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# On the irreducible components of the solutions of Matsuo's differential equations

Dedicated to Professor Kiyosato Okamoto on his sixtieth birthday

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## 0. Introduction

Studying the Knizhnik-Zamolodchikov equation in conformal field theory, Matsuo found a new system of differential equations of first order for a function taking values in the group algebra $\mathbf{C}[W]$ of the Weyl group $W$ associated with an arbitrary root system in [4]. His system is equivalent to the system of the differential equations given by Heckman and Opdam which is a deformation of the system satisfied by the zonal spherical function of the Riemannian symmetric space $G/K$ of non compact type ([4] Theorem 5.4.1).

Let $\Phi$ be a solution of Matsuo's equations (see (1.1)). $\hat{W}$ denotes the set of the equivalence classes of the irreducible representations of $W$. For $\delta \in \hat{W}$ let $E_\delta$ be a representation space of $\delta$ and $n_\delta = \dim E_\delta$. Then $\mathbf{C}[W] = \sum_{\delta \in \hat{W}} \mathbf{C}[W]_\delta$, where $\mathbf{C}[W]_\delta = \bigoplus_{i=1}^{n_\delta} E_{\delta,i}$ and $E_{\delta,i}$ is equivalent to $E_\delta$ $(1 \leq i \leq n_\delta)$. Let $\delta_0$ be the trivial representation and $\Phi_0$ be the $\mathbf{C}[W]_{\delta_0}$-component of $\Phi$. The Correspondence $\Phi \to \Phi_{\delta_0}$ gives the equivalence of the above two systems.

For $\delta \in \hat{W}$ We consider the other $\mathbf{C}[W]_\delta$-components $\Phi_\delta$ of $\Phi$. In this paper we obtain a system of differential equations satisfied by $\Phi_\delta$.

## 1. Preliminaries

Let $E$ be an n-Euclidean space with the inner product $(\ ,\ )$ and $E^*$ be the dual space of $E$. For $\alpha \in E$ with $\alpha \neq 0$ put $\alpha^\vee = 2(\alpha, \alpha)^{-1}\alpha$ and denote $s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha$ for the orthogonal reflection in the hyperplane perpendicular to $\alpha$ $(\lambda \in E)$. Let $\Sigma \subset E$ be a root system with rank $(\Sigma) = \dim E = n$. Fix a system of positive roots $\Sigma^+$ in $\Sigma$. Furthermore we put $\Sigma_0 = \{\alpha \in \Sigma;\ \alpha \notin 2\Sigma\}$ and $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$. Let $W$ be the Weyl group and $\mathbf{C}[W]$ be the group algebra of $W$. Put $\mathfrak{a} = E^*$, $\mathfrak{h} = E^* \oplus iE^*$. The inner product in $E$ and the reflections can be extended to $\mathfrak{h}^*$ naturally. We identify $\mathfrak{h}^*$ with $\mathfrak{h}$ via the inner product $(\ ,\ )$:

$$\lambda(u) = (\lambda, u) \qquad (\lambda \in \mathfrak{h}^*, \ u \in \mathfrak{h}).$$

We define the endomorphisms $\sigma_\alpha$ and $\varepsilon_\alpha$ of $\mathbf{C}[W]$ as follows:

$$\sigma_\alpha(w) = s_\alpha w$$

and

$$\varepsilon_\alpha(w) = \begin{cases} w & \text{if } w^{-1}\alpha \in \Sigma^+, \\ -w & \text{otherwise,} \end{cases}$$

where $w \in W$, $\alpha \in \Sigma$. Furthermore for any $\lambda \in \mathfrak{h}^*$ and $\xi \in \mathfrak{h}$ we define $e_\xi(\lambda) \in \mathrm{End}(\mathbf{C}[W])$ by

$$e_\xi(\lambda)(w) = (w\lambda, \xi)w.$$

Consider the following system of differential equations for a $\mathbf{C}[W]$-valued function $\Phi$ on $\mathfrak{h}$:

(1.1)
$$\begin{aligned} &\partial_\xi \Phi(u) \\ &= \{ \sum_{\alpha \in \Sigma^+} (k_\alpha/2)(\alpha, \xi)((e^{\alpha(u)} + 1)(e^{\alpha(u)} - 1)^{-1}(\sigma_\alpha - 1) + \sigma_\alpha \varepsilon_\alpha) \\ &\quad + e_\xi(\lambda)\} \Phi(u); \qquad \xi \in \mathfrak{h}, \\ &(u \in \mathfrak{h}), \end{aligned}$$

where $k_\alpha$ are given complex numbers such that $k_{w\alpha} = k_\alpha$ for all $\alpha \in \Sigma$ and $w \in W$ (see Matsuo [4]).

$\hat{W}$ denotes the set of the equivalence classes of the irreducible representations of $W$ and $\nu$ denotes the left regular representation of $W$. For $\delta \in \hat{W}$ let $E_\delta$ be a representation space of $\delta$ and $n_\delta = \dim E_\delta$. Then it is well known that $\mathbf{C}[W] = \sum_{\delta \in \hat{W}} \mathbf{C}[W]_\delta$, where $\mathbf{C}[W]_\delta = E_{\delta,1} \oplus \cdots \oplus E_{\delta,n_\delta}$ and $E_{\delta,i}$ is equivalent to $E_\delta$ $(i = 1, 2, \cdots, n_\delta)$. Since $E_{\delta,i}$ is an irreducible left ideal of $\mathbf{C}[W]$, there is some irreducible idempotent $\varepsilon_{\delta,i} \in \mathbf{C}[W]$ such that

(1.2)
$$E_{\delta,i} = \mathbf{C}[W]\varepsilon_{\delta,i} \qquad (i = 1, 2, \cdots, n_\delta).$$

$\chi_\delta$ denotes the character of $\delta$. We put

(1.3)
$$P_\delta = n_\delta |W|^{-1} \sum_{w \in W} \chi_\delta(w^{-1})\nu(w).$$

Then $P_\delta$ is the projection of $\mathbf{C}[W]$ onto $\mathbf{C}[W]_\delta$. We set

(1.4)
$$C_\xi = \sum_{\alpha \in \Sigma^+} (k_\alpha/2)(\alpha, \xi)\sigma_\alpha \varepsilon_\alpha + e_\xi(\lambda).$$

We have $v(w)C_\xi v(w)^{-1} = C_{w\xi}$ for any $w \in W$. Note that $\sum_{t\in W} C_{t\xi}^d$ commutes with the left regular representation of $W$ for any natural number $d$. Let $\mathscr{R}$ be the algebra of functions on $\{u \in \mathfrak{h},\ e^{\alpha(u)} \neq 1$ for any $\alpha \in \Sigma_0^+\}$ generated by $\{(1 - e^{\alpha(u)})^{-1};\ \alpha \in \Sigma^+\}$. $\mathfrak{A}(\mathfrak{h})$ denotes the set of all differential operators on $\mathfrak{h}$ with constant coefficients. If $P$ belongs to $\mathscr{R} \otimes \mathfrak{A}(\mathfrak{h})$, $P$ is expressed as

$$P = \sum_{\mu \in Q_+} e^\mu \partial(P^\mu), \tag{1.5}$$

where $P^\mu$ is some element of the symmetric algebra of $\mathfrak{h}$ and $Q_+ = \{\sum_{\alpha\in\Sigma^+} n_\alpha \alpha;$ $n_\alpha = 0, 1, 2, \cdots\}$. We denote by $\mathbf{C}[\mathfrak{h}^*]$ the polynomial algebra on $\mathfrak{h}^*$. For $P = \sum_{\mu\in Q_+} e^\mu \partial(P^{(\mu)}) \in \mathscr{R} \otimes \mathfrak{A}(\mathfrak{h})$ the Harish-Chandra homomorphism $r: \mathscr{R} \otimes \mathfrak{A}(\mathfrak{h}) \to \mathbf{C}[\mathfrak{h}^*]$ is the algebra homomorphism defined by

$$r(P)(\lambda) = P^{(0)}(\lambda + \rho), \tag{1.6}$$

where $\rho = \sum_{\alpha\in\Sigma^+} (k_\alpha/2)\alpha$, $\lambda \in \mathfrak{h}^*$. For $T \otimes P \in \mathrm{End}\,(\mathbf{C}[W]_\delta) \otimes (\mathscr{R} \otimes \mathfrak{A}(\mathfrak{h}))$ we define

$$r_\delta(T \otimes P)(\lambda) = r(P)(\lambda)T. \tag{1.7}$$

We define the differential operator $D_{\delta,\xi}^{(d)} \in \mathrm{End}\,(\mathbf{C}[W]_\delta) \otimes (\mathscr{R} \otimes \mathfrak{A}(\mathfrak{h}))$ for $\delta \in \hat{W}$, $\xi \in \mathfrak{h}$ and a nonnegative integer $d$ inductively by

$$\begin{aligned} D_{\delta,\xi}^{(d)} &= (1_\delta \otimes \partial_\xi) D_{\delta,\xi}^{(d-1)} \\ &\quad - \sum_{\alpha\in\Sigma^+} (k_\alpha/2)(\alpha, \xi)(e^\alpha + 1)(e^\alpha - 1)\{(v_\delta(s_\alpha) \otimes 1) D_{\delta,s_\alpha\xi}^{(d-1)} - D_{\delta,\xi}^{(d-1)}\}, \end{aligned} \tag{1.8}$$

$$D_{\delta,\xi}^{(0)} = 1_\delta \otimes 1, \tag{1.9}$$

where $1_\delta$ is the identity mapping on $\mathbf{C}[W]_\delta$ and $v_\delta = v|_{\mathbf{C}[W]_\delta}$. We set

$$\tilde{D}_{\delta,\xi}^{(d)} = \sum_{t\in W} D_{\delta,t\xi}^{(d)}. \tag{1.10}$$

## 2. The differential equations for the irreducible components

Our main theorem in this paper is the following

THEOREM 2.1. Suppose that $\Phi$ is a $\mathbf{C}[W]$-valued function and satisfies (1.1). Then $\Phi_\delta = P_\delta \circ \Phi$ satisfies the following formulas:

$$\tilde{D}^{(d)}_{\delta,\xi}\Phi_\delta = (\sum_{t\in W} C^d_{t\xi})\Phi_\delta \qquad (d = 0, 1, 2, \cdots). \tag{2.1}$$

In particular $\sum_{t\in W} C^2_{t\xi}$ is a scalar operator on $\mathbf{C}[W]_\delta$ and we have

$$\tilde{D}^{(2)}_{\delta,\xi}\Phi_\delta = r_\delta(\tilde{D}^{(2)}_{\delta,\xi})(\lambda)\Phi_\delta, \tag{2.2}$$

$$\begin{aligned} r_\delta(\tilde{D}^{(2)}_{\delta,\xi})(\lambda) \\ = \sum_{t\in W}(\lambda, t\xi)^2 - n_\delta^{-1}\sum_{t\in W}\sum_{\alpha,\beta\in\Sigma^+}(k_\alpha k_\beta/4)(\alpha, t\xi)(\beta, t\xi)\chi_\delta(s_\alpha s_\beta). \end{aligned} \tag{2.3}$$

We need the following lemmas to prove Theorem 2.1.

LEMMA 2.2 ([4] Lemma 4.1.1). If $\Phi(u)$ is a solution of (1.1), we have

$$D^{(d)}_{\delta,\xi}\Phi_\delta = P_\delta(C^d_\xi\Phi). \tag{2.4}$$

PROOF. We obtain (2.4) in the same way as [4] Lemma 4.1.1.

LEMMA 2.3. Let $A\in\mathrm{End}(\mathbf{C}[W])$. If $A$ commutes with the left regular representation of $W$ and $A(1)$ belongs to the center of $\mathbf{C}[W]$, then $A$ is a scalar operator on $\mathbf{C}[W]$.

PROOF. From the conditions on $A$

$$A(x) = xA(1) = A(1)x \tag{2.5}$$

for any $x\in\mathbf{C}[W]$. $A|_{E_{\delta,i}}$ is the endomorphism on $E_{\delta,i}$ from (2.5) and commutes with the left regular representation on $W$. So $A$ is a scalar operator on $E_{\delta,i}$ by Schur's lemma. There exists $f_{i,j}\in\mathbf{C}[W]$ such that

$$\varepsilon_{\delta,i}f_{i,j}\varepsilon_{\delta,j} \neq 0 \tag{2.6}$$

because $\varepsilon_{\delta,i}$ and $\varepsilon_{\delta,j}$ are equivalent $(i, j = 1, 2, \cdots, n_\delta)$. If $A|_{E_{\delta,i}} = \lambda_i\cdot 1$ $(\lambda_i\in\mathbf{C}$, $i = 1, 2, \cdots, n_\delta)$, we have

$$A\varepsilon_{\delta,i} = \lambda_i\varepsilon_{\delta,i}, \tag{2.7}$$

$$A\varepsilon_{\delta,j} = \lambda_j\varepsilon_{\delta,j}. \tag{2.8}$$

Then we have

$$A(\varepsilon_{\delta,i})f_{i,j}\varepsilon_{\delta,j} = \lambda_i\varepsilon_{\delta,i}f_{i,j}\varepsilon_{\delta,j}, \tag{2.9}$$

$$\varepsilon_{\delta,i}f_{i,j}A(\varepsilon_{\delta,j}) = \lambda_j\varepsilon_{\delta,i}f_{i,j}\varepsilon_{\delta,j}. \tag{2.10}$$

(2.5) gives

$$A(\varepsilon_{\delta,i})f_{i,j}\varepsilon_{\delta,j} = \varepsilon_{\delta,i}A(1)f_{i,j}\varepsilon_{\delta,j} = \varepsilon_{\delta,i}f_{i,j}\varepsilon_{\delta,j}A(1) = \varepsilon_{\delta,i}f_{i,j}A(\varepsilon_{\delta,j}) \tag{2.11}$$

and we obtain $\lambda_i = \lambda_j$ from (2.9)–(2.11). Hence we can see that $A$ is a scalar operator on $\mathbf{C}[W]_\delta$. q.e.d.

LEMMA 2.4. $\sum_{t\in W} C_{t\xi}^2(1)$ belongs to the center of $\mathbf{C}[W]$.

PROOF. By the definition of $C_\xi^2$ we get

$$\sum_{t\in W} C_{t\xi}^2(1) = \sum_{t\in W} (\lambda, t\xi)^2 \cdot 1 - \sum_{t\in W} \sum_{\alpha,\beta\in\Sigma^+} (k_\alpha/4)(\alpha, t\xi)(\beta, t\xi)s_\alpha s_\beta. \tag{2.12}$$

we set

$$C_0 = \sum_{t\in W} \sum_{\alpha,\beta\in\Sigma_0^+} (k_\alpha k_\beta/4)(\alpha, t\xi)(\beta, t\xi)s_\alpha s_\beta, \tag{2.13}$$

$$C_1 = \sum_{t\in W} \sum_{\alpha\in\Sigma^+\setminus\Sigma_0^+,\beta\in\Sigma_0^+} (k_\alpha k_\beta/4)(\alpha, t\xi)(\beta, t\xi)s_\alpha s_\beta + \sum_{t\in W} \sum_{\alpha\in\Sigma_0^+,\beta\in\Sigma^+\setminus\Sigma_0^+} (k_\alpha k_\beta/4)(\alpha, t\xi)(\beta, t\xi)s_\alpha s_\beta, \tag{2.14}$$

$$C_2 = \sum_{t\in W} \sum_{\alpha,\beta\in\Sigma^+\setminus\Sigma_0^+} (k_\alpha k_\beta/4)(\alpha, t\xi)(\beta, t\xi)s_\alpha s_\beta. \tag{2.15}$$

Suppose $\gamma\in\Sigma_0^+$ and $2\gamma\notin\Sigma^+$. Then we see

$$s_\gamma(\Sigma^+\setminus\Sigma_0^+) = \Sigma^+\setminus\Sigma_0^+, \tag{2.16}$$

$$s_\gamma(\Sigma_0^+) = (\Sigma_0^+\setminus\{\gamma\})\cup\{-\gamma\}. \tag{2.17}$$

Since $s_\gamma s_\alpha s_\gamma^{-1} = s_{s_\gamma(\alpha)}$ $(\alpha\in\Sigma)$, we get

$$\begin{aligned} s_\gamma C_1 s_\gamma^{-1} &= \sum_{t\in W} \sum_{\alpha\in\Sigma^+\setminus\Sigma_0^+,\beta\in\Sigma_0^+} (k_\alpha k_\beta/4)(\alpha, t\xi)(\beta, t\xi)s_{s_\gamma(\alpha)}s_{s_\gamma(\beta)}. \\ &+ \sum_{t\in W} \sum_{\alpha\in\Sigma_0^+,\beta\in\Sigma^+\setminus\Sigma_0^+} (k_\alpha k_\beta/4)(\alpha, t\xi)(\beta, t\xi)s_{s_\gamma(\alpha)}s_{s_\gamma(\beta)} \end{aligned} \tag{2.18}$$

If we replace $s_\gamma(\alpha)$ and $s_\gamma(\beta)$ with $\alpha$ and $\beta$, (2.15)–(2.17) imply

$$s_\gamma C_1 s_\gamma^{-1} = \sum_{t\in W} \sum_{\substack{\alpha\in\Sigma^+\setminus\Sigma_0^+ \\ \beta\in(\Sigma_0^+\setminus\{\gamma\})\cup\{-\gamma\}}} (k_\alpha k_\beta/4)(\alpha, t\xi)(\beta, t\xi)s_\alpha s_\beta \tag{2.19}$$

$$+ \sum_{t\in W} \sum_{\substack{\alpha\in(\Sigma_0^+\setminus\{\gamma\})\cup\{-\gamma\} \\ \beta\in\Sigma^+\setminus\Sigma_0^+}} (k_\alpha k_\beta/4)(\alpha, t\xi)(\beta, t\xi)s_\alpha s_\beta.$$

(2.19) gives

$$s_\gamma C_1 s_\gamma^{-1} - C_1 \tag{2.20}$$

$$= -2\sum_{t\in W} \sum_{\beta\in\Sigma^+\setminus\Sigma_0^+} (k_\gamma k_\beta/4)(\gamma, t\xi)(\beta, t\xi)s_\gamma s_\beta$$

$$-2\sum_{t\in W} \sum_{\alpha\in\Sigma^+\setminus\Sigma_0^+} (k_\alpha k_\gamma/4)(\alpha, t\xi)(\gamma, t\xi)s_\alpha s_\gamma.$$

If we put $\alpha = s_\gamma(\beta)$, we have $s_\gamma s_\beta = s_\alpha s_\gamma$, $k_\alpha = k_{s_\gamma(\beta)} = k_\beta$ and the second term of the right hand side of (2.20) is

$$\sum_{t\in W} \sum_{\alpha\in\Sigma^+\setminus\Sigma_0^+} (k_\alpha k_\gamma/4)(\alpha, t\xi)(\gamma, t\xi)s_\alpha s_\gamma \tag{2.21}$$

$$= \sum_{t\in W} \sum_{\beta\in\Sigma^+\setminus\Sigma_0^+} (k_\beta k_\gamma/4)(\beta, t\xi)(-\gamma, t\xi)s_\gamma s_\beta.$$

(2.20) and (2.21) imply $s_\gamma C_1 s_\gamma^{-1} = C_1$. we can see that $s_\gamma C_0 s_\gamma^{-1} = C_0$ and $s_\gamma C_2 s_\gamma^{-1} = C_2$ similarly.

Next suppose $\gamma\in\Sigma_0^+$ and $2\gamma\in\Sigma^+$. In this case we have

$$s_\gamma(\Sigma^+\setminus\Sigma_0^+\cup\{2\gamma\}) = \Sigma_0^+\cup\{2\gamma\}, \tag{2.22}$$

$$s_\gamma(2\gamma) = -2\gamma. \tag{2.23}$$

By using (2.22) and (2.23) we can prove $s_\gamma C_1 s_\gamma^{-1} = C_1$ similarly. Hence $C_1$ belongs to the center of $\mathbf{C}[W]$. In the same way we can see that $C_0$ and $C_2$ belongs to the center of $\mathbf{C}[W]$ and this proves the lemma. q.e.d.

LEMMA 2.5 (cf. [4] Lemma 4.1.2). For any $x\in\mathbf{C}[W]_\delta$ we have

$$r_\delta(D_{\delta,\xi}^{(d)})(\lambda)x = C_\xi^d(1)x. \tag{2.24}$$

PROOF. When $d = 0$, (2.24) is valid. We assume that (2.24) holds for $d-1$. By using $\nu(s_\alpha)C_{s_\alpha\xi}^{d-1}(1) = C_\xi^{d-1}(s_\alpha)$ we have

$$r_\delta(D_{\delta,\xi}^{(d)})(\lambda)x = r_\delta((1_\delta\otimes\partial_\delta)D_{\delta,\xi}^{(d-1)} \tag{2.25}$$

$$-\sum_{\alpha\in\Sigma^+}(k_\alpha/2)(e^\alpha+1)(e^\alpha+1)(e^\alpha-1)^{-1}\{(\nu(s_\alpha)\otimes 1)D_{\delta,s_\alpha\beta}^{(d-1)} - D_{\delta,\xi}^{(d-1)}\})(\lambda)x$$

$$= (\lambda, \xi) r_\delta(D_{\delta,\xi}^{(d-1)})(\lambda)x + \sum_{\alpha\in\Sigma^+} (k_\alpha/2)(\alpha, \xi) v(s_\alpha) r_\delta(D_{\delta,s_\alpha\xi}^{(d-1)})(\lambda)x$$

$$= (\lambda, \xi) C_\xi^{d-1}(1)x + \sum_{\alpha\in\Sigma^+} (k_\alpha/2)(\alpha, t\xi) v(s_\alpha) C_{s_\alpha\xi}^{d-1}(1)x$$

$$= \{(\lambda, \xi) C_\xi^{d-1}(1) + \sum_{\alpha\in\Sigma^+} (k_\alpha/2)(\alpha, \xi) C_\xi^{d-1}(s_\alpha)\} x$$

$$= C_\xi^{d-1}(C_\xi(1))x.$$

Therefore we get (2.25). q.e.d.

PROOF OF THEOREM 2.1. Suppose that $\Phi(u)$ is a solution of (1.1). Since $\sum_{t\in W} C_{t\xi}^2$ is a linear mapping and commutes with $v(w)$ for any $w\in W$, we have

$$P_\delta(\sum_{t\in W} C_{t\xi}^2) = (\sum_{t\in W} C_{t\xi}^2) P_\delta \tag{2.26}$$

from (1.3). (2.1) follows from (2.4) and (2.26). By Lemmas 2.3 and 2.4 we see that $\sum_{t\in W} C_{t\xi}^2$ is a scalar operator on $\mathbf{C}[W]_\delta$. Since $\sum_{t\in W} C_{t\xi}^2(1)$ belongs to the center of $\mathbf{C}[W]$ we get (2.2) from (2.1) and (2.25). We obtain (2.3) by calculations. q.e.d.

REMARK. Let $\delta_0$ and $\delta_1$ be the trivial representation and the alternative representation, respectively. Since $\mathbf{C}[W]_{\delta_0}$ and $\mathbf{C}[W]_{\delta_1}$ are 1-dimensional spaces, $\tilde{D}_{\delta_0,\xi}^{(d)}$ and $\tilde{D}_{\delta_1,\xi}^{(d)}$ belong to $\mathscr{R}\otimes\mathfrak{A}(\mathfrak{h})$. If $\Phi$ is a solution of (1.1), the following formulas are valid for $d = 0, 1, 2, \cdots$:

$$\tilde{D}_{\delta_0,\xi}^{(d)} \Phi_{\delta_0} = r(\tilde{D}_{\delta_0,\xi}^{(d)})(\lambda)\Phi_{\delta_0}, \tag{2.27}$$

$$\tilde{D}_{\delta_1,\xi}^{(d)} \Phi_{\delta_1} = r(\tilde{D}_{\delta_1,\xi}^{(d)})(\lambda)\Phi_{\delta_1}. \tag{2.28}$$

(2.27) is proved in Matsuo [4]. Since $\sum_{t\in W} C_{t\xi}^d(1)$ belongs to the center of $\mathbf{C}[W]_{\delta_1}$ we have (2.28) by (2.24).

## 3. An example of type $A_3$

In this section let $\Sigma$ be the $A_3$ type root ststem. We put $\mathfrak{a} = \{(t_1, t_2, t_3)\in R^3; t_1 + t_2 + t_3 = 0\}$ and $\mathfrak{h} = \mathfrak{a} + i\mathfrak{a}$. For $h = (h_1, h_2, h_3)\in\mathfrak{h}$ we define $\alpha_i\in\Sigma^+$ $(i = 1, 2, 3)$ as follows:

$$\begin{aligned} \alpha_1(h) &= h_1 - h_2, \\ \alpha_2(h) &= h_2 - h_3, \\ \alpha_3(h) &= \alpha_1(h) + \alpha_2(h). \end{aligned} \tag{3.1}$$

Let $s_i$ be the reflection along $\alpha_i$. We set

$$\begin{aligned}\varepsilon_0 &= (1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1)/6,\\ \varepsilon_1 &= (1 + s_1 - s_2 s_1 - s_1 s_2 s_1)/3,\\ \varepsilon_2 &= (1 - s_1 - s_1 s_2 + s_1 s_2 s_1)/3,\\ \varepsilon_3 &= (1 - s_1 - s_2 + s_1 s_2 + s_2 s_1 - s_1 s_2 s_1)/6.\end{aligned} \tag{3.2}$$

$\varepsilon_0, \varepsilon_1, \varepsilon_2,$ and $\varepsilon_3$ are irreducible idempotent elements of $\mathbf{C}[W]_\delta$ and $\mathbf{C}[W] = \bigoplus_{i=0}^{3} \mathbf{C}[W]\varepsilon_i$ is the irreducible decomposition of $\mathbf{C}[W]$. $v$ acts trivially on $\mathbf{C}[W]\varepsilon_0$ and alternatively on $\mathbf{C}[W]\varepsilon_3$. $\mathbf{C}[W]\varepsilon_1$ and $\mathbf{C}[W]\varepsilon_2$ are equivalent. Furthermore we have

$$\sum_{i=0}^{3} \varepsilon_i = 1, \qquad \varepsilon_i \varepsilon_j = \delta_{i,j}\varepsilon_i \ (i, j = 0, 1, 2, 3). \tag{3.3}$$

If we put

$$P_i x = x\varepsilon_i \ (i = 0, 1, 2, 3,\ x \in \mathbf{C}[W]), \tag{3.4}$$

then $P_i$ is the projection onto $\mathbf{C}[W]\varepsilon_i$.

For $\sum_{w\in W} a(w)w$ and $\sum_{w\in W} b(w)w \in \mathbf{C}[W]$ we define

$$\Big(\sum_{w\in W} a(w)w, \sum_{w\in W} b(w)w\Big) = \sum_{w\in W} a(w)b(w), \tag{3.5}$$

$(a(w), b(w) \in \mathbf{C})$. $(\ ,\ )$ is a non-degenerate bilinear form and for any $w \in W$ and $u, v \in \mathbf{C}[W]$ we have

$$(wv, u) = (v, w^{-1}u). \tag{3.6}$$

If $T$ is a linear mapping on $\mathbf{C}[W]$ and satisfies the formula $(Tx, y) = (x, Ty)$ (resp. $(Tx, y) = (x, -Ty)$), we call $T$ is symmetric (resp. anti symmetric) with respect to the bilinear form $(\ ,\ )$.

We put $v_i = v|_{\mathbf{C}[W]\varepsilon_i}$ and

$$D_{i,\xi}^{(d)} = (1 \otimes \partial_\xi) D_{i,\xi}^{(d-1)} - \sum_{\alpha\in\Sigma^+} (k_\alpha/2)(\alpha, \xi)(e^\alpha + 1)(e^\alpha - 1)\{(v_i(s_\alpha) \otimes 1) D_{i,s_\alpha\xi}^{(d-1)} - D_{i,\xi}^{(d-1)}\}, \tag{3.7}$$

$$D_{i,\xi}^{(0)} = 1 \otimes 1, \tag{3.8}$$

(3.9) $$\tilde{D}^{(d)}_{i,\xi} = \sum_{t \in W} D^{(d)}_{i,t\xi}.$$

For $T \otimes P \in \mathrm{End}\,(\mathbf{C}[W]\varepsilon_i) \otimes (\mathscr{R} \otimes \mathfrak{A}(\mathfrak{h}))$ we define $r_i(T \otimes P)$ in the same way as (1.7).

We shall prove the following theorem in this section.

THEOREM 3.1. If $\Phi$ is a solution of (1.1), we have

(3.9) $$\tilde{D}^{(d)}_{i,\xi} \Phi_i = r_i(\tilde{D}_{i,\xi})(\lambda)\Phi_i \qquad (d = 0, 1, 2, \cdots),$$

where we put $\Phi_i = P_i \Phi$.

We need the following lemma to prove Theorem 3.1.

LEMMA 3.2. $\sum_{t \in W} C^d_{t\xi}$ (1) belongs to the center of $\mathbf{C}[W]$ $(d = 0, 1, 2, \cdots)$.

PROOF. Since $\sigma_\alpha \varepsilon_\alpha$ is anti symmetric and $e_\xi(\lambda)$ is symmetric with respect to the bilinear form ( , ), $\sum_{t \in W} C^d_{t\xi}$ is expressed as follows:

(3.10) $$\sum_{t \in W} C^d_{t\xi} = A_{\xi,d} + B_{\xi,d},$$

where $A_{\xi,d}$ is symmetric and $B_{\xi,d}$ is anti symmetric with respect to the bilinear form ( , ) and $A_{\xi,d}(1)$ is a linear combination of even products of reflections and $B_{\xi,d}(1)$ is a linear combination of odd products of reflections. For any $w \in W$ we see that $\nu(w)A_{\xi,d}\nu(w)^{-1}$ is symmetric and $\nu(w)B_{\xi,d}\nu(w)^{-1}$ is anti symmetric by (3.6). Therefore $\nu(w)A_{\xi,d}\nu(w)^{-1} = A_{\xi,d}$ and $\nu(w)B_{\xi,d}\nu(w)^{-1} = B_{\xi,d}$ because $\nu(w)(\sum_{t \in W} C^d_{t\xi})\nu(w)^{-1} = \sum_{t \in W} C^d_{t\xi}$ for any $w \in W$. Then we have for any $w \in W$

(3.11) $$(A_{\xi,d}(1),\ w - w^{-1}) = 0$$

because $(A_{\xi,d}(w), 1) = (wA_{\xi,d}(1), 1) = (A_{\xi,d}(1), w^{-1})$ and $(A_{\xi,d}(w), 1) = (w, A_{\xi,d}(1)) = (A_{\xi,d}(1), w)$. Similarly we have for any $w \in W$

(3.12) $$(B_{\xi,d}(1),\ w + w^{-1}) = 0.$$

Since $\{1, s_1, s_2, s_1 s_2 s_1, s_1 s_2 + s_2 s_1, s_1 s_2 - s_2 s_1\}$ is a basis of $\mathbf{C}[W]$, $A_{\xi,d}(1)$ and $B_{\xi,d}(1)$ are expressed as follows:

(3.13) $$A_{\xi,d}(1) = a_0 \cdot 1 + a_1 s_1 + a_2 s_2 + a_3 s_1 s_2 s_1 + a_4(s_1 s_2 + s_2 s_1),$$

(3.14) $$B_{\xi,d}(1) = a_5(s_1 s_2 - s_2 s_1),$$

where $a_0, \cdots, a_5 \in \mathbf{C}$. Hence we get $B_{\xi,d}(1) = 0$ and $A_{\xi,d}(1) = a_0 \cdot 1 + a_4(s_1 s_2 + s_2 s_1)$. This shows that $\sum_{t \in W} C_{t\xi}^d(1)$ belongs to the center of $\mathbf{C}[W]$. q.e.d.

PROOF OF THEOREM 3.1. In the same way as Lemma 2.2 and Lemma 2.5 we have for $i = 0, 1, 2, 3$

$$\tilde{D}_{i,\xi}^{(d)} \Phi_i = \sum_{t \in W} C_{t\xi}^d \Phi_i, \tag{3.15}$$

$$r_i(D_{i,\xi}^{(d)})(\lambda) x = C_\xi^d(1) x \qquad ({}^\forall x \in \mathbf{C}[W]\varepsilon_i). \tag{3.16}$$

From (3.15), (3.16) and Lemma 3.2 we can prove (3.9) in the same way as the proof of Theorem 2.1. q.e.d.

## References

[1] I. V. Cherednik, Integration of quantum many-body problems by affine KZ equations. Preprint of RIMS, Kyoto, 1991.

[2] G. J. Heckman, Root systems and hypergeometric functions II, Comp. Math., **64** (1987), 353–373.

[3] G. J. Heckman and E. M. Opdam, Root systems and hypergeometric functions I, Comp. Math., **64** (1987), 329–352.

[4] A. Matsuo, Integrable connections related to zonal spherical functions, Invent. Math., **110** (1992), 95–121.

[5] E. M. Opdam, Root systems and hypergeometric functions III, Comp. Math., **67** (1988), 21–49.

[6] E. M. Opdam, Root systems and Hypergeometric functions IV, Comp. Math., **67** (1988), 191–207.

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