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Oscillation of differential equation of neutral type

Dedicated to Professor Takaŝi Kusano on his sixtieth birthday

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1. Introduction

Consider the neutral differential equation

$$\frac{d}{dt}[x(t) - p(t)x(\sigma(t))] + q(t)x(\tau(t)) = 0\,, \qquad t \geq t_0\,, \tag{1}$$

under the standing hypotheses that:

(a) $p \in C\ [[t_0, \infty), (0, \infty)]$;

(b) $\sigma \in C\ [[t_0, \infty), R]$, σ is strictly increasing and $\lim_{t\to\infty} \sigma(t) = \infty$;

(c) $q \in C\ [[t_0, \infty), R]$, $q(t) \not\equiv 0$;

(d) $\tau \in C\ [[t_0, \infty), R]$, $\lim_{t\to\infty} \tau(t) = \infty$.

Our aim in this paper is to obtain sufficient conditions for the oscillation of all solutions of equation (1). The asymptotic behaviour of the solutions of equation (1) is also studied.

By a solution of equation (1) we mean a continuous function $x: [T_x, \infty) \to R$ such that $x(t) - p(t)x(\sigma(t))$ is continuously differentiable and $x(t)$ satisfies equation (1) for all sufficiently large $t > T_x$. The solutions which vanish for all large t will be excluded from our consideration. A solution of (1) is said to be oscillatory if it has an infinite sequence of zeros tending to infinity; otherwise a solution is said to be nonoscillatory.

The problem of oscillation and nonoscillation for neutral differential equations has received considerable attention in recent years; see e.g. [1–7, 9] and the references cited therein. However some results in this paper are new and the other ones in many cases complete the previous ones.

2. Some basic lemmas

The following lemmas will be useful in the proofs of the main results.

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LEMMA 1. *In addition to the conditions* (a) *and* (b) *suppose that*

$$0 < p(t) \leq 1 \qquad \text{for } t \geq t_0 .$$

Let $x(t)$ be a continuous nonoscillatory solution of the functional inequality

$$x(t)[x(t) - p(t)x(\sigma(t))] < 0$$

defined in a neighborhood of infinity.

(i) *Suppose that $\sigma(t) < t$ for $t \geq t_0$. Then $x(t)$ is bounded. If, moreover,*

$$0 < p(t) \leq \lambda^* < 1 , \qquad t \geq t_0 ,$$

for some positive constant λ^*, then $\lim_{t\to\infty} x(t) = 0$.

(ii) *Suppose that $\sigma(t) > t$ for $t \geq t_0$. Then $x(t)$ is bounded away from zero, that is, there exists a positive constant c such that $|x(t)| \geq c$ for all large t.*

LEMMA 2. *In addition to* (a) *and* (b) *suppose that*

$$1 \leq p(t) \qquad \text{for } t \geq t_0 .$$

Let $x(t)$ be a continuous nonoscillatory solution of the functional inequality

$$x(t)[x(t) - p(t)x(\sigma(t))] > 0$$

defined in a neighborhood of infinity.

(i) *Suppose that $\sigma(t) > t$ for $t \geq t_0$. Then $x(t)$ is bounded. If, moreover,*

$$1 < \lambda_* \leq p(t) , \qquad t \geq t_0 ,$$

for some positive constant λ_*, then $\lim_{t\to\infty} x(t) = 0$.

(ii) *Suppose that $\sigma(t) < t$ for $t \geq t_0$. Then $x(t)$ is bounded away from zero.*

The above Lemmas and their proofs can be found in [6].

The next two Lemmas can be derived on the base of Theorem 2 in [8].

LEMMA 3. *Assume that*

$$g \in C[[t_0, \infty), [0, \infty)] , \qquad \delta \in C[[t_0, \infty), R] , \qquad \lim_{t\to\infty} \delta(t) = \infty \tag{2}$$

and

$$\delta(t) < t \qquad \text{for } t \geq t_0 ,$$

$$\liminf_{t\to\infty} \int_{\delta(t)}^{t} g(s)ds > \frac{1}{e} .$$

Then the functional inequality

$$x'(t) + g(t)x(\delta(t)) \leq 0 , \qquad t \geq t_0 ,$$

cannot have an eventually positive solution, and

$$x'(t) + g(t)x(\delta(t)) \geq 0\,, \qquad t \geq t_0\,,$$

cannot have an eventually negative solution.

LEMMA 4. *Assume that* (2) *is satisfied and*

$$\delta(t) > t \qquad \text{for } t \geq t_0\,,$$

$$\liminf_{t\to\infty} \int_t^{\delta(t)} g(s)ds > \frac{1}{e}\,.$$

Then the functional inequality

$$x'(t) - g(t)x(\delta(t)) \geq 0\,, \qquad t \geq t_0\,,$$

has no eventually positive solution, and

$$x'(t) - g(t)x(\delta(t)) \leq 0\,, \qquad t \geq t_0\,,$$

has no eventually negative solution.

3. Oscillation of equation (1)

We shall study the oscillation of all solutions of equation (1).

THEOREM 1. *Suppose that*

$0 < p(t) \leq \lambda^* < 1$, $\quad 0 \leq q(t)$, $\quad \sigma(t) < t$, $\quad \tau(t) < t \quad$ *for* $t \geq t_0$ *and some* $\lambda^* \in (0, 1)$ *and*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} q(s)ds > \frac{1}{e}\,. \tag{3}$$

Then every solution of equation (1) *is oscillatory.*

PROOF. Suppose that $x(t)$ is an eventually positive solution of equation (1). Then

$$\frac{d}{dt}[x(t) - p(t)x(\sigma(t))] \leq 0$$

for all large t. Thus for sufficiently large t_0 we have two cases:

1. $x(t) - p(t)x(\sigma(t)) > 0\,, \qquad t \geq t_0\,;$
2. $x(t) - p(t)x(\sigma(t)) < 0\,, \qquad t \geq t_0\,.$

Set

$$u(t) = x(t) - p(t)x(\sigma(t)).$$

Case 1. One can find that

$$q(t)u(\tau(t)) = q(t)x(\tau(t)) - p(\tau(t))q(t)x(\sigma(\tau(t)))$$

and according to (1) we obtain

$$u'(t) + q(t)u(\tau(t)) \le 0, \qquad t \ge t_0.$$

From Lemma 3 it follows that $u(t) \le 0$, which is a contradiction to $u(t) > 0$.

Case 2. From Lemma 1 it follows that

$$\lim_{t\to\infty} x(t) = 0.$$

This implies that $\lim_{t\to\infty} u(t) = 0$, which contradicts the fact that $u(t) < 0$ and $u'(t) \le 0$ for $t \ge t_0$. The proof is complete.

In the next theorems $\sigma^{-1}(t)$ will denote the inverse function of $\sigma(t)$.

THEOREM 2. *Suppose that*

$$1 \le p(t), \qquad 0 \le q(t), \qquad \sigma(t) < t, \qquad \sigma^{-1}(\tau(t)) > t \qquad \textit{for } t \ge t_0 \textit{ and}$$

$$\liminf_{t\to\infty} \int_t^{\sigma^{-1}(\tau(t))} \frac{q(s)}{p(\sigma^{-1}(\tau(s)))} ds > \frac{1}{e}. \tag{3'}$$

Then every solution of equation (1) *is oscillatory.*

PROOF. Assume for the sake of contradiction that $x(t)$ is an eventually positive solution of (1). Then

$$\frac{d}{dt}[x(t) - p(t)x(\sigma(t))] \le 0$$

for all large t. Thus for sufficiently large t_0 we have two cases:

1. $x(t) - p(t)x(\sigma(t)) > 0, \qquad t \ge t_0;$
2. $x(t) - p(t)x(\sigma(t)) < 0, \qquad t \ge t_0.$

Set

$$u(t) = x(t) - p(t)x(\sigma(t)).$$

Case 1. Integrating (1) from T to t, $T \ge t_0$, and then letting $t \to \infty$, we get

$$u(T) \ge \int_T^\infty q(t)x(\tau(t))dt. \tag{4}$$

By Lemma 2, $x(t)$ is bounded away from zero for all large t. The condition (3′) implies that

$$\int_T^\infty q(t)dt = \infty .$$

Otherwise if

$$\int_T^\infty q(t)dt < \infty ,$$

we can choose $t_1 \geq T$ such large that

$$\int_{t_1}^\infty q(t)dt < \frac{1}{e} .$$

The above facts and (4) give a contradiction.

Case 2. Set $\gamma(t) = \sigma^{-1}(\tau(t))$. Then

$$-\frac{q(t)}{p(\gamma(t))}u(\gamma(t)) = -\frac{q(t)}{p(\gamma(t))}x(\gamma(t)) + q(t)x(\tau(t))$$

and with regard to (1) we have

$$u'(t) - \frac{q(t)}{p(\gamma(t))}u(\gamma(t)) \leq 0 , \qquad t \geq t_0 ,$$

which implies by Lemma 4 that $u(t) \geq 0$ and this contradicts the fact that $u(t) < 0$. The proof is complete.

THEOREM 3. *Suppose that*

$$0 < p(t) , \qquad 0 \leq q(t) , \qquad \sigma^{-1}(\tau(t)) > t , \qquad \tau(t) < t \qquad \textit{for } t \geq t_0 ,$$

and conditions (3), (3′) *hold.*

Then every solution of equation (1) *is oscillatory.*

PROOF. Assume for the sake of contradiction that $x(t)$ is an eventually positive solution of equation (1). Then

$$\frac{d}{dt}[x(t) - p(t)x(\sigma(t))] \leq 0$$

for all large t. For sufficiently large t_0 we have two cases:

1. $x(t) - p(t)x(\sigma(t)) > 0 , \qquad t \geq t_0 ;$
2. $x(t) - p(t)x(\sigma(t)) < 0 , \qquad t \geq t_0 .$

Now we can treat the case 1 in the same way as in the proof of Theorem 1 and the case 2 in the same way as in the proof of Theorem 2. The proof is complete.

To the above theorems we can establish the next three theorems. Their proofs, with using Lemmas 1–4, are similar to the previous ones and will be omitted.

THEOREM 4. *Suppose that*

$$0 < p(t) \le 1\,, \qquad q(t) \le 0\,, \qquad \sigma(t) > t\,, \qquad \tau(t) > t \qquad \textit{for } t \ge t_0 \textit{ and}$$

$$\liminf_{t\to\infty} \int_t^{\tau(t)} |q(s)|\,ds > \frac{1}{e}\,. \tag{5}$$

Then every solution of equation (1) *is oscillatory.*

THEOREM 5. *Suppose that*

$$1 < \lambda_* \le p(t)\,, \qquad q(t) \le 0\,, \qquad \sigma(t) > t\,, \qquad \sigma^{-1}(\tau(t)) < t \qquad \textit{for } t \ge t_0$$

and some $\lambda_* \in (1, \infty)$,

$$\liminf_{t\to\infty} \int_{\sigma^{-1}(\tau(t))}^{t} \frac{|q(s)|}{p(\sigma^{-1}(\tau(t)))}\,ds > \frac{1}{e}\,. \tag{6}$$

Then every solution of equation (1) *is oscillatory.*

THEOREM 6. *Suppose that*

$$0 < p(t)\,, \qquad q(t) \le 0\,, \qquad \sigma^{-1}(\tau(t)) < t\,, \qquad \tau(t) > t \qquad \textit{for } t \ge t_0\,,$$

and conditions (5), (6) *hold.*

Then every solution of equation (1) *is oscillatory.*

4. Asymptotic behaviour

In this section we shall study the asymptotic behaviour of the non-oscillatory solutions of the equation (1).

THEOREM 7. *Suppose that*

$$1 < \lambda_* \le p(t)\,, \qquad q(t) \le 0\,, \qquad \sigma(t) > t \qquad \textit{for } t \ge t_0$$

and some $\lambda_* \in (1, \infty)$,

$$\int_{t_0}^{\infty} \frac{|q(t)|}{p(\sigma^{-1}(\tau(t)))}\,dt = \infty\,.$$

Then every nonoscillatory solution of equation (1) *tends to zero as* $t \to \infty$.

PROOF. We may assume that $x(t)$ is an eventually positive solution of (1). Then

$$\frac{d}{dt}[x(t) - p(t)x(\sigma(t))] \geq 0$$

for all large t. For sufficiently large t_0 we have two cases:

$$1. \quad x(t) - p(t)x(\sigma(t)) > 0\,, \qquad t \geq t_0\,;$$

$$2. \quad x(t) - p(t)x(\sigma(t)) < 0\,, \qquad t \geq t_0\,.$$

Set

$$u(t) = x(t) - p(t)x(\sigma(t))\,.$$

Case 1. From Lemma 2 it follows that $\lim_{t\to\infty} x(t) = 0$. This implies, with regard to the inequality $x(t) > p(t)x(\sigma(t))$, that $\lim_{t\to\infty} u(t) = 0$, which is impossible since $u(t) > 0$ and $u'(t) \geq 0$ for $t \geq t_0$.

Case 2. We have $u(t) \to L \leq 0$ as $t \to \infty$. Suppose that $L < 0$. We get

$$L \geq u(t) > -p(t)x(\sigma(t))$$

or

$$L > -p(t)x(\sigma(t))\,.$$

Then we have

$$L > -p(\sigma^{-1}(\tau(t)))x(\tau(t))$$

and

$$x(\tau(t)) > -\frac{L}{p(\sigma^{-1}(\tau(t)))}$$

for $t \geq t_1 \geq t_0$, where t_1 is sufficiently large.

From (1) it follows that

$$-u(T) > -\int_T^\infty q(t)x(\tau(t))dt > L\int_T^\infty \frac{q(t)}{p(\sigma^{-1}(\tau(t)))}dt\,,$$

$T \geq t_1$, which is impossible. Thus $u(t) \to 0$ as $t \to \infty$.

Now we show that $x(t)$ is bounded. Otherwise there exists a sequence $t_1, t_2, \ldots$ with the following properties:

(a) $t_m \to \infty$ as $m \to \infty$; (b) $x(t_m) \to \infty$ as $m \to \infty$;

(c) $x(\sigma(t_m)) = \max\{x(\sigma(s)) \colon t_0 \leq \sigma(s) \leq \sigma(t_m)\}$.

But for sufficiently large t_m we have

$$-1 < u(t_m) \le x(t_m) - \lambda_* x(\sigma(t_m)) \le x(t_m) - \lambda_* x(t_m),$$

from which it follows that

$$x(t_m) < \frac{1}{\lambda_* - 1}$$

which contradicts the above condition (b). Thus $x(t)$ is bounded.

Now suppose it is not true that $\lim_{t\to\infty} x(t) = 0$. Set $\limsup_{t\to\infty} x(t) = c > 0$. This ensures the existence of an increasing sequence $\{t_n\}$ with $t_n \to \infty$ and $x(t_n) \to c$ as $n \to \infty$. Then we have

$$u(t_n) = x(t_n) - p(t_n)x(\sigma(t_n)) \le x(t_n) - \lambda_* x(\sigma(t_n)).$$

Since $c > 0$ and $\lambda_* \in (1, \infty)$, there exists a positive number ε such that $\lambda_*(c - \varepsilon) > c + \varepsilon$ and with regard to the inequalities $x(t_n) < c + \varepsilon$, $x(\sigma(t_n)) > c - \varepsilon$ for all sufficiently large n, we have

$$u(t_n) < c + \varepsilon - \lambda_*(c - \varepsilon)$$

for all such n. Since $u(t_n) \to 0$ as $n \to \infty$, we obtain a contradiction by letting $n \to \infty$ in the last inequality. We conclude that $x(t) \to 0$ as $t \to \infty$. The proof is complete.

THEOREM 8. *Suppose that*

$$0 < p(t) \le \lambda^* < 1, \qquad 0 \le q(t), \qquad \sigma(t) < t \qquad \text{for } t \ge t_0$$

and some $\lambda^* \in (0, 1)$,

$$\int_{t_0}^{\infty} q(t)dt = \infty.$$

Then every nonoscillatory solution of equation (1) *tends to zero as* $t \to \infty$.

The proof of Theorem 8 is similar to the one of Theorem 7 and will be omitted.

THEOREM 9. *Suppose that*

$$1 < \lambda_* \le p(t), \qquad 0 \le q(t), \qquad \sigma(t) > t, \qquad \sigma^{-1}(\tau(t)) > t \qquad \text{for } t \ge t_0$$

and some $\lambda_* \in (1, \infty)$, *and condition* (3′) *holds.*

Then every nonoscillatory solution of equation (1) *tends to zero as* $t \to \infty$.

PROOF. Without lack of generality we may suppose that $x(t)$ is an eventually positive solution of equation (1). Then

$$\frac{d}{dt}[x(t) - p(t)x(\sigma(t))] < 0$$

for all large t. For sufficiently large t_0 we have two cases:

1. $x(t) - p(t)x(\sigma(t)) > 0\,, \qquad t \geq t_0\,;$
2. $x(t) - p(t)x(\sigma(t)) < 0\,, \qquad t \geq t_0\,.$

Case 1. From Lemma 2 it follows that

$$\lim_{t\to\infty} x(t) = 0\,.$$

Case 2. Set

$$u(t) = x(t) - p(t)x(\sigma(t))\,.$$

Then we have

$$u'(t) - \frac{q(t)}{p(\sigma^{-1}(\tau(t)))}u(\sigma^{-1}(\tau(t))) \leq 0\,, \qquad t \geq t_0\,.$$

From Lemma 4 it follows that $u(t) \geq 0$, which is a contradiction.

THEOREM 10. *Suppose that*

$$0 < p(t) \leq \lambda^* < 1\,, \qquad q(t) \leq 0\,, \qquad \sigma(t) < t\,, \qquad \tau(t) > t \qquad \textit{for } t \geq t_0$$

and some $\lambda^* \in (0, 1)$, *and condition* (5) *holds.*
Then every nonoscillatory solution of equation (1) *tends to zero as* $t \to \infty$.

The proof is similar to the one of Theorem 9 with using Lemmas 4 and 1.

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