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Simultaneous confidence procedures for multiple comparisons of mean vectors in multivariate normal populations

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0. Introduction

The study of the subjects of multiple comparisons under univariate and multivariate statistical analyses has been done by many authors. Reviews on some aspects of multiple comparison procedure have been given in Miller [22]. We refer to Miller [23, 24], Hochberg & Tamhane [11], etc. for the univariate case and to Roy & Bose [27], Krishnaiah & Reising [20] and Krishnaiah [18, 19], and so on for the multivariate case. This paper is concerned with multiple comparisons of correlated mean vectors under the multivariate normal populations. One of the important problems is to construct the simultaneous confidence intervals for the multivariate multiple comparisons with the given simultaneous confidence level in unbalanced models. In multivariate setting, however, it is difficult to find the exact simultaneous confidence intervals even in balanced models. In order to respond to the problem, we discuss the approximation procedures to obtain the good approximate simultaneous confidence intervals. We also discuss the approximation to the simultaneous confidence intervals in a GMANOVA model which is a useful basis in the analysis of data on growth curve. First, in order to achieve the purpose, it is necessary to find the upper percentiles of the key statistics which play an important role in constructing the simultaneous confidence intervals. This is what is called the generalized $T^2_{\max}$-type statistic. In Section 1, the approximate upper percentiles of the statistics are given for multiple comparisons among pairwise treatments and for comparisons among treatments with a control, respectively, and the accuracy of the estimated percentiles is investigted by Monte Carlo simulations. On the approximate procedure, we adopt the modified second approximation procedure which have been referred to Siotani [32, 33], Seo & Siotani [29]. The modified second approximation procedure based on Bonferroni's inequalities offers an attractive and an intuitive approach to produce fairly accurate approximations, though the accuracy of the approximations depends on parameters. Further, in Section 2, the approximate simultaneous confidence intervals for multiple comparisons among the components of treatment vectors are discussed. In Section 3, we consider a multivariate version of the generalized Tukey-type conjecture in which the procedure yields the conservative simultaneous confidence intervals. The Tukey-Kramer (T-K) procedure (Tukey [35], Kramer [16, 17]) offers a very simple and practicable solution to the problem of making pairwise comparisons in univariate unbalanced cases. Seo, Mano & Fujikoshi [31] have established the generalized Tukey conjecture for pairwise comparisons among mean vectors in multivariate setting, and they proved the conservativeness in the case of three correlated mean vectors. Here we consider the generalized Tukey-type conjecture in the comparisons with a control, and discuss the conservative simultaneous confidence intervals. In Section 4, we discussed multiple comparisons in the GMANOVA model which are useful in the analysis of data on growth curve.

1. Multiple comparisons among correlated mean vectors

Consider the simultaneous confidence intervals for multiple comparisons among mean vectors. Let $\boldsymbol{M} = [\boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_k]$ be the matrix of k p-dimensional mean vectors corresponding to the k treatments. Let $\hat{\boldsymbol{M}} = [\hat{\boldsymbol{\mu}}_1, \cdots, \hat{\boldsymbol{\mu}}_k]$ be the estimator of $\boldsymbol{M}$ such that $\mathrm{vec}(\hat{\boldsymbol{M}})$ is distributed as $N_{kp}(\mathrm{vec}(\boldsymbol{M}), \boldsymbol{V} \otimes \boldsymbol{\Sigma})$, where $\mathrm{vec}(\cdot)$ denotes the column vector formed by stacking the columns of the matrix under each other, and $\boldsymbol{V}: k \times k$ and $\boldsymbol{\Sigma}: p \times p$ are a known and an unknown positive definite matrices, respectively. Let $\boldsymbol{S}(= [s_{ij}])$ be an unbiased estimator of $\boldsymbol{\Sigma}$ such that $\nu \boldsymbol{S}$ is independent of $\hat{\boldsymbol{M}}$ and is distributed as a Wishart distribution $W_p(\boldsymbol{\Sigma}, \nu)$.

Then, the usual simultaneous confidence intervals for multiple comparisons can be written as the form

$$\boldsymbol{a}'\boldsymbol{M}\boldsymbol{b} \in [\boldsymbol{a}'\hat{\boldsymbol{M}}\boldsymbol{b} \pm t(\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b})^{1/2}(\boldsymbol{a}'\boldsymbol{S}\boldsymbol{a})^{1/2}], \qquad \forall \boldsymbol{a} \in \mathbb{R}^p,\ \forall \boldsymbol{b} \in \mathbb{B}^k, \tag{1.1}$$

where $\mathbb{R}^p$ is the set of any nonzero real p-dimensional vectors and $\mathbb{B}^k$ is a subset that consists of r vectors in the k-dimensional space.

In order to make the simultaneous confidence intervals (1.1) with the confidence level, it is necessary to find the value $t(>0)$, which is usually satisfying

$$\Pr\{T_{\max}^2 > t^2\} = \alpha,$$

for given $\alpha(0 < \alpha < 1)$, where

$$T_{\max}^2 = \max_{\boldsymbol{b} \in \mathbb{B}^k} \left(\frac{\boldsymbol{b}'\boldsymbol{X}'\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{b}}{\boldsymbol{b}'\boldsymbol{V}\boldsymbol{b}} \right), \qquad \boldsymbol{X} \equiv \hat{\boldsymbol{M}} - \boldsymbol{M}. \tag{1.2}$$

The generalized $T^2_{\max}$-type statistic (1.2) which is an extension of the multivariate Studentized range statistic (see, Seo & Siotani [29]) is fundamental for pairwise comparisons among correlated mean vectors, and is also used for comparisons with a control when $V = I_k$ (see, Seo & Siotani [30]). In general, however, it is difficult to find the exact percentiles of the generalized $T^2_{\max}$-type statistic. Here we adopt the modified second approximation procedure based on Bonferroni's inequalities, which is explained in the next subsection, in order to obtain a practically good approximation. The accuracy of the approximation is investigated by Monte Carlo simulations for the selected parameters.

1.1. Modified second approximation procedure

This subsection attempts to give the modified second approximation procedure for a general multiple comparison. For references for the idea of this approximation procedure, see Seo & Siotani [29, 30].

Put $z_i = (b_i' V b_i)^{-1/2} X b_i$, $i = 1, \cdots, r$, where b_i's $i = 1, \cdots, r$ are given vectors. Let $\mathbb{B}^k = \{b_1, \cdots, b_r\}$, and let $t^2 \equiv t^2(\alpha; p, k, r, v, V)$ be the exact upper α percentile of the generalized $T^2_{\max}$-type statistic. Then z_i has the p-dimensional normal distribution with mean vector $\mathbf{0}$ and covariance matrix Σ.

On the basis of Bonferroni's inequalities for the generalized $T^2_{\max}$-type statistic; that is,

$$\sum_{i=1}^{r} \Pr\{z_i' S^{-1} z_i > t^2\} > \Pr\{T^2_{\max} > t^2\} > \sum_{i=1}^{r} \Pr\{z_i' S^{-1} z_i > t^2\} - \beta,$$

where

$$\beta = \sum\sum_{i<j} \Pr\{z_i' S^{-1} z_i > t^2,\ z_j' S^{-1} z_j > t^2\},$$

we define the first approximation to t^2 by t_1^2 satisfying $\sum_{i=1}^{r} \Pr\{z_i' S^{-1} z_i > t_1^2\} = \alpha$. Such t_1^2 can be determined by using the fact that $z_i' S^{-1} z_i$ is the Hotelling T^2-statistic with v degrees of freedom (d.f.); that is,

$$t_1^2 = \frac{vp}{v - p + 1} F_{p, v-p+1}\left(\frac{\alpha}{r}\right), \tag{1.3}$$

where $F_{p, v-p+1}(\alpha/r)$ is the upper α/r percentile of F-distribution with p and $v - p + 1$ d.f.'s.

The first approximation (1.3) to the upper α percentile of the generalized $T^2_{\max}$-type statistic is an overestimate and a conservative approximation. Note also that the first approximation $t_1^2 \equiv t_1^2(\alpha; p, k, r, v, V)$ does not depend on V, and hence $t_1/\sqrt{2}$ coincides with the corresponding first approximation of the multivariate Studentized range in Seo & Siotani [29].

The *modified second approximation* to t^2 is defined by t_M^2 satisfying

$$\sum_{i=1}^{r} \Pr\{z_i' S^{-1} z_i > t_M^2\} = \alpha + \beta_1,$$

where

$$\beta_1 = \sum\sum_{i<j} \Pr\{z_i' S^{-1} z_i > t_1^2,\ z_j' S^{-1} z_j > t_1^2\}. \tag{1.4}$$

Hence the modified second approximation t_M^2 can be written as

$$t_M^2 = \frac{\nu p}{\nu - p + 1} F_{p,\nu-p+1}\left(\frac{\alpha + \beta_1}{r}\right).$$

The modified second approximation consists of the form of the second approximation with the use of the first approximation. It may be noted that the modified second approximation is larger than the second approximation andsmaller than the first approximation. However, there is no theory about conservativeness of the modified second approximation.

If the modified second approximation is still an overestimate with significant excess, we can obtain an accurate approximation by repeating once more this procedure. This approximation is called the *doubly modified second approximation* given by

$$t_{DM}^2 = \frac{\nu p}{\nu - p + 1} F_{p,\nu-p+1}\left(\frac{\alpha + \beta_2}{r}\right),$$

where

$$\beta_2 = \sum\sum_{i<j} \Pr\{z_i' S^{-1} z_i > t_M^2,\ z_j' S^{-1} z_j > t_M^2\}.$$

Though we have to evaluate β_1 in order to obtain the modified second approximation t_M^2, it is difficult to obtain the exact evaluation of the probabilities in (1.4). As the large sample approximations, however, an asymptotic expansion formula up to the term of order ν^{-2} can be obtained, and is derived for pairwise comparisons and comparisons with a control, respectively, in the next subsections.

1.2. Pairwise comparisons and comparisons with a control

In many experimental situations, pairwise comparisons and comparisons with a control are standard for multiple comparisons. In this section, we evaluate the probability in (1.4) by asymptotic expansions in general form, so that the approximate simultaneous confidence intervals by the modified second approximation procedure are given in the cases of pairwise comparisons and comparisons with a control.

To evaluate the probability in (1.4), the following results are needed. We use the bivariate χ^2 distribution with p d.f. and correlation parameter ρ_{ij} (see, e.g., Siotani [32], Krishnaiah [18]) which has the following density function

$$f(v_i, v_j) = (1-\rho_{ij}^2)^{p/2} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2}p+m)}{m!\,\Gamma(\frac{1}{2}p)} \rho_{ij}^{2m} g_{p+2m}^*(v_i) g_{p+2m}^*(v_j),$$

where $g_k^*(v) = [2^{k/2}\Gamma(k/2)]^{-1} v^{k/2-1} e^{-v/2}$. Then, we have the following lemma.

LEMMA 1.1. *Suppose that* $\mathrm{vec}\,(\boldsymbol{X}) \sim N_{kp}(\boldsymbol{0}, \boldsymbol{V} \otimes \boldsymbol{\Sigma})$, *and let* $\boldsymbol{z}_i = (\boldsymbol{b}_i'\boldsymbol{V}\boldsymbol{b}_i)^{-1/2}\boldsymbol{X}\boldsymbol{b}_i$, $\boldsymbol{z}_j = (\boldsymbol{b}_j'\boldsymbol{V}\boldsymbol{b}_j)^{-1/2}\boldsymbol{X}\boldsymbol{b}_j$, $i \neq j$. *Then*

$$\begin{pmatrix} \boldsymbol{z}_i \\ \boldsymbol{z}_j \end{pmatrix} \sim N_{2p}\left(\begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}, \begin{bmatrix} \boldsymbol{\Sigma} & \rho_{ij}\boldsymbol{\Sigma} \\ \rho_{ij}\boldsymbol{\Sigma} & \boldsymbol{\Sigma} \end{bmatrix}\right),$$

where

$$\rho_{ij} = \frac{\boldsymbol{b}_i'\boldsymbol{V}\boldsymbol{b}_j}{(\boldsymbol{b}_i'\boldsymbol{V}\boldsymbol{b}_i)^{1/2}(\boldsymbol{b}_j'\boldsymbol{V}\boldsymbol{b}_j)^{1/2}},$$

and

$$v_i = \frac{1}{1-\rho_{ij}^2} \boldsymbol{z}_i'\boldsymbol{\Sigma}^{-1}\boldsymbol{z}_i, \quad v_j = \frac{1}{1-\rho_{ij}^2} \boldsymbol{z}_j'\boldsymbol{\Sigma}^{-1}\boldsymbol{z}_j$$

have the bivariate χ^2 *distribution with* p *d.f. and parameter* ρ_{ij}.

Further, from Lemma 1.1, for any fixed q, we have

$$\Pr\{\boldsymbol{z}_i'\boldsymbol{\Sigma}^{-1}\boldsymbol{z}_i > q^2, \boldsymbol{z}_j'\boldsymbol{\Sigma}^{-1}\boldsymbol{z}_j > q^2\} = (1-\rho_{ij}^2)^{p/2} \sum_{m=0}^{\infty} \frac{\Gamma(\frac{1}{2}p+m)}{m!\,\Gamma(\frac{1}{2}p)} \rho_{ij}^{2m} G_{p+2m}^{*2}(2\eta_{ij}),$$

where

$$\eta_{ij} = \frac{1}{2(1-\rho_{ij}^2)} q^2, \quad G_{p+2m}^*(\eta_{ij}) \equiv \int_{\eta_{ij}}^{\infty} g_{2(p+2m)}^*(v)\,dv.$$

Note also that

$$\Pr\{\boldsymbol{z}_i'\boldsymbol{S}^{-1}\boldsymbol{z}_i > q^2, \boldsymbol{z}_j'\boldsymbol{S}^{-1}\boldsymbol{z}_j > q^2\} = E_S[\Pr\{\boldsymbol{z}_i'\boldsymbol{S}^{-1}\boldsymbol{z}_i > q^2, \boldsymbol{z}_j'\boldsymbol{S}^{-1}\boldsymbol{z}_j > q^2 \mid \boldsymbol{S} \text{ is fixed}\}].$$

According to Welch [36] and James [13], we can make the Taylor expansion of the function of S inside of above brackets about $S=\Sigma$ and take the expectation of the resultant with respect to S. Thus we can use the following formula.

LEMMA 1.2. *If $f(S)$ is a function of S, where vS is distributed as $W_p(\Sigma, v)$, $v \geq p$, then*

$$E_S[f(S)] = \Theta \cdot f(\Omega)|_{\Omega=\Sigma}$$
$$= \Theta \cdot \Pr\{z_i'\Omega^{-1}z_i > q^2, z_j'\Omega^{-1}z_j > q^2\}|_{\Omega=\Sigma},$$

where

$$\Theta = \left|I_p - \frac{2}{v}\Sigma\partial\right|^{-v/2} \mathrm{etr}(-\Sigma\partial)$$
$$= 1 + v^{-1}\,\mathrm{tr}(\Sigma\partial)^2 + v^{-2}\left[\frac{4}{3}\mathrm{tr}(\Sigma\partial)^3 + \frac{1}{2}\{\mathrm{tr}(\Sigma\partial)^2\}^2\right] + \mathrm{O}(v^{-3}),$$

$\partial: p \times p \equiv [(1/2)(1+\delta_{ij})\partial/\partial\omega_{ij}]$, $\Omega: p \times p = [\omega_{ij}]$.

By Hotelling and Frankel [12], the upper percentile of the Hotelling's T^2 statistic has an asymptotic expansion such that

$$t_1^2 = c_0 + v^{-1}c_1 + v^{-2}c_2 + \mathrm{O}(v^{-3}),$$

where

$$c_0 = \chi^2,$$
$$c_1 = \frac{1}{2}\chi^2(\chi^2 + p),$$
$$c_2 = \frac{1}{24}\chi^2\{4\chi^4 + (13p-2)\chi^2 + 7p^2 - 4\},$$

$\chi^2 \equiv \chi^2(\alpha/r; p)$ is the upper α percentile of the χ^2 distribution with p d.f. Hence it follows from Lemma 1.2 that

$$\Pr\{z_i'S^{-1}z_i > t_1^2, z_j'S^{-1}z_j > t_1^2\}$$
$$= \Big[1 + v^{-1}\{c_1D + \mathrm{tr}(\Sigma\partial)^2\}$$
$$+ v^{-2}\left[c_2D + \frac{1}{2}c_1^2D^2 + c_1\,\mathrm{tr}(\Sigma\partial)^2D + \frac{4}{3}\mathrm{tr}(\Sigma\partial)^3 + \frac{1}{2}\{\mathrm{tr}(\Sigma\partial)^2\}^2\right]$$
$$+ \mathrm{O}(v^{-3})\Big] \times \Pr\{z_1'\Omega^{-1}z_1 > u, z_2'\Omega^{-1}z_2 > u\}|_{\Omega=\Sigma, u=c_0},$$

where $D \equiv \partial/\partial u$.

When $\boldsymbol{V} = \boldsymbol{I}$, an asymptotic expansion of the probability in (1.4) was given by Siotani [32], and its simplified and practical formula was obtained in Seo & Siotani [29]. After a great deal of calculation, we can obtain the following theorem.

THEOREM 1.1. *With the notations*

$$\eta_{ij} = \frac{\chi^2}{2(1-\rho_{ij}^2)},$$

$$g_a(\eta_{ij}) = \frac{1}{\Gamma(a)} \eta_{ij}^{a-1} e^{-\eta_{ij}} (a > 0), \quad G_a(\eta_{ij}) = \int_{\eta_{ij}}^{\infty} g_a(t)dt,$$

$$g_{(p/2)-1}(\eta_{ij}) \equiv -\frac{1}{2\sqrt{\pi}} \eta_{ij}^{-3/2} e^{-\eta_{ij}} \textit{ for } p = 1; \equiv 0 \textit{ for } p = 2,$$

$$\left(\frac{1}{2}p\right)_m = \frac{p}{2}\cdot\left(\frac{p}{2}+1\right)\cdots\left(\frac{p}{2}+m-1\right),$$

it holds that

$$\Pr\{z_i \boldsymbol{S}^{-1} z_i > t_1^2,\ z_j \boldsymbol{S}^{-1} z_j > t_1^2\} = A_0(\rho_{ij}) + v^{-1} A_1(\rho_{ij}) + v^{-2} A_2(\rho_{ij}) + O(v^{-3}),$$

where

$$A_0(\rho_{ij}) = (1-\rho_{ij}^2)^{p/2} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}p)_m}{m!} \rho_{ij}^{2m} G_{p/2+m}^2(\eta_{ij}),$$

$$A_1(\rho_{ij}) = \frac{1}{2}(1-\rho_{ij}^2)^{p/2-2} \chi^2 \sum_{m=0}^{\infty} \frac{(\frac{1}{2}p)_m}{m!} \rho_{ij}^{2m} g_{p/2+m}(\eta_{ij}) \times \left[\{\rho_{ij}^2(\chi^2+2m) - 2m\} G_{p/2+m}(\eta_{ij}) + \frac{2m+1}{p+2m} \chi^2 g_{p/2+m}(\eta_{ij})\right],$$

$$A_2(\rho_{ij}) = \frac{1}{48}(1-\rho_{ij}^2)^{p/2-4} \chi^2 \sum_{m=0}^{\infty} \frac{(\frac{1}{2}p)_m}{m!} \rho_{ij}^{2m} \Big[a_1(\rho_{ij}) g_{p/2-1+m}(\eta_{ij}) G_{p/2+m}(\eta_{ij}) + a_2(\rho_{ij}) g_{p/2+m}(\eta_{ij}) G_{p/2+m}(\eta_{ij}) + a_3(\rho_{ij}) g_{p/2+m}^2(\eta_{ij})\Big]$$

with the coefficients $a_1(\rho_{ij})$, $a_2(\rho_{ij})$ *and* $a_3(\rho_{ij})$ *given by*

$$a_1(\rho_{ij}) = -3(1-\rho_{ij}^2)^2 \chi^2 (\chi^2+p)^2,$$

$$a_n(\rho_{ij}) = 3\rho_{ij}^4 \chi^6 + (1-\rho_{ij}^2)\chi^4 \{-8\rho_{ij}^4 - 2(3p+12m-2)\rho_{ij}^2 + 3(p+2m-2)\} + 2(1-\rho_{ij}^2)^2 \chi^2 \{-(3p^2-13p-12m^2+2)\rho_{ij}^2 + 3p^2 + 6(m-1)p + 2m(3m+1)\} + (1-\rho_{ij}^2)^3 \{3p^3 + 6(m-1)p^2 - 4m(3m+4)p - 8m^2(3m-4)\},$$

$$a_3(\rho_{ij}) = 3\chi^6 \left\{ \rho_{ij}^4 + \frac{4}{p+2m}(1+2m)\rho_{ij}^2 + \frac{2(2m+1)(2m+3)}{(p+2m)(p+2m+2)} \right\}$$
$$+ 12(1-\rho_{ij}^2)\chi^4 \left\{ -(1+3m)\rho_{ij}^2 - \frac{1}{p+2m}(4m+1)(2m+1) \right\}$$
$$+ 6(1-\rho_{ij}^2)^2\chi^2 \left\{ \frac{1}{p+2m}\{(14m^2+4m+3)p + 28m^3 - 2m - 2\} \right\}.$$

Therefore, the modified second approximation to the upper percentile of the generalized $T^2_{\max}$ statistic is given by

$$t_M^2 = \frac{vp}{v-p+1} F_{p,v-p+1}\left(\frac{\alpha+\beta_1}{r}\right), \tag{1.6}$$

where

$$\beta_1 = \sum\sum_{i<j} \{A_0(\rho_{ij}) + v^{-1}A_1(\rho_{ij}) + v^{-2}A_2(\rho_{ij}) + \mathrm{O}(v^{-3})\}. \tag{1.7}$$

Practically, we can numerically calculate an asymptotic expansion (1.7) of β_1 up to the order v^{-2} by computer. On pairwise comparisons and comparisons with a control, we discuss the accuracy of the approximate percentiles for each parameters α, p, k, v and V by the Monte Carlo simulations in final subsection.

In the case of pairwise comparisons, the generalized $T^2_{\max}$-type statistic can be reduced to

$$T^2_{\max\cdot p} = \max_{c\in\mathbb{C}^k}\left(\frac{c'X'S^{-1}Xc}{c'Vc}\right),$$

where $\mathbb{C}^k = \{c\in\mathbb{C}^k : c = e_\ell - e_m,\ 1 \le \ell < m \le k\}$ and e_ℓ is the ℓth unit vector of the k-dimensional space. It is easily noted that $r = k(k-1)/2$ in this case.

THEOREM 1.2. *The modified second approximation $t^2_{M\cdot p}$ for pairwise comparisons among the correlated mean vectors is given by* (1.6) *with* $r = k(k-1)/2$ *and*

$$\rho_{ij} = \frac{c_i'Vc_j}{(c_i'Vc_i)^{1/2}(c_j'Vc_j)^{1/2}},$$

and then, the approximate simultaneous confidence intervals are given by

$$a'(\mu_i - \mu_j) \in [a'(\hat{\mu}_i - \hat{\mu}_j) \pm t_{M\cdot p}(d_{ij}a'Sa)^{1/2}],\quad \forall a\in\mathbb{R}^p,\ 1\le i<j\le k,$$

where $d_{ij} = v_{ii} - 2v_{ij} + v_{jj}$.

We note that, when $V = I$, the result in Theorem 1.2 can be reduced to the one in Seo & Siotani [29].

Next, in order to discuss the case of comparisons with a control, we assume that k-th treatment is the control treatment with which the remaining $(k-1)$ treatments are to be compared. Note that $r = k - 1$, and that t_c^2 is the upper percentile of the generalized $T^2_{\max\cdot c}$ statistic given by

$$T^2_{\max\cdot c} = \max_{d\in\mathbb{D}^k}\left(\frac{d'X'S^{-1}Xd}{d'Vd}\right), \tag{1.8}$$

where $\mathbb{D}^k = \{d\in\mathbb{R}^k\colon d = e_i - e_k,\ i = 1,\cdots,k-1\}$. When the $(k-1)$ treatments are independent of the control, the generalized $T^2_{\max\cdot c}$ statistic (1.8) is reduced to $T^2_{\max}$ statistic with a control in Seo & Siotani [30]. Then we have the following theorem.

THEOREM 1.3. *The modified second approximation $t^2_{M\cdot c}$ for comparisons among the correlated mean vectors with a control is given by* (1.6) *with $r = k-1$ and*

$$\rho_{ij} = \frac{d_i'Vd_j}{(d_i'Vd_i)^{1/2}(d_j'Vd_j)^{1/2}},$$

and then, the approximate simultaneous confidence intervals are given by

$$a'(\mu_i - \mu_k)\in[a'(\hat{\mu}_i - \hat{\mu}_k) \pm t_{M\cdot c}(d_{ik}a'Sa)^{1/2}], \qquad \forall a\in\mathbb{R}^p,\ i = 1,\cdots,k-1,$$

where $d_{ik} = v_{ii} - 2v_{ik} + v_{kk}$.

We note that, when $V = I$, the result in Theorem 1.3 is reduced to the one in Seo & Siotani [30].

1.3. Monte Carlo simulation

In the case of pairwise comparisons, Monte Carlo simulation was done to compare the simulated values of the upper percentiles $t^2(\alpha; p, k, r, v, V)$ of the generalized $T^2_{\max}$ statistic with the approximations by the modified second approximation procedure. The programs for computing the approximations are written by FORTRAN, in which subroutines for the gamma density and distribution functions, and the percentiles of χ^2 and F distributions are taken from Statistical Tables and Formulas with Computer Applications JSA-1972 (edited by Yamauti [37]). Without any loss of generality, we may assume that $\Sigma = I_p$. To obtain precisely simulated values, the large scale Monte Carlo simulation was done; that is, 20,000 simulations for each set $(\alpha; p, v, V)$ of parameters based on normal random vectors from $N_{kp}(\mathbf{0}, V\otimes I_p)$. Note that the sample covariance matrix $\boldsymbol{S}$ is formed independently in each time. This process is repeated 100 times to obtain 100 estimates. The average of 100 estimates based on 20,000 simulations is used as the simulated value of generalized $T^2_{\max}$-type statistic. Computations are made for $p = 1, 2, 3, 5$; $k = 3, 4$; $\nu = 10, 20, 40, 60$; $\alpha = 0.10, 0.05, 0.01$; $\boldsymbol{V}$ is made for the following four cases when $k = 3$:

$$\boldsymbol{I},\ \boldsymbol{V}_1 = \begin{bmatrix} 1 & 0.5 & 1 \\ 0.5 & 2 & 1.5 \\ 1 & 1.5 & 3 \end{bmatrix},\ \boldsymbol{V}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},\ \boldsymbol{V}_3 = \begin{bmatrix} 1 & 0.25 & 0.45 \\ 0.25 & 1 & 0.7 \\ 0.45 & 0.7 & 1 \end{bmatrix}.$$

Note that $\boldsymbol{V}_1$ satisfies $d_{12} = d_{13} = d_{23}$, and d_{ij}'s are the same for $\boldsymbol{V}_2$ and $\boldsymbol{V}_3$ ($d_{12} = 1.5$, $d_{13} = 1.1$, $d_{23} = 0.6$). Table 1.1 gives simulated values with 10 ($S.D.$), where $S.D.$ means the estimated standard deviation of simulated values. From the magnitude of the standard deviation in Table 1.1, it can be seen that the simulated values are good enough for using as the reference values in our examination of precision of the approximations. Note that the first approximation are overestimate of the exact upper percentile of the statistic.

In the case of pairwise comparisons where $\boldsymbol{V} = \boldsymbol{I}$ and $p = 1$, the generalized $T^2_{\max}$-type statistic can be reduced to the univariate Studentized range distribution whose percentiles given by Harter [7]. The results in Table 1.1 suggest that the first approximations are not depend on $\boldsymbol{V}$, and the modified second approximations are the same when the corresponding d_{ij}'s are the same, which are shown in Section 3. Also, it can be seen in Table 1.1 that the modified second approximations are conservative for the case $k = 3$ and give the pretty better approximations than the first approximations. In general, however, the modified second approximation does not always give the conservative approximation. If $0.10 \le \alpha \le 0.01$, $1 \le p \le 5$, $k = 3$ and appropriate large values of ν are concerned, then the modified second approximations are at least good approximations with sufficient precision for the practical use.

Further, it may be noted that the modified second approximation procedure yields good approximations for the practical use to the case where $k \ge 4$. In addition, Monte Carlo simulation for the case of comparisons with a control was done for selected parameters in the same way as the case of pairwise comparisons. Consequently, it is also noted that the results similar to those for the case of pairwise comparisons are obtained.

Table 1.1. Simulation results and approximations ($\alpha = 0.10$)

$k=3$				$p=1$				
v	I	V_1	t_1	$t_{M\cdot p}$	V_2	V_3	t_1	$t_{M\cdot p}$
10	2.312 (0.013)	2.314 (0.013)	2.466	2.349	2.290 (0.013)	2.291 (0.013)	2.466	2.331
20	2.176 (0.011)	2.176 (0.012)	2.285	2.201	2.153 (0.011)	2.154 (0.011)	2.285	2.185
40	2.114 (0.009)	2.114 (0.010)	2.204	2.133	2.091 (0.010)	2.092 (0.010)	2.204	2.118
60	2.092 (0.010)	2.093 (0.009)	2.178	2.111	2.071 (0.009)	2.071 (0.009)	2.178	2.096
				$p=2$				
10	3.175 (0.016)	3.177 (0.014)	3.361	3.221	3.147 (0.016)	3.148 (0.015)	3.361	3.199
20	2.824 (0.011)	2.825 (0.011)	2.934	2.849	2.800 (0.013)	2.800 (0.011)	2.934	2.832
40	2.677 (0.010)	2.679 (0.010)	2.761	2.694	2.654 (0.011)	2.656 (0.010)	2.761	2.678
60	2.634 (0.009)	2.632 (0.010)	2.708	2.647	2.612 (0.009)	2.611 (0.010)	2.708	2.631
				$p=3$				
10	4.026 (0.021)	4.025 (0.019)	4.255	4.079	3.991 (0.021)	3.989 (0.021)	4.255	4.053
20	3.357 (0.012)	3.357 (0.014)	3.473	3.382	3.330 (0.012)	3.331 (0.012)	3.473	3.363
40	3.105 (0.010)	3.106 (0.011)	3.188	3.123	3.083 (0.011)	3.083 (0.010)	3.188	3.106
60	3.033 (0.011)	3.033 (0.010)	3.105	3.046	3.011 (0.010)	3.011 (0.010)	3.105	3.030
				$p=5$				
10	6.190 (0.034)	6.194 (0.033)	6.633	6.317	6.139 (0.033)	6.142 (0.031)	6.633	6.278
20	4.356 (0.015)	4.355 (0.016)	4.491	4.384	4.325 (0.014)	4.324 (0.015)	4.491	4.362
40	3.826 (0.010)	3.827 (0.010)	3.908	3.841	3.801 (0.009)	3.803 (0.011)	3.908	3.823
60	3.682 (0.009)	3.681 (0.010)	3.752	3.693	3.658 (0.010)	3.657 (0.009)	3.752	3.677

Table 1.1. Continued ($\alpha = 0.05$)

$k = 3$				$p = 1$				
v	I	V_1	t_1	$t_{M \cdot p}$	V_2	V_3	t_1	$t_{M \cdot p}$
10	2.740 (0.020)	2.743 (0.016)	2.870	2.767	2.715 (0.018)	2.715 (0.018)	2.870	2.748
20	2.528 (0.014)	2.530 (0.016)	2.613	2.545	2.506 (0.014)	2.506 (0.015)	2.613	2.529
40	2.436 (0.011)	2.435 (0.013)	2.499	2.445	2.415 (0.012)	2.413 (0.012)	2.499	2.430
60	2.404 (0.013)	2.405 (0.012)	2.463	2.413	2.383 (0.013)	2.384 (0.011)	2.463	2.398
				$p = 2$				
10	3.691 (0.021)	3.696 (0.023)	3.852	3.725	3.661 (0.022)	3.661 (0.022)	3.852	3.703
20	3.198 (0.015)	3.199 (0.015)	3.283	3.214	3.174 (0.018)	3.174 (0.013)	3.283	3.197
40	2.998 (0.013)	3.000 (0.013)	3.057	3.008	2.976 (0.013)	2.978 (0.012)	3.057	2.992
60	2.939 (0.011)	2.937 (0.012)	2.989	2.944	2.918 (0.010)	2.916 (0.012)	2.989	2.929
				$p = 3$				
10	4.665 (0.031)	4.662 (0.030)	4.868	4.703	4.624 (0.030)	4.622 (0.029)	4.868	4.677
20	3.759 (0.017)	3.759 (0.017)	3.849	3.776	3.732 (0.018)	3.732 (0.016)	3.849	3.757
40	3.432 (0.014)	3.433 (0.013)	3.491	3.443	3.412 (0.014)	3.410 (0.015)	3.491	3.426
60	3.337 (0.014)	3.339 (0.012)	3.387	3.345	3.317 (0.014)	3.317 (0.013)	3.387	3.329
				$p = 5$				
10	7.269 (0.052)	7.278 (0.047)	7.695	7.383	7.211 (0.054)	7.211 (0.049)	7.695	7.341
20	4.836 (0.021)	4.833 (0.020)	4.940	4.853	4.802 (0.021)	4.801 (0.020)	4.940	4.830
40	4.171 (0.016)	4.175 (0.014)	4.230	4.180	4.148 (0.013)	4.149 (0.013)	4.230	4.162
60	3.995 (0.012)	3.995 (0.013)	4.042	4.001	3.974 (0.014)	3.972 (0.013)	4.042	3.984

Table 1.1. Continued ($\alpha = 0.01$)

$k = 3$				$p = 1$				
ν	I	V_1	t_1	$t_{M\cdot p}$	V_2	V_3	t_1	$t_{M\cdot p}$
10	3.723 (0.044)	3.721 (0.045)	3.827	3.739	3.694 (0.043)	3.693 (0.040)	3.827	3.719
20	3.285 (0.033)	3.283 (0.034)	3.331	3.286	3.260 (0.033)	3.258 (0.028)	3.331	3.269
40	3.089 (0.027)	3.087 (0.028)	3.122	3.092	3.066 (0.024)	3.069 (0.026)	3.122	3.076
60	3.027 (0.024)	3.027 (0.024)	3.057	3.031	3.008 (0.022)	3.009 (0.025)	3.057	3.016
				$p = 2$				
10	4.913 (0.052)	4.913 (0.041)	5.052	4.936	4.876 (0.050)	4.876 (0.049)	5.052	4.912
20	4.005 (0.032)	4.005 (0.035)	4.057	4.011	3.981 (0.032)	3.981 (0.035)	4.057	3.993
40	3.655 (0.029)	3.655 (0.027)	3.687	3.659	3.639 (0.031)	3.636 (0.028)	3.687	3.643
60	3.555 (0.025)	3.550 (0.024)	3.577	3.554	3.534 (0.021)	3.532 (0.024)	3.577	3.540
				$p = 3$				
10	6.212 (0.082)	6.228 (0.077)	6.405	6.246	6.172 (0.074)	6.179 (0.083)	6.405	6.218
20	4.637 (0.039)	4.632 (0.037)	4.693	4.644	4.611 (0.038)	4.607 (0.037)	4.693	4.624
40	4.105 (0.026)	4.102 (0.025)	4.137	4.110	4.086 (0.026)	4.085 (0.029)	4.137	4.094
60	3.957 (0.025)	3.954 (0.023)	3.981	3.959	3.937 (0.026)	3.937 (0.023)	3.981	3.944
				$p = 5$				
10	10.102 (0.145)	10.127 (0.124)	10.555	10.215	10.001 (0.153)	10.036 (0.135)	10.555	10.164
20	5.889 (0.043)	5.891 (0.048)	5.966	5.904	5.857 (0.040)	5.861 (0.044)	5.966	5.879
40	4.891 (0.030)	4.889 (0.030)	4.920	4.893	4.870 (0.029)	4.867 (0.030)	4.920	4.876
60	4.634 (0.027)	4.632 (0.027)	4.655	4.634	4.614 (0.026)	4.616 (0.026)	4.655	4.618

2. Multiple comparisons among the components of the mean vector

In this section, we consider the simultaneous confidence procedures for multiple comparisons of the components of the mean vector in a multivariate normal distribution. Such a situation arises, for example, in multiple comparisons of the components of repeated measurements of the same quantity in different conditions. For the purpose of constructing of the intervals in this case, the upper percentiles of the statistic for comparing of the components of the mean vector are discussed. The form of this type statistic is given by

$$F_{\max}^2 = \max_{a \in \mathbb{A}^p} \left(\frac{a'xx'a}{a'Sa} \right), \tag{2.1}$$

where x corresponds to X with $p = 1$ and $\mathbb{A}^p$ is a subset that consists of s vectors in the p-dimensional space. Note that $F_{\max}^2$ statistic (2.1) is similar to the generalized $T_{\max}^2$-type statistic. We discuss the upper percentiles of $F_{\max}^2$ statistic, so that the simultaneous confidence intervals are obtained for the cases of pairwise comparisons and comparisons with a control.

2.1. Pairwise comparisons and comparisons with a control

Let $\hat{\mu}$ be the estimator of μ such that $\hat{\mu}$ is distributed as $N_p(\mu, \Sigma)$, where μ and Σ are unknown, and let $S(= [s_{ij}])$ be an unbiased estimator of Σ such that vS is independent of $\hat{\mu}$ and is distributed as a Wishart distribution $W_p(\Sigma, v)$. Let $x = \hat{\mu} - \mu$, and consider the statistic given by (2.1). The approximate upper percentiles of $F_{\max}^2$ statistic by the modified second approximation procedure are described as follows.

Let $F_i^2 = (a_i'xx'a_i)/(a_i'Sa_i)$, $w_1^2 = F_{1,v}(\alpha/s)$ and

$$\gamma_1 = \sum_{i<j}\sum \Pr\{F_i^2 > w_1^2, F_j^2 > w_1^2\}. \tag{2.2}$$

Then, by Bonferroni's inequality for $F_{\max}^2$ statistic, the modified second approximation is a solution of the equation

$$\sum_{i=1}^{s} \Pr\{F_i^2 > w_M^2\} = \alpha + \gamma_1.$$

Hence

$$w_M^2 = F_{1,v}\left(\frac{\alpha + \gamma_1}{s}\right).$$

Since the exact evaluation of the probability in (2.2) is also complicated in this case, we obtain an asymptotic expansion for the probability in (2.2) which is a large sample approximation.

Letting $y_i = a_i'x/(a_i\Sigma a_i)^{1/2}$ and $y_j = a_j'x/(a_j\Sigma a_j)^{1/2}$, $i \neq j$,

$$\begin{pmatrix} y_i \\ y_j \end{pmatrix} \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} 1 & \delta_{ij} \\ \delta_{ij} & 1 \end{bmatrix}\right),$$

where

$$\delta_{ij} = \frac{\boldsymbol{a}_i'\boldsymbol{\Sigma}\boldsymbol{a}_j}{(\boldsymbol{a}_i'\boldsymbol{\Sigma}\boldsymbol{a}_i)^{1/2}(\boldsymbol{a}_j'\boldsymbol{\Sigma}\boldsymbol{a}_j)^{1/2}}.$$

Then

$$f(y_i, y_j) = \frac{1}{2\pi\sqrt{1-\delta_{ij}^2}} \exp\left[-\frac{1}{2(1-\delta_{ij}^2)}(y_i^2 - 2\delta_{ij}y_iy_j + y_j^2)\right].$$

The following result corresponds to a univariate case of Lemma 1.1.

LEMMA 2.1. *Suppose that y_i and y_j are under the assumptions above. Then, for any fixed q,*

$$\Pr\{y_i^2 > q^2, y_j^2 > q^2\} = (1-\delta_{ij}^2)^{1/2} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m}{m!} \delta_{ij}^{2m} G_{m+1/2}^2(\eta_{ij}),$$

where

$$\eta_{ij} = \frac{1}{2(1-\delta_{ij}^2)} q^2, \quad G_{m+1/2}(\eta_{ij}) \equiv \frac{1}{\Gamma(m+\frac{1}{2})} \int_{\eta_{ij}}^{\infty} t^{m-1/2} e^{-t} dt.$$

Put $v_i = y_i^2/(1-\delta_{ij}^2)$ and $v_j = y_j^2/(1-\delta_{ij}^2)$. Then (v_i, v_j) is the bivariate χ^2 distribution with 1 d.f. and parameter δ_{ij}.

We give an asymptotic expansion for probability in (2.2) by using the following idea which is essentially due to Welch [36]. The method was used for the probabilities similar to (2.2) by Siotani [32]. Let $\boldsymbol{t}_i$ and $\tilde{\boldsymbol{x}}$ be the vectors such that $\boldsymbol{t}_i'\boldsymbol{t}_i = \boldsymbol{a}_i'\boldsymbol{\Sigma}\boldsymbol{a}_i$ and $\boldsymbol{t}_i'\tilde{\boldsymbol{x}} = \boldsymbol{a}_i'\boldsymbol{x}$, respectively. Consider for any fixed q,

$$J \equiv \Pr\left\{\frac{(\boldsymbol{t}_i'\tilde{\boldsymbol{x}})^2}{\boldsymbol{t}_i'(\boldsymbol{I}+\varepsilon)\boldsymbol{t}_i} > q^2, \frac{(\boldsymbol{t}_j'\tilde{\boldsymbol{x}})^2}{\boldsymbol{t}_j'(\boldsymbol{I}+\varepsilon)\boldsymbol{t}_j} > q^2\right\},$$

where $\varepsilon = [\varepsilon_{rs}]$ is a symmetric matrix consisting of small increments ε_{rs} to i_{rs} such that $\boldsymbol{I}+\varepsilon$ is still positive definite. Then

$$J = \left[1 + \sum_{rs} \varepsilon_{rs}\partial_{rs} + \frac{1}{2!}\sum_{rstu} \varepsilon_{rs}\varepsilon_{tu}\partial_{rs}\partial_{tu} + \cdots\right] \times \Pr\left\{\frac{(\boldsymbol{t}_i'\tilde{\boldsymbol{x}})^2}{\boldsymbol{t}_i'\boldsymbol{\Omega}\boldsymbol{t}_i} > q^2, \frac{(\boldsymbol{t}_j'\tilde{\boldsymbol{x}})^2}{\boldsymbol{t}_j'\boldsymbol{\Omega}\boldsymbol{t}_j} > q^2\right\}\Bigg|_{\Omega=I}, \tag{2.3}$$

where $\partial_{rs} = (1/2)(1+\delta_{rs})\partial/\partial w_{rs}$, and δ_{rs} is the Kronecker delta.

On the other hand, we can express J as

$$J = \int_Z \frac{1}{2\pi\sqrt{1-\delta_{ij}^2}} \exp\left[-\frac{1}{2(1-\delta_{ij}^2)}(y_i^2 - 2\delta_{ij}y_iy_j + y_j^2)\right] dy_i dy_j, \tag{2.4}$$

where

$$Z: \frac{t_i' t_i}{t_i'(I+\varepsilon)t_i} y_i^2 > q^2, \ \frac{t_j' t_j}{t_j'(I+\varepsilon)t_j} y_j^2 > q^2.$$

After a great deal of calculation, we obtain the following theorem by comparing (2.3) with (2.4).

THEOREM 2.1. *For any fixed q,*

$$\Pr(F_i^2 > q^2, \ F_j^2 > q^2\} = B_0(\delta_{ij}) + v^{-1}B_1(\delta_{ij}) + v^{-2}B_2(\delta_{ij}) + \mathrm{O}(v^{-3}),$$

where

$$B_0(\delta_{ij}) = (1-\delta_{ij}^2)^{1/2} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m}{m!} \delta_{ij}^{2m} G_{m+1/2}^2(\eta_{ij}),$$

$$B_1(\delta_{ij}) = (1-\delta_{ij}^2)^{1/2} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m}{m!} \delta_{ij}^{2m} [\{2\eta_{ij}^2 - (2m-1)\eta_{ij}\} g_{m+1/2}(\eta_{ij}) G_{m+1/2}(\eta_{ij}) + 2\eta_{ij}^2 g_{m+1/2}^2(\eta_{ij})],$$

$$B_2(\delta_{ij}) = (1-\delta_{ij}^2)^{1/2} \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m}{m!} \delta_{ij}^{2m} [b_1(\delta_{ij}) g_{m+1/2}(\eta_{ij}) G_{m+1/2}(\eta_{ij}) + b_2(\delta_{ij}) g_{m+1/2}^2(\eta_{ij})]$$

with the coefficients $b_1(\delta_{ij})$, $b_2(\delta_{ij})$ *given by*

$$b_1(\delta_{ij}) = \eta_{ij}^4 - \frac{1}{6}(18m+7)\eta_{ij}^3 + \frac{1}{12}(2m-1)(18m+5)\eta_{ij}^2 - \frac{1}{24}(2m-1)(2m-3)(6m+1)\eta_{ij},$$

$$b_2(\delta_{ij}) = (2\delta_{ij}^4 + 4\delta_{ij}^2 + 1)\eta_{ij}^4 - \{2(2m-1)\delta_{ij}^4 + 4(2m+1)\delta_{ij}^2 + (2m-1)\}\eta_{ij}^3 + \left\{\frac{1}{2}(2m-1)^2\delta_{ij}^4 + (2m-1)(2m+1)\delta_{ij}^2 + \frac{1}{4}(2m-1)^2\right\}\eta_{ij}^2.$$

When $\delta_{ij} = 0$, the above asymptotic expansion can be simplified. The following corollary is useful for evaluation of approximation by computer.

COROLLARY 2.1. *If* $\delta_{ij} = 0$, *then*

$$\begin{aligned}\Pr\{F_i^2 > q^2, \ F_j^2 > q^2\} &= G_{1/2}^2(\eta) + v^{-1}\{(2\eta+1)\eta \cdot g_{1/2}(\eta)G_{1/2}(\eta) + 2\eta^2 g_{1/2}^2(\eta)\} \\ &+ v^{-2}\left\{\frac{1}{24}(24\eta^3 - 28\eta^2 + 10\eta - 3)\eta \cdot g_{1/2}(\eta)G_{1/2}(\eta) + \frac{1}{4}(2\eta+1)^2\eta^2 g_{1/2}^2(\eta)\right\} \\ &+ \mathrm{O}(v^{-3}),\end{aligned}$$

where $\eta = q^2/2$.

For the details of derivation of the above result, see Seo [28]. In addition, in order to check this result, we consider an asymptotic expansion for the probability in (2.2) by a perturbation method which is different from the preceding method. However, it is also complicated to evaluate the asymptotic expanision up to the higher order in the case of the perturbation method. This is why we give the asymptotic expansion up to the term of order v^{-1} for the probability and then check previous result with this one.

Letting $\boldsymbol{\Lambda} = \boldsymbol{I} + \boldsymbol{V}/\sqrt{v}$, we have

$$\Pr\{F_i^2 > q^2, F_j^2 > q^2\} = E_{\Lambda}\left[(1-\delta_{ij}^2)^{1/2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m}{m!}\delta_{ij}^{2m}G_{m+1/2}(\eta_{ij}^{(i)})G_{m+1/2}(\eta_{ij}^{(j)})\right],$$

where $\eta_{ij}^{(k)} = r_k^2 q^2/2(1-\delta_{ij}^2)$, $k = i, j$, and $r_k^2 = \boldsymbol{t}_k'\boldsymbol{\Lambda}\boldsymbol{t}_k/\boldsymbol{t}_k'\boldsymbol{t}_k$.

Further, suppose that $\eta_{ij}^{(k)} = \eta_0 + \eta_1^{(k)}$, where

$$\eta_0 = q^2/2(1-\delta_{ij}^2) \quad \eta_1^{(k)} = (1/\sqrt{v})\eta_0 \boldsymbol{t}_k'\boldsymbol{V}\boldsymbol{t}_k/\boldsymbol{t}_k'\boldsymbol{t}_k.$$

Then we have

$$G_{m+1/2}(\eta_{ij}^{(i)})G_{m+1/2}(\eta_{ij}^{(j)}) = G_{m+1/2}^2(\eta_0) + G_{m+1/2}(\eta_0)G_{m+1/2}'(\eta_0)\{\eta_1^{(i)} + \eta_1^{(j)}\}$$
$$+\left[G_{m+1/2}'(\eta_0)\eta_1^{(i)}\eta_1^{(j)} + \frac{1}{2}G_{m+1/2}''(\eta_0)G_{m+1/2}(\eta_0)\{\eta_1^{(i)2} + \eta_1^{(j)2}\}\right] + \cdots.$$

Note that

$$G_{m+1/2}'(\eta_0) = -g_{m+1/2}(\eta_0),$$
$$G_{m+1/2}''(\eta_0) = \frac{1}{2}\{2-(2m-1)\eta_0^{-1}\}g_{m+1/2}(\eta_0).$$

Using the formulas for expectations about $\boldsymbol{V}$ and $\boldsymbol{V}^2$ (see, e.g., Fujikoshi [3]), we can obtain

$$\Pr\{F_i^2 > q^2, F_j^2 > q^2\} = (1-\delta_{ij}^2)^{1/2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m}{m!}\delta_{ij}^{2m}\Big[G_{m+1/2}^2(\eta_0)$$
$$+\frac{1}{v}\{\{2\eta_0^2-(2m-1)\eta_0\}g_{m+1/2}(\eta_0)G_{m+1/2}(\eta_0) + 2\eta_0^2 g_{m+1/2}^2(\eta_0)\}\Big] + O(v^{-2}).$$

This is the same as the result of Theorem 2.1 up to the term of order v^{-1}. The approximate upper α percentile of the statistic by modified second approximation procedure is given by

$$w_M^2 = F_{1,v}\left(\frac{\alpha+\gamma_1}{s}\right), \tag{2.5}$$

where

$$\gamma_1 = \sum\sum_{i<j} \{B_0(\delta_{ij}) + v^{-1}B_1(\delta_{ij}) + v^{-2}B_2(\delta_{ij}) + O(v^{-3})\}.$$

Applying the above results to the pairwise comparisons and comparisons with a control among the components of the mean vector, we have the following results.

In the case of pairwise comparisons, $F^2_{\max}$ statistic is reduced to

$$F^2_{\max\cdot p} = \max_{u \in \mathbb{U}^p} \left(\frac{u'xx'u}{u'Su} \right),$$

where $\mathbb{U}^p = \{u \in \mathbb{R}^p : u = e_i - e_j,\ 1 \le i < j \le p\}$ and e_i is the ith unit vector of the p-dimensional space.

THEOREM 2.2. *The modified second approximation $w^2_{M\cdot p}$ for the pairwise comparisons among the components of the mean vector is given by* (2.5) *with* $s = p(p-1)/2$ *and*

$$\delta_{ij} = \frac{u_i'\Sigma u_j}{(u_i'\Sigma u_i)^{1/2}(y_j'\Sigma u_j)^{1/2}},$$

and then, the approximate simultaneous confidence intervals with the confidence level $(1-\alpha)$ *are given by*

$$\mu_i - \mu_j \in [\hat{\mu}_i - \hat{\mu}_j \pm w_{M\cdot p}(s_{ii} - 2s_{ij} + s_{jj})^{1/2}], \qquad 1 \le i < j \le p.$$

In the case of comparisons with a control, assuming that p-th component is the control component with which the remaining $(p-1)$ components are to be compared, $F^2_{\max}$ statistic is reduced to

$$F^2_{\max\cdot c} = \max_{v \in \mathbb{V}^p} \left(\frac{v'xx'v}{v'Sv} \right),$$

where $\mathbb{V}^p = \{v \in \mathbb{R}^p : v = e_i - e_p,\ i = 1, \cdots, p-1\}$.

THEOREM 2.3. *The modified second approximation $w^2_{M\cdot c}$ for the comparisons with a control among the components of the mean vector is givne by* (2.5) *with* $s = p - 1$ *and*

$$\delta_{ij} = \frac{v_i'\Sigma v_j}{(v_i'\Sigma v_i)^{1/2}(v_j'\Sigma v_j)^{1/2}},$$

and then, the approximate simultaneous confidence intervals with the confidence level $(1-\alpha)$ *are given by*

$$\mu_i - \mu_p \in [\hat{\mu}_i - \hat{\mu}_p \pm w_{M\cdot c}(s_{ii} - 2s_{ip} + s_{pp})^{1/2}], \qquad i = 1, \cdots, p-1.$$

2.2. Monte Carlo simulation

A computer program for evaluating the approximate upper α percentile of the statistic by the modified second aproximation procedure is written in FORTRAN. Since the modified second approximation procedure produces the accurate approximations in most cases, it is expected that the approximations give good estimates. Here, as evaluation of the approximations in practical use, we use the extimate of ρ_{ij}, that is,

$$\hat{\rho}_{ij} = \frac{\boldsymbol{c}_i'\boldsymbol{S}\boldsymbol{c}_j}{(\boldsymbol{c}_i'\boldsymbol{S}\boldsymbol{c}_i)^{1/2}(\boldsymbol{c}_j'\boldsymbol{S}\boldsymbol{c}_j)^{1/2}}.$$

In the case of pairwise comparisons, Monte Carlo simulation based on 20,000 simulations was done for the parameters $\alpha = 0.10, 0.05, 0.01$; $p = 3, 5, 10$; $\nu = 10, 20, 30$, and repeated 100 times, where $\boldsymbol{\Sigma}$ is given by $\boldsymbol{I}$. It may be noted that the simultaneous confidence intervals which are constructed by using the upper percentile in the case $\boldsymbol{\Sigma} = \boldsymbol{I}$ in Table 2.1 are conservative in most cases. Also, it may be seen that the approximations are close to the simulated values as ν is large. It can be expected that the modified second approximations give good estimates if $\boldsymbol{\Sigma} \neq \boldsymbol{I}$.

In the case of comparisons with a control, the approximations by the modified second approximation procedure and simulation results for selected parameters which is $\boldsymbol{\Sigma} = \boldsymbol{I}$ and $p = 3, 5, 10$; $\nu = 10, 20, 30$; $\alpha = 0.10, 0.05, 0.01$ are given in Table 2.2. It may be noted that the approximations by the modified second approximation procedure are close to the exact critical points as p is large.

Table 2.1. Case of pairwise comparisons

		$p = 3$			$p = 5$			$p = 10$		
		α			α			α		
ν		0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
10	Approx. (MSA)	2.383	2.822	3.822	3.061	3.540	4.586	4.048	4.513	5.617
	Simulation	2.379	2.812	3.797	3.033	3.489	4.549	3.923	4.419	5.582
20	Approx. (MSA)	2.209	2.558	3.311	2.713	3.068	3.829	3.368	3.733	4.484
	Simulation	2.202	2.558	3.304	2.718	3.067	3.815	3.360	3.707	4.459
30	Approx. (MSA)	2.159	2.484	3.166	2.624	2.941	3.613	3.187	3.506	4.169
	Simulation	2.151	2.483	3.164	2.628	2.945	3.605	3.201	3.510	4.152

Table 2.2. Case of comparisons with a control

		$p=3$			$p=5$			$p=10$		
		α			α			α		
ν		0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
10	Approx. (MSA)	2.173	2.599	3.576	2.535	2.984	3.999	2.988	3.467	4.516
	Simulation	2.177	2.600	3.564	2.536	2.970	3.979	2.959	3.417	4.475
20	Approx. (MSA)	2.038	2.389	3.140	2.326	2.676	3.433	2.644	3.004	3.776
	Simulation	2.039	2.390	3.135	2.332	2.682	3.427	2.661	3.012	3.762
30	Approx. (MSA)	1.997	2.327	3.015	2.268	2.591	3.272	2.555	2.878	3.563
	Simulation	1.996	2.330	3.015	2.272	2.597	3.266	2.576	2.895	3.566

3. A multivariate generalized Tukey-type conjecture

In this section, we discuss a multivariate version of the generalized Tukey conjecture given as the statement that the Tukey-Kramer (T-K) procedure yields the conservative simultaneous confidence intervals. The T-K procedure (Tukey [35], Kramer [16, 17]) is an attractive and simple procedure for pairwise multiple comparisons (see, e.g., Hochberg & Tamhane [11]). On the univariate case, it is shown in Dunnett [2] that the generalized Tukey conjecture for pairwise comparisons by an extensive simulation study. Theoretical discussions related to this conjecture are referred to Hayter [8, 9], Brown [1]. It is known that this generalized conjecture is true for (i) $k=3$ (see, Brown [1]) and (ii) d_{ij}'s satisfy $d_{ij}=a_i+a_j$ for some positive numbers a_i and a_j (see, Hayter [9]), where $d_{ij}=v_{ii}-2v_{ij}+v_{jj}$. Thus even for the univariate case, there has been no analytical proof of the generalized Tukey conjecture except the special cases. Further, Lin, Seppänen & Uusipaikka [21] have discussed the generalized Tukey conjecture for pairwise comparisons among the components of the mean vector. Recently, Seo, Mano & Fujikoshi [31] have established the generalized Tukey conjecture for pairwise comparisons among mean vectors in multivariate setting, and they proved the conservativeness in the case of three correlated mean vectors. In this section, we consider the multivariate T-K type procedure including comparisons with a control. Some interesting properties of the procedure are presented; first, we describe the multivariate T-K procedure and give a reduction for the coverage probability related to the ultivariate T-K procedure. Secondly, a conservative procedure for comparisons with a control is derived, so that we show that the obtained procedure gives the conservative confidence intervals in the case of three mean vectors, and we give the conjecture in the case more than four correlated mean vectors. Further, we discuss the conservative procedure in the case of comparisons with a control among the components of the mean vector. Finally, we investigate the conservativeness of the procedure for some selected parameters by Monte Carlo simulations.

3.1. Multiple comparisons among mean vectors

Under the situations in Section 1, we consider T-K procedure which is often used for making the simultaneous confidence intervals for the case of pairwise multiple comparisons in unbalanced models. We also apply T-K type procedure to the problem of multiple comparisons with a control in multivariate setting.

Under the same notations as in Section 1, the following properties of generalized $T^2_{\max}$-type statistics (1.2) are obtained.

THEOREM 3.1. *The distribution of generalized $T^2_{\max}$-type statistics depend on V only through ρ_{ij}, $i<j$.*

Proof. Let $\boldsymbol{X}=[\boldsymbol{x}_1,\cdots,\boldsymbol{x}_k]$ and $\boldsymbol{Z}=[\boldsymbol{z}_1,\cdots,\boldsymbol{z}_r]$, where $\boldsymbol{x}_i=\hat{\boldsymbol{\mu}}_i-\boldsymbol{\mu}_i$, $i=1,\cdots,k$, and $\boldsymbol{z}_i=(\boldsymbol{X}\boldsymbol{b}_i)/(\boldsymbol{b}_i'\boldsymbol{V}\boldsymbol{b}_i)^{1/2}$, $i=1,\cdots,r$. Without loss of generality we can assume that $\boldsymbol{\Sigma}=\boldsymbol{I}$. Then the generalized $T^2_{\max}$ statistic depends on $\mathrm{vec}(\boldsymbol{Z})$ and $\boldsymbol{S}$. Then the distribution of $\mathrm{vec}(\boldsymbol{Z})$ is normal with means zero, and the covariance matrix of $\boldsymbol{z}_i$ and $\boldsymbol{z}_j$ is given by

$$\mathrm{Cov}(\boldsymbol{z}_i,\boldsymbol{z}_j)=\begin{cases}\boldsymbol{I} & \text{if } i=j,\\ \rho_{ij}\boldsymbol{I} & \text{if } i\neq j,\end{cases}$$

where

$$\rho_{ij}=\frac{\boldsymbol{b}_i'\boldsymbol{V}\boldsymbol{b}_j}{(\boldsymbol{b}_i'\boldsymbol{V}\boldsymbol{b}_i)^{1/2}(\boldsymbol{b}_j'\boldsymbol{V}\boldsymbol{b}_j)^{1/2}}.$$

Therefore, the covariance matrix of $\mathrm{vec}(\boldsymbol{Z})$ depends on V through the values of ρ_{ij}, $i<j$.

COROLLARY 3.1. *Suppose that $\boldsymbol{b}_1,\cdots,\boldsymbol{b}_r$ be the elements of $\mathbb{B}^k=\{\boldsymbol{b}\in\mathbb{R}^k:\ \boldsymbol{b}=\boldsymbol{e}_\ell-\boldsymbol{e}_m,\ 1\le\ell<m\le k\}$. If for all $1\le i<j\le k$, $d_{ij}=v_{ii}-2v_{ij}+v_{jj}=d$(constant), then the distributions of generalized $T^2_{\max}$-type statistics are the same as that of $T^2_{\max}$ statistic with $\boldsymbol{V}=\boldsymbol{I}$.*

Proof. Under the situation above, it is easily checked that the distribution of $\mathrm{vec}(\boldsymbol{Z})$ in the case $d_{ij}=v_{ii}-2v_{ij}+v_{jj}=d$ is the same as that of $\mathrm{vec}(\boldsymbol{Z})$ in the case $\boldsymbol{V}=\boldsymbol{I}$.

COROLLARY 3.2. *The generalized $T^2_{\max\cdot c}$ statistics with $d_{ij}=d_{ik}+d_{jk}$ have the same distribution for all $1\le i<j\le k-1$.*

Proof. When $d_{ij}=d_{ik}+d_{jk}$, the covariance matrix of $\boldsymbol{z}_{ik}$ and $\boldsymbol{z}_{jk}$ $(i\neq j)$ is given by 0. Hence we have the corollary.

The property of Theorem 3.1 in the case of pairwise comparisons is shown by Seo, Mano & Fujikoshi [31], and is an extension of Hochberg [10]. In that case, the simultaneous confidence intervals by the multivariate T-K procedure were proposed by replacing with the upper α percentile of generalized $T^2_{\max\cdot p}$ statistic with $V = I$. Similarly, in the case of multiple comparisons with a control, the simultaneous confidence intervals by the multivariate T-K type procedure may be obtained by replacing the upper α point of generalized $T^2_{\max\cdot c}$ statistic with that of $T^2_{\max\cdot c}$ statistic with $V = I$; that is,

$$a'Md \in [a'\hat{M}d \pm t_{c,I}(d'Vd)^{\frac{1}{2}}(a'Sa)^{\frac{1}{2}}], \qquad \forall a \in \mathbb{R}^p,\ \forall d \in \mathbb{D}^k,$$

where $t_{c,I} = t_c(\alpha;\, p,\, k,\, v,\, I)$, $\mathbb{D}^k = \{d \in \mathbb{R}^k\colon d = e_i - e_k,\ i = 1, \cdots, k-1\}$. However, we cannot expect the property that the multivariate T-K type procedure gives the conservative simultaneous confidence intervals in the case of multiple comparisons with a control. We shall show this fact by reduction for the coverage probability, and give T-K type procedure with another condition which satisfies the conservativeness. The following idea for the reduction is the same as that of Seo, Mano & Fujikoshi [31]. Consider the coverage probability

$$Q(q,\, V,\, \mathbb{D}^k) = \Pr\{(Xd)'(vS)^{-1}(Xd) \le q(d'Vd),\ \forall d \in \mathbb{D}^k\}, \tag{3.1}$$

where q is any fixed constant, and we can assume without loss of generality that $\Sigma = I$.

Let A and B be $k \times (k-1)$ matrix constructed from an orthonormal basis of space spanned by $\mathbb{D}^k$ and $(k-1) \times (k-1)$ nonsingular matrix such that $BB' = A'VA$, respectively. Then, by the transformation $Y = XAB^{-1}$, $\mathrm{vec}(Y) \sim N(\mathbf{0}, I_{k-1} \otimes I_p)$. Further, letting

$$\mathbb{G} = \{\gamma \in \mathbb{R}^{k-1};\ \gamma = (d'Vd)^{1/2} BA'd,\ \forall d \in \mathbb{D}^k\},$$

the coverage probability $Q(q,\, V,\, \mathbb{D}^k)$ can be written as

$$\begin{aligned} Q(q,\, V,\, \mathbb{D}^k) &= \Pr\{(YBA'd)'(vS)^{-1}(YBA'd) \le q(d'Vd),\ \forall d \in \mathbb{D}^k\} \\ &= \Pr\{(Y\gamma)'(vS)^{-1}Y\gamma \le q,\ \gamma \in \mathbb{G}\}. \end{aligned}$$

Further, we can write $vS = HLH'$ such that $L = \mathrm{diag}(\ell_1, \cdots, \ell_p)$, $\ell_1 \ge \cdots \ge \ell_p$, and H is an orthogonal matrix. Then L and H are independent and the pdf of L is given by

$$\frac{\pi^{p^2/2}}{2^{pv/2}\Gamma_p(v/2)\Gamma_p(p/2)} \exp\left(-\frac{1}{2}\sum_{i=1}^{p}\ell_i\right)\prod_{i=1}^{p}\ell_i^{(v-p-1)/2}\prod_{i<j}^{p}(\ell_i - \ell_j), \tag{3.2}$$

where $\Gamma_p(n) = \pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma(n-(i-1)/2)$ (see, e.g., Siotani, Hayakawa & Fujikoshi [34], p. 450). Then

$$Q(q, V, \mathbb{D}^k) = E_L[\Pr\{(Y\gamma)' L^{-1} Y\gamma \le q,\ \gamma \in \mathbb{G}\}].$$

Consider the special case $k = 3$. Since the dimension of the space spanned by $\mathbb{D}^k$ equals 2, there exists an orthogonal matrix H such that $\gamma_j' H = [\cos\beta_j, \sin\beta_j]$, $j = 1, 2$, where $0 \le \beta_1 \le \beta_2 \le \pi$. Further, we can write $YH = [u_1, \cdots, u_p]'$, where $u_i = r_i[\cos\theta_i, \sin\theta_i]'$, $i = 1, \cdots, p$, and r_i^2 and $\theta_i (0 \le \theta_i < 2\pi)$ are independently distributed as χ_2^2 and uniform distribution on $[0, 2\pi)$. Hence the coverage probability can be expressed as

$$Q(q, V, \mathbb{D}^k) = E_{L,R}\left[\Pr\left\{\sum_{i=1}^{p} \frac{r_i^2}{\ell_i} \cos^2(\theta_i - \beta_j) \le q \quad \text{for } j = 1, 2\right\}\right],$$

where $R = \mathrm{diag}(r_1, \cdots, r_p)$ is independent of $L = \mathrm{diag}(\ell_1, \cdots, \ell_p)$.

Consider the probability

$$G(\beta_1, \beta_2) = \Pr\left\{\sum_{i=1}^{p} \frac{r_i^2}{\ell_i} \cos^2(\theta_i - \beta_j) \le q \quad \text{for } j = 1, 2\right\}. \tag{3.3}$$

Then we have the following lemma by using Lemma 4.1 in Seo, Mano & Fujikoshi [31].

LEMMA 3.1. *For $0 \le \beta_1 \le \beta_2 \le \pi$, the probability* (3.3) *is minimized when* $\beta_2 - \beta_1 = \pi/2$.

THEOREM 3.2. *If $d_{12} = d_{13} + d_{23}$, where $d_{ij} = v_{ii} + v_{jj} - 2v_{ij}$, the coverage probability* (3.1) *is equal or greater than* $1 - \alpha$.

Proof. From Lemma 3.1, te coverage probability (3.1) has the minimum value when $\beta_2 - \beta_1 = \pi/2$ for $0 \le \beta_1 < \beta_2 \le \pi$. Further, since $\beta_2 - \beta_1 = \pi/2$ if and only if $\gamma_1'\gamma_2 = 0$, it follows from Corollary 3.2 that the coverage probability (3.1) is equal or greater than $1 - \alpha$.

From Theorem 3.2, it is noted that the multivariate T-K procedure with $t_{c,I} = t_c(\alpha; p, k, v, I)$ for multiple comparisons with a control does not always yield the conservativeness for general V when the case $k = 3$. For the case $k \ge 4$, it may be expected that the conservativeness of the multivariate T-K procedure does not hold, though the procedure may give good approximations. We can conjecture that conservative simultaneous confidence intervals are given by

$$a'Md \in [a'\hat{M}d \pm t_{c,V_0}(d'Vd)^{\frac{1}{2}}(a'Sa)^{\frac{1}{2}}], \qquad \forall a \in \mathbb{R}^p,\ \forall d \in \mathbb{D}^k, \tag{3.4}$$

with $t_{c,V_0} = t_c(\alpha; p, k, v, V_o)$ obtained under the conditions $d_{ij} = d_{ik} + d_{jk}$, $1 \le i < j \le k - 1$, where $d_{ij} = v_{ii} + v_{jj} - 2v_{ij}$.

In this subsection, we have proved that the obtained procedure defined by (3.4) yields conservativeness when the case $k = 3$. However, it is a future problem whether the procedure gives the conservative confidence intervals in the case more than four correlated mean vectors. In the next subsection, we discuss the conservative procedure in the case of comparisons with a control among the components of the mean vector.

3.2. Multiple comparisons among the components of the mean vector

Recall the situatin in Section 2. It is known (see, Lin, Seppänen & Uusipaikka [21]) that the T-K procedure yields the conservative simultaneous confidence intervals in the case of pairwise comparisons among the components of the mean vector. However, in the case of comparisons with a control, it may not hold this fact.

Consider the simultaneous confidence intervals for comparing with a control. Then the T-K type procedure can be written as

(3.5)

$$\Pr\{\mu_i - \mu_p \in [\hat{\mu}_i - \hat{\mu}_p \pm w_{c,I}(s_{ii} - 2s_{ip} + s_{pp})^{1/2}], \quad 1 \le i \le p-1,\} \ge 1 - \alpha$$

where $w_{c,I}^2 = w_c^2(\alpha; p, k, v, \boldsymbol{I})$ is the upper α point of $F_{\max\cdot c}^2$ statistic when $\Sigma = \boldsymbol{I}$. We discuss whether or not the statement (3.5) holds for the general covariance matrix $\boldsymbol{\Sigma}$.

Consider the coverage probability for (3.5) given by

$$Q(q, \boldsymbol{\Sigma}, \mathbb{V}) = \Pr\{(\boldsymbol{v}'\boldsymbol{x})^2 < q(\boldsymbol{v}'\boldsymbol{S}\boldsymbol{v}), \ \forall \boldsymbol{v} \in \mathbb{V}\}, \tag{3.6}$$

where q is any fixed constant. By the same idea as in the previous subsection, we can obtain a reduction for the coverage probability as follows.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $p \times (p-1)$ matrix constructed from an orthonormal basis of space spanned by $\mathbb{V}$ and $(p-1) \times (p-1)$ nonsingular matrix such that $\boldsymbol{B}\boldsymbol{B}' = \boldsymbol{A}'\boldsymbol{\Sigma}\boldsymbol{A}$, respectively. Using the transformation $\boldsymbol{y} = \boldsymbol{B}^{-1}\boldsymbol{A}'\boldsymbol{x}$ and $\boldsymbol{V} = v\boldsymbol{B}^{-1}\boldsymbol{A}'\boldsymbol{S}\boldsymbol{A}(\boldsymbol{B}')^{-1}$, then $\boldsymbol{y} \sim N(\boldsymbol{0}, \boldsymbol{I}_{p-1})$ and $\boldsymbol{V} \sim W_{p-1}(\boldsymbol{I}_{p-1}, v)$. The coverage probability $Q(q, \boldsymbol{\Sigma}, \mathbb{V})$ can be written as

$$E_L\left[\Pr\left\{\frac{(\boldsymbol{\eta}\boldsymbol{y})^2}{\boldsymbol{\eta}'\boldsymbol{L}\boldsymbol{\eta}} \le \frac{q}{v}, \ \boldsymbol{\eta} \in \mathbb{E}\right\}\right],$$

where

$$\mathbb{E} = \{\boldsymbol{\eta} \in \mathbb{R}^{p-1}; \boldsymbol{\eta} = (\boldsymbol{v}'\boldsymbol{\Sigma}\boldsymbol{v})^{-1/2}\boldsymbol{B}'\boldsymbol{A}'\boldsymbol{v}, \ \forall \boldsymbol{v} \in \mathbb{V}\},$$

$\boldsymbol{L} = \mathrm{diag}(\ell_1, \cdots, \ell_{p-1})$ such that $\boldsymbol{V} = \boldsymbol{H}\boldsymbol{L}\boldsymbol{H}'$, and $\boldsymbol{L}$ and an orthogonal matrix $\boldsymbol{H}$ are independent and the pdf of $\boldsymbol{L}$ is given by (3.2) with $q = p - 1$.

In the special case $p = 3$; that is, the dimension of the space spanned by $\mathbb{V}$

equals 2, we can write $\boldsymbol{H\eta}_j = [\cos\phi_j, \sin\phi_j]'$, $j = 1, 2$, and $\boldsymbol{Hy} = r[\cos\theta, \sin\theta]'$, where r^2 and θ are independently distributed as χ_2^2 and uniform distribution on $[0, 2\pi)$. Hence

$$Q(q, \boldsymbol{\Sigma}, \mathbb{V}) = E_L\left[\Pr\left\{\frac{r^2\cos^2(\theta-\phi_j)}{(\ell_1-\ell_2)\cos^2\phi_j+\ell_2} \le \frac{q}{v},\quad j = 1, 2\right\}\right]. \tag{3.7}$$

LEMMA 3.2. *If $\phi_1 - \phi_2 = \pi/2$, then the probability* (3.7) *achieves its minimum value for $0 \le \phi_1 < \phi_2 \le \pi$. Also, $\phi_1 - \phi_2 = \pi/2$ if and only if $d_{12} = d_{13} + d_{23}$, where $d_{ij} = \sigma_{ii} + \sigma_{jj} - 2\sigma_{ij}$.*

Proof. The probability in (3.7) is equal to

$$\Pr\{\cos^2(\theta-\phi_j) \le c[(v-1)\cos^2\phi_j + 1],\quad j = 1, 2\},$$

where

$$c = q/(r^2 v\ell_2),\quad v = \ell_1/\ell_2 (\ge 1).$$

Without loss of generality, we can assume that $\theta \in [0, \pi)$, since $\cos^2(\theta) = \cos^2(\theta + u\pi)$ for any integer u.

Letting $\Theta = \{\theta: 0 < \theta < \pi\}$ and $R_j = \{\theta \in \Theta: \cos^2(\theta - \phi_j) \le c[(v-1)\cos^2\phi_j + 1]\}$ for $j = 1, 2$, then (3.7) equals $(1 - \text{area}[R_1 \cup R_2]/\pi)$. Therefore, to minimize $Q(q, \boldsymbol{\Sigma}, \mathbb{V})$ is equivalent to maximizing the area of the union of R_1 and R_2. It may be noted that the area of the union of R_1 and R_2 becomes maximum when $\phi_1 - \phi_2 = \pi/2$. Further, since $\phi_1 - \phi_2 = \pi/2$ if and only if $\boldsymbol{\eta}_1'\boldsymbol{\eta}_2 = 0$, the probability (3.7) has a minimum value when $\boldsymbol{\eta}_1'\boldsymbol{\eta}_2 = 0$. The proof is similar to the one as in Lin, Seppänen & Uusipaikka [21] and Seo, Mano & Fujikoshi [31]. All the distributions of $F^2_{\max\cdot c}$ statistic with $d_{12} = d_{13} + d_{23}$ are the same, since $\boldsymbol{\eta}_1'\boldsymbol{\eta}_2 = \rho_{12} = 0$ if and only if $d_{12} = d_{13} + d_{23}$.

THEOREM 3.3. *If $d_{12} = d_{13} + d_{23}$, where $d_{ij} = \sigma_{ii} + \sigma_{jj} - 2\sigma_{ij}$, the coverage probability* (3.6) *is equal to or greater than $1 - \alpha$.*

It may be noted that there is no diagonal matrix satisfying $d_{12} = d_{13} + d_{23}$ in the class of positive definite matrix. It is noted that the case of $\boldsymbol{\Sigma} = \boldsymbol{I}$ isn't satisfied with $d_{12} = d_{13} + d_{23}$. From Theorem 3.3, T-K procedure for the case of comparison with a control is not always conservative for general $\boldsymbol{\Sigma}$ when $p = 3$. We can also conjecture that conservative simultaneous confidence intervals for comparing with a control are given by

$$\mu_i - \mu_p \in [\hat{\mu}_i - \hat{\mu}_p \pm w_{c,V_0}(s_{ii} - 2s_{ip} + s_{pp})^{1/2}],\qquad 1 \le i \le p-1,$$

with w_{c,V_0} obtained under the conditions $d_{ij} = d_{ip} + d_{jp}$, $1 \le i < j \le p-1$ where $d_{ij} = \sigma_{ii} + \sigma_{jj} - 2\sigma_{ij}$.

In conclusion, T-K procedure for the cases of comparisons with a control is not always conservative when $p = 3$. It may be implied that T-K procedure is not always conservative for $p > 3$ in those cases.

3.3. Monte Carlo simulation

In order to see the conservativeness of the simultaneous confidence intervals by the obtained procedure, the upper percentiles of the generalized $T^2_{\max\cdot c}$ statistic are computed for some selected parameters by Monte Carlo simulation. The Monte Carlo simulation is made in the same way as in Seo, Mano and Fujikoshi [31]. Computations are made for $k = 3, 4$; $p = 1, 2, 3, 5$; $\nu = 10, 20, 40$; $\alpha = 0.1, 0.05, 0.01, 0.001$; and for the cases of $\boldsymbol{I}$ and $\boldsymbol{V}_0$;

(i) $k = 3$,

$$\boldsymbol{V}_0 = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{bmatrix},$$

(ii) $k = 4$,

$$\boldsymbol{V}_0 = \begin{bmatrix} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}.$$

Table 3.1 gives the upper percentiles of generalized $T^2_{\max\cdot c}$ for each parameters. It is seen from some simulation results that the upper percentiles with $\boldsymbol{V} = \boldsymbol{V}_0$ are always largest values in general $\boldsymbol{V}$ for each parameters. It may be noted that the obtained procedure leads to the simultaneous confidence intervals with the conservative confidence level on multiple comparisons with a control.

Table 3.1. Simulation results

$k = 3$				$p = 1$				
α	0.10		0.05		0.01		0.001	
ν	$\boldsymbol{I}$	$\boldsymbol{V}_0$	$\boldsymbol{I}$	$\boldsymbol{V}_0$	$\boldsymbol{I}$	$\boldsymbol{V}_0$	$\boldsymbol{I}$	$\boldsymbol{V}_0$
10	2.148	2.195	2.566	2.610	3.525	3.563	4.964	5.003
20	2.027	2.063	2.376	2.409	3.123	3.142	4.116	4.133
40	1.968	2.005	2.291	2.321	2.947	2.967	3.771	3.781
				$p = 2$				
10	2.980	3.032	3.479	3.529	4.665	4.713	6.525	6.627
20	2.664	2.703	3.037	3.070	3.836	3.856	4.915	4.935
40	2.529	2.562	2.852	2.880	3.516	3.532	4.355	4.361

				$p=3$				
10	3.787	3.847	4.405	4.462	5.907	5.965	8.361	8.493
20	3.180	3.223	3.584	3.617	4.451	4.476	5.656	5.663
40	2.953	2.985	3.282	3.307	3.961	3.975	4.836	4.843
				$p=5$				
10	5.823	5.927	6.852	6.958	9.552	9.686	14.619	14.654
20	4.147	4.192	4.621	4.663	5.667	5.699	7.190	7.201
40	3.661	3.695	4.013	4.037	4.738	4.753	5.653	5.683
$k=4$				$p=1$				
α	0.10		0.05		0.01		0.001	
v	I	V_0	I	V_0	I	V_0	I	V_0
10	2.337	2.411	2.760	2.828	3.732	3.792	5.222	5.292
20	2.193	2.255	2.541	2.594	3.277	3.322	4.275	4.301
40	2.124	2.181	2.440	2.485	3.090	3.116	3.891	3.916
				$p=2$				
10	3.204	3.291	3.717	3.798	4.929	5.019	6.842	6.950
20	2.839	2.903	3.210	3.262	4.012	4.048	5.113	5.135
40	2.688	2.739	3.005	3.046	3.658	3.683	4.476	4.491
				$p=3$				
10	4.058	4.164	4.688	4.794	6.250	6.351	8.858	8.951
20	3.374	3.439	3.773	3.829	4.644	4.682	5.872	5.878
40	3.116	3.167	3.439	3.478	4.106	4.131	4.954	4.960
				$p=5$				
10	6.257	6.431	7.335	7.512	10.177	10.360	15.399	15.653
20	4.369	4.449	4.845	4.912	5.894	5.952	7.396	7.448
40	3.832	3.884	4.176	4.216	4.887	4.915	5.793	5.822

4. Multiple comparisons in the GMANOVA model

In this section, the simultaneous confidence intervals for multiple comparisons in a generalized multivariate analysis of variance (GMANOVA) model are discussed.

Consider the model equation proposed by Potthoff & Roy [26]

$$E[\underset{N\times p}{\boldsymbol{Y}}] = \underset{N\times k}{\boldsymbol{A}}\ \underset{k\times q}{\boldsymbol{\Xi}}\ \underset{q\times p}{\boldsymbol{B}}, \tag{4.1}$$

where the rows of $\boldsymbol{Y}: N \times p$ are independent multivariate normals with unknown common covariance matrix $\boldsymbol{\Sigma}: p \times p$. Suppose that $\boldsymbol{\Xi}: k \times q$ is an unknown matrix, and $\boldsymbol{A}: N \times k$ and $\boldsymbol{B}: q \times p (q \leq p)$ are assumed to be known such that the ranks of $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal to k and q, respectively. This model is called a generalized multivariate analysis of variance model. When $B = I_p$, the model (4.1) is reduced to the multivariate analysis of variance (MANOVA) model.

We discuss multiple comparisons in the GMANOVA model with a positive definite covariance matrix. Multiple comparisons related to the simultaneous test procedure under the model (4.1) were discussed by Khatri [15], Morrison [25] and Krishnaiah [18]. They discussed the confidence bounds for all non-null vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Recently, these confidence bounds in the growth curve model with the covariance structures have been discussed by Kanda [14]. In this section, we consider the simultaneous confidence intervals on $\boldsymbol{a}'\Xi'\boldsymbol{b}$ for all non-null vector $\boldsymbol{a}$ and pre-determined vector $\boldsymbol{b}$, for example, pairwise comparisons and comparisons with a control.

It is well known (see, e.g., Grizzle & Allen [6], Khatri [15]) that the maximum likelihood estimator of Ξ which is unbiased is given explicitly by

$$\hat{\Xi} = (A'A)^{-1}A'YS^{-1}B'(BS^{-1}B')^{-1},$$

where

$$S = \frac{1}{n} Y'(I_N - A(A'A)^{-1}A')Y, \qquad n = N - k.$$

Then, we note that $\mathrm{vec}(\hat{\Xi}')$ is asymptotically distributed as $N_{kq}(\mathrm{vec}(\Xi'), \tilde{V} \otimes \tilde{\Xi})$, where $\tilde{V} = (A'A)^{-1}$ and $\tilde{\Sigma} = (B\Sigma^{-1}B')^{-1}$. Further, let $\tilde{S} = (BS^{-1}B')^{-1}$ and $v = n - (p - q)$, then $(n/v)\tilde{S}$ is an unbiased estimator of $\tilde{\Sigma}$.

In relation to asymptotic distributions of $\boldsymbol{a}'(\hat{\Xi} - \Xi)'\boldsymbol{b}$ and their error bounds, Fujikoshi [5] has discussed a general approximation theory of a scale mixture of the standard normal distribution. Since the distribution of $\mathrm{vec}(\hat{\Xi})$ is not exactly but asymptotically normal, the result in the previous sections cannot be applied for constructing the simultaneous confidence intervals in this case. Here, we obtain the simultaneous confidence intervals for the mean parameters in the GMANOVA model similar to the ones in the previous sections by extending the distributional results in the previous sections.

4.1. $T^2_{\max}$-type statistic in the GMANOVA model

Based on an asymptotic distribution of $\mathrm{vec}(\Xi')$, we consider the simultaneous confidence intervals on $\boldsymbol{a}'\Xi'\boldsymbol{b}$ for all non-null $\boldsymbol{a} \in \mathbb{R}^q$ and $\boldsymbol{b} \in \mathbb{B}^k = \{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_r\}$ such that

$$\boldsymbol{a}'\Xi'\boldsymbol{b} \in [\boldsymbol{a}'\hat{\Xi}'\boldsymbol{b} \pm \tilde{t}(\boldsymbol{b}'\tilde{V}\boldsymbol{b})^{1/2}(\boldsymbol{a}'\tilde{S}\boldsymbol{a})^{1/2}], \qquad \forall \boldsymbol{a} \in \mathbb{R}^q,\ \forall \boldsymbol{b} \in \mathbb{B}^k. \tag{4.2}$$

Then we need to obtain the upper α percentiles $\tilde{t}^2$ of the following $\mathrm{T}^2_{\max}$-type statistic

$$\tilde{T}^2_{\max} = \max_{\boldsymbol{b} \in \mathbb{B}^k} \frac{\boldsymbol{b}'(\hat{\Xi} - \Xi)\tilde{S}^{-1}(\hat{\Xi} - \Xi)'\boldsymbol{b}}{\boldsymbol{b}'\tilde{V}\boldsymbol{b}}.$$

It is easily seen that the distribution of $\tilde{T}^2_{\max}$ is the same as the one of

$$\tilde{T}^2_{\max} = \max_i \{c_i' X \tilde{S}^{-1} X' c_i\}, \tag{4.3}$$

where $X = (A'A)^{1/2}(\hat{\Xi} - \Xi)$, $c_i = \tilde{b}_i/(\tilde{b}_i'\tilde{b}_i)^{1/2}$, and $\tilde{b}_i = (A'A)^{-1/2} b_i$. In the following we study the upper percentiles of this statistic. As stated in the previous sections, it is difficult to find the exact upper percentiles of this type statistic. We consider the approximate percentiles of the statistic (4.3) by modified second approximation procedure.

First, we derive the upper percentiles of $\tilde{T}^2 = c' X \tilde{S}^{-1} X' c$, where $c'c = 1$, by using a perturbation method. An asymptotic expansion of the distribution of $\tilde{T}^2$ has been essentially obtained by Kanda [14]. However, we give derivation method, since the method is also used for the joint distribution of $\tilde{T}_i^2$ and $\tilde{T}_j^2$, where $\tilde{T}_i^2 = c_i' X \tilde{S}^{-1} X' c_i$, $i = 1, 2$. The latter problem is a main interest in this section.

Let

$$U = (A'A)^{-1/2} A'(Y - A \Xi B) \Sigma^{-1/2},$$
$$V = \sqrt{n}(\Sigma^{-1/2} S \Sigma^{-1/2} - I_p),$$

respectively. Then, putting $z = (B \Sigma^{-1} B')^{1/2} X' c$, we have

$$z = z^{(1)} + \frac{1}{\sqrt{n}} z^{(2)} + \frac{1}{n} z^{(3)} + O_p(n^{-3/2}),$$

where

$$z^{(1)} = H U' c,$$
$$z^{(2)} = H V (P - I) U' c,$$
$$z^{(3)} = H V (P - I) V (P - I) U' c,$$

$H = (\tilde{B}\tilde{B}')^{-1/2} \tilde{B}$, $\tilde{B} = B \Sigma^{-1/2}$ and $P = H'H$.

Further, we can expand $\tilde{T}^2$ as

$$\tilde{T}^2 = \tilde{T}^{(1)} + \frac{1}{\sqrt{n}} \tilde{T}^{(2)} + \frac{1}{n} \tilde{T}^{(3)} + O_p(n^{-3/2}),$$

where

$$\begin{aligned} \tilde{T}^{(1)} &= z^{(1)\prime} z^{(1)}, \\ \tilde{T}^{(2)} &= 2 z^{(1)\prime} z^{(2)} - z^{(1)\prime} H V H' z^{(1)}, \\ \tilde{T}^{(3)} &= 2 z^{(1)\prime} z^{(3)} - 2 z^{(1)\prime} H V H' z^{(2)} + z^{(2)\prime} z^{(2)} + z^{(1)\prime} H V^2 H' z^{(1)}. \end{aligned} \tag{4.4}$$

Hence, the characteristc function $C_{\tilde{T}^2}(it)$ can be expressed as

$$\begin{aligned} C_{\tilde{T}^2}(it) &= E[\exp(it\tilde{T}^2)] \\ &= E\left[\exp(it\tilde{T}^{(1)})\left\{1 + \frac{it}{\sqrt{n}}\tilde{T}^{(2)} + \frac{it}{n}\tilde{T}^{(3)} + \frac{(it)^2}{2n}(T^{(2)})^2\right\}\right] + \mathrm{O}(n^{-3/2}). \end{aligned}$$

Since $\boldsymbol{U}$ and $\boldsymbol{V}$ are independent, first we calculate the expectation with respect to $\boldsymbol{V}$, and then we calculate the expectation with respect to $\boldsymbol{U}$. For the expectation with respect to $\boldsymbol{V}$, we have

$$E_V[\tilde{T}^{(2)}] = 0, \quad E_V[\tilde{T}^{(3)}] = (p+1)X + qY, \quad E_V[(\tilde{T}^{(2)})^2] = 4XY + 2X^2, \tag{4.5}$$

where $X = \boldsymbol{c}'\boldsymbol{U}\boldsymbol{P}\boldsymbol{U}'\boldsymbol{c}$, $Y = \boldsymbol{c}'\boldsymbol{U}(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{U}'\boldsymbol{c}$. Then we obtain

$$\begin{aligned} C_{\tilde{T}^2}(it) = E&\left[\exp(itX)\left\{1 + \frac{it}{n}\{(p+1)X + qY\} + \frac{(it)^2}{n}\{2XY + X^2\}\right\}\right] \\ &+ \mathrm{O}(n^{-3/2}). \end{aligned}$$

We note that X and Y are independently distributed as χ_q^2 and χ_{p-q}^2 distributions. Hence we can obtain

$$\begin{aligned} C_{\tilde{T}^2}(it) &= \varphi^{-q/2}\Big[1 + \frac{q}{n}(it\{p - q + (p+1)\varphi^{-1}\} \\ &\qquad + (it)^2\{2(p-q)\varphi^{-1} + (q+1)\varphi^{-2}\}\Big)\Big] + \mathrm{O}(n^{-2}) \\ &= \varphi^{-q/2}\left[1 + \frac{q}{4n}\{-(4p - 3q) + 2(2p - 2q - 1)\varphi^{-1} + (q+2)\varphi^{-2}\}\right] \\ &\qquad + \mathrm{O}(n^{-2}), \end{aligned}$$

where $\varphi = (1 - 2it)$.

Therefore, inverting this characteristic function $C_{\tilde{T}^2}(it)$, we have

$$\Pr(\tilde{T}^2 > x) = \Pr(\chi_q^2 > x) + \frac{q}{4n}\sum_{j=0}^{2} h_j \Pr(\chi_{q+2j}^2 > x) + \mathrm{O}(n^{-2}), \tag{4.6}$$

where $h_0 = -(4p - 3q)$, $h_1 = 2(2p - 2q - 1)$ and $h_2 = q + 2$. Further, we obtain an expansion for the upper percentiles by using a general formula (see, e.g., Fujikoshi [4]) for the upper percentiles of the statistic whose distribution has an expansion given by (4.6). Summarizing these results, we have the following results, which has been essentially obtained by Kanda [14].

THEOREM 4.1. *The distribution of* $\tilde{T}^2 = \boldsymbol{c}'\boldsymbol{X}\tilde{\boldsymbol{S}}^{-1}\boldsymbol{X}'\boldsymbol{c}$ *can be expanded as in* (4.6), *and its upper* α *percentiles can be expanded as*

$$x = \chi_q^2(\alpha) + \frac{1}{2n}\chi_q^2(\alpha)\{\chi_q^2(\alpha) + 4p - 3q\} + O(n^{-2}),$$

where $\chi_q^2(\alpha)$ is the upper α percentiles of χ^2 distribution with q d.f.

We note that, when $p = q$, the result in Theorem 4.1 is reduced to an asymptotic expansion for the upper percentiles of the Hotelling's T^2 statistic.

Next, we consider the following joint probability to give the modified second approximation to the upper percentiles of $\tilde{T}^2_{\max}$; that is,

$$\Pr\{\tilde{T}_1^2 > a_1^2,\ \tilde{T}_2^2 > a_2^2\}, \tag{4.7}$$

where $\tilde{T}_i^2 = c_i' X\tilde{S}^{-1}X' c_i$, $i = 1, 2$, a_1^2 and a_2^2 are given constants. In practice, we use the probability (4.7) with $a_1^2 = a_2^2 = \tilde{t}_1^2$ to obtain the modified second approximation $\tilde{t}_M^2$. By the same line as the previous first approximation, we consider the joint characteristic function of $\tilde{T}_1^2$ and $\tilde{T}_2^2$; that is,

$$C_{\tilde{T}^2}(it_1, it_2) = E[\exp(it_1\tilde{T}_1^2 + it_2\tilde{T}_2^2)].$$

By using the perturbation method, we have

$$\begin{aligned} C_{\tilde{T}^2}(it_1, it_2) = E[\exp(it_1\tilde{T}_1^{(1)} + it_2\tilde{T}_2^{(1)})]\bigg[1 &+ \frac{1}{\sqrt{n}}\{it_1\tilde{T}_1^{(2)} + it_2\tilde{T}_2^{(2)}\} \\ &+ \frac{1}{n}\bigg\{it_1\tilde{T}_1^{(3)} + \frac{(it_1)^2}{2}(\tilde{T}_1^{(2)})^2 + it_2\tilde{T}_2^{(3)} + \frac{(it_2)^2}{2}(\tilde{T}_2^{(2)})^2 \\ &+ (it_1)(it_2)\tilde{T}_1^{(2)}\tilde{T}_2^{(2)}\bigg\}\bigg] + O(n^{-3/2}), \end{aligned}$$

where $\tilde{T}_i^{(1)}$, $\tilde{T}_i^{(2)}$ and $\tilde{T}_i^{(1)}$, $i = 1, 2$, are given by (4.4) with $c = c_i$.

Using (4.5) with $c = c_i$ and the following results, which are obtained by calculating the expectation with respect to V,

$$E_V[\tilde{T}_1^{(2)}\tilde{T}_2^{(2)}] = 4c_1' UPU' c_2 \cdot c_1' U(I - P)U' c_2 + 2(c_1' UPU' c_2)^2,$$

these imply that

$$\begin{aligned} C_{\tilde{T}^2}(it_1, it_2) &= E[\exp(it_1X_1 + it_2X_2)] \\ &\times\bigg[1 + \frac{1}{n}\bigg\{it_1[q(p-q) + (p+1)X_1] + it_2[q(p-q) + (p+1)X_2] \\ &+ \frac{(it_1)^2}{2}[4(p-q)X_1 + 2X_1^2] + \frac{(it_2)^2}{2}[4(p-q)X_2 + 2X_2^2] \\ &+ (it_1)(it_2)[4(p-q)\rho c_1' UPU' c_2 + 2(c_1' UPU' c_2)^2\bigg\}\bigg] + O(n^{-3/2}), \end{aligned}$$

where $\rho = \rho_{12} = c_1' c_2$, $X_1 = c_1' UPU' c_1$ and $X_2 = c_2' UPU' c_2$. Further, we have

$$E[\exp(it_1 X_1 + it_2 X_2)] = \phi^{-q/2},$$
$$E[c_1' UPU' c_2 \exp(it_1 X_1 + it_2 X_2)] = \phi^{-q/2-1} q\rho,$$
$$E[(c_1' UPU' c_2)^2 \exp(it_1 X_1 + it_2 X_2)] = \phi^{-q/2-2} q\{(1-\rho^2)\phi + (q+2)\rho^2\},$$

where $\phi = (1-2it_1)(1-2it_2) - 4\rho^2 i^2 t_1 t_2$. From these results, putting

$$Q_1 = 4\lambda_1\lambda_2\{\lambda_1\lambda_2 - (\lambda_1 + \lambda_2)\} + (\lambda_1 + \lambda_2)^2,$$
$$Q_2 = 3\lambda_1\lambda_2 - 2(\lambda_1 + \lambda_2) + 1,$$
$$Q_3 = -2\lambda_1\lambda_2 + \lambda_1 + \lambda_2,$$

we have

$$C_{\tilde{T}^2}(it_1, it_2) = \phi^{-q/2}$$
$$\times \left[1 + \frac{q}{n}\left\{\frac{q+2}{4(1-\rho^2)^2} Q_1\phi^{-2} - \frac{q+1}{2(1-\rho^2)} Q_2\phi^{-1} + \frac{p-q}{1-\rho^2} Q_3\phi^{-1}\right\}\right]$$
$$+ O(n^{-3/2}),$$

where $\lambda_1 = 1 - 2(1-\rho^2)it_1$ and $\lambda_2 = 1 - 2(1-\rho^2)it_2$. We also note that

$$\phi^{-q/2} = (1-\rho^2)^{q/2} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}q)_m}{m!} \cdot \frac{\rho^{2m}}{\lambda_1^{q/2+m}\lambda_2^{q/2+m}}.$$

Inverting this characteristic function $C_{\tilde{T}^2}(it_1, it_2)$, the following result is obtained.

THEOREM 4.2. *With the notations*

$$f = \frac{q}{2} + m,\ \eta^{(i)} = \eta_{12}^{(i)} = \frac{a_i^2}{2(1-\rho_{12}^2)},\ i = 1, 2,$$
$$G_k^{(i)} = G_k(\eta^{(i)}) = \int_{\eta^{(i)}}^{\infty} g_k(t)dt,\ g_k(t) = \frac{1}{\Gamma(k)} t^{k-1} e^{-t},$$

it holds that

$$\Pr\{\tilde{T}_1^2 > a_1^2, \tilde{T}_2^2 > a_2^2\} = (1-\rho^2)^{q/2} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}q)_m}{m!} \rho^{2m}$$
$$\times \left[G_f^{(1)} G_f^{(2)} + \frac{1}{n}\left\{\frac{1}{4}(q+2m)(q+2m+2)\tilde{Q}_1 - \frac{1}{2}(q+1)(q+2m)\tilde{Q}_2\right.\right.$$
$$\left.\left. + (p-q)(q+2m)\tilde{Q}_3\right\}\right] + O(n^{-2})$$

with the coefficients $\tilde{Q}_1$, $\tilde{Q}_2$ *and* $\tilde{Q}_3$ *given by*

$$\tilde{Q}_1 = 4\{G_f^{(1)}G_f^{(2)} - (G_f^{(1)}G_{f+1}^{(2)} + G_f^{(2)}G_{f+1}^{(1)})\} + G_f^{(1)}G_{f+2}^{(2)} + G_f^{(2)}G_{f+2}^{(1)} + 2G_{f+1}^{(1)}G_{f+1}^{(2)},$$
$$\tilde{Q}_2 = 3G_f^{(1)}G_f^{(2)} - 2(G_f^{(1)}G_{f+1}^{(2)} + G_f^{(2)}G_{f+1}^{(1)}) + G_{f+1}^{(1)}G_{f+1}^{(2)},$$
$$\tilde{Q}_3 = -2G_f^{(1)}G_f^{(2)} + G_f^{(1)}G_{f+1}^{(2)} + G_f^{(2)}G_{f+1}^{(1)}.$$

Further, after a great deal of calculation, we have the following theorem.

THEOREM 4.3. *If* $a_1^2 = a_2^2 = a^2$, *then the joint probability is given by*

(4.8)
$$\Pr\{\tilde{T}_i^2 > a^2, \tilde{T}_j^2 > a^2\} = (1 - \rho_{ij}^2)^{q/2} \sum_{m=0}^{\infty} \frac{(\frac{1}{2}q)}{m!} \rho_{ij}^{2m}$$
$$\times \left[G_{q/2+m}^2(\eta_{ij}) + \frac{1}{n}\{\tilde{h}_1 g_{q/2+m}(\eta_{ij}) G_{q/2+m}(\eta_{ij}) + \tilde{h}_2 g_{q/2+m}^2(\eta_{ij})\} \right] + O(n^{-2}),$$

where $2\eta_{ij} = a^2/(1 - \rho_{ij}^2)$, *and the coefficients* $\tilde{h}_1$ *and* $\tilde{h}_2$ *are given by*

$$\tilde{h}_1 = \eta_{ij}(2\eta_{ij} + q - 2m) + 4\eta_{ij}(p - q),$$
$$\tilde{h}_2 = 2\eta_{ij}^2(2m + 1)(q + 2m)^{-1}.$$

4.2. Simultaneous confidence intervals in the GMANOVA model

We can construct the simultaneous confidence intervals (4.2) in the GMANOVA model by using the upper percentiles of $\tilde{T}_{\max}^2$ statistic (4.3), which can be approximated as follows.

From Theorems 4.1 and 4.3, we note that the modified second approximation to $\tilde{t}^2$ is given by

$$\tilde{t}_M^2 = \chi_q^2(\tilde{\alpha}) + \frac{1}{2n}\chi_q^2(\tilde{\alpha})\{\chi_q^2(\tilde{\alpha}) + 4p - 3q\} + O(n^{-2}),$$

where $\chi_q^2(\tilde{\alpha})$ is the upper $\tilde{\alpha} = (\alpha + \tilde{\beta}_1)/r$ percentiles of χ^2 distribution with q d.f., $\tilde{\beta}_1 = \sum\sum_{i<j} \Pr\{\tilde{T}_i^2 > \tilde{t}_1^2, \tilde{T}_j^2 > \tilde{t}_1^2\}$. In practice, each term of $\tilde{\beta}_1$ can be evaluated by using an asymptotic expansion up to the order n^{-1} of (4.8). An asymptotic expansion for $\tilde{t}_1^2$ is given by

(4.9)
$$\tilde{t}_1^2 = \chi_q^2(\alpha/r) + \frac{1}{2n}\chi_q^2(\alpha/r)\{\chi_q^2(\alpha/r) + 4p - 3q\} + O(n^{-2}).$$

Further, we have an asymptotic formula such that

$$G_{q/2+m}^2(\eta_{ij}) = G_{q/2+m}^2(\tilde{\eta}_{ij}) - \frac{1}{2n(1 - \rho_{ij}^2)}[\chi_q^2(\alpha/r)\{\chi_q^2(\alpha/r) + 4p - 3q\}]$$

$$\times\, g_{q/2+m}(\tilde{\eta}_{ij})G_{q/2+m}(\tilde{\eta}_{ij}) + \mathrm{O}(n^{-2}),$$

where

$$\eta_{ij} = \frac{\tilde{t}_1^2}{2(1-\rho_{ij}^2)}, \quad \tilde{\eta}_{ij} = \frac{\chi_q^2(\alpha/r)}{2(1-\rho_{ij}^2)}.$$

From these results, we obtain

(4.10)

$$\Pr\{\tilde{T}_i^2 > \tilde{t}_1^2,\ \tilde{T}_j^2 > \tilde{t}_1^2\} = (1-\rho_{ij}^2)^{q/2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}q)_m}{m!}\rho_{ij}^{2m}$$

$$\times\left[G_{q/2+m}^2(\tilde{\eta}_{ij}) + \frac{1}{n}\{\tilde{h}_1^* g_{q/2+m}(\tilde{\eta}_{ij})G_{q/2+m}(\tilde{\eta}_{ij}) + \tilde{h}_2^* g_{q/2+m}^2(\tilde{\eta}_{ij})\}\right] + \mathrm{O}(n^{-2}),$$

where

$$\tilde{h}_1^* = 2\tilde{\eta}_{ij}(\rho_{ij}^2\tilde{\eta}_{ij} - m) + 4\tilde{\eta}_{ij}(p-q),$$

$\tilde{h}_2^*$ is given by $\tilde{h}_2$ in Theorem 4.3 with $\eta_{ij} = \tilde{\eta}_{ij}$. We note that, when $p = q$, the result is reduced to the one in Theorem 1.1.

THEOREM 4.4. *The approximate simultaneous confidence intervals based on the modified second approximation procedure in the GMANOVA model are given by*

$$\boldsymbol{a}'\Xi'\boldsymbol{b} \in [\boldsymbol{a}'\hat{\Xi}'\boldsymbol{b} \pm \tilde{t}_M^*(\boldsymbol{b}'\tilde{V}\boldsymbol{b})^{1/2}(\boldsymbol{a}'\tilde{S}\boldsymbol{a})^{1/2}], \qquad \forall \boldsymbol{a} \in \mathbb{R}^q,\ \forall \boldsymbol{b} \in \mathbb{B}^k,$$

where $\tilde{t}_M^{*2}$ *is given by*

$$\tilde{t}_M^{*2} = \chi_q^2(\alpha^*) + \frac{1}{2n}\chi_q^2(\alpha^*)\{\chi_q^2(\alpha^*) + 4p - 3q\},$$

$\alpha^* = (\alpha + \tilde{\beta}_1^*)/r$, *and* $\tilde{\beta}_1^*$ *is an asymptotic expansion of* $\tilde{\beta}_1$ *up to the order* n^{-1} *based on* (4.10).

It is easily noted that the approximate simultaneous confidence intervals for pairwise comparisons and for comparisons with a control in the GMANOVA model can be constructed by Theorem 4.4. It may be expected that these intervals are shorter length intervals than the simultaneous confidence intervals on $\boldsymbol{a}'\Xi'\boldsymbol{b}$ for all non-null $\boldsymbol{a}$ and $\boldsymbol{b}'\boldsymbol{1} = 0$ in those cases.

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