mathscr{R}}\text{-}dim_{\lambda_N}(M)$$

holds.

PROOF. Since R is included in a ball with radius $|R|$, we have

$$C|R|^N \le \lambda_N(R) \le \Omega_N |R|^N \qquad \text{for all } R \in \mathscr{R} \tag{3.1}$$

by (2) of DEFINITION 2.1. Therefore we have

$$C^{\alpha/N} \cdot \sum_{i=1}^{\infty} |R_i|^\alpha \le \sum_{i=1}^{\infty} \lambda_N^{\alpha/N}(R_i) \le \Omega_N^{\alpha/N} \sum_{i=1}^{\infty} |R_i|^\alpha,$$

where $\{R_i \in \mathscr{R}\}_{i=1}^{\infty}$ satisfies $|R_i| < \rho$, $M \subseteq \bigcup_{i=1}^{\infty} R_i$. This implies

$$C^{\alpha/N} \mathscr{R}\text{-H}^\alpha(M) \le \mathscr{R}\text{-M}_{\lambda_N}^{\alpha/N}(M) \le \Omega_N^{\alpha/N} \mathscr{R}\text{-H}^\alpha(M).$$

Together with LEMMA 3.1, we have

$$\text{H-dim}\,(M) = N \cdot \text{H}_{\mathscr{R}}\text{-dim}_{\lambda_N}(M). \quad \square$$

Next theorem is *an extension of Billingsley's theorem.*

THEOREM 3.3. *Suppose that* $\mathscr{R}$ *is a* ν*-NET and also a* μ*-NET for* $K \subset \mathbb{R}^N$. *If* M *satisfies the following condition*

$$M \subseteq \left\{ \omega \in K_\nu \cap K_\mu : \liminf_{n \to \infty} \inf_{\omega \in R \in \mathscr{R}_\lambda^{(n)}} \frac{\log \nu(R)}{\log \mu(R)} \ge \delta \right\}, \tag{3.2}$$

then

$$H_{\mathscr{R}}\text{-}dim_\mu(M) \ge \delta \cdot H_{\mathscr{R}}\text{-}dim_\nu(M).$$

PROOF. We may assume $\delta > 0$. Set

$$M_{\rho,\varepsilon} = \{\omega \in M : \mu(R) \ge \rho \text{ or } \nu(R) \le \mu^{\delta-\varepsilon}(R) \text{ for any } R \in \mathscr{R} \text{ such that } \omega \in R\}.$$

Then for any $\varepsilon > 0$ and $\omega \in M$, there exists $N(\omega, \varepsilon)$ such that

$$\nu(R) \le \mu^{\delta-\varepsilon}(R) \qquad \text{for any } R \in \mathscr{R}_\lambda^{(n)} \text{ with } \omega \in R$$

for any $n \ge N(\omega, \varepsilon)$. Therefore if we take

$$\rho = \inf\{\mu^{\delta-\varepsilon}(R) : \omega \in R \in \mathscr{R}_\lambda^{(N(\omega,\varepsilon))}\} \ge \inf\{\mu^{\delta-\varepsilon}(\tilde{R}) : \omega \in \tilde{R} \in \tilde{\mathscr{R}}_\lambda^{(N(\omega,\varepsilon)+1)}\} > 0$$

then we see that $\omega \in M_{\rho,\varepsilon}$. That is to say, for any $\varepsilon > 0$,

$$M_{\rho,\varepsilon} \uparrow M \quad as \quad \rho \downarrow 0.$$

Since $M_{\rho,\varepsilon} \subseteq M$, for any given $(\rho >)\rho' > 0$, $\gamma > 0$ and $\gamma' > 0$, there exists $\{R_i\}_{i=1}^{\infty}$ satisfying the following conditions:

$$R_i \in \mathscr{R},\ \mu(R_i) < \rho',\ M_{\rho,\varepsilon} \subseteq \bigcup_{i=1}^{\infty} R_i,\ R_i \cap M_{\rho,\varepsilon} \neq \emptyset$$

and

$$\sum_{i=1}^{\infty} \mu^{\delta'}(R_i) < \gamma,$$

where

$$\delta' = H_{\mathscr{R}}\text{-dim}_{\mu}(M) + \gamma'.$$

Therefore we have

$$\mathscr{R}\text{-M}_{\nu,\rho'^{\delta-\varepsilon}}^{\frac{\delta'}{\delta-\varepsilon}}(M_{\rho,\varepsilon}) \leq \sum_{i=1}^{\infty} \nu^{\frac{\delta'}{\delta-\varepsilon}}(R_i) \leq \sum_{i=1}^{\infty} \mu^{\delta'}(R_i) < \gamma.$$

By letting $\rho' \downarrow 0$, we have $\mathscr{R}\text{-M}_{\nu}^{\frac{\delta'}{\delta-\varepsilon}}(M_{\rho,\varepsilon}) \leq \gamma$. Therefore $H_{\mathscr{R}}\text{-dim}_{\nu}(M_{\rho,\varepsilon}) \leq \dfrac{\delta'}{\delta-\varepsilon}$. Since $M_{\rho,\varepsilon} \uparrow M$ as $\rho \downarrow 0$, by (3) in Proposition 2.4, we have

$$H_{\mathscr{R}}\text{-dim}_{\nu}(M) \leq \frac{H_{\mathscr{R}}\text{-dim}_{\mu}(M) + \gamma'}{\delta - \varepsilon}.$$

Since $\varepsilon > 0$ and $\gamma' > 0$ are arbitrary, we have the conclusion. □

By Lemma 3.2 and Theorem 3.3, we have the following main result.

Theorem 3.4. *Suppose that $\mathscr{R}$ is a ν-NET for a bounded set $K \subset \mathbb{R}^N$.*
(*a*) *If M satisfies the condition*

$$M \subseteq \left\{\omega \in K_{\nu} \colon \liminf_{n\to\infty} \inf_{\omega \in R \in \mathscr{R}_K^{(n)}} \frac{\log \nu(R)}{\log |R|} \geq \delta\right\}, \tag{3.3}$$

then

$$H\text{-}dim\,(M) \geq \delta \cdot H_{\mathscr{R}}\text{-}dim_{\nu}(M).$$

(*b*) *If M satisfies the condition*

$$M \subseteq \left\{\omega \in K_{\nu} \colon \alpha \leq \liminf_{n\to\infty} \inf_{\omega \in R \in \mathscr{R}_K^{(n)}} \frac{\log \nu(R)}{\log |R|} \leq \limsup_{n\to\infty} \sup_{\omega \in R \in \mathscr{R}_K^{(n)}} \frac{\log \nu(R)}{\log |R|} \leq b\right\}, \tag{3.4}$$

then

$$a \cdot H_{\mathscr{R}}\text{-}dim_{\nu}(M) \le H\text{-}dim(M) \le b \cdot H_{\mathscr{R}}\text{-}dim_{\nu}(M).$$

Especially, if $\nu(M) > 0$ *then*

$$a \le H\text{-}dim(M) \le b.$$

PROOF. Let μ be λ_N. Then we can apply THEOREM 3.3 and see

$$\mathrm{H}_{\mathscr{R}}\text{-dim}_{\lambda_N}(M) \ge \frac{\delta}{N} \cdot \mathrm{H}_{\mathscr{R}}\text{-dim}_{\nu}(M).$$

Since $\mathscr{R}$ is a NET for K_ν, by LEMMA 3.2 we have (a) in the theorem. By interchanging ν and μ in THEOREM 3.3, we have (b) in the same way. The last part is seen by (2) of Proposition 2.4. □

Readers will find that this Theorem is similar to the result of L. S. Young (cited in Section 1). The following formulation which is due to Billingsley is more useful in calculation.

THEOREM 3.5. *Suppose that* $\mathscr{R}$ *is a* ν*-NET for* $K \subset \mathbb{R}^N$. *If* $\mathscr{R}$ *is a countable family and M satisfies the conditions*

$$M \subseteq \left\{\omega \in K_\nu \colon \liminf_{n\to\infty} \inf_{\omega\in R\in\mathscr{R}_\lambda^{(n)}} \frac{\log \nu(R)}{\log |R|} = \limsup_{n\to\infty} \sup_{\omega\in R\in\mathscr{R}_\lambda^{(n)}} \frac{\log \nu(R)}{\log |R|} = \delta\right\} \tag{3.4'}$$

$$H\text{-}dim(K \cap R) \le \delta \cdot H_{\mathscr{R}}\text{-}dim_\nu(M) \tag{3.5}$$

for any $R \in \mathscr{R}$ *with* $\nu(R) = 0$, *and*

$$H\text{-}dim(K \setminus \bigcup_{R\in\mathscr{R}_\lambda^{(n)}} R) \le \delta \cdot H_{\mathscr{R}}\text{-}dim_\nu(M) \tag{3.6}$$

for any $n \ge N_{\mathscr{R}}$, *then*

$$H\text{-}dim(M \cup (K \setminus K_\nu)) = \delta \cdot H_{\mathscr{R}}\text{-}dim_\nu(M) = H\text{-}dim(M).$$

COROLLARY. *Especially, if* $M = K_\nu$ *satisfies the conditions* (3.4)′, (3.5) *and* (3.6), *then we have*

$$H\text{-}dim(K) = \delta \cdot H_{\mathscr{R}}\text{-}dim_\nu(K_\nu).$$

PROOF OF THEOREM 3.5. Since $\mathscr{R}$ is a countable family and the equality

$$\mathrm{H}\text{-dim}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sup_i \mathrm{H}\text{-dim}(A_i) \tag{3.7}$$

holds, by (3.5) we see that

$$\text{H-dim}\,(K \cap \bigcup_{R\in\mathscr{R},\, v(R)=0} R) \le \delta \cdot \text{H}_{\mathscr{R}}\text{-dim}\,(M),$$

and by (3.6) we see that for all $n \ge N_{\mathscr{R}}$

$$\text{H-dim}\,((K \setminus \bigcup_{R\in\mathscr{R}_\lambda^{(n)}} R) \cup (K \cap \bigcup_{R\in\mathscr{R},\, v(R)=0} R)) \le \delta \cdot \text{H}_{\mathscr{R}}\text{-dim}\,(M).$$

Since

$$K \setminus K_v = \bigcup_{l=N_{\mathscr{R}}}^{\infty} (K \setminus \bigcup_{R\in\mathscr{R}_\lambda^{(l)}} R) \cup \bigcup_{R\in\mathscr{R},\, v(R)=0} (K \cap R),$$

again by (3.7) we have

$$\begin{aligned}\text{H-dim}\,(K \setminus K_v) &\le \text{H-dim}\,(\bigcup_{l=N_{\mathscr{R}}}^{\infty} (K \setminus \bigcup_{R\in\mathscr{R}_\lambda^{(l)}} R) \cup (K \cap \bigcup_{R\in\mathscr{R},\, v(R)=0} R))\\ &\le \delta \cdot \text{H}_{\mathscr{R}}\text{-dim}\,(M).\end{aligned}$$

Together with THEOREM 3.4 (b), we have

$$\text{H-dim}\,(K \setminus K_v) \le \delta \cdot \text{H}_{\mathscr{R}}\text{-dim}\,(M) = \text{H-dim}\,(M).$$

This means

$$\text{H-dim}\,(M \cup (K \setminus K_v)) = \text{H-dim}\,(M) = \delta \cdot \text{H}_{\mathscr{R}}\text{-dim}\,(M). \quad \square$$

An application of THEOREM 3.5 will appear in EXAMPLE 5.2 of Section 5.

LEMMA 3.6. *Suppose that $\mathscr{R}$ is a v-NET for $K \subset \mathbb{R}^N$. If M satisfies*

$$M \subseteq \left\{\omega \in K_v \colon a \le \liminf_{n\to\infty} \inf_{\omega\in R\in\mathscr{R}_\lambda^{(n)}} \frac{v(R)}{|R|^\delta} \le \limsup_{n\to\infty} \sup_{\omega\in R\in\mathscr{R}_\lambda^{(n)}} \frac{v(R)}{|R|^\delta} \le b\right\}, \tag{3.8}$$

then

$$b^{-1}\mathscr{R}\text{-}M_v^1(M) \le R\text{-}H^\delta(M) \le a^{-1}\mathscr{R}\text{-}M_v^1(M).$$

PROOF. If $a = 0$ then the lefthand side inequality is clear, so we assume $a > 0$. For $\rho > 0$, $\varepsilon > 0$, set

$$M_{\rho,\varepsilon} = \{\omega \in M \colon (a-\varepsilon)|R|^\delta \le v(R) \le (b+\varepsilon)|R|^\delta \text{ or } v(R) \ge \rho \text{ for any } R \in \mathscr{R} \text{ such that } \omega \in R\}.$$

By (3.8), for any $\varepsilon > 0$ and $\omega \in M$, there exists $N(\omega, \varepsilon) \ge N_{\mathscr{R}}$ such that

$$(a-\varepsilon)|R|^{\delta}\le \nu(R)\le (b+\varepsilon)|R|^{\delta}$$

for any $n\ge N(\omega,\varepsilon)$ and any $R\in\mathscr{R}_{\lambda}^{(n)}$ with $\omega\in R$. Similarly to the proof of Theorem 3.3, we can see $M_{\rho,\varepsilon}\uparrow M$ as $\rho\downarrow 0$.

Firstly we prove the lefthand side inequality. For any sufficiently small $\rho''>0$ there exists $0<\rho'<\rho$ such that

$$\nu(R)<\rho'' \quad \text{if} \quad |R|<\rho'. \tag{3.9}$$

And for any $\gamma>0$ there exists $\{R_i\}_{i=1}^{\infty}$ satisfying the conditions:

$$R_i\in\mathscr{R},\ |R_i|<\rho',\ M_{\rho,\varepsilon}\subseteq\bigcup_{i=1}^{\infty}R_i,\ \lambda_N(R_i\cap R_j)=0\quad(i\ne j),$$

$$R_i\cap M_{\rho,\varepsilon}\ne\emptyset,\ 0\le\sum_{i=1}^{\infty}|R_i|-\mathscr{R}\text{-H}_{\rho'}^{\delta}(M_{\rho,\varepsilon})<\gamma.$$

By (3.9) and the definition of $M_{\rho,\varepsilon}$, we have

$$(a-\varepsilon)|R_i|^{\delta}\le\nu(R_i)\le(b+\varepsilon)|R_i|^{\delta} \tag{3.10}$$

for any i. Therefore we have the following estimate

$$\begin{aligned}\mathscr{R}\text{-H}^{\delta}(M)\ge\mathscr{R}\text{-H}_{\rho'}^{\delta}(M_{\rho,\varepsilon})&\ge\sum_{i=1}^{\infty}|R_i|^{\delta}-\gamma\\&\ge(b+\varepsilon)^{-1}\sum_{i=1}^{\infty}\nu(R_i)-\gamma\\&\ge(b+\varepsilon)^{-1}\mathscr{R}\text{-M}_{\nu,\rho''}^{1}(M_{\rho,\varepsilon})-\gamma.\end{aligned}$$

By letting $\rho'',\gamma\downarrow 0$, we have

$$(b+\varepsilon)^{-1}\mathscr{R}\text{-M}_{\nu}^{1}(M_{\rho,\varepsilon})\le\mathscr{R}\text{-H}^{\delta}(M).$$

Since $\mathscr{R}$-$\mathrm{M}_{\nu}^{\alpha}$ is an outer measure, $\mathscr{R}\text{-M}_{\nu}^{\alpha}(M_{\rho,\varepsilon})\uparrow\mathscr{R}\text{-M}_{\nu}^{\alpha}(M)$ as $\rho\downarrow 0$ (see P. Halmos [4] p. 47). Therefore

$$(b+\varepsilon)^{-1}\mathscr{R}\text{-M}_{\nu}^{1}(M)\le\mathscr{R}\text{-H}^{\delta}(M).$$

Since $\varepsilon>0$ is arbitrary, we have the lefthand side inequality.

Secondly we prove the righthand side inequality. For any $0<\rho'<\rho$ there exists $\rho''>0$ such that

$$|R|<\rho' \quad \text{if} \quad \nu(R)<\rho''. \tag{3.11}$$

And for any $\gamma>0$ there exists $\{R_i\}_{i=1}^{\infty}$ satisfying the conditions:

$$R_i\in\mathscr{R},\ \nu(R_i)<\rho'',\ M_{\rho,\varepsilon}\subseteq\bigcup_{i=1}^{\infty}R_i,\ \lambda_N(R_i\cap R_j)=0\qquad(i\ne j),$$

$$R_i \cap M_{\rho,\varepsilon} \neq \emptyset,\ 0 \leq \sum_{i=1}^{\infty} \nu(R_i) - \mathscr{R}\text{-}\mathrm{M}^1_{\nu,\rho''}(M_{\rho,\varepsilon}) < \gamma.$$

Therefore together with (3.10)

$$\begin{aligned}\mathscr{R}\text{-}\mathrm{M}^1_\nu(M) \geq \mathscr{R}\text{-}\mathrm{M}^1_{\nu,\rho''}(M_{\rho,\varepsilon}) &\geq \sum_{i=1}^{\infty} \nu(R_i) - \gamma \\ &\geq (a-\varepsilon)\sum_{i=1}^{\infty} |R_i|^\delta - \gamma \\ &\geq (a-\varepsilon)\mathscr{R}\text{-}\mathrm{H}^\delta_{\rho'}(M_{\rho,\varepsilon}) - \gamma.\end{aligned}$$

By letting $\rho', \gamma \downarrow 0$, we have

$$\mathscr{R}\text{-}\mathrm{H}^\delta(M_{\rho,\varepsilon}) \leq (a-\varepsilon)^{-1}\mathscr{R}\text{-}\mathrm{M}^1_\nu(M).$$

Therefore we have the righthand side inequality similarly to the lefthand side inequality. □

LEMMA 3.7. *Suppose that $\mathscr{R}$ is a ν-NET for K, $E \subseteq K_\nu$ then we have the inequalities*

$$\nu^*(E) \leq \mathscr{R}\text{-}M^1_\nu(E) \leq L_{N,\lambda,C}\,\nu(\mathbb{R}^N),$$

where ν^* means the outer measure which is induced from measure ν. Especially, the condition

$$\nu(R_i \cap R_j) = 0 \quad \text{if} \quad \lambda_N(R_i \cap R_j) = 0 \qquad R_i, R_j \in \mathscr{R} \tag{3.12}$$

is satisfied then

$$\nu^*(E) = \mathscr{R}\text{-}M^1_\nu(E).$$

PROOF.

$$\nu^*(E) \leq \mathscr{R}\text{-}\mathrm{M}^1_\nu(E) \leq \tilde{\mathscr{R}}\text{-}\mathrm{M}^1_\nu(E)$$

is clear from definition. Therefore we have only to show that

$$\tilde{\mathscr{R}}\text{-}\mathrm{M}^1_\nu(E) \leq L_{N,\lambda,C}\,\nu(\mathbb{R}^N).$$

Put $\tilde{V}_n = \bigcup_{R \cap K_\nu \neq \emptyset, R \in \tilde{\mathscr{R}}^{(n)}_\lambda} R$ then $K_\nu \subseteq \tilde{V}_n$ for any $n \geq N_{\mathscr{R}}$. That is to say, K_ν is covered by the elements of $\tilde{\mathscr{R}}^{(n)}_\lambda$ for any fixed $n \geq N_{\mathscr{R}}$. By PROPOSITION 2.2, there exists $\{R_i\}_{i=1}^m$ *such that*

$$R_i \in \tilde{\mathscr{R}}^{(n)}_\lambda,\ E \subseteq \bigcup_{i=1}^{m} R_i,\ \lambda_N(R_i \cap R_j) = 0 \qquad (i \neq j).$$

For any fixed $\omega \in \bigcup_{i=1}^m R_i$, the number of R_i's containing ω is bounded by $L_{N,\lambda,C}$. Therefore

$$\mathscr{R}\text{-}\mathrm{M}_\nu^1(E) \le \sum_{i=1}^{m} \nu(R_i) \le L_{N,\lambda,C}\,\nu(\bigcup_{i=1}^{m} R_i) \le L_{N,\lambda,C}\,\nu(\mathbb{R}^N).$$

The last part is clear from the measurability of $\tilde{\mathscr{R}}$. □

By Lemma 3.1, Lemma 3.6 and Lemma 3.7, we have the following.

THEOREM 3.8. *Suppose that $\mathscr{R}$ is a ν-NET for $K \subset \mathbb{R}^N$.*

(*a*) *If M satisfies* (3.8) *then we have the inequalities*

$$b^{-1}\lambda^{-\delta}L_{N,\lambda,C}^{-1}\,\nu^*(M) \le H^\delta(M) \le a^{-1}L_{N,\lambda,C}\,\nu(\mathbb{R}^N).$$

(*b*) *If M satisfies* (3.8) *and $\mathscr{R}$ satisfies condition* (3.12) *then*

$$b^{-1}\lambda^{-\delta}L_{N,\lambda,C}^{-1}\,\nu^*(M) \le H^\delta(M) \le a^{-1}\nu^*(M).$$

COROLLARY. *If $\mathscr{R}$ is a countable family and there exists $\delta' < \delta$ such that*

$$H\text{-}dim\,(K \cap R) \le \delta' \cdot H_{\mathscr{R}}\text{-}dim_\nu\,(M), \tag{3.5$'$}$$

for any $R \in \mathscr{R}$ with $\nu(R) = 0$, and

$$H\text{-}dim\,(K \setminus \bigcup_{R \in \mathscr{R}_\lambda^{(n)}} R) \le \delta' \cdot H_{\mathscr{R}}\text{-}dim_\nu\,(M), \tag{3.6$'$}$$

for any $n \ge N_{\mathscr{R}}$, then

$$b^{-1}\lambda^{-\delta}L_{N,\lambda,C}^{-1}\,\nu^*(M) \le H^\delta(M) = H^\delta(M \cup (K \setminus K_\nu)) \le a^{-1}L_{N,\lambda,C}\,\nu(\mathbb{R}^N).$$

Furthermore, if $\mathscr{R}$ satisfies the condition (3.12) *then*

$$b^{-1}\lambda^{-\delta}L_{N,\lambda,C}^{-1}\,\nu^*(M) \le H^\delta(M) = H^\delta(M \cup (K \setminus K_\nu)) \le a^{-1}\nu^*(M).$$

Especially, if $M = K_\nu$ then

$$b^{-1}\lambda^{-\delta}L_{N,\lambda,C}^{-1}\,\nu^*(K_\nu) \le H^\delta(K_\nu) = H^\delta(K) \le a^{-1}\nu^*(K_\nu).$$

4. Strict NET

In this section, we introduce some additional results which hold under stronger conditions than in Section 3.

DEFINITION 4.1. For a given bounded set $K \subset \mathbb{R}^N$, a family $\mathscr{R}$ of bounded Borel subsets of $\mathbb{R}^N$ is called *an sNET for K* (a NET in strict sense) if $\mathscr{R}$ satisfies the following conditions:

(1) If $R_1, R_2 \in \mathscr{R}$ then $R_1 \subseteq R_2$, $R_2 \subseteq R_1$ or $R_1 \cap R_2 = \emptyset$ holds.
(2) There exists a positive constant C such that

$$\lambda_N(R) \geq C \cdot |R|^N \qquad \text{for any } R \in \mathscr{R}.$$

(3) There exist two constants $\lambda > 1$, $N_{\mathscr{R}} \in \mathbb{Z}$ satisfying the condition: There exists $R \in \mathscr{R}_\lambda^{(n)}$ with $\omega \in R$ for any $n \geq N_{\mathscr{R}}$ and any $\omega \in K$.

Similarly to the proof of PROPOSITION 2.2, we can prove PROPOSITION 4.2.

PROPOSITION 4.2. *If $\mathscr{R}$ is an sNET for $K \subset \mathbb{R}^N$, then it is a NET for K and there exist a constant $\lambda > 1$ and a sequence $\{E_n(\omega) \in \mathscr{R}\}_{n=N_{\mathscr{R}}}^{\infty}$ for any $\omega \in K$ such that*

$$\omega \in E_n(\omega),\ E_n(\omega) \supset E_{n+1}(\omega) \quad \textit{and} \quad E_n(\omega) \in \mathscr{R}_\lambda^{(n)}$$

and that if $\omega \neq \omega'$ then

$$E_n(\omega) = E_n(\omega') \quad \textit{or} \quad E_n(\omega) \cap E_n(\omega') = \emptyset$$

holds for any $n \geq N_{\mathscr{R}}$. Moreover, for $\omega \in K$ and $R \in \mathscr{R}_\lambda^{(n)}$ with $\omega \in R$, $E_{n+1}(\omega) \subset R \subset E_{n-1}(\omega)$ holds for any $n \geq N_{\mathscr{R}} + 1$.

DEFINITION 4.3. Suppose that $K \subset \mathbb{R}^N$ is a bounded set, $\mathscr{R}$ is a family of bounded Borel subsets of $\mathbb{R}^N$ and ν is a positive finite Borel measure on $\mathbb{R}^N$ without atoms. Then the family $\mathscr{R}$ is calle *a ν-sNET for K*, if $\mathscr{R}$ is an sNET for K_ν.

Through out this section, $\{E_n(\omega) \in \mathscr{R}\}_{n=1}^{\infty}$ means the same meaning in PROPOSITION 4.2 for a given NET for K. By PROPOSITION 4.2 and by definition of K_ν, if $\mathscr{R}$ is an sNET for K, then we have

$$K_\nu = \bigcap_{l=N_{\mathscr{R}}}^{\infty} \bigcup_{R \in \mathscr{R}_\lambda^{(l)},\, \nu(R) > 0} R \cap K.$$

LEMMA 4.4. *Suppose that $\mathscr{R}$ is a ν-sNET and also a μ-sNET for $K \subset \mathbb{R}^N$. If*

$$\lim_{n \to \infty} \frac{\log \nu(E_{n+1}(\omega))}{\log \nu(E_n(\omega))} = 1 \tag{4.1}$$

for all $\omega \in K_\nu \cap K_\mu$ then

$$\liminf_{n \to \infty} \frac{\log \nu(E_n(\omega))}{\log \mu(E_n(\omega))} = \liminf_{n \to \infty} \inf_{\omega \in R \in \mathscr{R}_\lambda^{(n)}} \frac{\log \nu(R)}{\log \mu(R)} = \liminf_{n \to \infty} \sup_{\omega \in R \in \mathscr{R}_\lambda^{(n)}} \frac{\log \nu(R)}{\log \mu(R)}. \tag{4.2}$$

PROOF. By PROPOSITION 4.2, for $\omega \in K_\nu \cap K_\mu$ and $R \in \mathscr{R}_\lambda^{(n)}$ with $\omega \in R$

$$E_{n+1}(\omega) \subset R \subset E_{n-1}(\omega)$$

holds for any $n \geq N_{\mathscr{R}} + 1$. Therefore we have

$$\frac{\log \nu(E_{n-1}(\omega))}{\log \mu(E_{n+1}(\omega))} \leq \frac{\log \nu(R)}{\log \mu(R)} \leq \frac{\log \nu(E_{n+1}(\omega))}{\log \mu(E_{n-1}(\omega))}$$

for any $R \in \mathscr{R}_\lambda^{(n)}$ with $\omega \in R$. Together with (4.1), we have (4.2). □

By using LEMMA 4.4, we can rewrite the results in Section 3.

LEMMA 4.5. *Suppose that $\mathscr{R}$ is a ν-sNET and also a μ-sNET for $K \subset \mathbb{R}^N$. If $\mathscr{R}$ satisfies the condition* (4.1) *and M satisfies*

$$M \subseteq \left\{\omega \in K_\nu \cap K_\mu : \liminf_{n\to\infty} \frac{\log \nu(E_n(\omega))}{\log \mu(E_n(\omega))} \geq \delta \right\},$$

then

$$H_{\mathscr{R}}\text{-}dim_\mu (M) \geq \delta \cdot H_{\mathscr{R}}\text{-}dim_\nu (M)$$

holds.

It is easy to check that λ_N satisfies the condition (4.1). Therefore we have

THEOREM 4.6. *Suppose that $\mathscr{R}$ is a ν-sNET for $K \subset \mathbb{R}^N$. If M satisfies the following condition*

$$M \subseteq \left\{\omega \in K_\nu : \liminf_{n\to\infty} \frac{\log \nu(E_n(\omega))}{\log |E_n(\omega)|} \geq \delta \right\}, \tag{4.3}$$

then

$$\text{H-dim}\,(M) \geq \delta \cdot \text{H}_{\mathscr{R}}\text{-dim}_\nu (M).$$

COROLLARY. *If we change the condition* (4.3) *for the following condition*

$$M \subseteq \left\{\omega \in K_\nu : a \leq \liminf_{n\to\infty} \frac{\log \nu(E_n(\omega))}{\log |E_n(\omega)|} \leq \limsup_{n\to\infty} \frac{\log \nu(E_n(\omega))}{\log |E_n(\omega)|} \leq b \right\},$$

then

$$a \cdot H_{\mathscr{R}}\text{-}dim_\nu (M) \leq H\text{-}dim\,(M) \leq b \cdot H_{\mathscr{R}}\text{-}dim_\nu (M).$$

Especially, if $\nu(M) > 0$ then

$$a \leq H\text{-}dim\,(M) \leq b.$$

THEOREM 4.7. *Suppose that $\mathscr{R}$ is a ν-sNET for $K \subset \mathbb{R}^N$. If $\mathscr{R}$ is a countable family and M satisfies the conditions* (3.5), (3.6) (or (3.5)′, (3.6)′) *and*

$$M \subseteq \left\{\omega \in K_\nu : \lim_{n\to\infty} \frac{\log \nu(E_n(\omega))}{\log |E_n(\omega)|} = \delta \right\},$$

then

$$H\text{-}dim\,(M \cup (K \setminus K_\nu)) = \delta \cdot H_{\mathscr{R}}\text{-}dim_\nu\,(M) = H\text{-}dim\,(M).$$

THEOREM 4.8. *Suppose that $\mathscr{R}$ is a ν-sNET for $K \subset \mathbb{R}^N$. If M satisfies*

$$M \subseteq \left\{\omega \in K_\nu \colon a \leq \liminf_{n\to\infty} \frac{\nu(E_n(\omega))}{|E_n(\omega)|^\delta} \leq \limsup_{n\to\infty} \frac{\nu(E_n(\omega))}{|E_n(\omega)|^\delta} \leq b\right\}, \tag{4.4}$$

then

$$b^{-1}\lambda^{-\delta}L_{N,\lambda,C}^{-1}\,\nu^*(M) \leq H^\delta(M) \leq a^{-1}\nu^*(M).$$

COROLLARY. *Suppose that $\mathscr{R}$ is a ν-sNET for $K \subset \mathbb{R}^N$. If $\mathscr{R}$ is a countable family and M satisfies the conditions* (3.5)′, (3.6)′ *and* (4.4) *then*

$$b^{-1}\lambda^{-\delta}L_{N,\lambda,C}^{-1}\,\nu^*(M) \leq H^\delta(M) = H^\delta(M \cup (K \setminus K_\nu)) \leq a^{-1}\nu^*(M).$$

Especially, if $M = K_\nu$ then

$$b^{-1}\lambda^{-\delta}L_{N,\lambda,C}^{-1}\,\nu^*(K_\nu) \leq H^\delta(K_\nu) = H^\delta(K) \leq a^{-1}\nu^*(K_\nu).$$

5. Examples

In this section, we will introduce two examples. The first example $K(P_1, P_2, P_3)$ is not a compact set. In case of $(P_1, P_2, P_3) = (1/3, 1/3, 1/3)$, we know that the Hausdorff measure is positive and finite by using Theorem 3.8. But in the other case, we don't know whether the Hausdorff measure is finite or not. In this example, we will calculate only the Hausdorff dimension of $K\ (P_1, P_2, P_3)$.

EXAMPLE 5.1. (Sierpiński Gasket) Let us define contraction maps

$$\varphi_i \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad i = 1, 2, 3,$$

$$\varphi_1 \colon (x, y) \longrightarrow \left(\frac{1}{2}x, \frac{1}{2}y + \frac{\sqrt{3}}{2}\right),$$

$$\varphi_2 \colon (x, y) \longrightarrow \left(\frac{1}{2}x - \frac{1}{2}, \frac{1}{2}y\right),$$

$$\varphi_3 \colon (x, y) \longrightarrow \left(\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y\right),$$

and let X be the regular triangle:

$$X = \{(x, y) \in \mathbb{R}^2 \colon y \geq 0,\ y \leq \sqrt{3}\,x + 3,\ y \leq -\sqrt{3}\,x + 3\}.$$

The Sierpiński gasket K is defined by

$$[i_1, i_2, \cdots, i_n] = \varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_n}(X),$$

$$K = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, i_2, \cdots, i_n) \in \{1, 2, 3\}^n} [i_1, i_2, \cdots, i_n].$$

We define a surjection map φ from $\{1, 2, 3\}^{\mathbb{N}}$ to K by

$$\omega = (\omega_1, \omega_2, \cdots) \in \{1, 2, 3\}^{\mathbb{N}} \longrightarrow \bigcap_{n=1}^{\infty} [\omega_1, \omega_2, \cdots, \omega_n].$$

Define

$$K(P_1, P_2, P_3) = \left\{\varphi(\omega): \frac{N_i(\omega, n)}{n} \longrightarrow P_i \text{ as } n \longrightarrow \infty\right\},$$

where

$$N_i(\omega, n) = {}^{\#}\{k: k \le n, \omega_k = i\} \qquad \text{for } \omega = (\omega_1, \omega_2, \cdot, \cdot) \in \{1, 2, 3\}^{\mathbb{N}},$$

$$\sum_{i=1}^{3} P_i = 1, \qquad 0 < P_i < 1.$$

Then

$$\text{H-dim}\,(K(P_1, P_2, P_3)) = -\frac{\sum_{i=1}^{3} P_i \log P_i}{\log 2}. \tag{5.1}$$

PROOF OF (5.1). Let us define

$$\mathscr{R} = \{[i_1, i_2, \cdots, i_n]; (i_1, i_2, \cdots, i_n) \in \{1, 2, 3\}^n, n = 1, 2, \cdots\}.$$

It is easy to check that $\mathscr{R}$ is a NET for K. Let μ be the (P_1, P_2, P_3)-Bernoulli measure on $\{1, 2, 3\}^{\mathbb{N}}$, that is to say

$$\mu(\{\omega = (\omega_1, \omega_2, \cdots) \in \{1, 2, 3\}^{\mathbb{N}}; \omega_1 = i_1, \omega_2 = i_2, \cdots, \omega_n = i_n\}) = P_{i_1} P_{i_2} \cdots P_{i_n}.$$

Then μ has no atoms. We can introduced a probability measure ν on $\mathbb{R}^2$ from the probability measure μ by

$$\nu(B) = \mu(\varphi^{-1}(B \cap K)) \text{ for any Borel set } B \text{ of } \mathbb{R}^2.$$

Since $[i_1, i_2, \cdots, i_n]$ and $[j_1, j_2, \cdots, j_m]$ have the property that one includes the other or their intersection consists of at most one point. Therefore we see that

$$\nu([i_1, i_2, \cdots, i_n]) = P_{i_1} P_{i_2} \cdots P_{i_n},$$

and that

$$\nu([i_1, i_2, \cdots, i_n] \cap [j_1, j_2, \cdots, j_m]) = 0$$

if their intersection consists of at most one point. Therefore $\mathscr{R}$ is a ν-NET for K. Furthermore for any $\omega \in \varphi^{-1}(K(P_1, P_2, P_3))$

$$\frac{\log \nu([\omega_1, \omega_2, \cdots, \omega_n])}{\log |[\omega_1, \omega_2, \cdots, \omega_n]|} = \frac{\sum_{i=1}^3 N_i(\omega, n) \log P_i}{\sum_{i=1}^3 N_i(\omega, n) \log 2^{-1}} \longrightarrow -\frac{\sum_{i=1}^3 P_i \log P_i}{\log 2}. \tag{5.2}$$

By (5.2), THEOREM 3.4 (b) and $\nu(K(P_1, P_2, P_3)) = 1$, we have

$$\begin{aligned} \text{H-dim}\,(K(P_1, P_2, P_3)) &= -\frac{\sum_{i=1}^3 P_i \log P_i}{\log 2} \text{H}_{\mathscr{R}}\text{-dim}_\nu\,(K(P_1, P_2, P_3)) \\ &= -\frac{\sum_{i=1}^3 P_i \log P_i}{\log 2} \quad \square \end{aligned}$$

The second example is an application of Theorem 4.7 and Theorem 4.8. In this example, we can calculate Hausdorff dimension of K without constructing any complete covering of K.

EXAMPLE 5.2. Let us define contraction maps $\varphi_i \colon \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, 3, 4$,

$$\varphi_1 : (x, y) \longrightarrow \left(\frac{x}{r} + \frac{1}{2} + \frac{1}{2r}, \frac{y}{r^H} + \frac{1}{2} + \frac{1}{2r^H}\right),$$

$$\varphi_2 : (x, y) \longrightarrow \left(\frac{x}{r} - \frac{1}{2} - \frac{1}{2r}, \frac{y}{r^H} + \frac{1}{2} + \frac{1}{2r^H}\right),$$

$$\varphi_3 : (x, y) \longrightarrow \left(\frac{x}{r} - \frac{1}{2} - \frac{1}{2r}, \frac{y}{r^H} - \frac{1}{2} - \frac{1}{2r^H}\right),$$

$$\varphi_4 : (x, y) \longrightarrow \left(\frac{x}{r} + \frac{1}{2} + \frac{1}{2r}, \frac{y}{r^H} - \frac{1}{2} - \frac{1}{2r^H}\right),$$

where

$$0 < H \le 1, \qquad 2 < r^H. \tag{5.3}$$

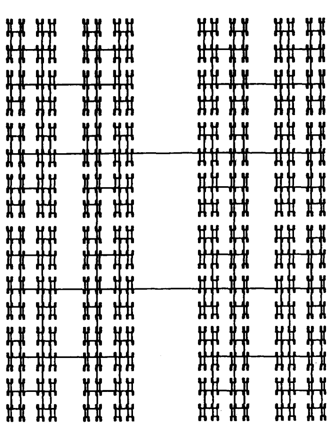

Figure 1. ($r = 2.5$, $H = .757$)

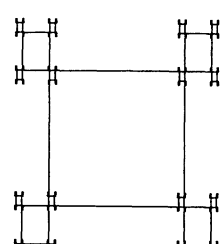

Figure 2. ($r = 5$, $H = .8$)

Put

$$X=\left\{(x, y)\in\mathbb{R}^2: y=\pm\frac{1}{2}, -\frac{1}{2}\le x\le\frac{1}{2}\right\}\cup$$

and

$$\left\{(x, y)\in\mathbb{R}^2: x=\pm\frac{1}{2}, -\frac{1}{2}\le y\le\frac{1}{2}\right\}$$

$$K=\text{the closure of }(X\cup\bigcup_{n=1}^{\infty}\bigcup_{(i_1,i_2,\dots,i_n)\in\{1,2,3,4\}^n}\varphi_{i_1}\circ\varphi_{i_2}\circ\cdots\circ\varphi_{i_n}(X)).$$

Then we have the following fact.

PROPOSITION 5.3. If $2<r^H<2^{H+1}$ then

$$\alpha=\text{H-dim}(K)=\frac{(1+H)\log 2}{H\log r},\qquad 0<\mathrm{H}^{\alpha}(K)<\infty.$$

If $2^{H+1}\le r^H$ then H-dim $(K)=1$ and

$$\mathrm{H}^1(K)=\begin{cases}\infty & \text{if } 2^{H+1}\le r^H\le 4,\\ \text{positive finite} & \text{if } 4<r^H.\end{cases}$$

PROOF. Suppose that $2<r^H$. Denote

$$E^*=\text{the closed convex hull of } E,$$

$$[0]=K^*,\ [0i_1i_2\cdots i_n]=(\varphi_{i_1}\circ\varphi_{i_2}\circ\cdots\circ\varphi_{i_n}(K))^* \tag{5.5}$$

and π_x and π_y be the projections onto x-coordinate and y-coordinate, respectively. Then $\{[0i_1i_2\cdots i_n]\}$ are rectangles. For any $(i_1, i_2,\cdots,i_n)$, $(j_1, j_2,\cdots,j_m)$, one of

$$[0i_1i_2\cdots i_n]\subseteq[0j_1j_2\cdots j_m],\ [0j_1j_2\cdots j_m]\subseteq[0i_1i_2\cdots i_n]\ \text{or}$$
$$[0i_1i_2\cdots i_n]\cap[0j_1j_2\cdots j_m]=\emptyset \tag{5.6}$$

holds. And their diameters are evaluated as

$$|\pi_x([0i_1i_2\cdots i_n])|=\left(\frac{r+1}{r-1}\right)r^{-n},\ |\pi_y([0i_1i_2\cdots i_n])|=\left(\frac{r^H+1}{r^H-1}\right)r^{-nH}.$$

Set

$$L([0i_1\cdots i_n])$$
$$=\left\{[0i_1i_2\cdots i_{[nH]}j_{[nH]+1}\cdots j_n]: j_m=\begin{cases}1\text{ or }2, & \text{if } i_m=1\text{ or }2\\ 3\text{ or }4, & \text{if } i_m=3\text{ or }4\end{cases},\ [nH]+1\le m\le n\right\},$$

$$S([0i_1i_2\cdots i_n])=(\bigcup_{L\in L([0i_1i_2\dots i_n])}L)^*. \tag{5.7}$$

Then $\{S([0i_1i_2\cdots i_n])\}$ are rectangles such that for any $(i_1, i_2,\cdots,i_n)$, $(j_1, j_2,\cdots, j_n)$, one of

$$S([0i_1i_2\cdots i_n]) \subseteq S([0j_1j_2\cdots j_m]),\ S([0j_1j_2\cdots j_m]) \subseteq S([0i_1i_2\cdots i_n]) \text{ or}$$
$$S([0i_1i_2\cdots i_n]) \cap S([0j_1j_2\cdots j_m]) = \emptyset \tag{5.8}$$

holds and that

$$|\pi_x(S([0i_1i_2\cdots i_n]))| = \left(\frac{r+1}{r-1}\right) r^{-[nH]},\ |\pi_y(S([0i_1i_2\cdots i_n]))| = \left(\frac{r^H+1}{r^H-1}\right) r^{-nH}. \tag{5.9}$$

Now we put

$$\mathscr{R} = \{S([0i_1i_2\cdots i_n]): (i_1, i_2,\cdots,i_n) \in \{1, 2, 3, 4\}^n,\ n = 1, 2,\cdots\}.$$

Since $[0i_1i_2\cdots i_n]$ is compact and

$$[0i_1i_2\cdots i_n i] \cap [0i_1i_2\cdots i_n j] = \emptyset \qquad (i \neq j)$$

and

$$[0i_1i_2\cdots i_n] \supset \bigcup_{j=1}^{4} [0i_1i_2\cdots i_n j] \qquad \text{for any } [0i_1i_2\cdots i_n],$$

there exists a probability measure ν on $\mathbb{R}^2$ such that

$$\nu([0i_1i_2\cdots i_n]) = 4^{-n} \qquad \text{for any } [0i_1i_2\cdots i_n].$$

We can easily see that $\mathscr{R}$ is an sNET for

$$K_\nu = K \cap \bigcap_{n=1}^{\infty} \bigcup_{(i_1, i_2,\ldots,i_n)\in\{1,2,3,4\}^n} S([0i_1i_2\cdots i_n]).$$

Since $\nu(S([0i_1i_2\cdots i_n])) = 2^{-(n+[nH])}$, we have

$$\frac{\log \nu(S([0i_1i_2\cdots i_n]))}{\log |S([0i_1i_2\cdots i_n])|} \longrightarrow \alpha = \frac{(1+H)\log 2}{H \log r} \tag{5.10}$$

for any $(0, i_1, i_2,\cdots,i_n,\cdots) \in \{1, 2, 3, 4\}^{\mathbb{N}}$. If $r^H < 2^{H+1}$ then $\alpha > 1$. Therefore together with (5.8), (5.9), $\mathscr{R}$ is a ν-sNET for K and $\mathscr{R}$ satisfies the condition (3.6)′. Since $\nu(R) > 0$ for any $R \in \mathscr{R}$, condition (3.5)′ holds obviously. Therefore by using Theorem 4.7, we have

$$\text{H-dim}(K) = \alpha = \frac{(1+H)\log 2}{H \log r}.$$

And

$$d_1^{-\alpha}2^{-n(1+H)}r^{n\alpha H} \leq \frac{\nu(S([0i_1i_2\cdots i_n]))}{|S([0i_1i_2\cdots i_n])|^\alpha} \leq 2d_2^{-\alpha}2^{-n(1+H)}r^{n\alpha H},$$

where

$$d_1 = \sqrt{\left(\frac{r+1}{r-1}r\right)^2 + \left(\frac{r^H+1}{r^H-1}\right)^2}, \quad d_2 = \sqrt{\left(\frac{r+1}{r-1}\right)^2 + \left(\frac{r^H+1}{r^H-1}\right)^2},$$

that is to say

$$d_1^{-\alpha} \leq \frac{\nu(S([0i_1i_2\cdots i_n]))}{|S([0i_1i_2\cdots i_n])|^\alpha} \leq 2d_2^{-\alpha} \qquad \text{for any } [0i_1i_2\cdots i_n].$$

Therefore by using THEOREM 4.8, we have $0 < \mathrm{H}^\alpha(K) < \infty$ for $2 < r^H < 2^{H+1}$.

On the other hand, if $2^{H+1} \leq r^H$ then $\alpha \leq 1$. Since H-dim$(K) \geq 1$ obviously, (5.10) implies that H-dim$(K) = 1$. Therefore evaluation of $\mathrm{H}^\alpha(K)$ is clear. □

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