HIROSHIMA MATH. J.
25 (1995), 83–96

Homotopy-normality of Lie groups III

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(Received February 18, 1993)
(Revised August 31, 1993)

1. Introduction

The notion of the homotopy normal subgroup was introduced by G.S.McCarty, Jr and I.M.James in the different ways (cf. [15, 8]). The definition of homotopy-normality adopted here is weakcr than those proposed by them. Thus, if a subgroup H of a topological group G is homotopy-normal in G in the sense of either McCarty or James then H is homotopy-normal in G in the sense of ours. However, the converse is not always true. The homotopy-normality of the classical groups in their senses was investigated by them and the author ([7, 8, 9, 15]).

Let

$$SU(3) \subset G_2 \subset Spin(7) \subset Spin(8) \subset Spin(9) \subset F_4 \subset E_6 \subset E_7 \subset E_8 \qquad (*)$$

be a chain of simply connected, compact, connected, simple Lie groups involving the exceptional Lie groups. These inclusions are uniquely defined to within conjugacy (cf. [6, especially p. 192]). In this paper, we consider only the inclusions which are the compositions of the successive natural inclusions in $(*)$. Then the following theorem is obtained.

THEOREM 1 ([7]). *Every subgroup except $E_7 \subset E_8$ of* $(*)$ *is not homotopy-normal in any group containing it in the sense of McCarty.*

The main purpose of this paper is to determine the case $E_7 \subset E_8$. This is done in §3 by using the Hopf algebra structures of E_7 and E_8 as in the following theorem.

THEOREM 2. *The group E_7 is not homotopy-normal in E_8 in the sense of ours* (*as in* 2.3), *hence also not in the sense of both McCarty and James.*

By the way we will improve some of the results in [7] in our sense by making use of the cohomological methods in §§4–7.

The author would like to express his gratitude to Professors T. Matumoto and A. Kono for their valuable suggestions. He also would like to thank Professor T. Yoshida who read the manuscript and suggested a number of improvements.

2. Preliminaries

First, we define three notions of homotopy-normality and discuss their relations.

DEFINITION 2.1 (McCarty [15]). A subgroup H of a topologicsl group G is *homotopy-normal* in G *in the sense of McCarty* if there exists a homotopy $v'_t: (G \times H, H \times H) \to (G, H)$ such that $v'_1(G \times H) \subset H$ and $v'_0(g, h) = ghg^{-1}$ for $g \in G$, $h \in H$.

DEFINITION 2.2 (James [8]). A subgroup H of a topological group G is *homotopy-normal* in G *in the sense of James* if the commutator map $G \wedge H \to G$ can be deformed into H where as usual $G \wedge H$ stands for the identification space $G \times H/G \times e \cup e \times H$, with newtral element $e \in H$.

DEFINITION 2.3. A subgroup H of a topological group G is *homotopy-normal* in G if there exists a homotopy $v_t: G \times H \to G$ such that $v_1(G \times H) \subset H$ and $v_0(g, h) = ghg^{-1}$ for $g \in G$, $h \in H$.

It is obious that homotopy-normality in the sense of McCarty implies the one in the sense of 2.3. Also, the homotopy-normality in the sense of James implies the one in 2.3. In fact if there exists a homotopy $c_t: G \wedge H \to G$ such that $c_1(G \wedge H) \subset H$ and $c_0(g \wedge h) = ghg^{-1}h^{-1}$ where $g \wedge h$ denotes the point $p(g, h)$ by the natural projection $p: G \times H \to G \wedge H$, then the map $v_t: G \times H \to G$ defined by $v_t(g, h) = c_t \circ p(g, h)h$ for $g \in G$, $h \in H$ gives the homotopy-normality in the sense of 2.3.

Before proving Theorem 2 we fix the notation and review the facts which will be used in §3. Let G be a Lie group with $\mu: G \times G \to G$ the group multiplication map and $\Delta: G \to G \times G$ the diagonal map. Then for a prime p, $H^*(G: \boldsymbol{Z}_p)$ is a Hopf algebra with the multiplication map $\Delta^*: H^*(G; \boldsymbol{Z}_p) \otimes H^*(G; \boldsymbol{Z}_p) \to H^*(G; \boldsymbol{Z}_p)$ and the diagonal map $\mu^*: H^*(G; \boldsymbol{Z}_p) \to H^*(G; \boldsymbol{Z}_p) \otimes H^*(G; \boldsymbol{Z}_p)$. Let $\mu^*(x) = x \otimes 1 + 1 \otimes x + \bar{\phi}(x)$ for $x \in H^*(G; \boldsymbol{Z}_p)$.

As for the cohomology mod 2 of the exceptional Lie groups E_7 and E_8 we will use the following theorem due to [3, 10, 14, 18, 19] and [16, Chapter 7, Theorems 6.24, 6.32 and 6.33]:

THEOREM 2.4. *As an algebra*

(i) $H^*(E_7; \boldsymbol{Z}_2) = \boldsymbol{Z}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Lambda(x_{15}, x_{17}, x_{23}, x_{27})$

where deg $x_i = i$ *and the generators are related by* $Sq^2 x_3 = x_5$, $Sq^4 x_5 = x_9$, $Sq^8 x_9 = x_{17}$, $Sq^8 x_{15} = x_{23}$, $Sq^4 x_{23} = x_{27}$. *The coalgebra structure is given by*

$$\bar{\phi}(x_i) = 0 \quad \textit{for} \quad i = 3, 5, 9, 17,$$

$$\bar{\phi}(x_{15}) = x_3^2 \otimes x_9 + x_5^2 \otimes x_5,$$

$$\bar{\phi}(x_{23}) = x_3^2 \otimes x_{17} + x_9^2 \otimes x_5,$$
$$\bar{\phi}(x_{27}) = x_5^2 \otimes x_{17} + x_9^2 \otimes x_9.$$

(ii) $H^*(E_8; \boldsymbol{Z}_2) = \boldsymbol{Z}_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \Lambda(x_{17}, x_{23}, x_{27}, x_{29})$

where deg $x_i = i$ *and the generators are related by* $Sq^1 x_3 = 0$, $Sq^1 x_{2i-1} = x_i^2$ *for* $i = 3, 5, 9, 15$, $Sq^1 x_{15} = x_5^2 x_3^2$, $Sq^1 x_{23} = x_9^2 x_3^2$, $Sq^1 x_{27} = x_9^2 x_5^2$, $Sq^2 x_3 = x_5$, $Sq^2 x_{15} = x_{17}$, $Sq^2 x_{27} = x_{29}$, $Sq^2 x_i = 0$ *for* $i = 5, 9, 17, 23, 29$, $Sq^4 x_5 = x_9$, $Sq^4 x_{23} = x_{27}$, $Sq^8 x_9 = x_{17}$, $Sq^8 x_{15} = x_{23}$, $Sq^4 x_i = Sq^8 x_i = Sq^{16} x_i = 0$ *for any other* $i > 0$. *The coalgebra structure is given by*

$$\bar{\phi}(x_i) = 0 \quad \textit{for} \ i = 3, 5, 9, 17,$$
$$\bar{\phi}(x_{15}) = x_3^2 \otimes x_9 + x_5^2 \otimes x_5 + x_3^4 \otimes x_3,$$
$$\bar{\phi}(x_{23}) = x_3^2 \otimes x_{17} + x_9^2 \otimes x_5 + x_5^4 \otimes x_3,$$
$$\bar{\phi}(x_{27}) = x_5^2 \otimes x_{17} + x_9^2 \otimes x_9 + x_3^8 \otimes x_3,$$
$$\bar{\phi}(x_{29}) = x_3^4 \otimes x_{17} + x_5^4 \otimes x_9 + x_3^8 \otimes x_5.$$

3. Proof of Theorem 2

Let $Ad: G \times G \to G$ be the map defined by $Ad(g, g') = gg'g^{-1}$ for $g, g' \in G$ and $\imath: G \to G$ be the map defined by $\imath(g) = g^{-1}$ for $g \in G$. Then we have

$$Ad = \mu \circ (\mu \times \mathrm{id}_G) \circ (\mathrm{id}_{G \times G} \times \imath) \circ \alpha \circ (\Delta \times \mathrm{id}_G) \tag{A}$$

where $\alpha(g_1, g_2, g_3) = (g_1, g_3, g_2)$ for $g_1, g_2, g_3 \in G$, and

$$\mu \circ (\mathrm{id}_G \times \imath) \circ \Delta = \text{the constant map} \tag{B}$$

LEMMA 3.1. *$\imath^* x_i = x_i$ in $H^i(E_8; \boldsymbol{Z}_2)$ for $i = 3, 5, 9, 17$.*

PROOF. Since x_i ($i = 3, 5, 9, 17$) are primitive (see Theorem 2.4 (ii)), we have by setting id $= \mathrm{id}_{E_8}$

$$\Delta^*(\mathrm{id} \times \imath)^* \mu^*(x_i) = \Delta^*(\mathrm{id} \times \imath)^*(x_i \otimes 1 + 1 \otimes x_i)$$
$$= \Delta^*(x_i \otimes 1 + 1 \otimes \imath^* x_i) = x_i + \imath^* x_i.$$

On the other hand, by (B) we have $x_i + \imath^* x_i = 0$. Hence $\imath^* x_i = x_i$. Q.E.D.

LEMMA 3.2. *$\imath^* x_{15} = x_{15} + x_3^5 + x_5^3 + x_3^2 x_9$ in $H^{15}(E_8; \boldsymbol{Z}_2)$.*

PROOF. By Theorem 2.4 (ii) and Lemma 3.1, we have by setting id $= \mathrm{id}_{E_8}$

$$
\begin{aligned}
&\Delta^*(\mathrm{id}\times\iota)^*\mu^*(x_{15})\\
&\quad=\Delta^*(\mathrm{id}\times\iota)^*(x_{15}\otimes 1+1\otimes x_{15}+x_3^2\otimes x_9+x_5^2\otimes x_5+x_3^4\otimes x_3)\\
&\quad=\Delta^*(x_{15}\otimes 1+1\otimes \iota^*x_{15}+x_3^2\otimes x_9+x_5^2\otimes x_5+x_3^4\otimes x_3)\\
&\quad=x_{15}+\iota^*x_{15}+x_3^2x_9+x_5^3+x_3^5.
\end{aligned}
$$

By (B), we have $\iota^*x_{15}=x_{15}+x_3^5+x_5^3+x_3^2x_9$. Q.E.D.

LEMMA 3.3. *$Ad^*(x_i)=1\otimes x_i$ in $H^0(E_8;\boldsymbol{Z}_2)\otimes H^i(E_8;\boldsymbol{Z}_2)$ for $i=3, 5, 9, 17$.*

PROOF. By (A), Theorem 2.4 (ii) and Lemma 3.1, we have by setting $\mathrm{id}=\mathrm{id}_{E_8}$

$$
\begin{aligned}
Ad^*(x_i)&=(\Delta\times\mathrm{id})^*\alpha^*(\mathrm{id}\times\mathrm{id}\times\iota)^*(\mu\times\mathrm{id})^*\mu^*(x_i)\\
&=(\Delta\times\mathrm{id})^*\alpha^*(\mathrm{id}\times\mathrm{id}\times\iota)^*\{\mu^*(x_i)\otimes 1+\mu^*(1)\otimes x_i\}\\
&=(\Delta\times\mathrm{id})^*\alpha^*(x_i\otimes 1\otimes 1+1\otimes x_i\otimes 1+1\otimes 1\otimes x_i)\\
&=(\Delta\times\mathrm{id})^*(x_i\otimes 1\otimes 1+1\otimes 1\otimes x_i+1\otimes x_i\otimes 1)\\
&=x_i\otimes 1+1\otimes x_i+x_i\otimes 1=1\otimes x_i.
\end{aligned}
$$

Q.E.D.

LEMMA 3.4. *$Ad^*(x_{15})=1\otimes x_{15}+x_3^4\otimes x_3+x_3\otimes x_3^4+x_5^2\otimes x_5+x_5\otimes x_5^2+x_3^2\otimes x_9+x_9\otimes x_3^2$ in $H^*(E_8;\boldsymbol{Z}_2)\otimes H^*(E_8;\boldsymbol{Z}_2)$.*

PROOF. By (A), Theorem 2.4 (ii), Lemmas 3.1 and 3.2, we have by setting $\mathrm{id}=\mathrm{id}_{E_8}$

$$
\begin{aligned}
Ad^*(x_{15})&=(\Delta\times\mathrm{id})^*\alpha^*(\mathrm{id}\times\mathrm{id}\times\iota)^*(\mu\times\mathrm{id})^*\mu^*(x_{15})\\
&=(\Delta\times\mathrm{id})^*\alpha^*(\mathrm{id}\times\mathrm{id}\times\iota)^*\{\mu^*(x_{15})\otimes 1+\mu^*(1)\otimes x_{15}+\mu^*(x_3^2)\otimes x_9\\
&\quad+\mu^*(x_5^2)\otimes x_5+\mu^*(x_3^4)\otimes x_3\}\\
&=(\Delta\times\mathrm{id})^*\alpha^*(\mathrm{id}\times\mathrm{id}\times\iota)^*\{(x_{15}\otimes 1+1\otimes x_{15}+x_3^2\otimes x_9+x_5^2\otimes x_5\\
&\quad+x_3^4\otimes x_3)\otimes 1+1\otimes 1\otimes x_{15}+(x_3^2\otimes 1+1\otimes x_3^2)\otimes x_9\\
&\quad+(x_5^2\otimes 1+1\otimes x_5^2)\otimes x_5+(x_3^4\otimes 1+1\otimes x_3^4)\otimes x_3\}\\
&=(\Delta\times\mathrm{id})^*\alpha^*\{(x_{15}\otimes 1\otimes 1+1\otimes x_{15}\otimes 1+x_3^2\otimes x_9\otimes 1\\
&\quad+x_5^2\otimes x_5\otimes 1+x_3^4\otimes x_3\otimes 1+1\otimes 1\otimes(x_{15}+x_3^5+x_5^3+x_3^2x_9)\\
&\quad+(x_3^2\otimes 1+1\otimes x_3^2)\otimes x_9+(x_5^2\otimes 1+1\otimes x_5^2)\otimes x_5\\
&\quad+(x_3^4\otimes 1+1\otimes x_3^4)\otimes x_3\}\\
&=(\Delta\times\mathrm{id})^*(x_{15}\otimes 1\otimes 1+1\otimes 1\otimes x_{15}+x_3^2\otimes 1\otimes x_9+x_5^2\otimes 1\otimes x_5\\
&\quad+x_3^4\otimes 1\otimes x_3+1\otimes x_{15}\otimes 1+1\otimes x_3^5\otimes 1+1\otimes x_5^3\otimes 1+1\otimes x_3^2x_9\otimes 1\\
&\quad+x_3^2\otimes x_9\otimes 1+1\otimes x_9\otimes x_3^2+x_5^2\otimes x_5\otimes 1
\end{aligned}
$$

$+ 1 \otimes x_5 \otimes x_5^2 + x_3^4 \otimes x_3 \otimes 1 + 1 \otimes x_3 \otimes x_3^4)$

$= x_{15} \otimes 1 + 1 \otimes x_{15} + x_3^2 \otimes x_9 + x_5^2 \otimes x_5 + x_3^4 \otimes x_3 + x_{15} \otimes 1$

$+ x_3^5 \otimes 1 + x_5^3 \otimes 1 + x_3^2 x_9 \otimes 1 + x_3^2 x_9 \otimes 1 + x_9 \otimes x_3^2 + x_5^3 \otimes 1$

$+ x_5 \otimes x_5^2 + x_3^5 \otimes 1 + x_3 \otimes x_3^4$

$= 1 \otimes x_{15} + x_3^4 \otimes x_3 + x_3 \otimes x_3^4 + x_5^2 \otimes x_5 + x_5 \otimes x_5^2 + x_3^2 \otimes x_9$

$+ x_9 \otimes x_3^2.$ Q.E.D.

By Lemma 3.4, we have

LEMMA 3.5. *$Ad^*(x_{15}^2) = 1 \otimes x_{15}^2 + x_3^8 \otimes x_3^2 + x_3^2 \otimes x_3^8 + x_5^4 \otimes x_5^2 + x_5^2 \otimes x_5^4 + x_3^4 \otimes x_9^2 + x_9^2 \otimes x_3^4$ in $H^*(E_8; \boldsymbol{Z}_2) \otimes H^*(E_8; \boldsymbol{Z}_2)$.*

PROOF OF THEOREM 2. The homotopy normality of E_7 in E_8 in the sense of 2.3 would imply the homotopy commutativity of the following diagram

$$\begin{array}{ccc} E_8 \times E_7 \underset{k}{\subset} E_8 \times E_8 & \xrightarrow[Ad]{} & E_8 \\ & \searrow^{\nu_1} & \uparrow j \\ & & E_7 \end{array}$$

where j is a natural inclusion and $k = \mathrm{id} \times j$.

Since $j^* x_{15} = x_{15}$, we have $j^* x_{15}^2 = 0$ by Theorem 2.4 (i). On the other hand, using $j^* x_i = x_i (i = 3, 5, 9)$ (see [14, Proposition 1.1 (2)]), Lemma 3.5 and Theorem 2.4, we have $k^* Ad^*(x_{15}^2) = x_3^8 \otimes x_3^2 + x_5^4 \otimes x_5^2 + x_3^4 \otimes x_9^2 \neq 0$ in $H^*(E_8; \boldsymbol{Z}_2) \otimes H^*(E_7; \boldsymbol{Z}_2)$. This is a contradiction. Q.E.D.

4. E_8 and its subgroups

The method we used in the proof of Theorem 2 can be applied to the other pair (G, H) of Lie groups in $(*)$. We recall the following from [5, 10, 11] and [16, Chapter 7, Lemma 6.5, Theorems 6.2, 6.6 and 6.18]:

THEOREM 4.1. *As an algebra*

(i) $H^*(E_6; \boldsymbol{Z}_2) = \boldsymbol{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_9, x_{15}, x_{17}, x_{23})$

where deg $x_i = i$ *and the generators are related by* $Sq^2 x_3 = x_5$, $Sq^4 x_5 = x_9$, $Sq^8 x_9 = x_{17}$, $Sq^8 x_{15} = x_{23}$, $Sq^1 x_i = 0$ *for* $i \neq 5$. *The coalgebra structure is given by*

$$\bar{\phi}(x_j) = 0 \qquad \textit{for } j = 3, 5, 9, 17,$$

$$\bar{\phi}(x_j) = x_3^2 \otimes x_{j-6} \qquad \textit{for } j = 15, 23.$$

(ii) $H^*(F_4; \boldsymbol{Z}_2) = \boldsymbol{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_{15}, x_{23})$

where deg $x_i = i$ *and the generators are related by* $Sq^2 x_3 = x_5$, $Sq^1 x_5 = Sq^3 x_3 = x_3^2$, $Sq^8 x_{15} = x_{23}$.

(iii) $H^*(G_2; \boldsymbol{Z}_2) = \boldsymbol{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5)$

where deg $x_i = i$ *and the generators are related by* $Sq^2 x_3 = x_5$, $Sq^1 x_5 = Sq^3 x_3 = x_3^2$, $Sq^i x_j = 0$ *otherwise.*

(iv) *The algebras* $H^*(F_4; \boldsymbol{Z}_2)$ *and* $H^*(G_2; \boldsymbol{Z}_2)$ *are primitively generated.*

Further we recall the following from [5, Theorem 12.1] and [16, Chapter 3, Theorem 6.5]:

THEOREM 4.2. *As an algebra*
(i) $H^*(Spin(9); \boldsymbol{Z}_2) = \boldsymbol{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_7, x'_{15})$,
(ii) $H^*(Spin(8); \boldsymbol{Z}_2) = \boldsymbol{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_7, x'_7)$,
(iii) $H^*(Spin(7); \boldsymbol{Z}_2) = \boldsymbol{Z}_2[x_3]/(x_3^4) \otimes \Lambda(x_5, x'_7)$,
(iv) $H^*(SU(n); \boldsymbol{Z}_2) = \Lambda(x_3, x_5, \ldots, x_{2n-1})$ *where* $x_{2i-1} = \sigma(c_i) \in H^{2i-1}(SU(n); \boldsymbol{Z}_2)$ *is the suspension image of the Chern class* c_i.

From these results we can prove the following.

THEOREM 4.3. *Let* H *be* $SU(3)$, G_2, $Spin(7)$, $Spin(8)$, $Spin(9)$, F_4 *or* E_6. *Then* $H \subset E_8$ *of* $(*)$ *is not homotopy normal in the sense of* 2.3, *hence also not in the sense of both McCarty and James.*

PROOF. The homotopy normality of H in E_8 in the sense of 2.3 would imply $Ad \circ k \simeq j \circ v_1$, where $j: H \to E_8$ is a natural inclusion and $k = \mathrm{id} \times j: E_8 \times H \to E_8 \times E_8$.

First, when $H = G_2$, $Spin(7)$, $Spin(8)$, $Spin(9)$, F_4 or E_6, we have $g^* x_{15} = x_{15}$, $f^* x_{15} = x_{15}$ and $h^* x_{15} = 0$ for the natural inclusions $G_2 \xrightarrow{h} F_4 \xrightarrow{f} E_6 \xrightarrow{g} E_8$ (cf. [14, Theorem 1.5], [11, p. 70], [5, Chapter V. (22.5)]). Therefore $j^* x_{15} = x_{15}$ when $H = F_4$, E_6 and $j^* x_{15} = 0$ when $H = G_2$. So $j^* x_{15}^2 = 0$ when $H = G_2$, F_4, E_6. Also $j^* x_{15}^2 = 0$ when $H = Spin(7)$, $Spin(8)$ and $Spin(9)$ because $x^2 = 0$ for all $x \in H^{15}(H; \boldsymbol{Z}_2)$ by Theorem 4.2. On the other hand, we have $k^* Ad^*(x_{15}^2) = x_3^8 \otimes x_3^2 \neq 0 \in H^{24}(E_8; \boldsymbol{Z}_2) \otimes H^6(H; \boldsymbol{Z}_2)$ by Theorems 2.4 (ii), 4.1, 4.2 and Lemma 3.5, because $j^* x_i = x_i$ $(i = 3, 5)$ (cf. [2, 5, 11, 14]), $j^* x_9 = j^* Sq^4 x_5 = Sq^4 j^* x_5 = Sq^4 x_5 = x_9$ when $H = E_6$ by Theorems 2.4 (ii), 4.1 (i), and $j^* x_9 = 0$ when $H = F_4$ by [10, Remark 5.6]. This is a contradiction.

Next, when $H = SU(3)$, we have $j^* x_{15} = 0$. On the other hand, we have $k^* Ad^*(x_{15}) = x_3^4 \otimes x_3 + x_5^2 \otimes x_5 \neq 0 \in H^*(E_8; \boldsymbol{Z}_2) \otimes H^*(H; \boldsymbol{Z}_2)$ by using $j^* x_3 = x_3$, $j^* x_5 = x_5$, $j^* x_9 = 0$ and Lemma 3.4. This is a contradiction. Q.E.D.

5. E_7 and its subgroups

Since $x_i \in H^i(E_7; \boldsymbol{Z}_2)$ ($i = 3, 5, 9, 17$) are primitive (see Theorem 2.4 (i)), we have

LEMMA 5.1. *$\iota^* x_i = x_i$ in $H^i(E_7; \boldsymbol{Z}_2)$ for $i = 3, 5, 9, 17$.*

By (A), Theorem 2.4 (i) and Lemma 5.1, we have

LEMMA 5.2. *$Ad^*(x_i) = 1 \otimes x_i$ in $H^0(E_7; \boldsymbol{Z}_2) \otimes H^i(E_7; \boldsymbol{Z}_2)$ for $i = 3, 5, 9, 17$.*

By Theorem 2.4 (i), Lemmas 3.2 and 3.4, we have

LEMMA 5.3. (i) *$\iota^* x_{15} = x_{15} + x_5^3 + x_3^2 x_9$ in $H^{15}(E_7; \boldsymbol{Z}_2)$,*
(ii) *$Ad^*(x_{15}) = 1 \otimes x_{15} + x_3^2 \otimes x_9 + x_5^2 \otimes x_5 + x_9 \otimes x_3^2 + x_5 \otimes x_5^2$ in $H^*(E_7; \boldsymbol{Z}_2) \otimes H^*(E_7; \boldsymbol{Z}_2)$.*

From Lemma 5.3 (ii) and Theorem 2.4 (i) we have

LEMMA 5.4. *$Ad^*(x_{15}^2) = 0$ in $H^*(E_7; \boldsymbol{Z}_2) \otimes H^*(E_7; \boldsymbol{Z}_2)$.*

Further we can show the following four lemmas:

LEMMA 5.5. *$\iota^* x_{23} = x_{23} + x_3^2 x_{17} + x_9^2 x_5$ in $H^{23}(E_7; \boldsymbol{Z}_2)$.*

PROOF. By Theorem 2.4 (i) and Lemma 5.1, we have by setting $\mathrm{id} = \mathrm{id}_{E_7}$

$$\begin{aligned}\Delta^*(\mathrm{id} \times \iota)^* \mu^*(x_{23}) &= \Delta^*(\mathrm{id} \times \iota)^*(x_{23} \otimes 1 + 1 \otimes x_{23} + x_3^2 \otimes x_{17} + x_9^2 \otimes x_5) \\ &= \Delta^*(x_{23} \otimes 1 + 1 \otimes \iota^* x_{23} + x_3^2 \otimes x_{17} + x_9^2 \otimes x_5) \\ &= x_{23} + \iota^* x_{23} + x_3^2 x_{17} + x_9^2 x_5.\end{aligned}$$

By (B), we have $\iota^* x_{23} = x_{23} + x_3^2 x_{17} + x_9^2 x_5$. Q.E.D.

LEMMA 5.6. *$Ad^*(x_{23}) = 1 \otimes x_{23} + x_3^2 \otimes x_{17} + x_9^2 \otimes x_5 + x_{17} \otimes x_3^2 + x_5 \otimes x_9^2$ in $H^*(E_7; \boldsymbol{Z}_2) \otimes H^*(E_7; \boldsymbol{Z}_2)$.*

PROOF. By (A), Theorem 2.4 (i), Lemmas 5.1 and 5.5, we have by setting $\mathrm{id} = \mathrm{id}_{E_7}$

$$\begin{aligned}&Ad^*(x_{23}) \\ &= (\Delta \times \mathrm{id})^* \alpha^* (\mathrm{id} \times \mathrm{id} \times \iota)^* (\mu \times \mathrm{id})^* \mu^*(x_{23}) \\ &= (\Delta \times \mathrm{id})^* \alpha^* (\mathrm{id} \times \mathrm{id} \times \iota)^* (\mu \times \mathrm{id})^* (x_{23} \otimes 1 + 1 \otimes x_{23} + x_3^2 \otimes x_{17} + x_9^2 \otimes x_5) \\ &= (\Delta \times \mathrm{id})^* \alpha^* (\mathrm{id} \times \mathrm{id} \times \iota)^* \{(x_{23} \otimes 1 + 1 \otimes x_{23} + x_3^2 \otimes x_{17} + x_9^2 \otimes x_5) \otimes 1 \\ &\quad + 1 \otimes 1 \otimes x_{23} + (x_3^2 \otimes 1 + 1 \otimes x_3^2) \otimes x_{17} + (x_9^2 \otimes 1 + 1 \otimes x_9^2) \otimes x_5\} \\ &= (\Delta \times \mathrm{id})^* \alpha^* \{x_{23} \otimes 1 \otimes 1 + 1 \otimes x_{23} \otimes 1 + x_3^2 \otimes x_{17} \otimes 1 + x_9^2 \otimes x_5 \otimes 1 \\ &\quad + 1 \otimes 1 \otimes (x_{23} + x_3^2 x_{17} + x_9^2 x_5) + x_3^2 \otimes 1 \otimes x_{17} + 1 \otimes x_3^2 \otimes x_{17} \\ &\quad + x_9^2 \otimes 1 \otimes x_5 + 1 \otimes x_9^2 \otimes x_5\}\end{aligned}$$

$= x_{23} \otimes 1 + 1 \otimes x_{23} + x_3^2 \otimes x_{17} + x_9^2 \otimes x_5 + x_{23} \otimes 1 + x_3^2 x_{17} \otimes 1$

$+ x_9^2 x_5 \otimes 1 + x_3^2 x_{17} \otimes 1 + x_{17} \otimes x_3^2 + x_9^2 x_5 \otimes 1 + x_5 \otimes x_9^2$

$= 1 \otimes x_{23} + x_3^2 \otimes x_{17} + x_9^2 \otimes x_5 + x_{17} \otimes x_3^2 + x_5 \otimes x_9^2.$ Q.E.D.

LEMMA 5.7. *$\iota^* x_{27} = x_{27} + x_5^2 x_{17} + x_9^3$ in $H^{27}(E_7; \boldsymbol{Z}_2)$.*

PROOF. By Theorem 2.4 (i) and Lemma 5.1, we have by setting $\mathrm{id} = \mathrm{id}_{E_7}$

$$\begin{aligned} \Delta^*(\mathrm{id} \times \iota)^* \mu^*(x_{27}) &= \Delta^*(\mathrm{id} \times \iota)^*(x_{27} \otimes 1 + 1 \otimes x_{27} + x_5^2 \otimes x_{17} + x_9^2 \otimes x_9) \\ &= \Delta^*(x_{27} \otimes 1 + 1 \otimes \iota^* x_{27} + x_5^2 \otimes x_{17} + x_9^2 \otimes x_9) \\ &= x_{27} + \iota^* x_{27} + x_5^2 x_{17} + x_9^3. \end{aligned}$$

By (B), we have $\iota^* x_{27} = x_{27} + x_5^2 x_{17} + x_9^3$. Q.E.D.

LEMMA 5.8. *$Ad^*(x_{27}) = 1 \otimes x_{27} + x_5^2 \otimes x_{17} + x_9^2 \otimes x_9 + x_{17} \otimes x_5^2 + x_9 \otimes x_9^2$ in $H^*(E_7; \boldsymbol{Z}_2) \otimes H^*(E_7; \boldsymbol{Z}_2)$.*

PROOF. By (A), Theorem 2.4 (i), Lemmas 5.1 and 5.7, we have by setting $\mathrm{id} = \mathrm{id}_{E_7}$

$Ad^*(x_{27})$

$= (\Delta \times \mathrm{id})^* \alpha^* (\mathrm{id} \times \mathrm{id} \times \iota)^* (\mu \times \mathrm{id})^* \mu^*(x_{27})$

$= (\Delta \times \mathrm{id})^* \alpha^* (\mathrm{id} \times \mathrm{id} \times \iota)^* (\mu \times \mathrm{id})^* (x_{27} \otimes 1 + 1 \otimes x_{27} + x_5^2 \otimes x_{17} + x_9^2 \otimes x_9)$

$= (\Delta \times \mathrm{id})^* \alpha^* (\mathrm{id} \times \mathrm{id} \times \iota)^* \{(x_{27} \otimes 1 + 1 \otimes x_{27} + x_5^2 \otimes x_{17} + x_9^2 \otimes x_9) \otimes 1$

$+ 1 \otimes 1 \otimes x_{27} + (x_5^2 \otimes 1 + 1 \otimes x_5^2) \otimes x_{17} + (x_9^2 \otimes 1 + 1 \otimes x_9^2) \otimes x_9\}$

$= (\Delta \times \mathrm{id})^* \alpha^* \{x_{27} \otimes 1 \otimes 1 + 1 \otimes x_{27} \otimes 1 + x_5^2 \otimes x_{17} \otimes 1 + x_9^2 \otimes x_9 \otimes 1$

$+ 1 \otimes 1 \otimes (x_{27} + x_5^2 x_{17} + x_9^3) + x_5^2 \otimes 1 \otimes x_{17} + 1 \otimes x_5^2 \otimes x_{17}$

$+ x_9^2 \otimes 1 \otimes x_9 + 1 \otimes x_9^2 \otimes x_9\}$

$= x_{27} \otimes 1 + 1 \otimes x_{27} + x_5^2 \otimes x_{17} + x_9^2 \otimes x_9 + x_{27} \otimes 1 + x_5^2 x_{17} \otimes 1 + x_9^3 \otimes 1$

$+ x_5^2 x_{17} \otimes 1 + x_{17} \otimes x_5^2 + x_9^3 \otimes 1 + x_9 \otimes x_9^2$

$= 1 \otimes x_{27} + x_5^2 \otimes x_{17} + x_9^2 \otimes x_9 + x_{17} \otimes x_5^2 + x_9 \otimes x_9^2.$ Q.E.D.

THEOREM 5.9. *Let H be $SU(3)$, G_2, $Spin(7)$, $Spin(8)$, $Spin(9)$ or E_6. Then $H \subset E_7$ of (*) is not homotopy normal in the sense of 2.3, hence also not in the sense of both McCarty and James.*

PROOF. The homotopy normality of H in E_7 in the sense of 2.3 would imply $Ad \circ k \simeq j \circ \nu_1$, where $j: H \to E_7$ is a natural inclusion and $k = \mathrm{id} \times j: E_7 \times H \to E_7 \times E_7$.

First, when $H = E_6$, we have $j^* x_{27} = 0$ for the natural inclusion

$j: E_6 \to E_7$ (cf. [10]). On the other hand, we have $k^* Ad(x_{27}) = x_5^2 \otimes x_{17} + x_9^2 \otimes x_9 \neq 0 \in H^*(E_7; \boldsymbol{Z}_2) \otimes H^*(E_6; \boldsymbol{Z}_2)$ by Theorems 2.4 (i), 4.1 (i) and Lemma 5.8, because $j^* x_i = x_i$ for $i = 3, 5, 9$ and 17 (cf. [14]). This is a contradiction.

Next, in the cases $H = Spin(9)$, $Spin(8)$ and $Spin(7)$, $j^* x_{23} = j^* Sq^8 x_{15} = Sq^8 j^* x_{15}$ hold. We show this is zero. For the case $H = Spin(9)$, one can write $j^* x_{15} = a x_3 x_5 x_7 + b x'_{15} \in H^{15}(Spin(9); \boldsymbol{Z}_2)$ with $a, b \in \boldsymbol{Z}_2$. Using the Cartan formula and Theorem 12.1 (c) of Borel [5], one obtains $Sq^8(x_3 x_5 x_7) = 0$. Also $Sq^8 x'_{15} = 0$ by Borel [5, §12]. Hence $j^* x_{23} = 0$. On the other hand, we have $k^* Ad^*(x_{23}) = x_9^2 \otimes x_5 + x_{17} \otimes x_3^2 \neq 0 \in H^*(E_7; \boldsymbol{Z}_2) \otimes H^*(Spin(9); \boldsymbol{Z}_2)$ by Theorem 4.2 (i) and Lemma 5.6, because $j^* x_i = x_i$ for $i = 3, 5$ (cf. [2, 11, 14]), $j^* x_9 = j^* Sq^4 x_5 = Sq^4 j^* x_5 = Sq^4 x_5 = 0$ (cf. [5, §12]) and $j^* x_{17} = 0$. This is a contradiction. For the case $H = Spin(8)$, one can write $j^* x_{15} = a x_3 x_5 x_7 + b x_3 x_5 x'_7 \in H^{15}(Spin(8); \boldsymbol{Z}_2)$ with $a, b \in \boldsymbol{Z}_2$. Since $Sq^8(x_3 x_5 x_7) = 0$ and $Sq^8(x_3 x_5 x'_7) = 0$ hold by the Cartan formula and [5, §12], we have $j^* x_{23} = 0$. For the case $H = Spin(7)$, one can write $j^* x_{15} = a x_3 x_5 x'_7 \in H^{15}(Spin(7); \boldsymbol{Z}_2)$ with $a \in \boldsymbol{Z}_2$. By $Sq^8(x_3 x_5 x'_7) = 0$, $j^* x_{23} = 0$ holds. This contradicts $k^* Ad^*(x_{23}) \neq 0 \in H^*(E_7; \boldsymbol{Z}_2) \otimes H^*(H; \boldsymbol{Z}_2)$ for $H = Spin(7)$ and $Spin(8)$ by a similar discussion as in the proof of $Spin(9)$.

The last cases are $SU(3)$ and G_2. For the natural inclusions $SU(3) \to G_2 \to E_7$, we have $j^* x_{15} = 0$ by the degree reason. On the other hand, we have $k^* Ad^*(x_{15}) = x_5^2 \otimes x_5$ (resp. $x_5^2 \otimes x_5 + x_9 \otimes x_3^2) \neq 0 \in H^*(E_7; \boldsymbol{Z}_2) \otimes H^*(H; \boldsymbol{Z}_2)$ if $H = SU(3)$ (resp. G_2) by Theorems 4.1 (iii), 4.2 and Lemma 5.3 (ii), because $j^* x_i = x_i$ for $i = 3, 5$, and $j^* x_9 = 0$ by the degree reason when $H = SU(3)$ and by [10, Remark 5.6] when $H = G_2$. This is a contradiction. Q.E.D.

6. E_6 and its subgroups

Since $x_i \in H^i(E_6; \boldsymbol{Z}_2)$ ($i = 3, 5, 9, 17$) are primitive (cf. Theorem 4.1 (i)), we have

LEMMA 6.1. *$\iota^* x_i = x_i$ in $H^i(E_6; \boldsymbol{Z}_2)$ for $i = 3, 5, 9, 17$.*

By Theorem 4.1 (i), Lemmas 5.3 (i) and 5.5, we have

LEMMA 6.2. *$\iota^* x_{15} = x_{15} + x_3^2 x_9$ in $H^{15}(E_6; \boldsymbol{Z}_2)$ and $\iota^* x_{23} = x_{23} + x_3^2 x_{17}$ in $H^{23}(E_6; \boldsymbol{Z}_2)$.*

From Theorem 4.1 (i), Lemmas 6.1, 5.3 (ii) and 5.6, we have

LEMMA 6.3. (i) *$Ad^*(x_i) = 1 \otimes x_i$ for $i = 3, 5, 9, 17$,*

(ii) *$Ad^*(x_{15}) = 1 \otimes x_{15} + x_3^2 \otimes x_9 + x_9 \otimes x_3^2$,*

(iii) *$Ad^*(x_{23}) = 1 \otimes x_{23} + x_3^2 \otimes x_{17} + x_{17} \otimes x_3^2$ in $H^*(E_6; \boldsymbol{Z}_2) \otimes H^*(E_6; \boldsymbol{Z}_2)$.*

THEOREM 6.4. *Let H be G_2, $Spin(7)$, $Spin(8)$ or $Spin(9)$. Then $H \subset E_6$ of (*) is not homotopy normal in the sense of* 2.3, *hence also not in the sense* *of both McCarty and James.*

PROOF. First, we consider the group G_2. For the natural inclusion $j: G_2 \to E_6$, $j^* x_{15} = 0$ holds by the degree reason. On the other hand, we have $k^* Ad^*(x_{15}) = x_9 \otimes x_3^2 \neq 0 \in H^9(E_6; \boldsymbol{Z}_2) \otimes H^6(G_2; \boldsymbol{Z}_2)$ by Theorem 4.1 (iii) and Lemma 6.3 (ii), because $j^* x_3 = x_3$ and $j^* x_9 = 0$ (see [10, Remark 5.6]). This is a contradiction.

Next, we consider the groups $H = Spin(9)$, $Spin(8)$ and $Spin(7)$. In these cases $j^* x_{23} = j^* Sq^8 x_{15} = Sq^8 j^* x_{15}$ hold. For the case $H = Spin(9)$, one can write $j^* x_{15} = a x_3 x_5 x_7 + b x'_{15}$ with $a, b \in \boldsymbol{Z}_2$. Using the Cartan formula and Theorem 12.1 (c) of [5], one obtains $Sq^8(x_3 x_5 x_7) = 0$. Also $Sq^8 x'_{15} = 0$ by Borel [5, §12]. Thus $j^* x_{23} = 0$. On the other hand, we have $k^* Ad^*(x_{23}) = x_{17} \otimes x_3^2 \neq 0 \in H^{17}(E_6; \boldsymbol{Z}_2) \otimes H^6(Spin(9); \boldsymbol{Z}_2)$ by Theorem 4.2 (i) and Lemma 6.3 (iii), because $j^* x_3 = x_3$ by [2, 11] and $j^* x_{17} = 0$ by the degree reason. This is a contradiction. For the case $H = Spin(8)$, one can write $j^* x_{15} = a x_3 x_5 x_7 + b x_3 x_5 x'_7$ with $a, b \in \boldsymbol{Z}_2$. Since $Sq^8(x_3 x_5 x_7) = 0$ and $Sq^8(x_3 x_5 x'_7) = 0$ hold by the Cartan formula and Theorem 12.1 of [5], one obtains $j^* x_{23} = 0$. For the case $H = Spin(7)$, one can write $j^* x_{15} = a x_3 x_5 x'_7$ with $a \in \boldsymbol{Z}_2$. Since $Sq^8(x_3 x_5 x'_7) = 0$, $j^* x_{23} = 0$ holds. Thus, by the same discussion as in the case of $Spin(9)$ we have $k^* Ad^*(x_{23}) \neq 0 \in H^*(E_6; \boldsymbol{Z}_2) \otimes H^*(H; \boldsymbol{Z}_2)$ for $H = Spin(7)$ and $Spin(8)$. Q.E.D.

7. The cases $(G, H) = (E_7, F_4)$, (E_6, F_4), $(E_6, SU(3))$, $(F_4, Spin(8))$, $(F_4, Spin(7))$ and $(F_4, SU(3))$

In this section we determine the homotopy normality of the subgroups of $E_i (i = 6, 7)$ which are not determined in Theorems 5.9 and 6.4, by the mod 3 cohomology algebra of the exceptional Lie groups. We also prove the cases (F_4, H) for $H = Spin(8)$, $Spin(7)$ *and* $SU(3)$.

We recall the following from [1, 5, 12, 13, 19]:

THEOREM 7.1. *As an algebra*

(i) $H^*(E_8; \boldsymbol{Z}_3) = \boldsymbol{Z}_3[x_8, x_{20}]/(x_8^3, x_{20}^3)$

$$\otimes \Lambda(x_3, x_7, x_{15}, x_{19}, x_{27}, x_{35}, x_{39}, x_{47})$$

where deg $x_i = i$ *and the generators are related by* $\beta x_7 = x_8$, $\beta x_{15} = -x_8^2$, $\beta x_{19} = x_{20}$, $\beta x_{27} = x_{20} x_8$, $\beta x_{35} = -x_{20} x_8^2$, $\beta x_{39} = -x_{20}^2$, $\beta x_{47} = x_{20}^2 x_8$, $\beta x_i = 0$ *for* $i = 3, 8, 20$; $\mathscr{P}^1 x_3 = x_7$, $\mathscr{P}^1 x_{15} = \varepsilon x_{19}$ *with* $\varepsilon = \pm 1$, $\mathscr{P}^1 x_{35} = \varepsilon x_{39}$ *with* $\varepsilon = \pm 1$, $\mathscr{P}^1 x_i = 0$ *for* $i = 7, 8, 19, 20, 27, 39, 47$; $\mathscr{P}^3 x_7 = x_{19}$, $\mathscr{P}^3 x_8 = x_{20}$, $\mathscr{P}^3 x_{15} = x_{27}$, $\mathscr{P}^3 x_{27} = -x_{39}$, $\mathscr{P}^3 x_{35} = x_{47}$, $\mathscr{P}^3 x_i = 0$ *for* $i = 3, 19, 20, 39, 47$; $\mathscr{P}^6 x_{15} = x_{39}$, $\mathscr{P}^6 x_i = 0$ *for* $i \neq 15$; $\mathscr{P}^j x_i = 0$ *for any other* $j > 0$. *The coalgebra structure is given by*

$$\bar{\phi}(x_i) = 0 \quad for \ i = 3, 7, 8, 19, 20,$$
$$\bar{\phi}(x_{15}) = x_8 \otimes x_7,$$
$$\bar{\phi}(x_{27}) = x_8 \otimes x_{19} + x_{20} \otimes x_7,$$
$$\bar{\phi}(x_{35}) = x_8 \otimes x_{27} - x_8^2 \otimes x_{19} + x_{20} \otimes x_{15} + x_{20}x_8 \otimes x_7,$$
$$\bar{\phi}(x_{39}) = x_{20} \otimes x_{19},$$
$$\bar{\phi}(x_{47}) = - x_8 \otimes x_{39} - x_{20} \otimes x_{27} - x_{20}x_8 \otimes x_{19} + x_{20}^2 \otimes x_7.$$

(ii) $H^*(E_7; \boldsymbol{Z}_3) = \boldsymbol{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35})$

where deg $x_i = i$ *and the generators are related by* $\beta x_7 = x_8$, $\beta x_{15} = -x_8^2$, $\beta x_i = 0$ *for* $i \neq 7, 15$; $\mathscr{P}^1 x_3 = x_7$, $\mathscr{P}^1 x_{11} = x_{15}$, $\mathscr{P}^1 x_{15} = \varepsilon x_{19}$ *with* $\varepsilon = \pm 1$, $\mathscr{P}^1 x_i = 0$ *for* $i \neq 3, 11, 15$; $\mathscr{P}^2 x_{11} = -\varepsilon x_{19}$ *with* $\varepsilon = \pm 1$, $\mathscr{P}^2 x_i = 0$ *for* $i \neq 11$; $\mathscr{P}^3 x_7 = x_{19}$, $\mathscr{P}^3 x_{15} = x_{27}$, $\mathscr{P}^3 x_i = 0$ *for* $i \neq 7, 15$; $\mathscr{P}^j x_i = 0$ *for any other* $j > 0$. *The coalgebra structure is given by*

$$\bar{\phi}(x_i) = 0 \quad for \ i = 3, 7, 8, 19,$$
$$\bar{\phi}(x_j) = x_8 \otimes x_{j-8} \quad for \ j = 11, 15, 27,$$
$$\bar{\phi}(x_{35}) = x_8 \otimes x_{27} - x_8^2 \otimes x_{19}.$$

(iii) $H^*(E_6; \boldsymbol{Z}_3) = \boldsymbol{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_9, x_{11}, x_{15}, x_{17})$

where deg $x_i = i$ *and the generators are related by* $\beta x_7 = x_8$, $\beta x_{15} = -x_8^2$, $\beta x_i = 0$ *for* $i \neq 7, 15$; $\mathscr{P}^1 x_3 = x_7$, $\mathscr{P}^1 x_{11} = x_{15}$, $\mathscr{P}^1 x_i = 0$ *for* $i \neq 3, 11$; $\mathscr{P}^j x_i = 0$ *for any other* $j > 1$. *The coalgebra structure is given by*

$$\bar{\phi}(x_i) = 0 \quad for \ i = 3, 7, 8, 9,$$
$$\bar{\phi}(x_j) = x_8 \otimes x_{j-8} \quad for \ j = 11, 15, 17.$$

(iv) $H^*(F_4; \boldsymbol{Z}_3) = \boldsymbol{Z}_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15})$

where deg $x_i = i$ *and the generators are related by* $\beta x_7 = x_8$, $\mathscr{P}^1 x_3 = x_7$, $\mathscr{P}^1 x_{11} = x_{15}$. *The coalgebra structuere is given by*

$$\bar{\phi}(x_i) = 0 \quad for \ i = 3, 7, 8,$$
$$\bar{\phi}(x_j) = x_8 \otimes x_{j-8} \quad for \ j = 11, 15.$$

(v) $H^*(G_2; \boldsymbol{Z}_3) = \Lambda(x_3, x_{11})$.

Further we recall the following from [5, 16]:

THEOREM 7.2. *As an algebra*

(i) $H^*(Spin(9); \boldsymbol{Z}_3) = \Lambda(x_3, x_7, x_{11}, x_{15})$

where deg $x_i = i$ *and the generators are related by* $\mathscr{P}^1 x_3 = x_7$, $\mathscr{P}^1 x_{11} = x_{15}$.

(ii) $H^*(Spin(8); \boldsymbol{Z}_3) = \Lambda(x_3, x_7, x_{11}, x_7')$,
(iii) $H^*(Spin(7); \boldsymbol{Z}_3) = \Lambda(x_3, x_7, x_{11})$,
(iv) $H^*(SU(n); \boldsymbol{Z}_3) = \Lambda(x_3, x_5, \dots, x_{2n-1})$.

By a similar argument as in the mod 2 case, we have

LEMMA 7.3. (i) $\iota^* x_{19} = -x_{19}$ *in* $H^{19}(E_7; \boldsymbol{Z}_3)$,

(ii) $\iota^* x_{27} = -x_{27} + x_8 x_{19}$ *in* $H^{27}(E_7; \boldsymbol{Z}_3)$.

From Theorem 7.1 (ii) and Lemma 7.3, we have

LEMMA 7.4. $Ad^*(x_{27}) = 1 \otimes x_{27} + x_8 \otimes x_{19} - x_{19} \otimes x_8$ *in* $H^*(E_7; \boldsymbol{Z}_3) \otimes H^*(E_7; \boldsymbol{Z}_3)$.

From these Lemmas, we can prove the following.

THEOREM 7.5. *The inclusion* $F_4 \subset E_7$ *of* (*) *is not homotopy normal in the sense of* 2.3, *hence also not in the sense of both McCarty and James.*

PROOF. For the inclusions $F_4 \xrightarrow{f} E_6 \xrightarrow{g} E_7$, $g^* x_{15} = x_{15}$ and $f^* x_{15} = x_{15}$ hold (see [1]). Therefore we have $j^* x_{27} = j^* \mathscr{P}^3 x_{15} = f^* g^* \mathscr{P}^3 x_{15} = f^* \mathscr{P}^3 g^* x_{15} = f^* \mathscr{P}^3 x_{15} = 0$, since $\mathscr{P}^3 x_{15} = 0$ in $H^{27}(E_6; \boldsymbol{Z}_3)$ (see Theorem 7.1 (ii) and (iii)). On the other hand, we have $k^* Ad^*(x_{27}) = -x_{19} \otimes x_8 \neq 0 \in H^{19}(E_7; \boldsymbol{Z}_3) \otimes H^8(F_4; \boldsymbol{Z}_3)$ by Theorem 7.1 (ii), (iv) and Lemma 7.4, because $j^* x_8 = x_8$ (cf. [1]) and $j^* x_{19} = j^* \varepsilon \mathscr{P}^1 x_{15} = \varepsilon f^* \mathscr{P}^1 g^* x_{15} = \varepsilon f^* \mathscr{P}^1 x_{15} = 0$ for $\varepsilon = \pm 1$, since $\mathscr{P}^1 x_{15} = 0$ in $H^{19}(E_6; \boldsymbol{Z}_3)$ (see Theorem 7.1 (ii) and (iii)). This is a contradiction. Q.E.D.

Similarly we obtain

LEMMA 7.6. (i) $\iota^* x_i = -x_i$ *in* $H^i(E_6; \boldsymbol{Z}_3)$ *for* $i = 3, 9$,
(ii) $\iota^* x_{11} = -x_{11} + x_8 x_3$ *in* $H^{11}(E_6; \boldsymbol{Z}_3)$,
(iii) $\iota^* x_{17} = -x_{17} + x_8 x_9$ *in* $H^{17}(E_6; \boldsymbol{Z}_3)$.

From Theorem 7.1 (iii) and Lemma 7.6, we have

LEMMA 7.7. (i) $Ad^*(x_{11}) = 1 \otimes x_{11} + x_8 \otimes x_3 - x_3 \otimes x_8$,
(ii) $Ad^*(x_{17}) = 1 \otimes x_{17} + x_8 \otimes x_9 - x_9 \otimes x_8$ *in* $H^*(E_6; \boldsymbol{Z}_3) \otimes H^*(E_6; \boldsymbol{Z}_3)$.

THEOREM 7.8. *The subgroups* $SU(3)$ *and* F_4 *of* E_6 *in* (*) *are not homotopy normal in the sense of* 2.3, *hence also not in the sense of both McCarty and James.*

PROOF. First, we consider the group F_4. For the inclusion $j: F_4 \to E_6$, $j^* x_{17} = 0$ holds by the degree reason. On the other hand, we have $k^* Ad^*(x_{17}) = -x_9 \otimes x_8 \neq 0 \in H^9(E_6; \boldsymbol{Z}_3) \otimes H^8(F_4; \boldsymbol{Z}_3)$ by Theorem 7.1 (iv) and Lemma 7.7 (ii), because $j^* x_8 = x_8$ by [1, p. 255] and $j^* x_9 = 0$ by the degree reason.

This is a contradiction.
Next, we put $H=SU(3)$. For the natural inclusion $j: H \rightarrow E_6, j^* x_{11}=0$ holds by the degree reason. On the other hand, we have $k^* Ad^*(x_{11})=x_8 \otimes x_3 \neq 0 \in H^8(E_6; \boldsymbol{Z}_3) \otimes H^3(H; \boldsymbol{Z}_3)$ by Theorem 7.2 (iv) and Lemma 7.7 (i), because $j^* x_3=x_3$ and $j^* x_8=0$ (cf. [1, 5]). This is a contradiction. Q.E.D.

Quite similarly we have

LEMMA 7.9. (i) $\iota^* x_i=-x_i$ *in* $H^i(F_4; \boldsymbol{Z}_3)$ *for* $i=3, 7$,
(ii) $\iota^* x_{11}=-x_{11}+x_8 x_3$ *in* $H^{11}(F_4; \boldsymbol{Z}_3)$,
(iii) $\iota^* x_{15}=-x_{15}+x_8 x_7$ *in* $H^{15}(F_4; \boldsymbol{Z}_3)$.

From Theorem 7.1 (iv) and Lemma 7.9, we have

LEMMA 7.10. (i) $Ad^*(x_{11})=1 \otimes x_{11}+x_8 \otimes x_3-x_3 \otimes x_8$,
(ii) $Ad^*(x_{15})=1 \otimes x_{15}+x_8 \otimes x_7-x_7 \otimes x_8$ *in* $H^*(F_4; \boldsymbol{Z}_3) \otimes H^*(F_4; \boldsymbol{Z}_3)$.

THEOREM 7.11. *The subgroups* $SU(3)$, $Spin(7)$ *and* $Spin(8)$ *of* F_4 *in* $(*)$ *are not homotopy-normal in the sense of* 2.3, *hence also not in the sense of both McCarty and James.*

PROOF. First, when $H=SU(3)$, we have $j^* x_{11}=0$ for the natural inclusion $j: H \rightarrow F_4$. On the other hand, we have $k^* Ad^*(x_{11})=x_8 \otimes x_3 \neq 0 \in H^8(F_4; \boldsymbol{Z}_3) \otimes H^3(H; \boldsymbol{Z}_3)$ by Lemma 7.10 (i), because $j^* x_3=x_3$ (cf. [5, §21]) and $j^* x_8=j^* \beta \mathscr{P}^1 x_3=\beta \mathscr{P}^1 j^* x_3=\beta \mathscr{P}^1 x_3=0$, since $H^7(H; \boldsymbol{Z}_3)=0$. This is a contradiction.
Next, when $H=Spin(7)$ or $Spin(8)$, we have $j^* x_{15}=0$ for the natural inclusion $j: H \rightarrow F_4$ by Theorem 7.2 (ii) and (iii). On the other hand, we have $k^* Ad^*(x_{15})=x_8 \otimes x_7 \neq 0 \in H^8(F_4; \boldsymbol{Z}_3) \otimes H^7(H; \boldsymbol{Z}_3)$ by Lemma 7.10 (ii), because $j^* x_7=x_7$ by [5] and $j^* x_8=0$ by the degree reason. This is a contradiction.
Q.E.D.

REMARK. If a cohomology algebra $H^*(G; \boldsymbol{Z}_p)$ is primitively generated as in the case of the classical group G and its subgroups, we cannot expect to obtain any informations about the homotopy-normality of the subgroups of G by the above methods.

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