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Existence theorems for a neutral functional differential equation whose leading part contains a difference operator of higher degree

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0. Introduction

In this paper we are concerned with the problem of existence of solutions for a neutral functional differential equation of the form

$$D^n \Delta_\lambda^m x(t) + f(t, x(g(t))) = 0\,, \qquad t \ge t_0\,, \tag{A}$$

where D^n and Δ_λ^m stand, respectively, for the n-th iterate of the differential operator D and the m-th iterate of the difference operator Δ_λ defined by

$$Dx(t) = \frac{d}{dt}x(t) \quad \text{and} \quad \Delta_\lambda x(t) = x(t) - \lambda x(t-\tau)\,. \tag{0.1}$$

In case $\lambda = 1$ use is made of the symbol Δ instead of Δ_1, i.e.,

$$\Delta x(t) = x(t) - x(t-\tau)\,. \tag{0.2}$$

The conditions always assumed for (A) are as follows:

(0.3) (a) $m \ge 1$, $n \ge 1$, $\lambda > 0$, $\tau > 0$ and $t_0 > 0$;
(b) $g \in C[t_0, \infty)$, and $\lim_{t\to\infty} g(t) = \infty$;
(c) $f \in C([t_0, \infty) \times \boldsymbol{R})$, and

$$|f(t, x)| \le F(t, |x|)\,, \qquad (t, x) \in [t_0, \infty) \times \boldsymbol{R}\,,$$

for some continuous function $F(t, u)$ on $[t_0, \infty) \times \boldsymbol{R}_+$, $\boldsymbol{R}_+ = [0, \infty)$, which is nondecreasing in u for each fixed $t \ge t_0$.

By a solution of (A) we mean a function $x \in C[T_x - m\tau, \infty)$ for some $T_x \ge t_0 + m\tau$ such that $\Delta_\lambda^m x(t)$ is n-times continuously differentiable and satisfies the equation on $[T_x, \infty)$. A solution of (A) is said to be oscillatory if it has an infinite sequence of zeros clustering at $t = \infty$; otherwise a solution is said to be nonoscillatory.

We observe that the associated unperturbed equation $D^n \Delta_\lambda^m x(t) = 0$ has the solutions

(0.4) $\omega(t)t^j,\ j = 0,\ 1,\ \dots,\ m-1;\qquad ct^k,\ k = m,\ m+1,\ \dots,\ m+n-1$

in the case $\lambda = 1$, and

(0.5) $\omega(t)t^j\lambda^{t/\tau},\ j = 0,\ 1,\ \dots,\ m-1;\qquad ct^k,\ k = 0,\ 1,\ \dots,\ n-1$

in the case $\lambda \neq 1$, where $\omega(t)$ is an arbitrary continuous τ-periodic function and c is an arbitrary constant. It is then natural to expect the occurrence of a situation in which the nonlinear term $f(t, x(g(t)))$ is so "small" that the equation (A) admits those solutions which behave like the functions in (0.4) or (0.5) at $t \to \infty$. Our objective here is to show that this expectation can actually be realized by establishing sufficient conditions under which, given any continuous τ-periodic function $\omega(t) \not\equiv 0$ and any nonzero constant c, (A) possesses solutions which are asymptotic to the functions (0.4) and (0.5) in the sense that

(I) $x(t) = t^j[\omega(t) + o(1)]$ as $t \to \infty$, $j = 0, 1, \dots, m-1$,

(II) $x(t) = t^k[c + o(1)]$ as $t \to \infty$, $k = m, m+1, \dots, m+n-1$,

in the case $\lambda = 1$, and

(I_λ) $x(t) = t^j\lambda^{t/\tau}[\omega(t) + o(1)]$ as $t \to \infty$, $j = 0, 1, \dots, m-1$,

(II_λ) $x(t) = t^k[c + o(1)]$ as $t \to \infty$, $k = 0, 1, \dots, n-1$,

in the case $\lambda \neq 1$. Note that, because of the arbitrariness of $\omega(t)$, the set of solutions of type (I) [or (I_λ)] contains both oscillatory and nonoscillatory solutions. Our results regarding type (I)-solutions [or type (I_λ)-solutions], therefore, establish the coexistence of oscillatory and nonoscillatory solutions for the neutral functional differential equation (A). No such coexistence result seems to be known for non-neutral equations of the form $D^n x(t) + f(t, x(g(t))) = 0$. It is obvious that the solutions of type (II) and (II_λ) are all nonoscillatory.

The construction of these types of solutions of (A) will be presented in Part 1 ($\lambda = 1$) and Part 2 ($\lambda \neq 1$). Our main tool is the fixed point theorem of Shauder-Tychonoff applied to nonlinear (functional) integral operators formed by suitably chosen "inverses" of Δ^m with Δ given by (0.2). In verifying the applicability of the fixed point theorem, a crucial role is played by some basic properties of the "inverses" of Δ^m, which will not be collected in one place but will be stated with proofs in several sections where they become necessary.

Qualitative theory of neutral functional differential equations has received wide attention in recent years because of its importance in various theoretical and practical problems. Needless to say, existence theory of solutions is a fundamental question to be investigated in depth for neutral equations. It seems, however, that the existence results obtained so far have been primarily concerned with the special case $m = 1$ of (A) (see e.g. the papers [2–8]) and nothing is known, except for [1], about the existence of solutions, oscillatory or nonoscillatory, for the case $m \geq 2$ of (A), that is, for neutral equations whose leading parts contain difference operators of higher degree applied to the unknown function. Motivated by this observation, the present work is so designed as to cover the equation (A) with $m \geq 1$, thereby extending and unifying all the basic results given in the above references.

Part 1. Existence of Solutions for the case $\lambda = 1$

1. Statement of Existence Theorems

We begin by considering the equation (A) with $\lambda = 1$, i.e.,

$$D^n \Delta^m x(t) + f(t, x(g(t))) = 0\,, \qquad t \geq t_0\,,$$

Δ being defined by (0.2), for which the conditions (0.3) are assumed to hold. The main existence theorems for this equation are as follows.

THEOREM I. *Let $j \in \{0, 1, \ldots, m-1\}$ and suppose that there is a constant $a > 0$ such that*

$$\int_{t_0}^{\infty} t^{m+n-j-1} F(t, a[g(t)]^j)\,dt < \infty\,. \tag{1.1}$$

Then, for any continuous τ-periodic function $\omega(t)$ such that $\max_t |\omega(t)| < a$, the equation (A) *with $\lambda = 1$ possesses a solution $x(t)$ with the property that*

$$x(t) = t^j[\omega(t) + o(1)] \qquad \text{as } t \to \infty\,. \tag{1.2}$$

THEOREM II. *Let $k \in \{m, m+1, \ldots, m+n-1\}$ and suppose that there is a constant $a > 0$ such that*

$$\int_{t_0}^{\infty} t^{m+n-k-1} F(t, a[g(t)]^k)\,dt < \infty\,. \tag{1.3}$$

Then, for any constant c such that $0 < |c| < a$, the equation (A) *with $\lambda = 1$ possesses a solution $x(t)$ with the property that*

$$x(t) = t^k[c + o(1)] \qquad \text{as } t \to \infty\,. \tag{1.4}$$

REMARK 1.1. The solution obtained in Theorem II is nonoscillatory, whereas the one obtained in Theorem I is oscillatory or nonscillatory according to whether the periodic function $\omega(t)$ involved is oscillatory or nonoscillatory. It is to be noted that the condition (1.1) which is independent of $\omega(t)$

guarantees the existence of both oscillatory and nonoscillatory solutions of (A). Thus one can easily speak of the phenomenon of coexistence of oscillatory and nonoscillatory solutions for neutral equations of the type (A). This is an aspect which is not shared by non-neutral equations of the form $D^n x(t) + f(t, x(g(t))) = 0$. Naturally a situation may occur in which the equation (A) with $\lambda = 1$ has all types of solutions listed in Theorems I and II.

EXAMPLE 1.1. For illustration of Theorems I and II we consider the equation

$$D^n[x(t) - 2x(t-1) + x(t-2)] + q(t)|x(t-3)|^\gamma \operatorname{sgn} x(t-3) = 0\,, \quad t \ge t_0\,, \tag{1.5}$$

where $\gamma > 0$, $t_0 > 3$, and $q: [t_0, \infty) \to \mathbf{R}$ is continuous. Since $\Delta^2 x(t) = x(t) - 2x(t-1) + x(t-2)$, this equation is a special case of (A) ($\lambda = 1$) in which $m = 2$, $\tau = 1$, $g(t) = t - 3$, and $f(t, x) = q(t)|x|^\gamma \operatorname{sgn} x$. Clearly the condition (0.3) is satisfied for (1.5) with $F(t, u) = |q(t)|u^\gamma$. The conditions (1.1) and (1.3) written for (1.5) reduce, respectively, to

$$\int_{t_0}^\infty t^{n+1+(\gamma-1)j}|q(t)|\,dt < \infty\,, \qquad j \in \{0, 1\} \tag{1.6}$$

and

$$\int_{t_0}^\infty t^{n+1+(\gamma-1)k}|q(t)|\,dt < \infty\,, \qquad k \in \{2, 3, \dots, n+1\}\,, \tag{1.7}$$

which are sufficient for the existence of solutions of (1.5) having the asymptotic behaviors (1.2) and (1.3), respectively.

If in particular

$$\int_{t_0}^\infty t^{n+1}|q(t)|\,dt < \infty \qquad \text{for the case } \gamma \le 1\,,$$

$$\int_{t_0}^\infty t^{\gamma(n+1)}|q(t)|\,dt < \infty \qquad \text{for the case } \gamma > 1\,,$$

then (1.5) possesses solutions of the type (I)

$$x_0(t) = \omega(t) + o(1),\ x_1(t) = t[\omega(t) + o(1)] \text{ as } t \to \infty$$

as well as solutions of the type (II)

$$x_2(t) = t^2[c + o(1)],\ x_3(t) = t^3[c + o(1)], \dots, x_{n+1}(t) = t^{n+1}[c + o(1)] \text{ as } t \to \infty$$

for any nonzero constant c and any continuous periodic function $\omega(t)$ of period 1. Typical examples of such $\omega(t)$ are $\cos 2l\pi t$, $\sin 2l\pi t$, $l = 0, 1, 2, \dots$.

2. Proof of Theorem I (The case $j = 0$)

A) LEMMAS. The purpose of this section is to give a proof of Theorem I for the case $j = 0$. To this end we require some basic results regarding a type of "inverse" of the difference operator Δ and its iterates.

We denote the $S[T, \infty)$ the set of all functions $\xi \in C[T, \infty)$ such that the sequence

$$\eta(t) = \sum_{i=1}^{\infty} \xi(t + i\tau), \qquad t \geq T - \tau, \tag{2.1}$$

converges uniformly on compact subintervals of $[T - \tau, \infty)$. We define Ψ to be the mapping which assigns to each $\xi \in S[T, \infty)$ a function $\eta(t)$ defined by (2.1). Further, for $l \in N$ we denote by Ψ^l the l-th iterate of Ψ which is defined on the set

$$S^l[T, \infty) = \{\xi \in S^{l-1}[T, \infty) : \Psi^{l-1}\xi \in S[T - (l-1)\tau, \infty)\}, \qquad l = 1, 2, \ldots,$$

where it is understood that $\Psi^0 = id$ (identity mapping) and $S^0[T, \infty) = C[T, \infty)$.

LEMMA 2.1. *Let $l \in N$. If $\xi \in S^l[T, \infty)$, then $\Psi^l\xi$ is a solution of the difference equation*

$$\Delta^l x(t) = (-1)^l \xi(t), \qquad t \geq T, \tag{2.2}$$

and satisfies

$$\Psi^l \xi(t) = o(1) \qquad \text{as } t \to \infty. \tag{2.3}$$

PROOF. Let $l = 1$ and $\xi \in S[T, \infty)$. That $\Psi\xi$ solves the difference equation $\Delta x(t) = -\xi(t)$, $t \geq T$, follows immediately from the definition of Ψ, so that (2.2) holds for $l = 1$. Let $\varepsilon > 0$ be given arbitrarily. Since (2.1) converges uniformly on $[T - \tau, T)$ by hypothesis, there exists $P \in N$ such that

$$\left| \sum_{i=p+1}^{\infty} \xi(t + i\tau) \right| < \varepsilon \qquad \text{for all } t \in [T - \tau, T) \text{ and } p \geq P. \tag{2.4}$$

Let $t \geq t_1 = T + P\tau$ and choose $p \in N$ so that $t - p\tau \in [T - \tau, T)$. Then

$$p > \frac{t - T}{\tau} \geq \frac{t_1 - T}{\tau} = P$$

and we have in view of (2.4)

$$|\Psi\xi(t)| = \left| \sum_{i=1}^{\infty} \xi(t + i\tau) \right| = \left| \sum_{i=1}^{\infty} \xi(t - p\tau + (i + p)\tau) \right|$$

$$= \left| \sum_{i=p+1}^{\infty} \xi(t - p\tau + i\tau) \right| < \varepsilon,$$

which shows that $\Psi\xi(t) \to 0$ as $t \to \infty$. Thus (2.3) holds for $l = 1$. This proves the lemma for the case $l = 1$. The proof for a general $l \geq 1$ is done by an inductive argument.

LEMMA 2.2. *Let $l \in N$. If $\xi \in S^l[T, \infty)$ and $\xi(t) \geq 0$ for $t \geq T$, then*

$$\Psi^l\xi(t) = \sum_{i=l}^{\infty} \binom{i-1}{l-1} \xi(t + i\tau), \qquad t \geq T - l\tau. \tag{2.5}$$

PROOF. Assume that (2.5) holds for some $l \in N$. Let $\xi \in S^{l+1}[T, \infty)$ be nonnegative for $t \geq T$. From the definition of Ψ we then see that

$$\begin{aligned}\Psi^{l+1}\xi(t) &= \sum_{i=1}^{\infty} \sum_{j=l}^{\infty} \binom{j-1}{l-1} \xi(t + (i+j)\tau) \\ &= \sum_{k=l+1}^{\infty} \left\{ \sum_{j=l}^{k-1} \binom{j-1}{l-1} \right\} \xi(t + k\tau), \qquad t \geq T - (l+1)\tau.\end{aligned}$$

Since

$$\sum_{j=l}^{k-1} \binom{j-1}{l-1} = \binom{k-1}{l},$$

it follows that

$$\Psi^{l+1}\xi(t) = \sum_{k=l+1}^{\infty} \binom{k-1}{l} \xi(t + k\tau), \qquad t \geq T - (l+1)\tau,$$

proving the truth of (2.5) with l replaced by $l + 1$. Since (2.5) trivially holds for $l = 1$, the induction completes the proof.

LEMMA 2.3. *Let $l \in N$ and $p \in N \cup \{0\}$. If $F \in C[T, \infty)$ and $F(t) \geq 0$ for $t \geq T$, then $\int_T^\infty t^{l+p}F(t)dt < \infty$ implies that $\int_t^\infty (s-t)^p F(s)ds \in S^l[T, \infty)$, and*

$$\Psi^l\left(\int_t^\infty (s-t)^p F(s)ds\right) \leq \frac{1}{\tau^l} \int_{t+l\tau}^\infty s^{l+p}F(s)ds, \qquad t \geq T - l\tau. \tag{2.6}$$

PROOF. Applying Lemma 2.2, we have

$$\begin{aligned}\Psi^l\left(\int_t^\infty (s-t)^p F(s)ds\right) &= \sum_{i=l}^{\infty} \binom{i-1}{l-1} \int_{t+i\tau}^\infty (s - t - i\tau)^p F(s)ds \\ &\leq \sum_{i=l}^{\infty} \binom{i-1}{l-1} \sum_{j=i}^{\infty} \int_{t+j\tau}^{t+(j+1)\tau} (s-t)^p F(s)ds \equiv I.\end{aligned}$$

Interchanging the order of summation in I, we see that

$$I = \sum_{j=l}^{\infty} \int_{t+j\tau}^{t+(j+1)\tau} \left\{ \sum_{i=l}^{j} \binom{i-1}{l-1} \right\} (s-t)^p F(s)ds,$$

from which, noting that

$$\sum_{i=l}^{j} \binom{i-1}{l-1} = \binom{j}{l} \leq j^l \leq \tau^{-l}(s-t)^l \qquad \text{for } s \in [t+j\tau, t+(j+1)\tau],$$

we conclude that

$$\Psi^l\left(\int_t^\infty (s-t)^p F(s)ds\right) \leq \tau^{-l} \sum_{j=l}^{\infty} \int_{t+j\tau}^{t+(j+1)\tau} (s-t)^{l+p} F(s)ds$$

$$= \tau^{-l} \int_{t+l\tau}^{\infty} (s-t)^{l+p} F(s)ds, \qquad t \geq T - l\tau.$$

This completes the proof.

Lemma 2.4. *Let $l \in N$ and $v \in S^l[T, \infty)$. Suppose that $v(t) \geq 0$ for $t \geq T$ and define*

$$U = \{u \in C[T, \infty) : |u(t)| \leq v(t), t \geq T\}.$$

Then the following statements hold.

(i) *Ψ^l is continuous on U in the $C[T - l\tau, \infty)$-topology.*

(ii) *If U is locally equicontinuous on $[T, \infty)$, then $\Psi^l(U)$ is locally equicontinuous on $[T - l\tau, \infty)$.*

Proof. We need only to give a proof for the case $l = 1$.

(i) Suppose that $v \in S[T, \infty)$ and $v(t) \geq 0$ for $t \geq T$. Let $\{u_\nu\}$ be a sequence in U converging to $u \in U$ in $C[T, \infty)$. Take an arbitrary compact subinterval I of $[T - \tau, \infty)$. Since $v \in S[T, \infty)$, given any $\varepsilon > 0$, there is $p \in N$ such that

$$\sum_{i=p+1}^{\infty} v(t + i\tau) < \tfrac{1}{3}\varepsilon, \qquad t \in I. \tag{2.7}$$

There is $\nu_0 \in N$ such that

$$\sum_{i=1}^{p} |u_\nu(t+i\tau) - u(t+i\tau)| < \tfrac{1}{3}\varepsilon, \qquad t \in I, \ \nu \geq \nu_0,$$

because of the uniform convergence $u_\nu(t) \to u(t)$ on I. It follows that

$$|\Psi u_\nu(t) - \Psi u(t)|$$

$$\leq \sum_{i=1}^{p} |u_\nu(t+i\tau) - u(t+i\tau)| + \sum_{i=p+1}^{\infty} |u_\nu(t+i\tau)| + \sum_{i=p+1}^{\infty} |u(t+i\tau)|$$

$$< \tfrac{1}{3}\varepsilon + 2\sum_{i=p+1}^{\infty} v(t+i\tau) < \varepsilon, \qquad t \in I, \ \nu \geq \nu_0,$$

implying that $\Psi u_\nu(t) \to \Psi u(t)$ uniformly on I. Since I is arbitrary, this means the convergence $\Psi u_\nu \to \Psi u$ in the topology of $C[T-\tau, \infty)$. Thus Ψ is continuous on U.

(ii) Let $I \subset [T-\tau, \infty)$ be any compact interval. Let $\varepsilon > 0$ be given. Choose $p \in N$ so that (2.7) holds. By hypothesis, U is equicontinuous on I, and so there is a constant $\delta > 0$ such that

$$\sum_{i=1}^{p} |u(t+i\tau) - u(s+i\tau)| < \tfrac{1}{3}\varepsilon \qquad \text{for all } u \in U$$

provided $|t-s| < \delta$, t, $s \in I$. Using this inequality and (2.7), we see that $|t-s| < \delta$, t, $s \in I$, implies that

$$\begin{aligned} &|\Psi u(t) - \Psi u(s)| \\ &\le \sum_{i=1}^{p} |u(t+i\tau) - u(s+i\tau)| + \sum_{i=p+1}^{\infty} |u(t+i\tau)| + \sum_{i=p+1}^{\infty} |u(s+i\tau)| \\ &< \tfrac{1}{3}\varepsilon + \sum_{i=p+1}^{\infty} v(t+i\tau) + \sum_{i=p+1}^{\infty} v(s+i\tau) < \varepsilon \qquad \text{for all } u \in U\,, \end{aligned}$$

which shows that $\Psi(U)$ is equicontinuous on I. Because of the arbitrariness of I it follows that $\Psi(U)$ is locally equicontinuous on $[T-\tau, \infty)$. This finishes the proof.

B) PROOF OF THEOREM I ($j = 0$). Put $\varepsilon_0 = a - \max_t |\omega(t)| > 0$. Choose $T > t_0$ large enough so that

$$T_* = \min\{T - m\tau, \inf_{t \ge T} g(t)\} > t_0 \tag{2.8}$$

and

$$\Psi^m\left(\int_t^\infty s^{n-1} F(s, a)ds\right) < \varepsilon_0\,, \qquad t \ge T - m\tau\,. \tag{2.9}$$

That (2.9) is possible is a consequence of the condition (1.1) and Lemma 2.3 ($l = m, p = n-1$). We consider the sets $X \subset C[T_*, \infty)$ and $Y \subset C[T, \infty)$ defined by

$$\begin{aligned} X &= \{x \in C[T_*, \infty) : |x(t)| \le a, t \ge T_*\}\,, \\ Y &= \{y \in C[T, \infty) : |y(t)| \le v(t), |y(t) - y(s)| \le |v(t) - v(s)|, s, t \ge T\}\,, \end{aligned} \tag{2.10}$$

where $v(t)$ is given by

$$v(t) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} F(s, a)ds\,, \qquad t \ge T. \tag{2.11}$$

Further, we consider the mappings $\mathscr{F}_1: Y \to C[T_*, \infty)$ and $\mathscr{F}_2: X \to C[T, \infty)$ defined by

$$
\begin{aligned}
&\mathscr{F}_1 y(t) = \begin{cases} \omega(t) + (-1)^m \Psi^m y(t), & t \geq T - m\tau, \\ \mathscr{F}_1 y(T - m\tau), & T_* \leq t \leq T - m\tau, \end{cases} \\
(2.12)\qquad & \\
&\mathscr{F}_2 x(t) = (-1)^{n-1} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x(g(s)))ds, \qquad t \geq T,
\end{aligned}
$$

and define the mapping $\mathscr{F}: X \times Y \to C[T_*, \infty) \times C[T, \infty)$ by

$$
(2.13)\qquad \mathscr{F}(x, y) = (\mathscr{F}_1 y, \mathscr{F}_2 x), \qquad (x, y) \in X \times Y.
$$

It can be shown that the mapping $\mathscr{F}$ is continuous and maps $X \times Y$ into a relatively compact subset of $X \times Y$.

(i) *$\mathscr{F}$ maps $X \times Y$ into itself.* It suffices to prove that $\mathscr{F}_1(Y) \subset X$ and $\mathscr{F}_2(X) \subset Y$. Let $y \in Y$. Then, in view of (2.12), (2.11) and (2.9), we have

$$
|\mathscr{F}_1 y(t)| \leq |\omega(t)| + |\Psi^m v(t)| < \max_t |\omega(t)| + \varepsilon_0 = a, \qquad t \geq T - m\tau,
$$

and

$$
|\mathscr{F}_1 y(t)| = |\mathscr{F}_1 y(T - m\tau)| \leq a, \qquad T_* \leq t \leq T - m\tau.
$$

This shows that $\mathscr{F}_1(Y) \subset X$. Now let $x \in X$. Then, by (2.12) and (2.11), we see that

$$
|\mathscr{F}_2 x(t)| \leq \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} F(s, a)ds = v(t), \qquad t \geq T,
$$

and that for $s,\ t \geq T$

$$
\begin{aligned}
|\mathscr{F}_2 x(t) - \mathscr{F}_2 x(s)| &= \left| \int_s^t f(r, x(g(r)))dr \right| \leq \left| \int_s^t F(r, a)dr \right| \\
&= |v(t) - v(s)| \qquad \text{if } n = 1,
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathscr{F}_2 x(t) - \mathscr{F}_2 x(s)| &= \left| \int_s^t \int_r^\infty \frac{(u-r)^{n-2}}{(n-2)!} f(u, x(g(u)))dudr \right| \\
&\leq \left| \int_s^t \int_r^\infty \frac{(u-r)^{n-2}}{(n-2)!} F(u, a)dudr \right| \\
&= |v(t) - v(s)| \qquad \text{if } n \geq 2.
\end{aligned}
$$

This implies that $\mathscr{F}_2(X) \subset Y$, thereby completing the proof that $\mathscr{F}(X \times Y) \subset X \times Y$.

(ii) *$\mathscr{F}$ is continuous.* Let $\{x_\nu\}$ be a sequence in X converging to $x \in X$ in $C[T_*, \infty)$. Then, using the Lebesgue dominated convergence theorem, we see that

$$\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x_\nu(g(s)))ds \to \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x(g(s)))ds$$

uniformly on $[T, \infty)$, so that $\mathscr{F}_2 x_\nu \to \mathscr{F}_2 x$ in $C[T, \infty)$. This shows the continuity of $\mathscr{F}_2$ in the $C[T, \infty)$-topology. The continuity of $\mathscr{F}_1$ in the topology of $C[T_*, \infty)$ is an immediate consequence of the first statement of Lemma 2.4. It follows that $\mathscr{F}$ is continuous on $X \times Y$ in the topology of $C[T_*, \infty) \times C[T, \infty)$.

(iii) *$\mathscr{F}(X \times Y)$ is relatively compact.* The relative compactness of $\mathscr{F}_1(Y)$ in $C[T_*, \infty)$ follows from the second statement of Lemma 2.4, while that of $\mathscr{F}_2(X)$ in $C[T, \infty)$ follows from the inequality

$$|(\mathscr{F}_2 x)'(t)| \leq |v'(t)|, \qquad t \geq T,$$

holding for all $x \in X$. It follows that $\mathscr{F}(X \times Y)$ is relatively compact in $C[T_*, \infty) \times C[T, \infty)$.

Thus all the hypotheses of the Schauder-Tychonoff fixed point theorem are satisfied, and so there exists an element $(x, y) \in X \times Y$ such that $(x, y) = \mathscr{F}(x, y)$, i.e., $x = \mathscr{F}_1 y$ and $y = \mathscr{F}_2 x$ by (2.13). In view of (2.12) this implies that

$$x(t) = \omega(t) + (-1)^m \Psi^m y(t), \qquad t \geq T - m\tau \tag{2.14}$$

and

$$y(t) = (-1)^{n-1} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x(g(s)))ds, \qquad t \geq T. \tag{2.15}$$

Operating Δ^m on (2.14), we see from Lemma 2.1 that $\Delta^m x(t) = y(t)$, $t \geq T$. Combining this equation with $D^n y(t) = -f(t, x(g(t)))$, $t \geq T$, which follows from (2.15), we conclude that $x(t)$ is a solution of the neutral equation (A) with $\lambda = 1$ for $t \geq T$. This solution has the required asymptotic property (1.2) since $x(t) - \omega(t) \to 0$ as $t \to \infty$ because of (2.14) and Lemma 2.1. This completes the proof of Theorem I for the case $j = 0$.

3. Proof of Theorem I (The case $1 \leq j \leq m - 1$)

A) LEMMAS. To construct solutions of type (I) for the case $1 \leq j \leq m - 1$ we need, along with the operator Ψ used before, another type of "inverse" of the difference operator Δ.

For a function $\xi \in C[T, \infty)$ we define

$$(3.1)\quad \eta(t)=\begin{cases}\displaystyle\sum_{i=0}^{N((t-T)/\tau)}\xi(t-i\tau)+\left[\frac{t-T}{\tau}-N\left(\frac{t-T}{\tau}\right)\right]\xi(T), & t\geq T,\\[2ex] \displaystyle\left[\frac{t-T}{\tau}-N\left(\frac{t-T}{\tau}\right)\right]\xi(T), & T-\tau\leq t<T,\end{cases}$$

where $N(u)$ denotes the largest integer not exceeding u. It is easy to see that $\eta(t)$ is continuous on $[T-\tau,\infty)$. Let Φ denote the mapping which assignes to each $\xi\in C[T,\infty)$ a function $\eta\in C[T-\tau,\infty)$ defined by (3.1), and let Φ^l, $l\in N$, be the l-th iterate of Φ. Clearly, Φ^l is defined on $C[T,\infty)$ and sends it into $C[T-l\tau,\infty)$. The following three lemmas describing basic properties of Φ^l will be required in completing the proof of Theorem I.

LEMMA 3.1. *Let $l\in N$ and $\xi\in C[T,\infty)$. Then, $\Phi^l\xi$ is a solution of the difference equation*

$$(3.2)\qquad \Delta^l x(t)=\xi(t),\qquad t\geq T.$$

This is an immediate consequence of the definition of Φ.

LEMMA 3.2. *Let $l\in N$ and $p\in N\cup\{0\}$. If $G\in C[T,\infty)$ satisfies $G(t)=o(t^p)$ as $t\to\infty$, then $\Phi^l G(t)=o(t^{l+p})$ as $t\to\infty$.*

This is an immediate consequence of the following lemma which is a difference version of l'Hospital's rule well-known in elementary calculus.

LEMMA 3.3. *Let α, $\beta\in C[T-\tau,\infty)$ be functions such that*

$$\Delta\beta(t)\neq 0\qquad \textit{and}\qquad \lim_{t\to\infty}\beta(t)=\infty\ (\textit{or}\ -\infty).$$

Then

$$\lim_{t\to\infty}\frac{\Delta\alpha(t)}{\Delta\beta(t)}=c\in \boldsymbol{R}\qquad \textit{implies}\ \lim_{t\to\infty}\frac{\alpha(t)}{\beta(t)}=c\,.$$

In fact, using Lemma 3.3 and noting that

$$\Delta^l t^{l+p}=(l+p)(l+p-1)\dots(p+1)\tau^l t^p+o(t^p)\qquad \text{as } t\to\infty\,,$$

we obtain

$$\lim_{t\to\infty}\frac{\Phi^l G(t)}{t^{l+p}}=\lim_{t\to\infty}\frac{\Delta^l\Phi^l G(t)}{\Delta^l t^{l+p}}=\lim_{t\to\infty}\frac{G(t)}{\Delta^l t^{l+p}}=0\,.$$

PROOF OF LEMMA 3.3. With no loss of generality we may suppose that $\Delta\beta(t)>0$, so that $\lim_{t\to\infty}\beta(t)=\infty$. Let T be such that $\beta(t)>0$ for $t\geq T-\tau$. Let $t\geq T$ be fixed and choose r and s so that $r\geq s\geq t+\tau$. Putting

$n(r) = N((r - t)/\tau) + 1$, we define the sequences $\{a_k\}$, $\{b_k\}$ by

$$a_k = \alpha(r + (k - n(r))\tau), \quad b_k = \beta(r + (k - n(r))\tau), \qquad k = 0, 1, 2, \ldots.$$

Since the sequence $\{b_k\}$ is strictly increasing, we have

$$\inf_{l \geq 1} \frac{a_l - a_{l-1}}{b_l - b_{l-1}} \leq \frac{a_{n(r)} - a_0}{b_{n(r)} - b_0} \leq \sup_{l \geq 1} \frac{a_l - a_{l-1}}{b_l - b_{l-1}}$$

(see Lemma 7 of [1]), from which, noting that $r + (l - n(r))\tau \geq t$ for $l \geq 1$ and

$$a_l - a_{l-1} = \Delta\alpha(r + (l - n(r))\tau), \qquad b_l - b_{l-1} = \Delta\beta(r + (l - n(r))\tau),$$

we see that

$$\inf_{u \geq t} \frac{\Delta\alpha(u)}{\Delta\beta(u)} \leq \frac{a_{n(r)} - a_0}{b_{n(r)} - b_0} \leq \sup_{u \geq t} \frac{\Delta\alpha(u)}{\Delta\beta(u)}. \tag{3.3}$$

Since $t - \tau \leq r - n(r)\tau < t$ and since

$$a_0 = \alpha(r - n(r)\tau), \qquad b_0 = \beta(r - n(r)\tau), \qquad b_{n(r)} = \beta(r),$$

we have

$$\left|\frac{a_0}{b_{n(r)}}\right| \leq \frac{A_t}{\inf\limits_{u \geq s} \beta(u)}, \qquad 0 < \frac{b_0}{b_{n(r)}} \leq \frac{B_t}{\inf\limits_{u \geq s} \beta(u)}, \tag{3.4}$$

where

$$A_t = \sup_{t - \tau \leq u \leq t} |\alpha(u)|, \qquad B_t = \sup_{t - \tau \leq u \leq t} \beta(u).$$

Let $S > t + \tau$ be such that

$$\frac{B_t}{\inf\limits_{u \geq s} \beta(u)} < 1 \qquad \text{for } s \geq S, \tag{3.5}$$

which is possible because $\beta(s) \to \infty$ as $s \to \infty$.

Let $r \geq s \geq S$. Using (3.3), (3.4) and (3.5) in the relation

$$\frac{a_{n(r)}}{b_{n(r)}} = \frac{a_{n(r)} - a_0}{b_{n(r)} - b_0}\left(1 - \frac{b_0}{b_{n(r)}}\right) + \frac{a_0}{b_{n(r)}},$$

we find that

$$\begin{aligned} \frac{\alpha(r)}{\beta(r)} = \frac{a_{n(r)}}{b_{n(r)}} &\leq \sup_{u \geq t} \frac{\Delta\alpha(u)}{\Delta\beta(u)} \cdot \left(1 - \frac{b_0}{b_{n(r)}}\right) + \frac{a_0}{b_{n(r)}} \\ &\leq \sup_{u \geq t} \frac{\Delta\alpha(u)}{\Delta\beta(u)} \cdot \left(1 + \frac{\sigma(t) B_t}{\inf\limits_{u \geq s} \beta(u)}\right) + \frac{A_t}{\inf\limits_{u \geq s} \beta(u)} \end{aligned}$$

and

$$\frac{\alpha(r)}{\beta(r)} \geq \inf_{u \geq t} \frac{\Delta\alpha(u)}{\Delta\beta(u)} \cdot \left(1 - \frac{b_0}{b_{n(r)}}\right) + \frac{a_0}{b_{n(r)}}$$
$$\geq \inf_{u \geq t} \frac{\Delta\alpha(u)}{\Delta\beta(u)} \cdot \left(1 - \frac{\rho(t)B_t}{\inf\limits_{u \geq s} \beta(u)}\right) + \frac{A_t}{\inf\limits_{u \geq s} \beta(u)}$$

where $\sigma(t) = \operatorname{sgn}\left[\sup\limits_{u \geq t} \Delta\alpha(u)/\Delta\beta(u)\right]$ and $\rho(t) = \operatorname{sgn}\left[\inf\limits_{u \geq t} \Delta\alpha(u)/\Delta\beta(u)\right]$. It follows therefore that

$$\sup_{r \geq s} \frac{\alpha(r)}{\beta(r)} \leq \sup_{u \geq t} \frac{\Delta\alpha(u)}{\Delta\beta(u)} \cdot \left(1 + \frac{\sigma(t)B_t}{\inf\limits_{u \geq s} \beta(u)}\right) + \frac{A_t}{\inf\limits_{u \geq s} \beta(u)}$$

and

$$\inf_{r \geq s} \frac{\alpha(r)}{\beta(r)} \geq \inf_{u \geq t} \frac{\Delta\alpha(u)}{\Delta\beta(u)} \cdot \left(1 - \frac{\rho(t)B_t}{\inf\limits_{u \geq s} \beta(u)}\right) - \frac{A_t}{\inf\limits_{u \geq s} \beta(u)}$$

for any r and s such that $r \geq s \geq S$, which implies in the limit as $s \to \infty$

$$\limsup_{s \to \infty} \frac{\alpha(s)}{\beta(s)} \leq \sup_{u \geq t} \frac{\Delta\alpha(u)}{\Delta\beta(u)} \tag{3.6}$$

and

$$\liminf_{s \to \infty} \frac{\alpha(s)}{\beta(s)} \geq \inf_{u \geq t} \frac{\Delta\alpha(u)}{\Delta\beta(u)}. \tag{3.7}$$

Here use is made of the fact that $\inf\limits_{u \geq s} \beta(u) \to \infty$ as $s \to \infty$. Letting $t \to \infty$ in (3.6) and (3.7), we conclude that

$$\lim_{t \to \infty} \frac{\alpha(t)}{\beta(t)} = \lim_{t \to \infty} \frac{\Delta\alpha(t)}{\Delta\beta(t)} = c.$$

This completes the proof.

LEMMA 3.4. *Let $l \in N$ and $Y \subset C[T, \infty)$. Then the following statements hold.*

(i) *Φ^l is continuous in the topology of $C[T - l\tau, \infty)$.*

(ii) *If Y is locally equicontinuous on $[T, \infty)$ and if*

$$\sup\{|y(T)| : y \in Y\} < \infty, \tag{3.8}$$

then $\Phi^l(Y)$ is locally equicontinuous on $[T - l\tau, \infty)$.

PROOF. It suffices to prove the lemma for the case $l=1$. For simplicity we use the notation $N_t = N((t-T)/\tau)$.

(i) Let $\{y_\nu\}$ be a sequence in Y converging to $y \in Y$ in $C[T, \infty)$. Choose an arbitrary compact interval $J \subset [T-\tau, \infty)$ and put $N^* = \max\{N_t : t \in J\}$. Let $\varepsilon > 0$ be given. Choose $\nu_0 \in N$ so that

$$|y_\nu(T) - y(T)| < \tfrac{1}{2}\varepsilon \qquad \text{for } \nu > \nu_0 \tag{3.9}$$

and

$$\sum_{i=0}^{N^*} |y_\nu(t-i\tau) - y(t-i\tau)| < \tfrac{1}{2}\varepsilon \qquad \text{for } \nu > \nu_0 \text{ and } t \in J \cap [T, \infty). \tag{3.10}$$

(3.10) follows from the uniform convergence of $\{y_\nu\}$ on $J \cap [T, \infty)$. Using the definition of Φ, (3.9) and (3.10), we have

$$\begin{aligned} |\Phi y_\nu(t) - \Phi y(t)| &= \left|\frac{t-T}{\tau} - N_t\right| |y_\nu(T) - y(T)| \\ &\leq |y_\nu(T) - y(T)| < \tfrac{1}{2}\varepsilon \qquad \text{if } t \in J \cap [T-\tau, T) \end{aligned}$$

and

$$\begin{aligned} &|\Phi y_\nu(t) - \Phi y(t)| \\ &= \left|\sum_{i=0}^{N_t} [y_\nu(t-i\tau) - y(t-i\tau)] + \left(\frac{t-T}{\tau} - N_t\right)[y_\nu(T) - y(T)]\right| \\ &\leq \sum_{i=0}^{N^*} |y_\nu(t-i\tau) - y(t-i\tau)| + |y_\nu(T) - y(T)| < \varepsilon \qquad \text{if } t \in J \cap [T, \infty), \end{aligned}$$

which implies that $\Phi y_\nu \to \Phi y$ in $C[T-\tau, \infty)$. Thus Φ is a continuous mapping.

(ii) Suppose that Y is locally equicontinuous on $[T, \infty)$ and satisfies (3.8). Let $J \subset [T-\tau, \infty)$ be any compact interval and put $N^* = \sup\{N_t : t \in J\}$ as before. Let $M = \sup\{|y(T)| : y \in Y\}$. Then, in view of the equicontinuity of Y on $J \cap [T, \infty)$, we see that, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < \delta < \tau$,

$$\frac{M\delta}{\tau} < \tfrac{1}{3}\varepsilon,$$

$$|y(t - N_t\tau - T) - y(T)| < \tfrac{1}{3}\varepsilon \qquad \text{for all } y \in Y$$

$$\text{if } 0 \leq t - N_t\tau - T < \delta,\ t \in J \cap [T, \infty),$$

and

$$\sum_{i=0}^{N^*} |y(t-i\tau) - y(s-i\tau)| < \tfrac{1}{3}\varepsilon \qquad \text{for all } y \in Y$$
$$\text{if } 0 \le t-s < \delta\,,\ s,\ t \in J \cap [T, \infty)\,.$$

Using these inequalities it can be shown that

(3.11) $|\Phi y(t) - \Phi y(s)| < \varepsilon$ for all $y \in Y$ provided $0 < t-s < \delta$, s, $t \in J$.

In fact, choose s, $t \in J$ such that $0 < t - s < \delta$. Since $\delta < \tau$, we have either $N_t = N_s$ or $N_t = N_s + 1$. Suppose first that $N_t = N_s$. In this case, if $s < T$, then

$$|\Phi y(t) - \Phi y(s)| = \left|\frac{t-s}{\tau} y(T)\right| < \frac{M\delta}{\tau} < \tfrac{1}{3}\varepsilon\,, \qquad y \in Y,$$

and if $s \ge T$, then

$$\begin{aligned} |\Phi y(t) - \Phi y(s)| &= \left|\sum_{i=0}^{N_s} [y(t-i\tau) - y(s-i\tau)] + \frac{t-s}{\tau} y(T)\right| \\ &\le \sum_{i=0}^{N^*} |y(t-i\tau) - y(s-i\tau)| + \frac{M\delta}{\tau} < \tfrac{2}{3}\varepsilon\,, \qquad y \in Y. \end{aligned}$$

Suppose next that $N_t = N_s + 1$. In this case, if $s < T$, then, noting that $0 \le t - T < \delta$, we have

$$\begin{aligned} |\Phi y(t) - \Phi y(s)| &= \left|y(t) + \frac{t-s}{\tau} y(T) - y(T)\right| \\ &\le |y(t) - y(T)| + \frac{M\delta}{\tau} < \tfrac{2}{3}\varepsilon\,, \qquad y \in Y, \end{aligned}$$

and if $s \ge T$, then, noting that $0 \le t - N_t\tau - T < \delta$, we have

$$\begin{aligned} &|\Phi y(t) - \Phi y(s)| \\ &= \left|\sum_{i=0}^{N_s} [y(t-i\tau) - y(s-i\tau)] + y(t-N_t\tau) + \frac{t-s}{\tau} y(T) - y(T)\right| \\ &\le \sum_{i=0}^{N^*} |y(t-i\tau) - y(s-i\tau)| + |y(t-N_t\tau) - y(T)| + \frac{M\delta}{\tau} < \varepsilon\,, \qquad y \in Y. \end{aligned}$$

We have thus proved (3.11), which is nothing else but the equicontinuity of $\Phi(Y)$ on J. Since J is an arbitrary compact subinterval of $[T-\tau, \infty)$, $\Phi(Y)$ is locally equicontinuous on $[T-\tau, \infty)$ as desired. Thus finishes the proof of Lemma 3.4.

B) Proof of Theorem I ($1 \le j \le m-1$). We introduce the abbreviation

$F(t) = F(t, a[g(t)]^j)$. Let $\varepsilon_0 = a - \max_t |\omega(t)|$. Take $T > t_0$ large enough so that (2.8) holds and

$$\Phi^j \Psi^{m-j}\left(\int_t^\infty s^{n-1}F(s)ds\right) < \varepsilon_0 t^j\,, \qquad t \geq T - m\tau\,. \tag{3.12}$$

That (3.12) is possible is shown as follows. Because of (1.1) it follows from Lemma 2.3 ($l = m - j, p = n - 1$) that $\int_t^\infty s^{n-1}F(s)ds \in S^{m-j}[T, \infty)$, so that

$$\Psi^{m-j}\left(\int_t^\infty s^{n-1}F(s)ds\right) = o(1) \qquad \text{as } t \to \infty \tag{3.13}$$

by Lemma 2.1. Operating Φ^j on (3.13) and using Lemma 3.2 ($l = j, p = 0$), we have

$$\Phi^j \Psi^{m-j}\left(\int_t^\infty s^{n-1}F(s)ds\right) = o(t^j) \qquad \text{as } t \to \infty\,, \tag{3.14}$$

which implies the truth of (3.12) for a sufficiently large T.

We now define the set $X \times Y \subset C[T_*, \infty) \times C[T, \infty)$ by

$$\begin{aligned} X &= \{x \in C[T_*, \infty) : |x(t)| \leq at^j, t \geq T_*\}, \\ Y &= \{y \in C[T, \infty) : |y(t)| \leq v(t), |y(t) - y(s)| \leq |v(t) - v(s)|, s, t \geq T\}, \end{aligned} \tag{3.15}$$

where

$$v(t) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}F(s)ds\,, \qquad t \geq T. \tag{3.16}$$

Let $\mathscr{F}$ denote the mapping which assigns to each $(x, y) \in X \times Y$ an element $(\mathscr{F}_1 y, F_2 x) \in C[T_*, \infty) \times C[T, \infty)$ given by

$$\begin{aligned} \mathscr{F}_1 y(t) &= \begin{cases} \omega(t)t^j + (-1)^{m-j}\Phi^j\Psi^{m-j}y(t)\,, & t \geq T - m\tau\,, \\ \mathscr{F}_1 y(T - m\tau)\dfrac{t^j}{(T - m\tau)^j}\,, & T_* \leq t \leq T - m\tau\,, \end{cases} \\ \mathscr{F}_2 x(t) &= (-1)^{n-1}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}f(s, x(g(s)))ds\,, \qquad t \geq T. \end{aligned} \tag{3.17}$$

(i) Let $y \in Y$. By the definitions of Y and $\mathscr{F}_1$ and (3.12) we have

$$|\mathscr{F}_1 y(t)| \leq |\omega(t)|t^j + |\Phi^j\Psi^{m-j}v(t)| \leq (|\omega(t)| + \varepsilon_0)t^j \leq at^j$$

for $t \geq T - m\tau$. We also have $|\mathscr{F}_1 y(t)| \leq at^j$ for $T_* \leq t \leq T - m\tau$ since $|\mathscr{F}_1 y(T - m\tau)/(T - m\tau)^j| \leq a$. Hence $\mathscr{F}_1(Y) \subset X$. It can be shown that $\mathscr{F}_2(X) \subset Y$ exactly as in the case $j = 0$. It follows that $\mathscr{F}$ maps $X \times Y$ into itself.

(ii) $\mathscr{F}$ is continuous in the topology of $C[T_*, \infty) \times C[T, \infty)$. In fact, the continuity of $\mathscr{F}_1$ in the $C[T_*, \infty)$-topology follows from combination of the first statements of Lemmas 2.4 and 3.4, while that of $\mathscr{F}_2$ in the $C[T, \infty)$-topology can be proved exactly as in the case $j = 0$.

(iii) $\mathscr{F}(X \times Y)$ is realtively compact. In fact, the relative compactness of $\mathscr{F}_1(Y)$ in $C[T_*, \infty)$ follows from combination of the second statements of Lemmas 2.4 and 3.4, and that of $\mathscr{F}_2(X)$ can be proved exactly as in the case $j = 0$.

Consequently, there exists an element $(x, y) \in X \times Y$ such that $(x, y) = \mathscr{F}(x, y)$, which satisfies

$$x(t) = \omega(t)t^j + (-1)^{m-j}\Phi^j\Psi^{m-j}y(t), \qquad t \geq T - m\tau,$$

$$y(t) = (-1)^{n-1}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x(g(s)))ds, \qquad t \geq T.$$

Since $\Delta^m x(t) = y(t)$ and $D^n y(t) = -f(t, x(g(t)))$ for $t \geq T$, we see that $x(t)$ is a solution of the equation (A) for $t \geq T$. From (3.14) it follows that $x(t) = \omega(t)t^j + o(t^j)$ as $t \to \infty$, that is, $x(t)$ satisfies the asymptotic relation (1.2). This completes the proof of Theorem I for the case $1 \leq j \leq m - 1$.

4. Proof of Theorem II

Let us now give a proof of Theorem II. Let $k \in \{m, m+1, \ldots, m+n-1\}$. Put $F(t) = F(t, a[g(t)]^k)$. For a function $\varphi \in C[T, \infty)$, $T \geq t_0$, and $i, j \in N \cup \{0\}$ we introduce the notation

$$(4.1) \quad I_{i,j}(t, T; \varphi) = \begin{cases} \displaystyle\int_T^t \frac{(t-s)^{i-1}}{(i-1)!}\varphi(s)ds & \text{for } i \neq 0,\ j = 0, \\ \displaystyle\int_t^\infty \frac{(s-t)^{j-1}}{(j-1)!}\varphi(s)ds & \text{for } i = 0,\ j \neq 0, \\ \displaystyle\int_T^t \frac{(t-s)^{i-1}}{(i-1)!}\int_s^\infty \frac{(r-s)^{j-1}}{(j-1)!}\varphi(r)drds & \text{for } i \neq 0,\ j \neq 0. \end{cases}$$

The condition (1.3) implies that $I_{k-m,m+n-k}(t, T; F)$ is well defined for any $T \geq t_0$ and

$$I_{k-m,m+n-k}(t, T; F) = o(t^{k-m}) \qquad \text{as } t \to \infty,$$

and so from Lemma 3.2 ($l = m$, $p = k - m$) it follows that

$$\Phi^m(I_{k-m,m+n-k}(t, T; F)) = o(t^k) \qquad \text{as } t \to \infty.$$

Hence one can choose $T > t_0$ so that (2.8) holds and

$$(4.2) \qquad \Phi^m(I_{k-m,m+n-k}(t, T; F)) < \varepsilon_0 t^k\,, \qquad t \geq T - m\tau\,,$$

where $\varepsilon_0 = a - |c| > 0$. With this choice of T one defines the set $X \times Y \subset C[T_*, \infty) \times C[T, \infty)$ and the mapping $\mathscr{F}: X \times Y \to C[T_*, \infty) \times C[T, \infty)$ by the formulas:

$$(4.3) \qquad \begin{aligned} X &= \{x \in C[T_*, \infty) : |x(t)| \leq at^k, t \geq T_*\}\,, \\ Y &= \{y \in C[T, \infty) : |y(t)| \leq v(t), |y(t) - y(s)| \leq |v(t) - v(s)|, s, t \geq T\}\,, \end{aligned}$$

where

$$(4.4) \qquad v(t) = I_{k-m,m+n-k}(t, T; F)\,,$$

$$(4.5) \qquad \mathscr{F}(x, y) = (\mathscr{F}_1 y, \mathscr{F}_2 x)\,, \qquad (x, y) \in X \times Y,$$

where

$$(4.6) \qquad \begin{aligned} \mathscr{F}_1 y(t) &= \begin{cases} ct^k + \Phi^m y(t)\,, & t \geq T - m\tau\,, \\ \mathscr{F}_1 y(T - m\tau)\dfrac{t^k}{(T - m\tau)^k}\,, & T_* \leq t \leq T - m\tau\,, \end{cases} \\ \mathscr{F}_2 x(t) &= (-1)^{m+n-k-1} I_{k-m,m+n-k}(t, T; f(t, x(g(t))))\,, \qquad t \geq T. \end{aligned}$$

Then one verifies without difficulty that $\mathscr{F}$ is well defined on $X \times Y$ and maps it continuously into a relative compact subset of $X \times Y$. The Schauder-Tychonoff theorem then eneures the existence of a fixed point $(x, y) \in X \times Y$ of $\mathscr{F}$. Since

$$x(t) = ct^k + \Phi^m y(t)\,, \qquad t \geq T - m\tau\,,$$

$$y(t) = (-1)^{m+n-k-1} I_{k-m,m+n-k}(t, T; f(t, x(g(t))))\,, \qquad t \geq T,$$

one sees that

$$D^n \Delta^m x(t) = D^n \Delta^m (ct^k) + D^n y(t) = -f(t, x(g(t)))\,, \qquad t \geq T$$

and

$$x(t) = ct^k + o(t^k) \qquad \text{as } t \to \infty\,,$$

concluding that $x(t)$ is a solution of the neutral equation (A) having the required asymptotic behavior (1.4). This sketches the proof of Theorem II. The details are left to the reader.

Part 2. Existence of Solutions for the case $\lambda \neq 1$

5. Statement of Existence Theorems

We now turn to the case $\lambda \neq 1$ of (A) and prove the following existence theorems.

THEOREM I_λ. *Let $j \in \{0, 1, \dots, m-1\}$ and suppose that there is a constant $a > 0$ such that*

$$(5.1) \qquad \int_{t_0}^{\infty} t^{m-j-1} \lambda^{-t/\tau} F(t, a[g(t)]^j \lambda^{g(t)/\tau}) dt < \infty .$$

Then, for any continuous τ-periodic function $\omega(t)$ such that $\max_t |\omega(t)| < a$, the equation (A) *with $\lambda \neq 1$ possesses a solution $x(t)$ with the property that*

$$(5.2) \qquad x(t) = t^j \lambda^{t/\tau}[\omega(t) + o(1)] \qquad \text{as } t \to \infty .$$

THEOREM II_λ. *Let $k \in \{0, 1, \dots, n-1\}$ and suppose that there is a constant $a > 0$ such that*

$$(5.3) \qquad \int_{t_0}^{\infty} t^{n-k-1} F(t, a[g(t)]^k) dt < \infty .$$

Then, for any constant c such that $0 < |c| < a$, the equation (A) *with $\lambda \neq 1$ possesses a solution $x(t)$ with the property that*

$$(5.4) \qquad x(t) = t^k[c + o(1)] \qquad \text{as } t \to \infty .$$

REMARK 5.1. The solution obtained in Theorem I_λ is oscillatory or nonoscillatory according to whether the periodic function $\omega(t)$ is oscillatory or nonoscillatory. In either case the solution is unbounded if $\lambda > 1$ and is decaying to zero as $t \to \infty$ if $\lambda < 1$. Since $\omega(t)$ does not appear explicitly in the condition (5.1), it guarantees the coexistence of oscillatory and nonoscillatory solutions for the equation (A) with $\lambda \neq 1$. The solution obtained in Theorem II_λ is clearly nonoscillatory.

EXAMPLE 5.1. Consider the neutral equation

$$(5.5) \qquad \begin{aligned} &D^n[x(t) - 2\lambda x(t-1) + \lambda^2 x(t-2)] \\ &\quad + q(t)|x(t-3)|^\gamma \operatorname{sgn} x(t-3) = 0 , \qquad t \geq t_0 , \end{aligned}$$

where $\lambda > 0$, $\neq 1$, $\gamma > 0$, $t_0 > 3$, and $q: [T_0, \infty) \to \boldsymbol{R}$ is continuous. Since $\Delta_\lambda^2 x(t) = x(t) - 2\lambda x(t-1) + \lambda^2 x(t-2)$, (5.5) is a special case of (A) ($\lambda \neq 1$) in which $m = 2$, $\tau = 1$, $g(t) = t - 3$ and $f(t, x) = q(t)|x|^\gamma \operatorname{sgn} x$. The function $F(t, u)$ in (0.3) can be taken to be $F(t, u) = |q(t)|u^\gamma$. The conditions (5.1) and (5.3) for this equation reduce, respectively, to

$$(5.6) \qquad \int_{t_0}^{\infty} t^{1+(\gamma-1)j} \lambda^{(\gamma-1)t} |q(t)| dt < \infty , \qquad j \in \{0, 1\} ,$$

and

(5.7) $$\int_{t_0}^{\infty} t^{n-1+(\gamma-1)k}|q(t)|\,dt < \infty\,, \qquad k \in \{0, 1, \ldots, n-1\}\,.$$

Suppose that

$$\int_{t_0}^{\infty} t^{n-1}|q(t)|\,dt < \infty \qquad \text{for the case } \gamma \le 1\,,$$

$$\int_{t_0}^{\infty} t^{\gamma}\lambda^{(\gamma-1)t}|q(t)|\,dt < \infty \qquad \text{for the case } \gamma > 1\,.$$

Then all the integrals listed in (5.6) and (5.7) converge, and so, by Theorem I_λ and II_λ, the equation (5.5) has solutions of type (I_λ)

$$x_0(t) = \lambda^t[\omega(t) + o(1)]\,, \qquad x_1(t) = t\lambda^t[\omega(t) + o(1)] \qquad \text{as } t \to \infty$$

as well as solutions of the type (II_λ)

$$y_0(t) = c + o(1)\,,\ y_1(t) = t[c + o(1)]\,,\ \ldots,\ y_{n-1}(t) = t^{n-1}[c + o(1)] \qquad \text{as } t \to \infty$$

for any nonzero constant c and any continuous periodic function $\omega(t)$ of period 1.

6. Proof of Theorem I_λ (The case $\lambda > 1$)

A) PRELIMINARY REMARK. In view of the proofs of Theorems I and II given in Part 1 one would be tempted to make use of appropriate "inverses" of the difference operator Δ_λ^m ($\lambda \neq 1$) to prove Theorem I_λ and II_λ. Such an attempt, however, is unnecessary; in fact, the "inverses" of Δ^m already employed, that is, suitable combinations of Φ and Ψ, are sufficient for our purposes. To see this, we observe that

$$\Delta_\lambda x(t) = \lambda^{t/\tau}\Delta[\lambda^{-t/\tau}x\ t)]\,($$

so that

$$\Delta_\lambda^m x(t) = \lambda^{t/\lambda}\Delta^m[\lambda^{-t/\tau}x(t)]\,, \qquad m = 1,\ 2,\ 3,\ \ldots,$$

and the equation (A) with $\lambda \neq 1$ can be expressed as

$$D^n[\lambda^{t/\tau}\Delta^m[\lambda^{-t/\tau}x(t)]] + f(t, x(g(t))) = 0\,, \qquad t \ge t_0\,.$$

We will rewrite the above equation as

(A*) $$D^n[\lambda^{t/\tau}\Delta^m x^*(t)] + f^*(t, x^*(g(t))) = 0\,, \qquad t \ge t_0\,,$$

by introducing the new functions

(6.1) $$x^*(t) = \lambda^{-t/\tau}x(t)\,, \qquad f^*(t, x^*) = f(t, \lambda^{g(t)/\tau}x^*)\,.$$

Since a solution $x^*(t)$ of (A*) gives rise to a solution $x(t) = \lambda^{t/\tau}x^*(t)$ of (A), in order to prove Theorem I_λ it suffices to show the existence of a solution $x^*(t)$ of (A*) such that $x^*(t) = t^j[\omega(t) + o(1)]$ as $t \to \infty$ for a given τ-periodic function $\omega(t)$ provided (5.1) is satisfied. Likewise, Theorem II_λ is proved if it is verified that (5.3) implies the existence of a solution $x^*(t)$ of (A*) such that $x^*(t) = \lambda^{-t/\tau}t^k[c + o(1)]$ as $t \to \infty$. A close look at the proofs of Theorems I and II suggests us to obtain a solution $x^*(t)$ of (A*) from a pair of functions $(x^*(t), y^*(t))$ satisfying

$$\text{(B*)} \qquad \Delta^m x^*(t) = y^*(t) \qquad \text{and} \qquad D^n(\lambda^{t/\tau}y^*(t)) = -f^*(t, x^*(g(t)))$$

for all sufficiently large t. In what follows we make an effort to solve the system (B*) by overcoming the difficulty caused by the presence of the factor $\lambda^{t/\tau}$.

B) LEMMA. We need the following lemma.

LEMMA 6.1. *Suppose that $\lambda > 1$. Let $l \in N$ and let $F \in C[T, \infty)$ be non-negative for $t \geq T$, $T \geq 0$. If $\int_T^\infty t^{l-1}\lambda^{-t/\tau}F(t)dt < \infty$, then $\lambda^{-t/\tau}\int_T^t (t-s)^p F(s)ds \in S^l[T, \infty)$ for any $p \in N \cup \{0\}$.*

PROOF. By Lemma 2.2 we have

$$\text{(6.2)} \quad \Psi^l\left(\lambda^{-t/\tau}\int_T^t (t-s)^p F(s)ds\right)$$
$$= \sum_{i=l}^\infty \binom{i-1}{l-1}\lambda^{-(t+i\tau)/\tau}\int_T^{t+i\tau}(t-s+i\tau)^p F(s)ds$$
$$= \sum_{i=l}^\infty \binom{i-1}{l-1}\lambda^{-(t+i\tau)/\tau}\int_T^{t+(l-1)\tau}(t-s+i\tau)^p F(s)ds$$
$$+ \sum_{i=l}^\infty \binom{i-1}{l-1}\lambda^{-(t+i\tau)/\tau}\sum_{j=l}^{i}\int_{t+(j-1)\tau}^{t+j\tau}(t-s+i\tau)^p F(s)ds \equiv I_1 + I_2\,.$$

In order to estimate I_1, I_2 we rewrite them as

$$I_1 = \int_T^{t+(l-1)\tau}\left\{\sum_{i=l}^\infty \binom{i-1}{l-1}(t-s+i\tau)^p\lambda^{-(t-s+i\tau)/\tau}\right\}\lambda^{-s/\tau}F(s)ds\,,$$
$$I_2 = \sum_{j=l}^\infty \int_{t+(j-1)\tau}^{t+j\tau}\left\{\sum_{i=j}^\infty \binom{i-1}{l-1}(t-s+i\tau)^p\lambda^{-(t-s+i\tau)/\tau}\right\}\lambda^{-s/\tau}F(s)ds\,,$$

and put

$$L = \sum_{i=0}^\infty \binom{i+l-1}{l-1}\lambda^{-i/2} \qquad \text{and} \qquad M = \sup\{u^p\lambda^{-u/2\tau} : u \geq 0\}\,.$$

Because of $\lambda > 1$, L and M are finite. Noting that $s \in [T, t + (l-1)\tau)$ implies

$$t - s + i\tau \geq (i - l + 1)\tau \geq (i - l)\tau\,,$$

we see that

$$\sum_{i=l}^{\infty} \binom{i-1}{l-1} (t - s + i\tau)^p \lambda^{-(t-s+i\tau)/\tau} \leq M \sum_{i=l}^{\infty} \binom{i-1}{l-1} \lambda^{-(t-s+i\tau)/2\tau}$$

$$\leq M \sum_{i=l}^{\infty} \binom{i-l+l-1}{l-1} \lambda^{-(i-l)/2} = ML \leq MLT^{1-l} s^{l-1}\,,$$

so that

$$I_1 \leq MLT^{1-l} \int_T^{t+(l-1)\tau} s^{l-1} \lambda^{-s/\tau} F(s) ds\,. \tag{6.3}$$

Let $s \in [t + (j-1)\tau, t + j\tau]$. Then,

$$t - s + i\tau \geq (i - j)\tau\,, \qquad j - 1 \leq \tau^{-1}(s - t)\,,$$

and so we have for $i \geq j \geq l$

$$\binom{i-1}{l-1} \Big/ \binom{i-j+l-1}{l-1} = \prod_{q=1}^{l-1} \frac{i-q}{i-j+l-q} = \prod_{q=1}^{l-1} \left(1 + \frac{j-l}{i-j+l-q}\right)$$

$$\leq \prod_{q=1}^{l-1} \left(1 + \frac{j-l}{l-q}\right) = \prod_{q=1}^{l-1} \frac{j-q}{l-q} = \binom{j-1}{l-1}.$$

Using this inequality we find that

$$\sum_{i=j}^{\infty} \binom{i-1}{l-1} (t - s + i\tau)^p \lambda^{-(t-s+i\tau)/\tau} \leq M \sum_{i=j}^{\infty} \binom{i-1}{l-1} \lambda^{-(t-s+i\tau)/2\tau}$$

$$\leq M \sum_{i=j}^{\infty} \binom{i-1}{l-1} \lambda^{-(i-j)/2} \leq M \binom{j-1}{l-1} \sum_{i=j}^{\infty} \binom{i-j+l-1}{l-1} \lambda^{-(i-j)/2}$$

$$= ML \binom{j-1}{l-1} \leq ML(j-1)^{l-1} \leq ML\tau^{1-l} s^{l-1}$$

for $s \in [t + (j-1)\tau, t + j\tau]$. Consequently, we have

$$I_2 \leq ML\tau^{1-l} \sum_{j=l}^{\infty} \int_{t+(j-1)\tau}^{t+j\tau} s^{l-1} \lambda^{-s/\tau} F(s) ds \tag{6.4}$$

$$= ML\tau^{1-l} \int_{t+(l-1)\tau}^{\infty} s^{l-1} \lambda^{-s/\tau} F(s) ds\,.$$

Using (6.3) and (6.4) in (6.2), we conclude that

$$\Psi^l\left(\lambda^{-t/\tau}\int_T^t (t-s)^p F(s)ds\right) \le ML \max\{T^{1-l}, \tau^{1-l}\}\int_T^\infty s^{l-1}\lambda^{-s/\tau}F(s)ds\,,$$

completing the proof.

C) PROOF OF THEOREM I_λ ($\lambda > 1$). Let $j \in \{0, 1, \dots, m-1\}$. Define the function

$$F^*(t, x^*) = F(t, \lambda^{g(t)/\tau}x^*)\,.$$

Then the condition (5.1) is written as

$$\int_{t_0}^\infty t^{m-j-1}\lambda^{-t/\tau}F^*(t, a[g(t)]^j)dt < \infty\,. \tag{6.5}$$

For simplicity we put $F^*(t) = F^*(t, a[g(t)]^j)$. From (6.5) and Lemma 6.1 ($l = m-j$, $p = n-1$) we have

$$\Psi^{m-j}(\lambda^{-t/\tau}I_{n,0}(t, T; F^*)) = o(1) \qquad \text{as } t \to \infty\,, \tag{6.6}$$

where $I_{n,0}(t, T; F^*)$ is defined by (4.1), i.e.,

$$I_{n,0}(t, T; F^*) = \int_T^t \frac{(t-s)^{n-1}}{(n-1)!}F^*(s)ds\,, \qquad t \ge T.$$

Applying Φ^j on both sides of (6.6) and using Lemma 3.2 ($l = j$, $p = 0$), we obtain

$$\Phi^j\Psi^{m-j}(\lambda^{-t/\tau}I_{n,0}(t, T; F^*)) = o(t^j) \qquad \text{as } t \to \infty\,,$$

whence it follows that $T > t_0$ can be chosen so large that (2.8) holds and

$$\Phi^j\Psi^{m-j}(\lambda^{-t/\tau}I_{n,0}(t, T; F^*)) < \varepsilon_0 t^j\,, \qquad t \ge T - m\tau\,,$$

where $\varepsilon_0 = a - \max_t |\omega(t)| > 0$.

Now we define the set $X^* \times Y^* \subset C[T_*, \infty) \times C[T, \infty)$ by

$$X^* = \{x^* \in C[T_*, \infty) : |x^*(t)| \le at^j, t \ge T_*\}\,,$$

$$Y^* = \{y^* \in C[T, \infty) : |y^*(t)| \le v^*(t), |y^*(t) - y^*(s)| \le |v^*(t) - v^*(s)|, s, t, \ge T\}\,,$$

where

$$v^*(t) = \lambda^{-t/\tau}I_{n,0}(t, T; F^*)\,, \qquad t \ge T,$$

and the mapping $\mathscr{F}^*: X^* \times Y^* \to C[T_*, \infty) \times C[T, \infty)$ by

$$\mathscr{F}^*(x^*, y^*) = (\mathscr{F}_1^* y^*, \mathscr{F}_2^* x^*)\,, \qquad (x^*, y^*) \in X^* \times Y^*\,,$$

where

$$\mathscr{F}_1^* y^*(t) = \begin{cases} \omega(t)t^j + (-1)^{m-j}\Phi^j\Psi^{m-j}y^*(t)\,, & t \geq T - m\tau\,, \\ \mathscr{F}_1^* y^*(T - m\tau)\dfrac{t^j}{(T - m\tau)^j}\,, & T_* \leq t \leq T - m\tau\,, \end{cases}$$

$$\mathscr{F}_2^* x^*(t) = -\lambda^{-t/\tau} I_{n,0}(t, T; f^*(t, x^*(g(t))))\,, \qquad t \geq T.$$

Then it is verified exactly as in the proof of Theorem 1 (§3) that there exists a fixed point $(x^*, y^*) \in X^* \times Y^*$ of $\mathscr{F}^*$. Since, in view of the definition of $\mathscr{F}^*$, x^* and y^* satisfy the equations (B*), the function $x^*(t)$ is shown to satisfy the equation (A*) for $t \geq T$, so that $x(t) = \lambda^{t/\tau}x^*(t)$ gives a solution of the equation (A) for $t \geq T$. That $x(t)$ has the desired asymptotic behavior (5.2) follows from the fact that $x^*(t) = t^j[\omega(t) + o(1)]$ as $t \to \infty$.

7. Proof of Theorem I_λ (The case $0 < \lambda < 1$)

A) LEMMA. The proof of Theorem I_λ for the case $0 < \lambda < 1$ requires a counterpart of Lemma 6.1 stated below.

LEMMA 7.1. *Suppose that* $0 < \lambda < 1$. *Let* $l \in N$ *and let* $F \in C[T, \infty)$ *be nonnegative for* $t \geq T$, $T \geq 0$. *If* $\int_T^\infty t^{l-1}\lambda^{-t/\tau}F(t)dt < \infty$, *then* $\lambda^{-t/\tau}\int_t^\infty (s - t)^p F(s)ds \in S^l[T, \infty)$ *for any* $p \in N \cup \{0\}$.

PROOF. Using Lemma 2.2 we have

$$\Psi^l\left(\lambda^{-t/\tau}\int_t^\infty (s - t)^p F(s)ds\right) = \sum_{i=l}^\infty \binom{i-1}{l-1}\lambda^{-(t+i\tau)/\tau}\int_{t+i\tau}^\infty (s - t - i\tau)^p F(s)ds$$

$$= \sum_{i=l}^\infty \binom{i-1}{l-1}\lambda^{-(t+i\tau)/\tau}\sum_{j=i}^\infty \int_{t+j\tau}^{t+(j+1)\tau} (s - t - i\tau)^p F(s)ds$$

$$= \sum_{j=l}^\infty \int_{t+j\tau}^{t+(j+1)\tau}\left\{\sum_{i=l}^j \binom{i-1}{l-1}(s - t - i\tau)^p\lambda^{(s-t-i\tau)/\tau}\right\}\lambda^{-s/\tau}F(s)ds\,, \qquad t \geq T.$$

Putting

$$L = \sum_{i=0}^\infty \lambda^{i/2} \qquad \text{and} \qquad M = \sup\{u^p\lambda^{t/2\tau} : u \geq 0\}$$

and noting that $s \in [t + j\tau, t + (j + 1)\tau]$ implies

$$s - t - i\tau \geq (j - i)\tau \qquad \text{and} \qquad j \leq \tau^{-1}(s - t)\,,$$

we see that

$$\sum_{i=l}^{j}\binom{i-1}{l-1}(s-t-i\tau)^p\lambda^{(s-t-i\tau)/\tau}\le M\sum_{i=l}^{j}\binom{i-1}{l-1}\lambda^{(s-t-i\tau)/2\tau}$$
$$\le M\binom{j-1}{l-1}\sum_{i=l}^{j}\lambda^{(j-i)/2}\le ML\binom{j-1}{l-1}$$
$$\le MLj^{l-1}\le ML\tau^{l-1}(s-t)^{l-1}\qquad\text{for } s\in[t+j\tau, t+(j+1)\tau]\,.$$

It follows therefore that

$$\Psi^l\left(\lambda^{-t/\tau}\int_t^\infty(s-t)^pF(s)ds\right)\le ML\tau^{l-1}\sum_{j=l}^{\infty}\int_{t+j\tau}^{t+(j+1)\tau}(s-t)^{l-1}\lambda^{-s/\tau}F(s)ds$$
$$\le ML\tau^{l-1}\int_{t+l\tau}^{\infty}s^{l-1}\lambda^{-s/\tau}F(s)ds\,,\qquad t\ge T.$$

B) Proof of Theorem I_λ $(0<\lambda<1)$. We make use of the same notation as in the proof for the case $\lambda>1$ (§6). Let $j\in\{0,1,\dots,m-1\}$. Since (6.5) holds, by Lemma 7.1 ($l=m-j$, $p=n-1$) we have

$$\Psi^{m-j}(\lambda^{-t/\tau}I_{0,n}(t,T;F^*))=o(1)\qquad\text{as } t\to\infty\,,$$

where

$$I_{0,n}(t,T;F^*)=\int_t^\infty\frac{(s-t)^{n-1}}{(n-1)!}F^*(s)ds\,,\qquad t\ge T,$$

which implies (apply Lemma 3.2 with $l=j$, $p=0$)

$$\Phi^j\Psi^{m-j}(\lambda^{-t/\tau}I_{0,n}(t,T;F^*))=o(t^j)\qquad\text{as } t\to\infty\,.$$

Thus $T>t_0$ can be chosen so that (2.8) holds and

$$\Phi^j\Psi^{m-j}(\lambda^{-t/\tau}I_{0,n}(t,T;F^*))<\varepsilon_0t^j\,,\qquad t\ge T-m\tau\,,$$

where $\varepsilon_0=a-\max_t|\omega(t)|>0$. From this point one proceeds exactly as in Subsection C of §6, except that the function $v^*(t)$ defining Y^* is replaced by $v^*(t)=\lambda^{-t/\tau}I_{0,n}(t,T;F^*)$ and $\mathscr{F}_2^*$ is given by

$$\mathscr{F}_2^*x(t)=(-1)^{n-1}\lambda^{-t/\tau}I_{0,n}(t,T;f^*(t,x^*(g(t))))\,,\qquad t\ge T,$$

to prove the existence of a fixed point of $\mathscr{F}^*$ in $X^*\times Y^*$, the first component of which gives rise to a solution of the equation (A) satisfying (5.2). The details may be omitted.

8. Proof of Theorem II_λ

A) Lemmas. We shall prove the existence of nonoscillatory solutions of type (II_λ) of (A) via Schauder-Tychonoff fixed point theorem. The following lemma is needed for this purpose.

LEMMA 8.1. *Let* $l \in N$, $p \in N \cup \{0\}$ *and* $G \in C[T, \infty)$.
(i) *If* $\lambda > 1$ *and* $G(t) = o(t^p \lambda^{-t/\tau})$ *as* $t \to \infty$, *then*

$$\Psi^l G(t) = o(t^p \lambda^{-t/\tau}) \qquad \text{as } t \to \infty .$$

(ii) *If* $0 < \lambda < 1$ *and* $G(t) = o(t^p \lambda^{-t/\tau})$ *as* $t \to \infty$, *then*

$$\Phi^l G(t) = o(t^p \lambda^{-t/\tau}) \qquad \text{as } t \to \infty .$$

The second statement of Lemma 8.1 follows from Lemma 3.3. In fact, repeated application of Lemma 3.3 shows that

$$\lim_{t \to \infty} \frac{\Phi^l G(t)}{t^p \lambda^{-t/\tau}} = \lim_{t \to \infty} \frac{\Delta^l \Phi^l G(t)}{\Delta^l (t^p \lambda^{-t/\tau})} = \lim_{t \to \infty} \frac{G(t)}{(1-\lambda)^l t^p \lambda^{-t/\tau}} = 0 ,$$

since $\Delta^l(t^p \lambda^{-t/\tau}) = (1-\lambda)^l t^p \lambda^{-t/\tau} + o(t^p \lambda^{-t/\tau})$ as $t \to \infty$.

To prove the first statement of Lemma 8.1 we need another l'Hospital's rule for differences.

LEMMA 8.2. *Let* α, $\beta \in C[T-\tau, \infty)$ *be functions such that*

$$\Delta\beta(t) \neq 0 \qquad \text{and} \qquad \lim_{t \to \infty} \alpha(t) = \lim_{t \to \infty} \beta(t) = 0 .$$

Then

$$\lim_{t \to \infty} \frac{\Delta\alpha(t)}{\Delta\beta(t)} = c \in \boldsymbol{R} \qquad \text{implies} \ \lim_{t \to \infty} \frac{\alpha(t)}{\beta(t)} = c .$$

Suppose that $\lambda > 1$ and $G(t) = o(t^p \lambda^{-t/\tau})$ as $t \to \infty$. Then it is clear that $G \in S[T, \infty)$ and so $\Psi G(t) = o(1)$ as $t \to \infty$ by Lemma 2.1. Applying Lemma 8.2 and noting that

$$\Delta(t^p \lambda^{-t/\tau}) = -(\lambda - 1) t^p \lambda^{-t/\tau} + o(t^p \lambda^{-t/\tau}) \qquad \text{as } t \to \infty ,$$

we see that

$$\lim_{t \to \infty} \frac{\Psi G(t)}{t^p \lambda^{-t/\tau}} = \lim_{t \to \infty} \frac{\Delta \Psi G(t)}{\Delta(t^p \lambda^{-t/\tau})} = \lim_{t \to \infty} \frac{G(t)}{(\lambda - 1) t^p \lambda^{-t/\tau}} = 0 ,$$

which implies that $\Psi G(t) = o(t^p \lambda^{-t/\tau})$ as $t \to \infty$. The above argument applied to $\Psi G(t)$ shows that $\Psi^2 G(t) = o(t^p \lambda^{-t/\tau})$ as $t \to \infty$. Thus we are led to the desired conclusion of (i) of Lemma 8.1 in finite steps.

PROOF OF LEMMA 8.2. We may assume that $\Delta\beta(t) < 0$ without loss of generality. Let $t \geq T$ be fixed and put

$$a_n = \alpha(t + n\tau) , \qquad b_n = \beta(t + n\tau) , \qquad n = 0, 1, 2, \ldots .$$

Then $\{b_n\}$ is strictly decreasing and $\lim_{t\to\infty} a_n = \lim_{t\to\infty} b_n = 0$. From Lemma 7 of [1] it follows that

$$\inf_{l\geq 1} \frac{a_l - a_{l-1}}{b_l - b_{l-1}} \leq \frac{a_n - a_0}{b_n - b_0} \leq \sup_{l\geq 1} \frac{a_l - a_{l-1}}{b_l - b_{l-1}}, \qquad n \geq 2\,. \tag{8.1}$$

Choose $p \in \boldsymbol{N}$ so that $\sup_{n\geq p} (b_n/b_0) < 1$. From (8.1) and the relation

$$\frac{a_0}{b_0} = \frac{a_n - a_0}{b_n - b_0}\left(1 - \frac{b_n}{b_0}\right) + \frac{a_n}{b_0}, \tag{8.2}$$

we have

$$\frac{a_0}{b_0} \leq \sup_{l\geq 1} \frac{a_l - a_{l-1}}{b_l - b_{l-1}} \cdot \left(1 - \frac{b_n}{b_0}\right) + \left|\frac{a_n}{b_0}\right|, \qquad n \geq p\,,$$

which gives in the limit as $n \to \infty$

$$\frac{a_0}{b_0} \leq \sup_{l\geq 1} \frac{a_l - a_{l-1}}{b_l - b_{l-1}}\,.$$

Since $a_0 = \alpha(t)$, $b_0 = \beta(t)$, $a_l - a_{l-1} = \Delta\alpha(t + l\tau)$, $b_l - b_{l-1} = \Delta\beta(t + l\tau)$, this implies

$$\frac{\alpha(t)}{\beta(t)} \leq \sup_{s\geq t} \frac{\Delta\alpha(s)}{\Delta\beta(s)},$$

so that

$$\sup_{s\geq t} \frac{\alpha(s)}{\beta(s)} \leq \sup_{s\geq t} \frac{\Delta\alpha(s)}{\Delta\beta(s)}\,. \tag{8.3}$$

On the other hand, letting $n \to \infty$ in the inequality

$$\frac{a_0}{b_0} \leq \inf_{l\geq 1} \frac{a_l - a_{l-1}}{b_l - b_{l-1}} \cdot \left(1 - \frac{b_n}{b_0}\right) - \left|\frac{a_n}{b_0}\right|, \qquad n \geq p\,,$$

which follows from (8.2), we obtain

$$\frac{a_0}{b_0} \geq \inf_{l\geq 1} \frac{a_l - a_{l-1}}{b_l - b_{l-1}},$$

which shows that

$$\inf_{s\geq t} \frac{\alpha(s)}{\beta(s)} \geq \inf_{s\geq t} \frac{\Delta\alpha(s)}{\Delta\beta(s)}\,. \tag{8.4}$$

From (8.3) and (8.4) it follows that

$$\liminf_{t\to\infty} \frac{\Delta\alpha(t)}{\Delta\beta(t)} \le \liminf_{t\to\infty} \frac{\alpha(t)}{\beta(t)} \le \limsup_{t\to\infty} \frac{\alpha(t)}{\beta(t)} \le \limsup_{t\to\infty} \frac{\Delta\alpha(t)}{\Delta\beta(t)},$$

which implies the truth of the first statement of Lemma 8.1.

B) PROOF OF THEOREM II_λ (The case $\lambda > 1$). Suppose that $\lambda > 1$. Let $k \in \{0, 1, \dots, n-1\}$. Put $F^*(t) = F^*(t, a[g(t)]^k \lambda^{-g(t)/\tau})$, where $F^*(t, x^*) = F(t, \lambda^{g(t)/\tau} x^*)$. The condition (5.3) then becomes

$$\int_{t_0}^{\infty} t^{n-k-1} F^*(t) dt < \infty,$$

and so the function $\lambda^{-t/\tau} I_{k,n-k}(t, T; F^*)$ (cf. (4.1)) has the property

$$\lambda^{-t/\tau} I_{k,n-k}(t, T; F^*) = o(t^k \lambda^{-t/\tau}) \qquad \text{as } t \to \infty.$$

From the first statement of Lemma 8.1 ($l = m$, $p = k$) we see that there is $T > t_0$ such that (2.8) holds and

$$\Psi^m(\lambda^{-t/\tau} I_{k,n-k}(t, T; F^*)) < \varepsilon_0 t^k \lambda^{-t/\tau}, \qquad t \ge T - m\tau,$$

where $\varepsilon_0 = a - |c|$.

Let us define the sets $X^* \subset C[T_*, \infty)$, $Y^* \subset C[T, \infty)$ and the mappings $\mathscr{F}_1^*: Y^* \to C[T_*, \infty)$, $\mathscr{F}_2^*: X^* \to C[T, \infty)$ as follows:

$$X^* = \{x^* \in C[T_*, \infty): |x^*(t)| \le a t^k \lambda^{-t/\tau}, t \ge T_*\},$$

$$Y^* = \{y^* \in C[T, \infty): |y^*(t)| \le |v^*(t)|, |y^*(t) - y^*(s)| \le |v^*(t) - v^*(s)|, s, t, \ge T\},$$

where $v^*(t) = \lambda^{-t/\tau} I_{k,n-k}(t, T; F^*)$,

$$\mathscr{F}_1^* y^*(t) = \begin{cases} c t^k \lambda^{-t/\tau} + (-1)^m \Psi^m y^*(t), & t \ge T - m\tau, \\ \mathscr{F}_1^* y^*(T - m\tau) \dfrac{t^k \lambda^{-t/\tau}}{(T - m\tau)^k \lambda^{-(T-m\tau)/\tau}}, & T_* \le t \le T - m\tau, \end{cases}$$

$$\mathscr{F}_2^* x^*(t) = (-1)^{n-k-1} \lambda^{-t/\tau} I_{k,n-k}(t, T; f^*(t, x^*(g(t)))), \qquad t \ge T.$$

Then, proceeding as before, we are able to apply the Schauder-Tychonoff theorem to conclude that the mapping $\mathscr{F}^*: X^* \times Y^* \to C[T_*, \infty) \times C[T, \infty)$ defined by

$$\mathscr{F}^*(x^*, y^*) = (\mathscr{F}_1^* y^*, \mathscr{F}_2^* x^*), \qquad (x^*, y^*) \in X^* \times Y^*,$$

possesses a fixed element $(x^*, y^*) \in X^* \times Y^*$ which satisfies

$$x^*(t) = c t^k \lambda^{-t/\tau} + (-1)^m \Psi^m y^*(t), \qquad t \ge T - m\tau,$$

$$y^*(t) = (-1)^{n-k-1} \lambda^{-t/\tau} I_{k,n-k}(t, T; f^*(t, x^*(g(t)))), \qquad t \ge T,$$

it follows that $x^*(t)$ and $y^*(t)$ satisfy (B*), and so the function $x(t) = \lambda^{t/\tau}x^*(t)$ is a solution of (A) for $t \geq T$. Since $\lambda^{t/\tau}\Psi^m y^*(t) = o(t^k)$ as $t \to \infty$ by (i) of Lemma 8.1, the solution $x(t)$ has the asymptotic property (5.4).

C) PROOF OF THEOREM II_λ (The case $0 < \lambda < 1$). Suppose that $0 < \lambda < 1$ and let $k \in \{0, 1, \dots, n-1\}$. Let $F^*(t)$ be as in the preceding subsection and take $T > t_0$ so that (2.8) holds and

$$\Phi^m(\lambda^{-t/\tau} I_{k,n-k}(t, T; F^*)) < \varepsilon_0 t^k \lambda^{-t/\tau}, \qquad t \geq T - m\tau,$$

where $\varepsilon_0 = a - |c|$. Such a choice of T is possible because of the second statement of Lemma 8.1 ($l = m, p = k$).

Let X^* and Y^* be the sets of continuous functions defined exactly as above. If we define the mappings $\mathscr{F}_1^*: Y^* \to C[T_*, \infty)$ and $\mathscr{F}_2^*: X^* \to C[T, \infty)$ by

$$\mathscr{F}_1^* y^*(t) = \begin{cases} ct^k\lambda^{-t/\tau} + \Phi^m y^*(t), & t \geq T - m\tau, \\ \mathscr{F}_1^* y^*(T - m\tau)\dfrac{t^k\lambda^{-t/\tau}}{(T - m\tau)^k\lambda^{-(T-m\tau)/\tau}}, & T_* \leq t \leq T - m\tau, \end{cases}$$

$$\mathscr{F}_2^* x^*(t) = (-1)^{n-k-1}\lambda^{-t/\tau} I_{k,n-k}(t, T; f^*(t, x^*(g(t)))), \qquad t \geq T,$$

then it can be shown in a routine manner that there exist functions $x^* \in X^*$ and $y^* \in Y^*$ such that $x^* = \mathscr{F}_1^* y^*$ and $y^* = \mathscr{F}_2^* x^*$ and that the function x^* gives rise to a solution $x(t) = \lambda^{t/\tau}x^*(t)$ of the equation (A) for $t \geq T$. From (ii) of Lemma 8.1 it follows that

$$x(t) = ct^k + \lambda^{t/\tau}\Phi^m y^*(t) = t^k[c + o(1)) \qquad \text{as } t \to \infty.$$

This completes the proof of Theorem II_λ for the case $0 < \lambda < 1$.

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